Arch. Rational Mech. Anal. 246 (2022) 561–602
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-022-01821-0

Cut Locus on Compact Manifolds and Uniform Semiconcavity Estimates for a Variational Inequality

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Communicated by F. Lin

Abstract

We study a family of gradient obstacle problems on a compact Riemannian manifold. We prove that the solutions of these free boundary problems are uniformly semiconcave and, as a consequence, we obtain some fine convergence results for the solutions and their free boundaries. More precisely, we show that the elastic and the λ-elastic sets of the solutions Hausdorff converge to the cut locus and the λ-cut locus of the manifold.

1. Introduction

Let M be a smooth n-dimensional compact Riemannian manifold without boundary. Let $b \in M$ be a fixed point. We denote by $d_b : M \to \mathbb{R}$ the distance function to b, and by $\mathrm{Cut}_b(M)$ the *cut locus*, that is the set of points (*cut points*) $p \in M$ for which there exists a geodesic γ, starting from b and passing through p, which is length minimizing between b and p, but not after p. The cut locus inherits much of the topology of M. This is a deformation retract of $M \setminus \{b\}$ and has the same homotopy type (see for instance [25, Chapter III, Section 4]). Moreover, it is also related to the global geometry of M, for instance, to the geodesic spectrum (every close geodesics starting from b crosses $\mathrm{Cut}_b(M)$) and the Ambrose's problem (see [17]).

The local structure of the cut locus can be very rich and at the same time complicated, as it seems to be closely related to the regularity of g. A stratification theorem is available only when the metric g is analytic (see [22] and [5]), while in general, it is known that $\mathrm{Cut}_b(M)$ must have an integer Hausdorff dimension (when g is C^∞) that might even become fractional when g is C^k (see [18] and the references therein). The sensitivity with respect to the regularity of the manifold (M, g) makes the cut locus difficult to recover by numerical methods involving discrete structures. A more stable object from this point of view is the so-called λ-*cut locus* $\mathrm{Cut}_b^\lambda(M)$, which we introduce in this paper in analogy with the λ-medial axis of Chazal and Lieutier, which is a widely studied object in Computational Geometry (see Section 1.1). We refer to [12] for a detailed account on the impact of our study to the numerical methods for the computation of the cut locus.

For any $\lambda > 0$, the λ-cut locus is defined as

$$\mathrm{Cut}_b^\lambda(M) := \left\{ p \in M \setminus \{b\} : |\nabla d_b(p)|^2 \leqq 1 - \frac{\lambda^2}{d_b^2(p)} \right\}, \tag{1.1}$$

the norm of the generalized gradient $|\nabla d_b|$ being defined at every point $p \in M \setminus \{b\}$ as

$$|\nabla d_b|(p) := \max \big\{ 0, \sup_{v \in T_x M, |v|=1} \partial_v^+ d_b(p) \big\}, \tag{1.2}$$

where $\partial_v^+ d_b(p)$ is the derivative of d_b in the direction v (see Section 2). The λ-cut locus approximates the cut locus in the following sense: for every $\lambda > 0$, we have $\mathrm{Cut}_b^\lambda(M) \subset \mathrm{Cut}_b(M)$, while the closure of the union of $\mathrm{Cut}_b^\lambda(M)$ over $\lambda > 0$ is precisely $\mathrm{Cut}_b(M)$ (see Proposition 2.9). In particular, just like the cut locus, the λ-cut locus is a non-smooth set, with a potentially very wild structure, even when M is smooth.

In this paper we study the asymptotic behavior of a family of gradient obstacle problems on the manifold M and we prove that both $\mathrm{Cut}_b(M)$ and $\mathrm{Cut}_b^\lambda(M)$ can be recovered from the solutions of these problems. Moreover, even if our study is purely theoretical, it leads to a new method for the numerical approximation of the cut locus and the λ-cut locus on a compact manifold (see Remark 1.2).

For any $m > 0$, we consider the variational minimization problem

$$\min \left\{ \int_M |\nabla u|^2 - mu \ : \ u \in H^1(M),\ |\nabla u| \leqq 1,\ u(b) = 0 \right\}. \tag{1.3}$$

This problem has a unique minimizer, which we will denote by u_m. We consider the sets

$$E_m := \{ p \in M \setminus \{b\} : |\nabla u_m(p)| < 1 \},$$
$$\text{and} \quad E_{m,\lambda} := \left\{ p \in M \setminus \{b\} : |\nabla u_m(p)|^2 \leqq 1 - \frac{\lambda^2}{u_m^2(p)} \right\}. \tag{1.4}$$

Our main result is the following:

Theorem 1.1. (Approximation of $Cut_b(M)$ and $Cut_b^\lambda(M)$) *Let M be a compact Riemannian manifold of dimension n and let $b \in M$ and $\lambda > 0$ be fixed. Then,*

$$E_m \underset{m \to +\infty}{\longrightarrow} Cut_b(M) \quad \text{in the Hausdorff sense.} \tag{1.5}$$

Moreover, for any fixed $\varepsilon > 0$, we have that

$$\sup_{p \in E_{m,\lambda}} d\big(p,\ Cut_b^\lambda(M)\big) \underset{m \to +\infty}{\longrightarrow} 0, \quad \text{and} \quad \sup_{p \in Cut_b^{\lambda+\varepsilon}(M)} d(p, E_{m,\lambda}) \underset{m \to +\infty}{\longrightarrow} 0 \tag{1.6}$$

Remark 1.2. (About the numerical computation of the cut locus) We notice that the direct numerical approximation of the cut locus and the λ-cut locus is difficult and requires significant computational resources. Conversely, the variational problem (1.3) consists in minimizing a convex functional under a convex constraint, which considerably simplifies this task. The numerical approach based on solving (1.3) is discussed in [12].

In order to prove Theorem 1.1, we have to study the regularity of the solutions u_m and the convergence of the sequence $(u_m)_m$ as $m \to \infty$. We gather our main results about the solutions of (1.3) in the following theorem, and we notice that Theorem 1.1 is in fact an immediate consequence of the claims (T5) and (T6) of Theorem 1.3 below (see Section 1.3):

Theorem 1.3. *(Regularity and convergence of u_m) Let M be a compact Riemannian manifold of dimension n and let $b \in M$ be fixed. Then, the following holds:*

(T1) **Regularity of u_m.** *There exists a constant $m_0 > 0$, depending only on the manifold M, such that for every $m > m_0$, the minimizer u_m of (1.3) is locally $C^{1,1}$ on $M \setminus \{b\}$.*

(T2) **Properties of E_m.** *For every $m \geq m_0$, E_m is an open subset of M and coincides with the set $\{u_m < d_b\}$. Moreover, E_m contains $Cut_b(M)$ and is at positive distance from b, that is $u_m = d_b$ in a neighborhood of b.*

(T3) **Monotonicity of u_m and E_m.** *For every $m \geq m' \geq m_0$, we have $u_m \geq u'_m$. In particular, $E_m \subset E_{m'}$.*

(T4) **Semiconcavity of u_m.** *For every $\rho > 0$, there are constants $C > 0$ and $m_1 > 0$, depending on ρ and on the manifold M, such that*

$$u_m \text{ is } C-\text{semiconcave on} M \setminus B_\rho(b), \tag{1.7}$$

for every $m \geq m_1$.

(T5) **Convergence of u_m.** *The sequence u_m converges uniformly on M to the distance function d_b.*

(T6) **Convergence of the gradients.** *Let $p_\infty \in M \setminus \{b\}$. Then*

- *for every sequence $p_m \to p_\infty$, we have*

$$|\nabla d_b|(p_\infty) \leq \liminf_{m\to\infty} |\nabla u_m|(p_m)\,; \tag{1.8}$$

- *there exists a sequence $p_m \to p_\infty$ such that*

$$|\nabla d_b|(p_\infty) = \lim_{m\to\infty} |\nabla u_m|(p_m)\,. \tag{1.9}$$

Remark 1.4. The semiconcavity of u_m (T4) and the convergence of the gradients (T6) are the most technical part of the proof and are precisely the properties that allow to approximate the λ-cut locus with the sets $E_{m,\lambda}$.

Remark 1.5. If we replace the manifold M with a smooth domain $\Omega \subset \mathbb{R}^n$ and d_b with the distance to the boundary of Ω, the problem (1.3) becomes the classical elastic-plastic torsion problem, which we discuss in detail in Section 1.1. We notice that, for this problem, the claims (T1), (T2), (T3) and (T5) are well-known. The

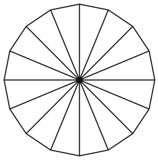

Fig. 1. A polygonal approximation of a circle, with its medial axis

elastic-plastic torsion problem has a long history and inspired the study of numerous problems involving more general (even fully nonlinear) operators. The crucial point in all these problems is that the gradient constraint in (1.3) can be transformed into an obstacle constraint on the function (see Section 1.1). Until now, this property was exclusive for the Euclidean setting and for operators depending only on ∇u and u, but not on the points $x \in \Omega$ (in fact, for operators with variable coefficients, this equivalence is known to be false). A consequence of our analysis is that this crucial equivalence is not exclusively Euclidean but is a property of the underlying Riemannian structure of the manifold (see Proposition 1.8).

The rest of the introduction is organized as follows: in Section 1.1 we will discuss the relation of the λ-cut locus and the problem (1.3) to the λ-medial axis of Chazal-Lieutier and the classical elastic-plastic torsion problem. In Section 1.2 we will discuss the key points in the proof of Theorem 1.3 and the plan of the paper.

1.1. Medial axis and λ-medial axis in a domain Ω

This section is dedicated to the Euclidean counterpart of Theorem 1.1. We go through the definitions of the medial axis and the λ-medial axis of a domain in the euclidean space. Then, we discuss the approximation theorem of Caffarelli and Friedman and its relation to Theorem 1.1. Throughout this section, we will use the following notation: Ω is a bounded open set with C^2 regular boundary in $\mathbb{R}^n$ and $d_{\partial\Omega} : \Omega \to \mathbb{R}$ is the distance function to the boundary of Ω,

$$d_{\partial\Omega}(x) := \min \big\{ |x-y| \ : \ y \in \partial\Omega \big\}.$$

1.1.1. Definition of medial axis and λ-medial axis The *medial axis* $\mathcal{M}(\Omega)$ is defined as the set of points of Ω with at least two different projections on the boundary $\partial\Omega$,

$$\mathcal{M}(\Omega) := \Big\{ x \in \Omega \ : \ \exists y, z \in \partial\Omega,\ \text{such that}\ \ y \neq z\ \ \text{and}\ d_{\partial\Omega}(x) = |x-y| = |x-z| \Big\}.$$

One crucial geometric property of the medial axis $\mathcal{M}(\Omega)$ is that it is unstable with respect to small perturbations of the boundary of Ω. For instance, the medial axis of the circle consists of its center only, while the medial axis of a polygonal approximation (the regularity of the approximating sets can be improved to C^∞ by rounding the corners) is the star-shaped set on Fig. 1. We refer to [2] for a detailed account on medial axis, stability and computability. This instability makes computing numerically $\mathcal{M}(\Omega)$ quite tricky. Indeed, any numerical approximation of Ω (for instance, with polygons) might introduce an artificial (and large) medial set. In order to deal with this problem, in [11], Chazal and Lieutier defined the so called λ-*medial axis* of Ω by setting, for any $\lambda > 0$,

$$\mathcal{M}_\lambda(\Omega) := \{x \in \Omega : r(x) \geqq \lambda\}, \tag{1.10}$$

where $r(x)$ is the radius of the smallest ball containing all the projections of x on the boundary $\partial\Omega$, *i.e.* the set $\{z \in \partial\Omega : |x - z| = d_{\partial\Omega}(x)\}$. It is known that, for λ small enough, $\mathcal{M}_\lambda(\Omega)$ has the same homotopy type as $\mathcal{M}(\Omega)$ (see [11, section 3, theorem 2]) and that

$$\mathcal{M}(\Omega) = \bigcup_{\lambda>0} \mathcal{M}_\lambda(\Omega).$$

These facts justify the that $\mathcal{M}_\lambda(\Omega)$ is a good approximation of $\mathcal{M}(\Omega)$, for λ small enough. The crucial difference though is that $\mathcal{M}_\lambda(\Omega)$ is stable with respect to small variations of Ω, whereas $\mathcal{M}(\Omega)$ is not (we refer to [11, section 4] for precise statements and proofs). Finally, we notice that the λ-medial axis can be equivalently defined (see [11, section 2.1]) as

$$\mathcal{M}_\lambda(\Omega) = \left\{x \in \Omega : |\nabla d_{\partial\Omega}(x)|^2 \leqq 1 - \frac{\lambda^2}{d^2_{\partial\Omega}(x)}\right\}, \tag{1.11}$$

where $\nabla d_{\partial\Omega}$ denotes the generalized gradient wherever $d_{\partial\Omega}$ is not differentiable.

1.1.2. Approximation of the medial axis Given a constant $m > 0$ and a domain Ω, as above, we consider the following *elastic-plastic torsion problem*

$$\min\left\{\int_\Omega \Big(|\nabla v|^2 - mv\Big)\,dx \;:\; v \in H^1_0(\Omega),\; |\nabla v| \leqq 1\right\}. \tag{1.12}$$

As in the case of (1.3), the problem (1.12) has a unique minimizer, which we will denote by v_m. Physically speaking, v_m represents the stress function of a long bar of cross section Ω, twisted with an angle m. The elastic-plastic torsion problem and the properties of its minimizer v_m have been studied by various authors in the 60's and 70's (see for instance [3,4,8,9,14,26,27] and [7]). In particular, in [4], Brezis and Sibony proved that the gradient constraint in (1.12) can be replaced with an obstacle-type constraint on the function. Precisely, the minimizer v_m of (1.12) is also the (unique) minimizer of

$$\min\left\{\int_\Omega \Big(|\nabla v|^2 - mv\Big)\,dx \;:\; v \in H^1_0(\Omega),\; v \leqq d_{\partial\Omega}\right\}. \tag{1.13}$$

Notice that this result was later generalized to a broader class of variational problems with convex constraints on the gradient (see [21,28] and [24]). However, none of these will apply to our variant of the problem on manifolds, for which the equivalence of constraints fails in general (see Section Appendix B).

Finally, using the equivalence of (1.12) and (1.13), Caffarelli and Friedman (see [6]) proved that the sequence of *elastic sets* $\{|\nabla v_m| < 1\}$ Hausdorff converges, as $m \to +\infty$, to the medial axis $\mathcal{M}(\Omega)$. To be precise, in [6], it was showed that the elastic sets converge to the so-called *ridge* $\mathcal{R}(\Omega)$ which coincides with the closure of $\mathcal{M}(\Omega)$, when Ω has a C^2 regular boundary. This result from [6] is the euclidean counterpart of the first part of Theorem 1.1. Nevertheless, the strategies from [4] and [6] cannot be reproduced on a manifold and do not imply the convergence of the λ-medial axis. In the proof of our Theorem 1.1, we still aim at replacing the constraint on the gradient with a constraint on the function, but our approach is different and allows us to deal with the presence of the manifold and to treat both the cut locus and the λ-cut locus. In particular, we obtain the following approximation result for the λ-medial axis:

Theorem 1.6. (Approximation of $\mathcal{M}_\lambda(\Omega)$) *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^2 regular boundary. Then, setting*

$$E_m^\Omega = \left\{x \in \Omega : |\nabla v_m(x)| < 1\right\} \text{ and } E_{m,\lambda}^\Omega = \left\{x \in \Omega : |\nabla v_m(x)|^2 \leqq 1 - \frac{\lambda^2}{v_m^2(x)}\right\},$$

we have that, for any fixed $\varepsilon > 0$,

$$\sup_{x \in E_{m,\lambda}^\Omega} d\big(x, \mathcal{M}_\lambda(\Omega)\big) \underset{m \to +\infty}{\longrightarrow} 0, \text{ and } \sup_{x \in \mathcal{M}_{\lambda+\varepsilon}(\Omega)} d(x, E_{m,\lambda}^\Omega) \underset{m \to +\infty}{\longrightarrow} 0. \tag{1.14}$$

Remark 1.7. We do not exclude that the convergence rates in (1.6) and (1.14) can be improved; for instance, it is natural to expect that there is a modulus of continuity $f : [0, +\infty) \to [0, +\infty)$ for which

$$\sup_{x \in \mathcal{M}_{f(\lambda)}(\Omega)} d(x, E_{m,\lambda}^\Omega) \underset{m \to +\infty}{\longrightarrow} 0.$$

1.2. Proof of Theorem 1.3 and plan of the paper

We consider the variational problem

$$\min\left\{\int_M |\nabla u|^2 - mu \ : \ u \in H^1(M),\ u \leqq d_b\right\}. \tag{1.15}$$

We can immediately check that (1.15) admits a minimizer and that this minimizer is unique (this follows by the convexity of the functional and the constraint). We will denote by $u_m^d : M \to \mathbb{R}$ ('d' stands for the 'distance' constraint) the unique minimizer of (1.15).

1.2.1. Part I. Equivalence of (1.3) and (1.15) Our first aim is to show that the problems (1.3) and (1.15) are equivalent, that is the minimizers u_m and u_m^d are the same. Now, since every function which is 1-Lipschitz and is zero in b stands below the distance function b, it is clear that u_m can be used to test the optimality of u_m^d, that is, we have

$$\int_M \big(|\nabla u_m^d|^2 - m u_m^d\big) \le \int_M \big(|\nabla u_m|^2 - m u_m\big).$$

Notice that, if we are able to prove that the minimizer u_m^d is 1-Lipschitz, then we can use u_m^d to test the minimality of u_m, i.e.

$$\int_M \big(|\nabla u_m^d|^2 - m u_m^d\big) \ge \int_M \big(|\nabla u_m|^2 - m u_m\big).$$

This gives that both u_m and u_m^d are solutions of (1.3) (and also of (1.15)), which means that they have to coincide. Thus, in order to prove that (1.3) and (1.15) are equivalent, we have to prove that

$$|\nabla u_m^d| \le 1 \quad \text{on} \quad M. \tag{1.16}$$

In order to prove this, we proceed as follows:

- First, we prove that u_m^d is C^1-regular locally in $M \setminus \{b\}$ (see Proposition 3.4).
- Then, from Lemma 3.3 and Lemma 3.1, we deduce that

  $$\mathrm{Cut}_b(M) \subset \{u_m^d < d_b\} \subset M \setminus \{b\}.$$

  In particular, since d_b is smooth away from $\{b\}$ and $\mathrm{Cut}_b(M)$, we get that on the boundary $\partial\{u_m^d < d_b\}$ both the distance function d_b and the solution u_m^d are differentiable and have the same gradient, which entails that $|\nabla u_m^d| = 1$ on $\partial\{u_m^d < d_b\}$.
- Finally, we use the fact that u_m^d solves the PDE

  $$\Delta u_m^d = m \quad \text{in} \quad \{u_m^d < d_b\}, \qquad |\nabla u_m^d| = 1 \quad \text{on} \quad \{u_m^d = d_b\}$$

  to deduce that $|\nabla u_m^d| \le 1$ also in the set $\{u_m^d < d_b\}$. Now, in the flat (Euclidean) case, this inequality is an immediate consequence of the fact that $|\nabla u_m^d|^2$ is subharmonic. On a general manifold M the situation is more complicated as the curvature comes into play in the computation of $\Delta\big(|\nabla u_m^d|^2\big)$. For this reason we are able to prove the bound $|\nabla u_m^d| \le 1$ on M (and so the equivalence of the two problems) only in the case when m is large enough. Before we give the precise statement of this result (see Proposition 1.8), let us emphasize that this is not a mere technical assumption, but a consequence of the geometry of the manifold. In fact, in the appendix (Theorem B.1), we give an example of a 2-manifold M for which the bound on the gradient fails when m is small.

The following is the main result of this first part of the paper (the proof is given in Section 4):

Proposition 1.8. (Equivalence of (1.3) and (1.15)) *Let M be an n-dimensional compact Riemannian manifold and let the constant $K \geqq 0$ be a lower bound for the Ricci curvature*

$$\mathrm{Ric} \geqq -K, \tag{1.17}$$

where Ric *denotes the Ricci curvature tensor of M. Then, for every*

$$m \geqq \frac{1}{2} \max \left\{ \sqrt{nK(1 + K\operatorname{diam}(M)^2)},\ nK\operatorname{diam}(M) \right\}, \tag{1.18}$$

we have that

$$\left|\nabla u_m^d\right| = 1 \quad \text{on} \quad \{d_b = u_m^d\}, \qquad \text{and} \qquad \left|\nabla u_m^d\right| < 1 \quad \text{in} \quad E_m^d := \{u_m^d < d_b\}. \tag{1.19}$$

In particular, for m as in (1.18), we have that $u_m^d = u_m$, where u_m is the minimizer of (1.3).

Finally, as a corollary of Proposition 1.8, we obtain the first two claims of Theorem 1.3.

Proof of Theorem 1.3 (T1) and (T2). By Proposition 1.8 we have that $u_m = u_m^d$. From the regularity of u_m^d (Proposition 3.4, Lemma 3.3 and Lemma 3.1), we obtain (T1) and (T2). □

Moreover, as in the classical case of the elastic-plastic torsion problem (see [6]), we can now use the structure of (1.15) to obtain information about the monotonicity of E_m and the uniform convergence of u_m.

Proof of Theorem 1.3 (T3) and (T5). The uniform convergence $u_m^d \to d_b$ on M, as $m \to \infty$, is proved in Lemma 5.1. The monotonicity of u_m and E_m, and the Hausdorff convergence of E_m to $\mathrm{Cut}_b(M)$, now follow from Proposition 5.2. □

1.2.2. Part II: Uniform semiconcavity and convergence of the gradients We recall that our final objective is to prove the convergence of the sets $E_{m,\lambda}$ (Theorem 1.1) and $E_{m,\lambda}^{\Omega}$ (Theorem 1.6). Now, from the definition of $E_{m,\lambda}$, it is clear that this boils down to proving a convergence result for the gradients $|\nabla u_m|$. On the other hand, we cannot expect any uniform estimate on the modulus of continuity of $|\nabla u_m|$; in fact, the sequence u_m converges (uniformly) to the distance function d_b, which is not even differentiable at all points. Thus, we adopt a different strategy and we prove that the solutions are uniformly semiconcave, where our definition of semiconcavity is the following:

Definition 1.9. (C-semiconcavity) Given a constant $C > 0$, a function u is said to be C-semiconcave on M if and only if for any unit speed geodesic $\gamma : [a, b] \to M$, the function $t \mapsto Ct^2 - u(\gamma(t))$ is convex. Moreover,

- we say that u is semiconcave if it is C-semiconcave for some constant $C > 0$;
- we say that u is locally semiconcave if for any $p \in M$, u is semiconcave in a neighborhood of p.

The main result of the paper is Theorem 1.3 (T4), which we prove in Section 6. The key result is Proposition 6.1 and applies to both Theorem 1.3 and Theorem 1.6. Let us briefly give the idea of the proof of this proposition here, directly in the setting of Theorem 1.3 (T4).

Sketch of the proof of Theorem 1.3 (T4) First, we fix a constant C_d such that the distance function d_b is C_d-semiconcave on $M \setminus B_\rho(b)$. Then, for every unit speed geodesic $\gamma : [a, b] \to M$, and every $\lambda \in [0, 1]$, we define the function

$$\begin{aligned} &c(\gamma, \lambda)\\ &:= \lambda(1-\lambda)(C_d+1)(b-a)^2\\ &\quad - \Big((1-\lambda)u_m(\gamma(a)) + \lambda u_m(\gamma(b)) - u_m(\gamma(\lambda_{ab}))\Big), \end{aligned}$$

where $\lambda_{ab} = (1-\lambda)a + \lambda b$. We will show that the minimum of this function over all geodesics γ and all λ is positive, which will give that u is (C_d+1)-semiconcave. First, we show that for any unit speed geodesic γ and $\lambda \in (0, 1)$, we can build a unit speed geodesic $\widehat{\gamma} : [a, b] \to M$ and $\widehat{\lambda} \in (0, 1)$, such that

$$c(\widehat{\gamma}, \widehat{\lambda}, u_m) \leqq c(\gamma, \lambda, u_m) \quad \text{and} \quad \widehat{\gamma}(a, b) \subset E_m = \{u_m < d_b\}.$$

This follows from the semiconcavity of d_b and the inequality $u_m \leq d_b$ (this is explained in detail in the proof of Proposition 6.1). Thus, we only need to show the semiconcavity of u_m in the non-contact region E_m. Since u_m is smooth in E_m, we need to prove that (see Proposition 2.2)

$$D^2 u_m \leqq (C_d + 1) Id \quad \text{in} \quad E_m.$$

In order to prove this inequality, for every $p \in E_m$ and $X \in \mathbb{S}^{n-1}(T_p M)$ we consider an auxiliary function of the form

$$f_\varepsilon(p, X) := D^2 u_m(X, X) + \varepsilon \left(C_1 \left|\nabla u_m\right|^2(p) + C_2 u_m^2(p) - C_3 u_m(p) \right),$$

and we show that for $\varepsilon > 0$ small enough and m large enough, we have $f_\varepsilon \leqq C_d + 1/2$. We suppose that the maximum of f_ε is achieved for some $q \in E_m$ and some $Y \in \mathbb{S}^{n-1}(T_q M)$ (the case when the minimum is achieved for $q \in \partial E_m$ is a consequence of known estimates for the solutions of the obstacle problem with variable coefficients, see Section 7). Then, we construct, locally around q, a function of the form

$$p \mapsto f_\varepsilon(p, X(p)) \quad \text{where} \quad X(p) \in \mathbb{S}^{n-1}(T_p M),$$

and we compute its Laplacian in the variable p (notice that in the flat euclidean case we can simply take the section $p \mapsto X(p)$ to be constant). Finally, we obtain that for an appropriate choice of ε and m, the Laplacian of this function has to be positive, which contradicts the minimality of q and concludes the proof. □

The main part of the proof of Theorem 1.3 (T4) is contained in Proposition 6.1, which applies to both Theorem 1.3 and Theorem 1.6. In the proof of Proposition 6.1, the function c is the Riemannian counterpart of the Korevaar's convexity function (see [19]); in computing the Laplacian of $f_\varepsilon(p, X(p))$ we use some of Guan's second order estimates for Hessian equations in Riemannian manifolds (see [15]).

At this point, the convergence of the gradients $|\nabla u_m|$ (Theorem 1.3 (T6)) follows from the uniform semiconcavity of u_m by a general argument (we give the proof of this fact in Section 7). We are now in position to prove Theorem 1.1.

1.3. Proof of Theorem 1.1

The Hausdorff convergence of the elastic sets E_m to $\mathrm{Cut}_b(M)$ is a consequence of the uniform convergence (Theorem 1.3 (T5)) of the solutions u_m to the distance function d_b, as explained in Proposition 5.2. Let us now prove the first claim in (1.6). Suppose by contradiction that there are a constant $\delta > 0$, a sequence $m_k \to \infty$ and a sequence of points p_k such that

$$p_k \in E_{m_k,\lambda} \quad \text{and} \quad d\big(p_k,\ \mathrm{Cut}_b^\lambda(M)\big) > \delta. \tag{1.20}$$

By the facts that M is compact and that u_{m_k} coincides with the distance function d_b in a neighborhood of b (that does not depend on k), we may suppose that p_k converges to some $p_\infty \in M \setminus \{b\}$. Now, from the uniform convergence of u_{m_k} and Theorem 1.3 (T6), we get that

$$|\nabla d_b|(p_\infty) \le \liminf_{k\to\infty} |\nabla u_{m_k}|(p_k) \le \lim_{k\to\infty} \left(1 - \frac{\lambda^2}{u_{m_k}^2(p_k)}\right) = 1 - \frac{\lambda^2}{d_b^2(p_\infty)},$$

which means that $p_\infty \in \mathrm{Cut}_b^\lambda(M)$, in contradiction with (1.20).

Suppose now that the second claim in (1.6) does not hold. Then, there are a constant $\delta > 0$, a sequence $m_k \to \infty$ and a sequence of points $p_k \in M \in \{b\}$ such that

$$p_k \in \mathrm{Cut}_b^{\lambda+\varepsilon}(M) \qquad \text{and} \qquad d\big(p_k, E_{m_k,\lambda}\big) > \delta \quad \text{for every} \quad k \ge 0.$$

Up to extracting a subsequence, we may suppose that p_k converges to a point p_∞ such that

$$p_\infty \in \mathrm{Cut}_b^{\lambda+\varepsilon}(M) \qquad \text{and} \qquad d\big(p_\infty, E_{m_k,\lambda}\big) > \frac{\delta}{2} \quad \text{for every} \quad k \ge 0. \tag{1.21}$$

Now, by Theorem 1.3 (T6), there is a sequence $q_k \to p_\infty$ such that

$$|\nabla d_b|(p_\infty) = \lim_{k\to\infty} |\nabla u_{m_k}|(q_k).$$

In particular, since $p_\infty \in \mathrm{Cut}_b^{\lambda+\varepsilon}(M)$, we have that

$$\lim_{k\to\infty}\left(|\nabla u_{m_k}|(q_k) - 1 + \frac{\lambda^2}{u_{m_k}^2(q_k)}\right) = |\nabla d_b|(p_\infty) - 1 + \frac{\lambda^2}{d_b^2(p_\infty)} \le -\frac{2\varepsilon\lambda + \varepsilon^2}{d_b^2(p_\infty)}.$$

Thus, the left-hand side is negative for k large enough and so, we have that $q_k \in E_{m_k,\lambda}$, which stands in contradiction with (1.21). This concludes the proof of Theorem 1.1. $\square$

1.4. Proof of Theorem 1.6

As shown in Section 6, we may apply Proposition 6.1 to get that the functions v_m are uniformly semiconcave on Ω. It is already known that the solution v_m of (1.12) and (1.13) is locally $C^{1,1}$ on Ω. It is also well-known that v_m converges uniformly to $d_{\partial\Omega}$ as $m \to \infty$. As a consequence, reasoning as in Section 7, we get that, for every $x_\infty \in \Omega$, the following holds:

- if $x_m \to x_\infty$, then $|\nabla d_{\partial\Omega}|(x_\infty) \leq \liminf_{m\to\infty} |\nabla v_m|(x_m)\,;$
- there exists a sequence $x_m \to x_\infty$ such that $|\nabla d_{\partial\Omega}|(x_\infty) = \lim_{m\to\infty} |\nabla v_m|(x_m)\,.$

Now, the conclusion follows as in the proof of Theorem 1.1. □

2. Notation, Definitions and Preliminary Results

2.1. General notation

We will denote by g the metric on M. TM denotes the tangent bundle of M and T_pM the tangent space of M at p. By $\mathbb{S}^{n-1}(T_pM)$ we will denote the unit sphere in T_pM, that is,

$$\mathbb{S}^{n-1}(T_pM) := \big\{X \in T_pM \ : \ g(X, X) = 1\big\}.$$

$\mathrm{Exp} : TM \to M$ is the global exponential map, while $\exp_p$ is its restriction to T_pM. Finally, given a function u on M, Du is the differential of u, ∇u is the gradient, and $D^k u$ is the k-th covariant derivative (in particular, by D we denote also the Riemannian connection on M). Thus, for smooth vector fields $X, Y : M \to TM$, we have

$$g(\nabla u, X) := Du(X) = D_X u = Xu \qquad \text{and} \qquad D^2u(X, Y) = g(D_X(\nabla u), Y).$$

We will also use the notation $|\nabla u|^2$ for $g(\nabla u, \nabla u)$, and Δu for the Laplace-Beltrami operator on M. We notice that $-\Delta$ is positive, that is, we have the integration by parts formula

$$\int_M g(\nabla u, \nabla v) = \int_M (-\Delta u)v$$

for every $u, v \in C^2(M)$. Unless otherwise specified, all the integrals will be taken with respect to the volume form associated to the Riemannian metric g. Finally, we recall that $H^1(M)$ denotes the usual space of Sobolev functions on M, which is the closure of $C^1(M)$ with respect to the H^1-norm defined as

$$\|u\|_{H^1}^2 = \int_M |\nabla u|^2 + \int_M u^2.$$

2.2. Semiconcave functions

In this section, we gather some of the main properties of semiconcave functions on smooth Riemannian manifolds, which we will need in the proof of Theorem 1.3. Some of these results can be found in [23], in the context of Alexandrov spaces, while for a more detailed introduction to semiconcave functions in the framework of euclidean spaces we refer to [10].

Let M be a Riemannian manifold, $u : M \to \mathbb{R}$ a given function and $\gamma : [a,b] \to M$ be a curve in M. It is immediate to check that the function

$$t \mapsto Ct^2 - u(\gamma(t))$$

is convex on $[a,b]$ if and only if

$$\begin{aligned}(1-\lambda)u(\gamma(a)) + \lambda u(\gamma(b)) - u(\gamma(\lambda_{ab}))&\\ \leqq C\lambda(1-\lambda)(b-a)^2 \quad \text{for any} \quad \lambda \in [0,1],&\end{aligned} \tag{2.1}$$

where here and throughout the paper, we use the notation

$$\lambda_{ab} := (1-\lambda)a + \lambda b \qquad \text{for any} \qquad a, b, \lambda \in \mathbb{R}. \tag{2.2}$$

In particular, this means that the function u is C-semiconcave on M if and only if (2.1) holds for any unit speed geodesic $\gamma : [a,b] \to M$. Analogously, u is locally semiconcave if for every $p \in M$ there is a geodesic ball $B_\rho(p)$ and a constant $C_p > 0$ such that (2.1) holds (with $C = C_p$) for every unit speed geodesic $\gamma : [a,b] \to B_\rho(p)$.

Remark 2.1. On a compact Riemannian manifold, semiconcavity and local semiconcavity are the same.

Proposition 2.2. (Semiconcavity in terms of D^2u) *Let $u : M \to \mathbb{R}$ be C^2-regular. Then*

$$D^2u \leqq 2C \ \text{ on } \ M \qquad \text{if and only if} \qquad u \text{ is } C\text{-semiconcave on } M.$$

Proof. Let $\gamma : [a,b] \to M$ be a unit speed geodesic. Then the function $t \mapsto Ct^2 - u(\gamma(t))$ is convex if and only if

$$\begin{aligned}0 \leqq 2C - \frac{d^2}{dt^2}u(\gamma(t)) &= 2C - \frac{d}{dt}Du(\dot\gamma(t))\\ &= 2C - \Big(D^2u(\dot\gamma(t),\dot\gamma(t)) + Du(D_{\dot\gamma(t)}\dot\gamma(t))\Big)\\ &= 2C - D^2u\big(\dot\gamma(t),\dot\gamma(t)\big).\end{aligned}$$

The claim follows. □

The semiconcavity can also be read in local coordinates as follows:

Proposition 2.3. (Semiconcavity in local coordinates) *Let $u : M \to \mathbb{R}$ be a locally Lipschitz function on a Riemannian manifold M. Then, u is locally semiconcave if and only if for any chart ψ of M, $u \circ \psi^{-1}$ is locally semiconcave as a function on $\mathbb{R}^n$.*

We postpone the proof of this proposition to Appendix A. We next show that we can define the gradient of a semiconcave function at every point.

Proposition 2.4. (The generalized gradient of a semiconcave function) *Let $u : M \to \mathbb{R}$ be a locally Lipschitz and semiconcave function. Then, at every point $p \in M$, u admits a directional derivative $\partial_v^+ u(p)$ in any direction $v \in T_pM \setminus \{0\}$; it is defined by*

$$\partial_v^+ u(p) := \frac{\mathrm{d}}{\mathrm{d}t}[u(\gamma(t))]_{t=0} = \lim_{t\to 0^+} \frac{u(\gamma(t)) - u(p)}{t},$$

where $\gamma : [0, 1] \to M$ is any curve such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Moreover, the map $v \mapsto \partial_v^+ u(p)$ is 1-homogeneous and concave on T_pM. Thus, it attains a unique maximum in the closed unit ball of T_pM at a unique vector v_p.

Proof. By Proposition 2.3, we can suppose that $M = \mathbb{R}^n$, $p = 0$ and that $\gamma(t) = tv$. Then the function $w(x) = C|x|^2 - u(x)$ is convex for C large enough and so, the function $t \mapsto \frac{w(\gamma(t))-w(0)}{t}$ is non-decreasing in t, so the limit $\partial_v^+ u(p) = -\partial_v^+ w(0)$ exists and is finite. The convexity of the function $v \mapsto \partial_v^+ w(0)$ is a consequence from the convexity of w. The existence of a maximum of $v \mapsto \partial_v^+ u(p)$ follows. □

If $\partial_{v_p}^+ u(p) > 0$, then the 1-homogeneity implies that v_p has norm one, and we define

$$\nabla u(p) := \partial_{v_p}^+ u(p) v_p \qquad \text{and} \qquad |\nabla u(p)| = \partial_{v_p}^+ u(p).$$

If $\partial_{v_p}^+ u(p) = 0$, then we set $\nabla u(p) = 0$. Thus, the norm of $\nabla u(p)$ is given by the following formula:

$$|\nabla u(p)| = \max\left\{0, \max_{v\in T_pM, |v|=1} \partial_v^+ u(p)\right\}. \tag{2.3}$$

2.3. Distance function, cut locus and cut points

Let M be a compact Riemannian manifold, $b \in M$ and $d_b : M \to \mathbb{R}$ be the distance function to b. Here we recall the definition and some of the main properties of the cut locus.

Definition 2.5. (Cut points) Let $T > 0$ and $\gamma : [0, T] \to M$ be a unit speed geodesic such that $\gamma(0) = b$, $t_0 \in (0, T)$ and $p = \gamma(t_0)$. We say that p is a *cut point* of b along γ if γ is length minimizing between b and p, but not after p, *i.e* $d_b(\gamma(t)) = t$ for $t \leqq t_0$, and $d_b(\gamma(t)) < t$ for $t > t_0$.

Definition 2.6. (Cut locus) The *cut locus* of b in M, $\mathrm{Cut}_b(M)$, is defined as the set of all cut points of b.

The following well-known facts about the cut locus can all be found in [25, Chapter III, Section 4]:

- $\mathrm{Cut}_b(M)$ is the closure of the set of points p in M, for which there are at least two minimizing geodesics connecting b and p;
- the distance function d_b is smooth outside $\mathrm{Cut}_b(M) \cup \{b\}$ and

$$|\nabla d_b| = 1 \quad \text{in} \quad M \setminus \Big(\mathrm{Cut}_b(M) \cup \{b\}\Big)\,;$$

- d_b is differentiable at $p \in M$ if and only if there is a unique minimizing geodesics between b and p;
- in particular, $\mathrm{Cut}_b(M) \cup \{b\}$ is the closure of the set of points of non differentiability of d_b;
- the exponential map $\exp_b : T_bM \to M$ is a diffeomorphism from an open set of T_bM onto $M \setminus \mathrm{Cut}_b(M)$;
- $\mathrm{Cut}_b(M)$ is a deformation retract of $M \setminus \{b\}$. In particular, these two sets have the same homotopy type, and so $\mathrm{Cut}_b(M)$ inherits much of the topology of M (like homology groups, for instance). See [25, Chapter III, Section 4, Proposition 4.5] for a precise statement.

We next recall that in [20, Proposition 3.4], it was proved that, for any chart ψ on $M \setminus \{b\}$, the function $d_b \circ \psi^{-1}$ is locally semiconcave on $\mathbb{R}^n$. Thus, by Proposition 2.3, d_b is locally semiconcave on $M \setminus \{b\}$ in the sense of Definition 1.9. More precisely, we have

Proposition 2.7. (Semiconcavity of the distance function) *Let M be a compact Riemannian manifold of dimension n and $b \in M$ be a given point. Then, for every $\rho > 0$, there is a constant $C > 0$ such that the distance function d_b is C-semiconcave on $M \setminus B_\rho(b)$.*

In particular, by Proposition 2.4, for any point $p \in M \setminus \{b\}$ and any direction $v \in T_pM$, d_b admits the directional derivative $\partial_v^+ d_b(p)$ and so we can define ∇d_b and $|\nabla d_b|$ at every point as in (2.3). In Lemma 2.8 we give a geometric interpretation of $|\nabla d_b|\,(p)$ in terms of the geodesics connecting p to b. We notice that similar results holds also in the more general framework of Alexandrov spaces, but with some additional restrictions on the curvature of the ambient space (see [1, Theorem 4.5.6] and also [1, Lemma 3.2] for the statement in the Riemannian context). We give the proof directly for the distance function to a compact subset K of M.

Lemma 2.8. (Geometric interpretation of the generalized gradient) *Let M be a smooth Riemannian manifold without boundary, K a compact subset of M, and d_K the distance function to K. Let p be a point of M such that there exist several minimizing geodesics from p to K. We denote the set of unit speed geodesics from p to K that are minimizing between p and K by* $\mathrm{geod}(p, K)$. *For any $v \in T_pM$, we have that*

$$\partial_v^+ d_K(p) = \min_{\gamma \in geod(p,K)} -\dot{\gamma}(0) \cdot v. \tag{2.4}$$

In particular,

$$|\nabla d_K|\,(p) = \max\{0, \max_{v \in T_pM, |v|=1} \min_{\gamma \in geod(p,K)} -\dot{\gamma}(0) \cdot v\}. \tag{2.5}$$

In particular, if $\gamma_1 : [0, d_K(p)] \to M$ and $\gamma_2 : [0, d_K(p)] \to M$ are two minimizing geodesics from p to K, then

$$|\nabla d_K(p)| \leq \sqrt{\frac{1 + \dot\gamma_1(0) \cdot \dot\gamma_2(0)}{2}}. \tag{2.6}$$

Proof. Let $\gamma : [0, d_K(p)] \to M$ be a geodesic of $geod(p, K)$. Let $a = \gamma(d_K(p)/2)$. As γ is minimizing between p and $\gamma(d_K(p))$, we have $a \notin Cut_p(M)$, and so $p \notin Cut_a(M)$. In particular, the function d_a is differentiable at p, and $\nabla d_a(p) = -\dot\gamma(0)$. Thus, for every $t > 0$, we have that

$$\begin{aligned}\frac{d_K(\exp_p(tv)) - d_K(p)}{t} &\leqq \frac{d_a(\exp_p(tv)) + d_K(a) - d_K(p)}{t}\\ &= \frac{d_a(\exp_p(tv)) - d_a(p)}{t}\end{aligned}$$

Passing to the limit as $t \to 0$, we get

$$\partial_v^+ d_K(p) \leqq \min_{\gamma \in geod(p,K)} -\dot\gamma(0) \cdot v. \tag{2.7}$$

Now, for every $t > 0$, let $\gamma_t \in geod(\exp_p(tv), K)$. For t small enough, the length of γ_t is bounded by $d_K(p) + 1$. By compactness of the set of geodesics of length bounded by a given constant, there exists a sequence of positive numbers $(t_n)_{n \geqq 0}$ that converges to 0, such that $\gamma_n := \gamma_{t_n}$ converges to a unit speed geodesic γ as $n \to +\infty$. As K is closed, γ is a geodesic from p to K. What is more, we have that

$$\text{length}(\gamma) = \lim_{n\to\infty} \text{length}(\gamma_n) = \lim_{n\to\infty} d_K(\exp_p(t_n v)) = d_K(p),$$

so $\gamma \in geod(p, K)$. Let $R = \min\{\text{inj}(M), d_K(p)/2\}$, where $\text{inj}(M)$ is the injectivity radius of M. In particular for any (x, y) such that $d(x, y) < R$ and $x \neq y$, the distance function $d(\cdot, \cdot)$ is smooth in a neighborhood of (x, y) in $M \times M$. For $n \in \mathbb{N}$, let $b_n := \gamma_n(R)$, and $b_\infty = \gamma(R)$. Let $U, V \subset M$ be precompact neighborhoods of p and b_∞ respectively such that $d(\cdot, \cdot)$ is smooth on $\overline{U} \times \overline{V}$. For n big enough, we have $\exp_p(t_n v) \in U$ and $b_n \in V$, and so

$$\begin{aligned} d_K(p) &\leqq d_K(b_n) + d(b_n, p)\\ &= d_K(\exp_p(t_n v)) - d(b_n, \exp_p(t_n v)) + d(b_n, p)\\ &= d_K(\exp_p(t_n v)) - \nabla_2 d(b_n, p) \cdot v + o(t_n), \end{aligned} \tag{2.8}$$

where ∇_2 is the gradient with respect to the second variable. We have that

$$\nabla_2 d(b_n, p) \xrightarrow[n\to\infty]{} \nabla_2 d(b_\infty, p) = -\dot\gamma(0)$$

because $d(\cdot, \cdot)$ is smooth on $U \times V$. Thus (2.8) yields

$$\liminf_{n\to\infty} \frac{d_K(\exp_p(t_n v)) - d_K(p)}{t_n} \geqq -\dot\gamma(0) \cdot v.$$

In particular,

$$\partial_v^+ d_K(p) \geqq \min_{\gamma \in geod(p,K)} -\dot{\gamma}(0) \cdot v.$$

With (2.7), this concludes the proof of (2.4). Now, (2.5) follows from (2.4) and the definition of the generalized gradient (2.3) of semiconcave functions. Finally, in order to prove (2.6), we consider the vector v that realizes the maximum in (2.5) and we write it as $v = -\alpha\dot{\gamma}_1(0) - \beta\dot{\gamma}_2(0) + v_\perp$, where $v_\perp$ is orthogonal to $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$. Then, we have that

$$-v \cdot \dot{\gamma}_1(0) = \alpha + \beta\dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0) \quad \text{and} \quad -v \cdot \dot{\gamma}_2(0) = \beta + \alpha\dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0).$$

In particular,

$$\begin{aligned} \min_{\gamma \in geod(p,K)} -\dot{\gamma}(0) \cdot v &\leq \frac{1}{2}\Big(-v \cdot \dot{\gamma}_1(0) - v \cdot \dot{\gamma}_2(0)\Big) \\ &\leq \frac{1}{2}(\alpha + \beta)\Big(1 + \dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0)\Big). \end{aligned} \tag{2.9}$$

Now, using the fact that

$$\alpha^2 + \beta^2 + 2\alpha\beta\dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0) \leq \|v\|^2 = 1,$$

we get that

$$(\alpha + \beta)^2 \leq 1 + 2\alpha\beta\Big(1 - \dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0)\Big) \leq 1 + \frac{1}{2}(\alpha + \beta)^2\Big(1 - \dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0)\Big),$$

which implies that

$$(\alpha + \beta)^2 \leq \frac{2}{1 + \dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0)},$$

which, together with (2.9), gives (2.6). □

As a consequence of Lemma 2.8 and in particular of (2.6), we obtain the λ-cut locus approximates the cut locus in the following sense:

Proposition 2.9. *Suppose that M is a compact Riemannian manifold, the point $b \in M$ is fixed and that d_b is the distance function to b. Then, for every $\lambda > 0$, $Cut_b^\lambda(M) \subset Cut_b(M)$. Moreover, the cut locus $Cut_b(M)$ is the closure of the union $\bigcup_{\lambda > 0} Cut_b^\lambda(M)$.*

Proof. The inclusion $\mathrm{Cut}_b^\lambda(M) \subset \mathrm{Cut}_b(M)$ follows from the fact that d_b is differentiable and $|\nabla d_b| = 1$ outside $\mathrm{Cut}_b(M) \cup \{b\}$. In order to prove the second claim, we fix a point $p \in \mathrm{Cut}_b(M)$. Then, there is a sequence of points $p_n \in \mathrm{Cut}_b(M)$ for each of which there are at least two different minimizing geodesics from p_n to b. Now, from (2.6), we have that $p_n \in \mathrm{Cut}_b^{\lambda_n}(M)$ for some $\lambda_n > 0$. This concludes the proof. □

3. Regularity of u_m^d

This section is dedicated to the $C^{1,1}$ regularity of the minimizer u_m^d of (1.15). We recall the following result:

Lemma 3.1. *(Regularization of the obstacle, [16]) For any $m > 0$, there exists a function $\widetilde{d}_b$ which is smooth on $M \setminus \{b\}$, such that*

$$u_m^d \leqq \widetilde{d}_b \leqq d_b \quad on \quad M, \qquad and \qquad \widetilde{d}_b < d_b \quad on \quad Cut_b(M).$$

In particular, u_m^d is also the solution of the obstacle problem

$$\min\left\{\int_M |\nabla u|^2 - mu \ : \ u \in H^1(M),\ u \leqq \widetilde{d}_b\right\}. \tag{3.1}$$

One could adapt to the manifold framework the regularity theorems for the classical obstacle problem on a euclidean domain and, with the preceding lemma, deduce the regularity of u_m^d. Rather than doing that, we will use Lemma 3.1 to reduce our problem to a classical obstacle problem on a euclidean domain. Let us start with the following regularity lemma:

Lemma 3.2. (Continuity of u_d^m) *For any $m > 0$, the function u_m^d is continuous on M.*

Proof. We will reduce our problem to a classical obstacle type variational problem on an open subset of $\mathbb{R}^n$, by a series of elementary modifications, and apply a classical $W^{2,p}$ regularity theorem.

From Lemma 3.1, we know that there exists an open set $U \subset M$ and $\varepsilon > 0$ such that

$$\mathrm{Cut}_b(M) \subset U \qquad \text{and} \qquad u_m^d \leqq d_b - \varepsilon \quad \text{on} \quad U.$$

As a consequence, on the set U, u_m^d verifies the Euler-Lagrange equation of (1.15), *i.e* $\Delta u_m^d = -2m$. In particular, it is C^∞ smooth on U. Let $\Omega \subset M$ be a smooth open set such that

$$U^c \subset \Omega, \quad \partial\Omega \subset U \quad \text{and} \quad \mathrm{Cut}_b(M) \cap \overline{\Omega} = \emptyset.$$

As $U^c \subset \Omega$, it suffices to show that u_m^d is continuous on Ω. As $\partial\Omega \subset U$, u_m^d is smooth on $\partial\Omega$, so there exists a smooth function v_m on $\overline{\Omega}$ such that $v_m = u_m^d$ on $\partial\Omega$. Then, one can check that u_m^d is a solution of the variational problem

$$\min\left\{\int_\Omega |\nabla u|^2 - mu \ : \ u \in H^1(\Omega),\ u \leqq d_b \text{ in } \Omega,\ u = v_m \text{ on } \partial\Omega\right\}.$$

As a consequence, $u_m^d - v_m$ is a solution of the variational problem

$$\min\left\{\int_\Omega |\nabla v|^2 - (m + \Delta v_m)v \ : \ v \in H_0^1(\Omega),\ v \leqq d_b - v_m \text{ in } \Omega\right\}. \tag{3.2}$$

Because we have $\mathrm{Cut}_b(M)\cap\overline{\Omega}=\emptyset$, the exponential map at b is a diffeomorphism onto Ω. Let $\phi:\Omega\to\widetilde{\Omega}\subset\mathbb{R}^n$ be a normal coordinates chart centered at b. Let $g=(g^{ij})$ denotes the metric of M in the coordinates defined by ϕ, and $\det g$ its determinant. We recall that the Riemannian volume measure is given in coordinates by $\sqrt{\det g}\,\mathrm{d}x$. Thus we have that

$$\int_\Omega\Big(|\nabla v|^2-(m+\Delta v_m)v\Big)=\int_{\widetilde{\Omega}}\Big(g^{ij}\partial_i(v\circ\phi^{-1})\partial_j(v\circ\phi^{-1})\sqrt{\det g}\\ -\big((m+\Delta v_m)\circ\phi^{-1}\big)\big(v\circ\phi^{-1}\big)\sqrt{\det g}\Big)\,dx,$$

so $(u_m^d-v_m)\circ\phi^{-1}$ is a minimizer of

$$\min\left\{\int_{\widetilde{\Omega}}\Big(g^{ij}\sqrt{\det g}\,\partial_i w\,\partial_j w-Fw\Big)\,dx\ :\ w\in H_0^1(\widetilde{\Omega}),\ w\leqq\psi\right\},\tag{3.3}$$

where we have set $\psi:=(d_b-v_m)\circ\phi^{-1}$ and $F:=(m+\Delta v_m)\circ\phi^{-1}\sqrt{\det g}$. We want to apply [29, Theorem 4.32]. For this we need to write the above variational problem into a variational inequality. Let w be a competitor in (3.3). Writing down the minimality of $w_m:=(u_m^d-v_m)\circ\phi^{-1}$ against the competitor $w_m+t(w-w_m)$, for $t\in(0,1)$ small, we find that

$$\langle Aw_m,w_m-w\rangle\geqq\langle F,w_m-w\rangle,$$

where A is the elliptic operator defined on $H_0^1(\widetilde{\Omega})$ by $Aw:=-\partial_j(g^{ij}\sqrt{\det g}\,\partial_i w)$. From there, we can apply [29, Theorem 4.32] to deduce that, for any $p<n$, if $A\psi\wedge F\in L^p(\widetilde{\Omega})$, then $Aw_m\in L^p(\widetilde{\Omega})$. To check that $A\psi\wedge F\in L^p(\widetilde{\Omega})$, it is enough to check that $A(d_b\circ\phi^{-1})\in L^p(\widetilde{\Omega})$. As d_b is smooth except at b, it is enough to check that $(A(d_b\circ\phi^{-1}))^p$ is integrable at 0. But this is a consequence of the fact that $-\Delta d_b\circ\phi^{-1}=\frac{1}{\sqrt{\det g}}A(d_b\circ\phi^{-1})$, and Lemma 3.5 below, from which we deduce that $A(d_b\circ\phi^{-1})(x)$ is equivalent to $\frac{n-1}{|x|}$ when x goes to 0. Therefore, for $p<n$, $(A(d_b\circ\phi^{-1}))^p$ is integrable at 0, and so $Aw_m\in L^p(\widetilde{\Omega})$. By elliptic regularity, this implies $w_m\in W^{2,p}(\widetilde{\Omega})$, for any $p<n$. By the Sobolev embeddings, w_m is then continuous on $\widetilde{\Omega}$, and so u_m^d is continuous on Ω. This concludes the proof. □

We can now define the set $E_m^d:=\{u_m^d<d_b\}$, for any $m>0$. It is an open subset of M, on which u_m^d solves the equation $\Delta u_m^d=-2m$. We can now prove.

Lemma 3.3. *For any $m>0$, we have $u_m^d=d_b$ in a neighborhood of b.*

Proof. Let us assume that we have constructed a C^1 function v on $\overline{B}_R(b)$ for some $R>0$, such that

$$\begin{cases} v\leqq d_b & \text{in } B_R(b), & (3.4)\\ v=d_b & \text{in } B_\varepsilon(b) \text{ for some } \varepsilon\in(0,R), & (3.5)\\ v<0 & \text{in } \partial B_R(b), & (3.6)\\ \Delta v\geqq -m & \text{in } B_R(b) \text{ in the distributional sense.} & (3.7)\end{cases}$$

We will then show that we have $u_m^d \geqq v$. The construction of v is postponed to the end of the proof. From Lemma 3.2, we know that the function $v - u_m^d$ is continuous. Let us first assume that $v - u_m^d$ attains a positive maximum at a point $x \in \overline{B}_R(b)$. We have that

$$0 < v(x) - u_m^d(x) \leqq d_b(x) - u_m^d(x),$$

so $x \in E_m^d$. Moreover, we have $u_m^d \geqq 0$ since $\max(u_m^d, 0)$ is a better competitor than u_m^d in (1.15), so

$$v - u_m^d \leqq v < 0 \quad \text{on} \quad \partial B_R(b),$$

and so $x \in B_R(b)$. Hence the function $v - u_m^d$ attains a positive maximum inside the open set $E_m^d \cap B_R(b)$, but its Laplacian verifies in the distributional sense:

$$\Delta(v - u_m^d) = \Delta v + m \geqq 0. \tag{3.8}$$

This yields a contradiction, by the maximum principle. Then, the maximum of $v - u_m^d$ on $\overline{B}_R(b)$ is non-positive, and we get

$$u_m^d \geqq v = d_b \quad \text{in} \quad B_\varepsilon(b),$$

which concludes the this proof.

Let us now construct the function v that was used above. Let $R > 0$ be small enough so that $\overline{B}_R(b)$ is contained in a normal neighborhood of b. In polar coordinates around b, we define v as a radial function. For $\varepsilon > 0$ to be chosen small enough later, let $f : [0, R] \to [0, \infty)$ be the C^1 function such that

$$\begin{cases} f(r) = r & \text{if} \quad r \leqq \varepsilon, \\ f''(r) + \dfrac{n-1}{r} f'(r) = -\dfrac{m}{2} & \text{if} \quad r > \varepsilon. \end{cases} \tag{3.9}$$

If $n = 2$, the unique C^1 solution to this system is given by

$$\begin{cases} f(r) = r & \text{if} \quad r \leqq \varepsilon, \\ f(r) = \varepsilon + \dfrac{m}{8}\left(\varepsilon^2 - r^2\right) + \left(\varepsilon + \dfrac{m}{4}\varepsilon^2\right) \ln(\dfrac{r}{\varepsilon}) & \text{if} \quad r > \varepsilon. \end{cases} \tag{3.10}$$

If $n \geq 3$, then the solution is

$$\begin{cases} f(r) = r & \text{if} \quad r \leqq \varepsilon, \\ f(r) = \varepsilon + \dfrac{m}{4n}\left(\varepsilon^2 - r^2\right) & \\ \qquad + \left(\varepsilon^{n-1} + \dfrac{m}{2n}\varepsilon^n\right) \dfrac{1}{n-2} \left(\dfrac{1}{\varepsilon^{n-2}} - \dfrac{1}{r^{n-2}}\right) & \text{if} \quad r > \varepsilon. \end{cases} \tag{3.11}$$

Then, we set in standard polar coordinates $v(x) = f(r)$ for $x \in B_R(b)$. For $r \leqq \varepsilon$, the constraint (3.5) is verified by definition. (For $r > \varepsilon$, we chose f so that Δv is small, but still bigger than $-m$.)

Let us show that (3.4) holds. Let us set $g(r) := f(r) - r$ and prove that $g \leqq 0$. We have $g(r) = 0$ for $r \leqq \varepsilon$ so it is sufficient to prove that $g'(r) \leqq 0$ for $r \geqq \varepsilon$. However, as f verifies (3.9), g verifies

$$g'' + \frac{n-1}{r} g' = -m - \frac{n-1}{r} \quad \text{for} \quad r \geqq \varepsilon.$$

In particular, whenever $g'(r) = 0$, we have $g''(r) < 0$. This implies $g'(r) \leqq 0$ for $r \geqq \varepsilon$, and so (3.4) is verified.

Now let us show that (3.7) holds if R has been taken small enough. We use the following expression of the Laplacian in coordinates:

$$\Delta v = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, g^{ij} \partial_j v \right).$$

Here $g = (g^{ij})$ is the metric of the manifold M, and $\det g$ its determinant. We apply this formula to polar coordinates to find that, on $B_R(b) \setminus \overline{B}_\varepsilon(b)$, we have, in the classical sense,

$$\begin{aligned} \Delta v = \frac{1}{\sqrt{\det g}} \partial_r \big(\sqrt{\det g}\, f'(r) \big) &= f'' + \frac{\partial_r \det g}{2 \det g} f' \\ &= f'' + \frac{n-1}{r} f' + \left(\frac{\partial_r \det g}{2 \det g} - \frac{n-1}{r} \right) f' \\ &= -\frac{m}{2} + \left(\frac{\partial_r \det g}{2 \det g} - \frac{n-1}{r} \right) f'. \end{aligned} \tag{3.12}$$

Note that by applying the Laplacian formula in polar coordinates to the distance function $d_b(x) = r$, we find that

$$\Delta d_b = \frac{\partial_r \det g}{2 \det g}. \tag{3.13}$$

Because of Lemma 3.5, we also have that

$$\Delta d_b(x) = \frac{n-1}{r} + o(1).$$

With (3.12) and (3.13), this last equation yields, in the classical sense,

$$\Delta v = -\frac{m}{2} + o(1) f'(r) \quad \text{on} \quad B_R(b) \setminus \overline{B}_\varepsilon(b). \tag{3.14}$$

Moreover, it is clear from the following expression that f' is bounded on $[\varepsilon, R]$, by a constant independent of R, as long as we choose $R \leqq 1$:

$$f'(r) = -\frac{m}{n} r + \left(\varepsilon^{n-1} + \frac{m}{n} \varepsilon^n \right) \frac{1}{r^{n-1}} \quad \text{for} \quad \varepsilon \leqq r \leqq R.$$

Hence from (3.14) we see that by taking R small enough (independently of ε), we can ensure that

$$\Delta v \geqq -m \quad \text{on} \quad B_R(b) \setminus \overline{B}_\varepsilon(b).$$

But from (3.14), we see that the above is also true on $B_\varepsilon(b)$ if ε is small enough. Thus the function v is C^1 on $B_R(b)$ and verifies $\Delta v(x) \geqq -m$ when $x \notin \partial B_\varepsilon(b)$, hence (3.7) holds. It is also clear from (3.10) and (3.11) that the constraint (3.6) is verified if ε is taken small enough. This concludes the proof. □

We can now prove the $C^{1,1}$ regularity of u_m^d.

Proposition 3.4. *For any $\varepsilon > 0$, the function u_m^d belongs to $C^{1,1}(M \setminus B_\varepsilon(b))$.*

Proof. We reproduce the proof of Lemma 3.2, but we replace the open set Ω with $\widehat{\Omega} := \Omega \setminus B_\varepsilon(b)$, and the function v_m with a function $\widehat{v_m}$ that is smooth and such that $w_m = u_m$ on $\partial\widehat{\Omega}$. We know that such a function exists because u_m is smooth on $\partial B_\varepsilon(b)$ for ε small enough, as it can be seen from Lemma 3.3. This way, we can apply the stronger $W^{2,\infty}$ regularity result for the obstacle problem [29, Theorem 4.38], since d_b is smooth on $\widehat{\Omega}$. We get that u_m^d belongs to $W^{2,\infty} = C^{1,1}(\widehat{\Omega})$. As u_m^d is smooth on E_m^d and $\partial\widehat{\Omega} \subset E_m^d$, then u_m^d is $C^{1,1}$ on $\widehat{\Omega} \cup E_m = M \setminus B_\varepsilon(b)$. □

We end this section with the following computational lemma, which we used in the proof of Lemma 3.3.

Lemma 3.5. *We have*

$$\Delta d_b(p) \underset{p\to b}{=} \frac{n-1}{d_b(p)} + o(1). \tag{3.15}$$

Proof. We compute Δd_b in normal coordinates centered at b. Let $g = (g^{ij})$ be the metric of M in these coordinates. We have

$$\Delta d_b(x) = \frac{1}{\sqrt{\det g}}\, \partial_i \left(\sqrt{\det g}\, g^{ij} \partial_j d_b\right)(x).$$

In normal coordinates, the metric is euclidean up to order 1 as x goes to 0. So we have

$$g^{ij}(x) = \delta^{ij} + o(x), \quad \partial_i \left(\sqrt{\det g}\, g^{ij}\right)(x) = o(1) \quad \text{and} \quad \frac{1}{\sqrt{\det g}} = 1 + o(x).$$

Moreover, in normal coordinates, we have $d_b(x) = |x|$, and so

$$\delta^{ij}\partial_{ij} d_b(x) = \frac{n-1}{|x|},$$

which gives, precisely, (3.15). □

4. Equivalence of the Two Constraints

Proof of Proposition 1.8. As above, we denote by u_m^d the minimizer of (1.15). In order to show that u_m^d solves (1.3), it is sufficient to show that u_m^d is an admissible competitor in (1.3), that is, $|u_m^d| \leq 1$ on M. Recall that the function u_m^d is C^1 except at b, by Proposition 3.4.

First, suppose that $x \neq b$ is in the contact set $P_m^d := \{u_m^d = d_b\}$. By Lemma 3.1, we have $x \notin \operatorname{Cut}_b(M)$, and so the distance function d_b is differentiable at x. It is a simple consequence of the constraint $u_m^d \leqq d_b$ and the equality $u_m^d(x) = d_b(x)$ that we have $\nabla u_m^d(x) = \nabla d_b(x)$. The desired inequality $\left|\nabla u_m^d(x)\right| \leqq 1$ follows.

In the non-contact set $E_m^d = \{u_m^d < d_b\}$, the function u_m^d solves the PDE

$$\Delta u_m^d = -2m. \tag{4.1}$$

In particular it is smooth, and we may apply the Bochner-Weitzenböck formula

$$\Delta\left(\left|\nabla u_m^d\right|^2\right) = 2\mathrm{Ric}(\nabla u_m^d, \nabla u_m^d) + 2\left|D^2 u_m^d\right|^2 + 2(\nabla \Delta u_m^d, \nabla u_m^d), \tag{4.2}$$

where Ric denotes the Ricci curvature tensor on the manifold M and $D^2 u_m^d$ is the second covariant derivative of u_m^d. The last term is 0 because of (4.1). As for the second term, we have that

$$\left|D^2 u_m^d\right|^2 \geqq \frac{1}{n}\left(\mathrm{Trace}(D^2 u_m^d)\right)^2 = 4\frac{m^2}{n}, \tag{4.3}$$

where the last inequality is due to (4.1). As the manifold M is compact, there exists a constant $K > 0$ (depending on M only) such that the Ricci curvature is bounded from below by $-K$. In the end, (4.2) yields

$$\Delta\left(\left|\nabla u_m^d\right|^2\right) + 2K\left|\nabla u_m^d\right|^2 \geqq \frac{8}{n}m^2. \tag{4.4}$$

Now notice that, by (4.1),

$$\Delta\left((u_m^d)^2\right) = 2u_m^d \Delta u_m^d + 2\left|\nabla u_m^d\right|^2 = -4m u_m^d + 2\left|\nabla u_m^d\right|^2,$$

so (4.4) gives

$$\begin{aligned}\Delta\left(\left|\nabla u_m^d\right|^2 + K(u_m^d)^2\right) &= \frac{8}{n}m^2 - 4Km\, u_m^d \\ &\geqq \frac{8}{n}m^2 - 4Km\, d_b \geqq \frac{8}{n}m^2 - 4Km\,\mathrm{diam}(M).\end{aligned}$$

Thus, if $m \geqq \frac{n}{2}K\mathrm{diam}(M)$, the function $\left|\nabla u_m^d\right|^2 + K(u_m^d)^2$ is subharmonic in the non-contact set E_m^d. From Lemma 3.3, we have $\overline{E_m^d} \subset M \setminus \{b\}$, and with Proposition

3.4, we get that the function $\left|\nabla u_m^d\right|^2 + K(u_m^d)^2$ is continuous on $\overline{E_m^d} \subset M \setminus \{b\}$. Therefore we may apply the maximum principle to get

$$\left|\nabla u_m^d\right|^2 \leqq \left|\nabla u_m^d\right|^2 + K(u_m^d)^2 \leqq \sup_{\partial E_m^d} \left(\left|\nabla u_m^d\right|^2 + K(u_m^d)^2\right) = 1 + K \sup_{\partial E_m^d} (u_m^d)^2$$
$$\leqq 1 + K \sup_{\partial E_m^d} (d_b)^2 \leqq 1 + K \operatorname{diam}(M)^2$$

With (4.4), this last inequality gives

$$\Delta\left(\left|\nabla u_m^d\right|^2\right) \geqq \frac{8}{n} m^2 - 2K(1 + K\operatorname{diam}(M)^2)$$

Thus, whenever the right-hand side is nonnegative, the maximum principle applied to the function $\left|\nabla u_m^d\right|^2$ on the open set E_m^d implies that $\left|\nabla u_m^d\right|^2 < 1$ on this set. This concludes the proof. □

5. Convergence of the Non-contact Set

In this section we show that the non-contact set $E_m^d = \{u_m^d < d_b\}$ (which coincides with E_m, for m large enough, as we showed in the previous section) Hausdorff-converges to $\mathrm{Cut}_b(M)$.

Lemma 5.1. *We have $\|d_b - u_m^d\|_{L^\infty(M)} \leqq \dfrac{C}{m}$, for some positive constant C depending on M only.*

Proof. We only need to prove the proposition for m large enough. Therefore, thanks to Proposition 1.8, we will assume that m is large enough so that $\left|\nabla u_m^d\right| \leqq 1$. We only need to show the estimate on E_m^d since outside this set, u_m^d and d_b are the same. We will show that for m large enough, we have that

$$\forall \overline{p} \in E_m^d,\ \exists p \in (E_m^d)^c \quad \text{such that} \quad d(p, \overline{p}) < 5n/m. \tag{5.1}$$

This will conclude the proof since by the 1-Lipschitzianity of u_m^d and d_b, we then have that

$$\left|d_b(\overline{p}) - u_m^d(\overline{p})\right| \leqq \left|d_b(p) - u_m^d(p)\right| + 2d(\overline{p}, p) = 0 + 2d(\overline{p}, p) \leqq \frac{10n}{m},$$

which is what we need. In order to prove (5.1), we argue by contradiction and assume that $B_{5n/m}(\overline{p}) \subset E_m^d$. We want to apply the maximum principle to the function v defined on $B_{5n/m}(\overline{p})$ by the formula

$$v(p) := u_m^d(p) - \inf_{\partial B_{\frac{5n}{m}}(\overline{p})} u_m^d + \frac{m}{2n}\left(d_{\overline{p}}(p)^2 - \left(\frac{5n}{m}\right)^2\right).$$

For any $p \in B_{5n/m}(\overline{p})$, we have $\Delta u_m^d(p) = -2m$ because we have assumed $B_{5n/m}(\overline{p}) \subset E_m^d$. To estimate the Laplacian of $d_{\overline{p}}^2$, we use some normal coordinates

(x^i) centered at $\overline{p}$. In these coordinates, the metric is euclidean up to order 1, uniformly in $\overline{p}$ since M is compact, and $d_{\overline{p}}(x) = |x|$ (see Lemma 3.5). We get that, for m large enough, independently of $\overline{p}$,

$$\forall p \in B_{5n/m}(\overline{p}), \quad \Delta d^2_{\overline{p}}(p) \leqq 2(2n).$$

All in all, we obtain, on $B_{5n/m}(\overline{p}) \subset E^d_m$, that

$$\Delta v \leqq -2m + \frac{m}{2n} 2(2n) = 0,$$

so we can apply the maximum principle to v to get

$$v(\overline{p}) \geqq \inf_{\partial B_{\frac{5n}{m}}(\overline{p})} v,$$

i.e.

$$u^d_m(\overline{p}) - \inf_{\partial B_{\frac{5n}{m}}(\overline{p})} u^d_m - \frac{m}{4n}\left(\frac{5n}{m}\right)^2 \geqq 0. \tag{5.2}$$

As we have taken m large enough so that $\left|\nabla u^d_m\right| \leqq 1$, we also have

$$u^d_m(\overline{p}) - \inf_{\partial B_{\frac{5n}{m}}(\overline{p})} u^d_m \leqq \frac{5n}{m} < \frac{m}{4n}\left(\frac{5n}{m}\right)^2,$$

which contradicts the estimate (5.2). This concludes the proof. □

Proposition 5.2. (Monotonicity of u^d_m and E^d_m, and convergence of E^d_m) *For any $m > m' > 0$, we have*

$$u^d_{m'} \leq u^d_m \leq d_b \quad \textit{and} \quad Cut_b(M) \subset E^d_m \subset E^d_{m'}.$$

Moreover,

$$E^d_m \underset{m\to\infty}{\longrightarrow} Cut_b(M) \quad \textit{in the Hausdorff sense.}$$

Proof. The fact that, for any $m > 0$, $\mathrm{Cut}_b(M) \subset E^d_m$, is a direct consequence of Lemma 3.1. Let us prove the second inclusion. For $m > m' > 0$, note that by the respective minimality of u^d_m and $u^d_{m'}$, we have

$$\int_M \left|\nabla \max(u^d_{m'}, u^d_m)\right|^2 - m\int_M \max(u^d_{m'}, u^d_m) \geqq \int_M \left|\nabla u^d_m\right|^2 - m\int_M u^d_m,$$

$$\text{and} \quad \int_M \left|\nabla \min(u^d_{m'}, u^d_m)\right|^2 - m'\int_M \min(u^d_{m'}, u^d_m) \geqq \int_M \left|\nabla u^d_{m'}\right|^2 - m'\int_M u^d_{m'}.$$

Using the formulas

$$\nabla \max(u^d_{m'}, u^d_m) = \nabla u^d_{m'} \mathbb{1}_{\{u^d_{m'} > u^d_m\}} + \nabla u^d_m \mathbb{1}_{\{u^d_{m'} \leqq u^d_m\}},$$

$$\nabla \min(u^d_{m'}, u^d_m) = \nabla u^d_m \mathbb{1}_{\{u^d_{m'} > u^d_m\}} + \nabla u^d_{m'} \mathbb{1}_{\{u^d_{m'} \leqq u^d_m\}},$$

we obtain

$$\int_{\{u^d_{m'}>u^d_m\}} \left(\left|\nabla u^d_{m'}\right|^2 - \left|\nabla u^d_m\right|^2\right) \geqq -m \int_{\{u^d_{m'}>u^d_m\}} \left(u^d_m - u^d_{m'}\right),$$

$$\text{and} \quad \int_{\{u^d_{m'}>u^d_m\}} \left(\left|\nabla u^d_m\right|^2 - \left|\nabla u^d_{m'}\right|^2\right) \geqq -m' \int_{\{u^d_{m'}>u^d_m\}} \left(u^d_{m'} - u^d_m\right).$$

Summing these two inequalities, we get

$$0 \geqq (m - m') \int_{\{u^d_{m'}>u^d_m\}} \left(u^d_{m'} - u^d_m\right),$$

and so $u^d_m \geqq u^d_{m'}$. In particular, $E^d_m \subset E^d_{m'}$.

We are left to show the Hausdorff convergence in E^d_m to $\mathrm{Cut}_b(M)$. Given $\varepsilon > 0$, let us set

$$\Omega_\varepsilon := \left\{x \in M : d(x,\ \mathrm{Cut}_b(M)) > \varepsilon\right\}.$$

We will show that for m large enough we have $E^d_m \subset (\Omega_{2\varepsilon})^c$, which will conclude the proof. Let $\phi : M \to \mathbb{R}$ be a function such that $\phi \leqq d_b$ on M, $\phi = d_b$ on $\Omega_{2\varepsilon}$, $\phi < d_b$ on $\partial\Omega_\varepsilon$, and ϕ is smooth on M except at b. We want to apply the maximum principle to the function $\phi - u^d_m$ on $E^d_m \cap \Omega_\varepsilon$. We have that

$$\Delta(\phi - u^d_m) = \Delta\phi + 2m \quad \text{on} \quad E^d_m \cap \Omega_\varepsilon,$$

so for m large enough the function $\phi - u^d_m$ is subharmonic on $E^d_m \cap \Omega_\varepsilon$. On $\partial\Omega_\varepsilon$, we have $\phi < d_b$ and u^d_m converges uniformly to d_b as m tends to $+\infty$ (Lemma 5.1) so $\phi - u^d_m \leqq 0$, for m large enough. On ∂E^d_m, we have $\phi - u^d_m = \phi - d_b \leqq 0$. Thus the maximum principle implies that for m large enough, we have $\phi - u^d_m \leqq 0$ on $E^d_m \cap \Omega_\varepsilon$. As $\phi = d_b$ on $\Omega_{2\varepsilon}$, we get $u^d_m \geqq d_b$ on $E^d_m \cap \Omega_{2\varepsilon}$. Since by definition we have $u^d_m < d_b$ on E^d_m, we get $E^d_m \subset (\Omega_{2\varepsilon})^c$, which concludes the proof. □

6. Semiconcavity

This section is dedicated to the semiconcavity of the solutions to the obstacle problems (1.15) and (1.13). The key result is Proposition 6.1, which applies to both Theorem 1.3 and Theorem 1.6.

In the case of Theorem 1.6, we have $\mathring{M} = \Omega$ and $\partial M = \partial\Omega$.

Proposition 6.1. *Let $M = \mathring{M} \sqcup \partial M$ be a smooth compact Riemanniannian manifold, with (possibly empty) boundary ∂M. Suppose that, for some constants $L > 0$ and $C > 0$, we are given the following:*

(a) *a function $d : \mathring{M} \to \mathbb{R}$, which is bounded and C-semiconcave on $\mathring{M}$;*
(b) *a family of functions $u_m : \mathring{M} \to \mathbb{R}^n$, for $m > 0$, such that:*
  (b.1) *for every $m > 0$, $u_m \leq d$ on $\mathring{M}$;*
  (b.2) *for every $m > 0$, u_m is L-Lipschitz on $\mathring{M}$;*

(b.3) *on the set* $E_m := \{u_m < d\}$, u_m *is* C^∞ *smooth and*

$$-\Delta u_m = 2m \quad in \quad E_m\,;$$

(b.4) E_m *is precompact in* $\mathring{M}$*;*

(b.5) *for every* $\eta > 0$*, for every* $m > 0$*, there is a neighborhood* $\mathcal{N}_{\eta,m}$ *of* ∂E_m *in* $\mathring{M}$ *such that*

$$D^2 u_m \leq (C+\eta)\, Id \quad in \quad E_m \cap \mathcal{N}_{\eta,m}.$$

Then, for every $\eta > 0$*, there exists* $m_0 > 0$ *such that*

$$u_m \text{ is } (C+\eta)\text{-semiconcave on } \mathring{M}, \text{ for every } m \geq m_0.$$

Application to Theorem 1.6. In order to apply Proposition 6.1 to Theorem 1.6, we take $\mathring{M} = \Omega$ and $\partial M = \partial\Omega$. The function d is the distance function $d_{\partial\Omega}$ to the boundary of Ω, while u_m is the solution v_m of (1.13) (thus, the Lipschitz constant from (b.2) is $L = 1$), which means that the conditions (b.1), (b.2) and (b.3) are fullfilled. When Ω is C^2 regular, the set $\overline{\mathcal{M}(\Omega)}$ is contained in Ω. Now, as the elastic sets $\{u_m < d\}$ Hausdorff-converge to $\overline{\mathcal{M}(\Omega)}$ (see [6]) we get that, for large m, u_m coincides with d in a neighborhood of $\partial\Omega$. Thus, (b.4) is fullfilled. As Ω is C^2, the function $d_{\partial\Omega}$ is known to be C-semiconcave in Ω for some $C > 0$ (see [10, (iii) of Proposition 2.2.2]), so (a) is fulfilled. Finally, condition (b.5) is a consequence of [13, Chapter 2, Theorem 3.8]. Thus, there exists a constant $C > 0$ such that for m big enough, v_m is C-semiconcave in Ω.

Application to Theorem 1.3 **(T4).** In the case of Theorem 1.3, we take $\mathring{M} = M \setminus \overline{B}_\rho(x_0)$ and $\partial M = \partial B_\rho(b)$, where $B_\rho(b)$ is a small geodesic ball centered at the base point b. The function d is the distance function d_b to the base point, while u_m is the solution of (1.15). The semiconcavity of the distance function d in $M \setminus B_\rho(b)$ was proved in [20], see Proposition 2.7. By Proposition 1.8, for large m, the problems (1.15) and (1.3) are equivalent and so we can take $L = 1$ in (b.2), and we also have that (b.1) are (b.3) are fullfilled. Next, we notice that by Lemma 3.3 we have that $u_m = d$ in a neighborhood of b, which proves (b.4) by choosing the radius ρ small enough. Finally, in Lemma 6.2 we will prove that also the condition (b.5) is fullfilled.

Proof of Proposition 6.1. First, we notice that by dividing all the functions by L, we can assume that $L = 1$. Let $\eta > 0$. As in Definition 1.9, for $a, b \in \mathbb{R}$ and $\lambda \in (0,1)$, we will use the notation

$$\lambda_{ab} := (1-\lambda)a + \lambda b.$$

For any unit speed geodesic $\gamma : [a,b] \to \mathring{M}$, $\lambda \in (0,1)$ and v a function on $\mathring{M}$, let us define

$$c(\gamma,\lambda,v) := \lambda(1-\lambda)(C+\eta)(b-a)^2 \\ - \Big((1-\lambda)v(\gamma(a)) + \lambda v(\gamma(b)) - v(\gamma(\lambda_{ab}))\Big).$$

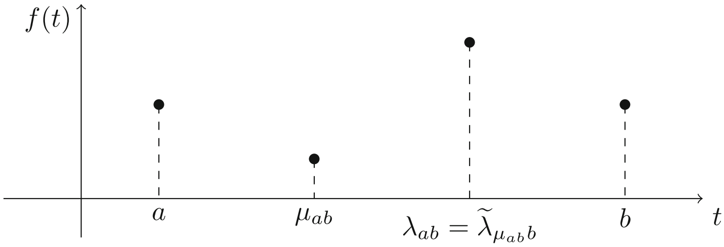

Fig. 2. Construction of $\widetilde{\gamma}$ and $\widetilde{\lambda}$

We aim to show that:

$$\inf_{\gamma,\lambda} c(\gamma, \lambda, u_m) \geqq 0, \tag{6.1}$$

where the infimum is taken over unit speed geodesics defined over finite intervals. Let us argue by contradiction and assume that (6.1) does not hold.

Let us show that we may assume that the infimum is actually taken over unit speed geodesics $\gamma : [a, b] \to \mathring{M}$ such that

$$\gamma\big((a, b)\big) \subset E_m = \{u_m < d\}. \tag{6.2}$$

Let $\gamma : [a, b] \to \mathring{M}$ be a unit speed geodesic, and $\lambda \in (0, 1)$, such that $c(\gamma, \lambda, u_m) < 0$. Let us assume that γ does not verify (6.2). We will build a geodesic $\widehat{\gamma}$ that does verify (6.2), and $\widehat{\lambda} \in (0, 1)$, such that

$$c(\widehat{\gamma}, \widehat{\lambda}, u_m) < c(\gamma, \lambda, u_m).$$

First, notice that if $\gamma(\lambda_{ab}) \notin E_m$, then we have $u_m(\gamma(\lambda_{ab})) = d(\gamma(\lambda_{ab}))$, $u_m(\gamma(a)) \leqq d(\gamma(a))$ and $u_m(\gamma(b)) \leqq d(\gamma(b))$, and so

$$c(\gamma, \lambda, u_m) \geqq c(\gamma, \lambda, d) > 0, \tag{6.3}$$

where the last inequality comes from the C-semiconcavity of d. This is contradictory, so $\gamma(\lambda_{ab}) \in E_m$. As γ does not verify (6.2), there exists $t \in (0, \lambda_{ab}) \cup (\lambda_{ab}, 1)$, such that $\gamma(t_{ab}) \notin E_m$. Up to reparametrization of γ, we may assume that $t \in (0, \lambda_{ab})$. We can define

$$\mu := \min\left\{s \in (0, \lambda) : \forall r \in (s, \lambda), \quad \gamma(r_{ab}) \in E_m\right\}.$$

We have $\gamma(\mu_{ab}) \notin E_m$, and $\gamma((\mu_{ab}, \lambda_{ab}]) \subset E_m$. Figure 2 may help justify intuitively the following construction. Let $\widetilde{\lambda} \in (0, 1)$ be such that

$$\widetilde{\lambda}_{\mu_{ab}b} = \lambda_{ab}. \tag{6.4}$$

Let $\widetilde{\gamma}$ be the unit speed geodesic defined by $\widetilde{\gamma} := \gamma_{|[\mu_{ab}, b]}$. Let us set $f(t) := (C + \eta)t^2 - u_m(\gamma(t))$. Then

$$\begin{aligned} c(\widetilde{\gamma}, \widetilde{\lambda}, u_m) &= (1 - \widetilde{\lambda}) f(\mu_{ab}) + \widetilde{\lambda} f(b) - f(\widetilde{\lambda}_{\mu_{ab}b}) \\ &= (1 - \widetilde{\lambda}) f(\mu_{ab}) + \widetilde{\lambda} f(b) - f(\lambda_{ab}) \\ &= c(\gamma, \lambda, u_m) - (1 - \lambda) f(a) + (\widetilde{\lambda} - \lambda) f(b) + (1 - \widetilde{\lambda}) f(\mu_{ab}). \end{aligned} \tag{6.5}$$

Now after some elementary calculations, (6.4) translates into

$$\begin{cases} 1-\lambda & = (1-\widetilde{\lambda})(1-\mu), \\ \widetilde{\lambda}-\lambda & = -(1-\widetilde{\lambda})\mu, \end{cases}$$

so (6.5) becomes

$$\begin{aligned} c(\widetilde{\gamma}, \widetilde{\lambda}, u_m) &= c(\gamma, \lambda, u_m) - (1-\widetilde{\lambda})\big((1-\mu)f(a) + \mu f(b) - f(\mu_{ab})\big) \\ &= c(\gamma, \lambda, u_m) - (1-\widetilde{\lambda})c(\gamma, \mu, u_m). \end{aligned}$$

Using the fact that $\gamma(\mu_{ab}) \notin E_m$, we deduce, as in (6.3), that

$$c(\gamma, \mu, u_m) \geqq c(\gamma, \mu, d) > 0.$$

This yields

$$c(\widetilde{\gamma}, \widetilde{\lambda}, u_m) < c(\gamma, \lambda, u_m).$$

Moreover the unit speed geodesic $\widetilde{\gamma} : [\mu_{ab}, b] \to \mathring{M}$ verifies $\widetilde{\gamma}((\mu_{ab}, \widetilde{\lambda}_{\mu_{ab}b}]) \subset E_m$. Now, arguing as above, if there exists $t \in (\widetilde{\lambda}, 1)$ such that $\widetilde{\gamma}(t_{\mu_{ab}b}) \notin E_m$, then we may build two numbers $\nu \in (\widetilde{\lambda}, 1)$ and $\widehat{\lambda} \in (0,1)$ such that the unit speed geodesic $\widehat{\gamma} := \widetilde{\gamma}_{|[\mu_{ab}, \nu_{ab}]}$ verifies

$$c(\widehat{\gamma}, \widehat{\lambda}, u_m) < c(\widetilde{\gamma}, \widetilde{\lambda}, u_m),$$

and

$$\widehat{\gamma}((\mu_{ab}, \nu_{ab})) \subset E_m.$$

Now we need to show that

$$\inf_{\gamma, \lambda} c(\gamma, \lambda, u_m) \geqq 0, \tag{6.6}$$

where the infimum is taken over unit speed geodesics $\gamma : [a, b] \to \mathring{M}$ such that $\gamma\big((a, b)\big) \subset E_m$.

By continuity of u_m, (6.6) is equivalent to simply saying that u_m is $(C+\eta)$-semiconcave on E_m. Therefore, as u_m is smooth on E_m, by Proposition 2.2, we only need to show the pointwise condition

$$D^2 u_m \leqq (C+\eta) Id \quad \text{on} \quad E_m. \tag{6.7}$$

Now, let $C_1, C_2, C_3 > 0$ be some constants to be determined later, and $\varepsilon > 0$ to be chosen small enough later. For $p \in E_m$ and $X \in \mathbb{S}^{n-1}(T_pM)$, we define

$$f_\varepsilon(p, X) := D^2 u_m(X, X) + \varepsilon \Big(C_1 \left|\nabla u_m\right|^2 (p) + C_2 u_m^2(p) - C_3 u_m(p) \Big). \tag{6.8}$$

We will show that for a good choice of constants C_1, C_2, C_3, depending only of M and $|d|_{L^\infty}$, for any $\varepsilon > 0$ small enough, depending only on M, $|d|_{L^\infty}$ and η, we have for any m large enough,

$$f_\varepsilon(p, X) \leqq C + \frac{2\eta}{3} \quad \text{for every} \quad p \in E_m \quad \text{and every} \quad X \in \mathbb{S}^{n-1}(T_pM). \tag{6.9}$$

This will conclude the proof since, as u_m is bounded by $|d|_{L^\infty}$ and 1-Lipschitz, we will then get

$$D^2u_m(X,X) \leqq C + \frac{2\eta}{3} + \varepsilon C(M, |d|_{L^\infty}),$$

where $C(M, |d|_{L^\infty}) > 0$ is a constant depending on M and $|d|_{L^\infty}$ only. But this implies (6.7) if ε has been taken small enough.

Suppose, by contradiction, that

$$\sup_{\substack{p \in E_m \\ X \in \mathbb{S}^{n-1}(T_pM)}} f_\varepsilon(p, X) > C + \frac{2\eta}{3}. \tag{6.10}$$

Let us assume that m is large enough so that $D^2u_m \leq (C + \eta/3)Id$ in a neighborhood of ∂E_m. In particular, we get that for ε small enough, depending only on M, $|d|_{L^\infty}$ and η,

$$f_\varepsilon < C + \frac{2\eta}{3} \quad \text{in a neighborhood of} \quad \partial E_m.$$

Thus, by (6.10) and the precompactness of E_m, there exist $q \in E_m$ and $Y \in \mathbb{S}^{n-1}(T_qM)$ such that

$$f_\varepsilon(q, Y) = \sup_{\substack{p \in E_m \\ X \in \mathbb{S}^{n-1}(T_pM)}} f_\varepsilon(p, X). \tag{6.11}$$

In the following, C^M will denote any constant that depends only on M. Let us pick some normal coordinates at q such that $\partial_1(q) = Y$. We then extend the vector Y into a vector field (still denoted by Y) in a neighborhood of q, by setting $Y := \partial_1 / |\partial_1|$. As $D\partial_1(q) = 0$, we also have $DY(q) = 0$. Moreover, as the manifold M is compact, $D^2Y(q)$ is bounded by a constant that depends on M only: we have that

$$\left| D^2Y(q) \right| \leqq C^M. \tag{6.12}$$

We will show that the Laplacian of $p \mapsto f_\varepsilon(p, Y(p))$ is positive at q, which contradicts the maximality of $(q, Y(q))$ in (6.11). Let us estimate $\Delta(D^2u_m(Y,Y))$ at the point q, using the abstract index notation.

$$\begin{aligned}\Delta(D^2u_m(Y,Y)) &= g^{ab}D_aD_b(D^2_{cd}u_mY^cY^d)\\ &= g^{ab}\Big(D^4_{abcd}u_mY^cY^d + D^3_{acd}u_mD_b(Y^cY^d)\\ &\qquad + D^3_{bcd}u_mD_a(Y^cY^d) + D^2_{cd}u_mD^2_{ab}(Y^cY^d)\Big).\end{aligned} \tag{6.13}$$

We may divide the right-hand side into four terms and estimate them at the point q individually. The second term is null because it contains $D_b(Y^cY^d) = (D_bY^c)Y^d +$

$Y^c D_b Y^d$, and $DY = 0$. The third term is also null, for the same reason. By (6.12), we can estimate the fourth term as follows:

$$g^{ab} D^2_{cd} u_m D^2_{ab}(Y^c Y^d) \geqq -C^M \left| D^2 u_m \right| \geq -\frac{C^M}{\varepsilon} - \varepsilon \left| D^2 u_m \right|^2. \tag{6.14}$$

It now remains to estimate the first term of (6.13). Using the notation

$$D_{[ab]} := D_a D_b - D_b D_a,$$

we compute that

$$\begin{aligned} D_a D_b D_c D_d u_m = {} & D_a D_{[bc]} D_d u_m + D_{[ac]} D_b D_d u_m \\ & + D_c D_a D_{[bd]} u_m + D_c D_{[ad]} D_b u_m + D_c D_d D_a D_b u_m. \end{aligned}$$

By definition of the Riemann tensor we have

$$D_a D_{[bc]} D_d u_m = D_a(R_{bced} D^e u_m) = (D_a R_{bced}) D^e u_m + R_{bced} D_a D^e u_m,$$

and so

$$\left| D_a D_{[bc]} D_d u_m \right| \geqq -C^M \left| \nabla u_m \right| - C^M \left| D^2 u_m \right|. \tag{6.15}$$

Likewise,

$$\left| D_c D_{[ad]} D_b u_m \right| \geqq -C^M \left| \nabla u_m \right| - C^M \left| D^2 u_m \right|. \tag{6.16}$$

To compute the term $D_{[ac]} D_b D_d u_m$, let us pick some coordinates (x^i) and write $D_b D_d u_m = D^2_{ij} u_m \mathrm{d}x^i_b \mathrm{d}x^j_d$. Then, we have that

$$\begin{aligned} & D_{[ac]} D_b D_d u_m \\ & \quad = D_{[ac]}(D^2_{ij} u_m \mathrm{d}x^i_b \mathrm{d}x^j_d) \\ & \quad = (D_{[ac]} D^2_{ij} u_m) \mathrm{d}x^i_b \mathrm{d}x^j_d + D^2_{ij} u_m (D_{[ac]} \mathrm{d}x^i_b) \mathrm{d}x^j_d + D^2_{ij} u_m \mathrm{d}x^i_b (D_{[ac]} \mathrm{d}x^j_d) \\ & \quad = 0 + D^2_{ij} u_m R_{aceb} (\mathrm{d}x^i)^e \mathrm{d}x^j_d + D^2_{ij} u_m \mathrm{d}x^i_b R_{aced} (\mathrm{d}x^j)^e \\ & \quad = R_{aceb} D^e D_d u_m + R_{aced} D_b D^e u_m, \end{aligned}$$

and so

$$\left| D_{[ac]} D_b D_d u_m \right| \geqq -C^M \left| D^2 u_m \right|. \tag{6.17}$$

By symmetry of the tensor $D^2 u_m$, we have that

$$D_c D_a D_{[bd]} u_m = 0. \tag{6.18}$$

Putting (6.15), (6.16), (6.17) and (6.18) together, we find that

$$\left| g^{ab} D_a D_b D_c D_d u_m \right| \geqq -C^M \left| \nabla u_m \right| - C^M \left| D^2 u_m \right| - \left| g^{ab} D_c D_d D_a D_b u_m \right|.$$

In E_m, u_m has constant Laplacian, so

$$g^{ab} D_c D_d D_a D_b u_m = D_c D_d g^{ab} D_a D_b u_m = D_c D_d \Delta u_m = 0,$$

and we get that

$$\left| g^{ab} D_a D_b D_c D_d u_m \right| \geqq -C^M \left|\nabla u_m\right| - C^M \left|D^2 u_m\right|.$$

From this and the fact Y has norm 1, we deduce

$$\left| g^{ab} D_a D_b D_c D_d u_m Y^c Y^d \right| \geqq -C^M \varepsilon^{-1} - \varepsilon \left|\nabla u_m\right|^2 - \varepsilon \left|D^2 u_m\right|^2.$$

Combining this equation with (6.13) and (6.14), we obtain at the point q,

$$\Delta(D^2 u_m(Y, Y)) \geqq -C^M \varepsilon^{-1} - 2\varepsilon \left|D^2 u_m\right|^2 - \varepsilon \left|\nabla u_m\right|^2. \tag{6.19}$$

We recall the Bochner-Weitzenböck formula:

$$\Delta\big(\left|\nabla u_m\right|^2\big) = 2\mathrm{Ric}(\nabla u_m, \nabla u_m) + 2\left|D^2 u_m\right|^2 + 2(\nabla \Delta u_m, \nabla u_m).$$

As M is compact, there exists a constant $K > 0$ such that Ric $\geqq -K$. Using the fact that u_m has constant Laplacian in E_m, we get

$$\Delta\big(\left|\nabla u_m\right|^2\big) \geqq 2\left|D^2 u_m\right|^2 - 2K\left|\nabla u_m\right|^2. \tag{6.20}$$

Furthermore, using the fact that $\Delta u_m = -2m$ in E_m again, we find

$$\begin{aligned} \Delta\big(u_m^2\big) &= 2\left|\nabla u_m\right|^2 - 2m u_m \\ &\geqq 2\left|\nabla u_m\right|^2 - 2m\left|d\right|_{L^\infty}, \qquad (6.21) \\ \Delta u_m &= -2m. \qquad (6.22) \end{aligned}$$

Using (6.20), (6.21) and (6.22), we get

$$\begin{aligned} &\Delta\left(\varepsilon\left|\nabla u_m\right|^2 + (K+1)\varepsilon u_m^2 - ((K+1)\left|d\right|_{L^\infty} + 1)\varepsilon u_m\right) \\ &\qquad \geqq 2\varepsilon\left|D^2 u_m\right|^2 + \varepsilon\left|\nabla u_m\right|^2 + \varepsilon m. \end{aligned}$$

Setting $(C_1, C_2, C_3) = (1, K+1, ((K+1)\left|d\right|_{L^\infty} + 1))$, and recalling the definition of f_ε (6.8), we obtain, thanks to (6.19),

$$\Delta(f_\varepsilon(p, Y(p)))_{p=q} \geqq -C^M \varepsilon^{-1} + \varepsilon m.$$

In particular, if m is large enough, depending on M and ε, this contradicts the maximality of $(q, Y(q))$ in (6.11).

This concludes the proof of (6.9) and Proposition 6.1. □

In order to apply Proposition 6.1 to problem (1.15), we will need the following lemma:

Lemma 6.2. (Bound of D^2u_m near ∂E_m) *Let u_m be the solution of* (1.3)*, as in Theorem 1.3. Let $\varepsilon > 0$ be smaller than the distance from b to $\mathrm{Cut}_b(M)$. Let $E_m := \{u_m < d_b\}$. From Proposition 5.2, we know that for m large enough, we have $\overline{E_m} \subset M \backslash B(b, \varepsilon)$. Let $C > 0$ be such that d_b is C-semiconcave on $M \backslash B(b, \varepsilon)$. Then, for any m large enough, for any $\eta > 0$, there is a neighborhood $\mathcal{N}_{\eta,m}$ of ∂E_m in $M \setminus B(b, \varepsilon)$ such that*

$$D^2u_m \leq (C + \eta)\, Id \quad in \quad E_m \cap \mathcal{N}_{\eta,m}. \tag{6.23}$$

Proof. We will use a theorem for obstacle problems on $\mathbb{R}^n$. Let us show that u_m is the solution of an obstacle problem on an open subset of $\mathbb{R}^n$. Then, we will apply [13, Chapter 2, Theorem 3.8] to conclude that (6.23) holds.

The minimality of u_m in (1.15) implies

$$-\Delta u_m - 2m \geqq 0, \quad u_m \leqq d_b \quad \text{and} \quad (-\Delta u_m - 2m)(u_m - d_b) = 0. \tag{6.24}$$

Let $\widetilde{\Omega}$ be defined as in the proof of Proposition 3.4. Let $\phi : \widetilde{\Omega} \to \widetilde{U}$ be a normal coordinates chart. Writing down (6.24) in these coordinates, we find

$$A\widetilde{u_m} - 2m \geqq 0, \quad \widetilde{u_m} \leqq \psi \quad \text{and} \quad (A\widetilde{u_m} - 2m)(\widetilde{u_m} - \psi) = 0,$$

where A is the Laplacian of M in the coordinates defined by ϕ, $\widetilde{u_m} = u_m \circ \phi^{-1}$ and $\psi = d_b \circ \phi^{-1}$. This is the form of [13, Chapter 2, equation (3.16)], so we can apply [13, Chapter 2, Theorem 3.8], to deduce that

$$\forall p \in \partial E_m, \forall X \in \mathbb{R}^n \lim_{\substack{q \to p \\ q \in E_m}} D^2\widetilde{u_m}(\phi(q))(X, X) \leqq D^2\psi(\phi(p))(X, X). \tag{6.25}$$

Moreover, we have that

$$\begin{aligned} D^2\widetilde{u_m} &= D^2u_m \circ (D\phi^{-1}, D\phi^{-1}) + Du_m \circ D^2\phi^{-1}, \\ D^2\psi &= D^2d_b \circ (D\phi^{-1}, D\phi^{-1}) + Dd_b \circ D^2\phi^{-1}, \end{aligned}$$

and $Du_m = Dd_b$ on ∂E_m because u_m is C^1. Thus, (6.25) yields

$$\forall p \in \partial E_m, \forall X \in \mathbb{R}^n \lim_{\substack{q \to p \\ q \in E_m}} D^2u_m(q)(X_q, X_q) \leqq D^2d_b(p)(X_p, X_p),$$

where we have set $X_q := D\phi^{-1}(\phi(q))X$. As d_b is C-semiconcave, with Proposition 2.2, we get

$$\forall p \in \partial E_m, \forall X \in \mathbb{R}^n \lim_{\substack{q \to p \\ q \in E_m}} D^2u_m(q)(X_q, X_q) \leqq C\left|X_p\right|^2. \tag{6.26}$$

From there, we deduce that

$$\text{for } q \in E_m \text{ close enough to } \partial E_m, \text{ we have } D^2u_m(q) \leqq C + \eta. \tag{6.27}$$

Indeed, if not, there exist a sequence (q_k) of points of E_m whose distance to ∂E_m goes to 0, and a sequence (X_k) of unit vectors of $\mathbb{R}^2$ such that for any $k \in \mathbb{N}$,

$$D^2u_m(q_k)\big((X_k)_{q_k}, (X_k)_{q_k}\big) > C + \eta. \tag{6.28}$$

As E_m is precompact, up to extracting a subsequence, we can assume that (q_k) converges to a point $p \in \partial E_m$, and (X_k) converges to a vector $Y \in \mathbb{R}^n$. Because of (6.26), we have

$$\lim_{k\to\infty} D^2 u_m(q_k)(Y_{q_k}, Y_{q_k}) \leqq C. \tag{6.29}$$

Furthermore, we know from Proposition 3.4 that $D^2 u_m$ is locally bounded. As $(X_k)_{q_k} - Y_{q_k}$ converges to 0 when k goes to ∞, this implies

$$\lim_{k\to\infty} D^2 u_m(q_k)\big((X_k)_{q_k}, (X_k)_{q_k}\big) - D^2 u_m(q_k)(Y_{q_k}, Y_{q_k}) = 0. \tag{6.30}$$

Inequalities (6.28), (6.29) and (6.30) yield a contradiction. So (6.27) is true. This concludes the proof. □

7. Convergence of the Gradients

In this section, we show that the uniform semiconcavity of u_m implies the convergence of the gradients in the sense of Theorem 1.3 (T6). We notice that the results from this section also apply to more general sequences of semiconcave functions.

7.1. Lower semicontinuity

The main result of this section is Proposition 7.2, which proves the first inequality in Theorem 1.3 (T6). We start with the following lemma:

Lemma 7.1. *Let $u : M \to \mathbb{R}$ be a C-semiconcave function. Let $p, q \in M$ be such that there exists a geodesic from p to q. Then,*

$$u(q) \leqq u(p) + |\nabla u(p)|\, d(p,q) + \frac{C}{2} d(p,q)^2,$$

where $|\nabla u|(p)$ is the norm of the generalized gradient, defined in (2.3).

Proof. Let $\gamma : [0, d(p,q)] \to M$ be a geodesic from p to q. Consider the function $f(t) = \frac{1}{2}Ct^2 - u(\gamma(t))$. By the semiconcavity of u, we know that f is convex. Thus, we have

$$f(d(p,q)) \geqq f(0) + f'(0) d(p,q).$$

On the other hand, setting $\dot\gamma(0) := v \in T_p(M)$, by construction, we have

$$f(0) = -u(p)\,, \quad f(d(p,q)) = \frac{C}{2} d(p,q)^2 - u(q)\,, \quad \text{and} \quad f'(0) = -\partial_v^+ u(p).$$

Thus, we obtain

$$u(q) \leqq u(p) + d(p,q)\partial_v^+ u(p) + \frac{C}{2} d(p,q)^2 \leq u(p) + |\nabla u(p)|\, d(p,q) + \frac{C}{2} d(p,q)^2.$$

□

Proposition 7.2. *Let M be a Riemannian manifold and let $C > 0$ be a fixed constant. Let $u_k : M \to \mathbb{R}$ be a sequence of C-semiconcave continuous functions that converges locally uniformly to a continuous function $u_\infty : M \to \mathbb{R}$. Then, u_∞ is also C-semiconcave, and for any sequence of points $p_k \to p_\infty \in M$, we have*

$$|\nabla u_\infty|(p_\infty) \leq \liminf_{k\to\infty} |\nabla u_k|(p_k). \tag{7.1}$$

Proof. First, notice that the C-semiconcavity of u_∞ is an immediate consequence of the pointwise convergence and the C-semiconcavity of u_k. In particular, the generalized gradients $\left|\nabla u_{m_k}\right|(p_k)$ and $|\nabla u_\infty|(p_\infty)$ are well-defined by Proposition 2.4. Thus, we only need to prove (7.1). We notice that (7.1) is trivial if $|\nabla u_\infty|(p_\infty) = 0$. Thus, we suppose that $|\nabla u_\infty|(p_\infty) > 0$. In particular, there are a vector $v \in \mathbb{S}^{n-1}(T_{p_\infty}M)$ and a unit speed geodesic γ with $\gamma(0) = p_\infty$ and $\dot\gamma(0) = v$ such that

$$|\nabla u_\infty(p_\infty)| = \lim_{t\to 0^+} \frac{u_\infty(\gamma(t)) - u_\infty(p_\infty)}{t}.$$

In particular, for any $\varepsilon > 0$, we can find $q \in M$ such that $d(p_\infty, q) \leq \varepsilon$ and

$$|\nabla u_\infty(p_\infty)| \leqq \frac{u_\infty(q) - u_\infty(p_\infty)}{d(q, p_\infty)} + \varepsilon.$$

Then, by the uniform convergence of u_k and Lemma 7.1, we get

$$\begin{aligned}|\nabla u_\infty(p_\infty)| \leqq \liminf_{k\to\infty} \frac{u_k(q) - u_k(p_k)}{d(q, p_k)} + \varepsilon &\leqq \liminf_{k\to\infty} |\nabla u_k(p_k)| + \frac{C}{2} d(q, p_k) + \varepsilon \\ &\leq \liminf_{k\to\infty} |\nabla u_k(p_k)| + (C+1)\varepsilon,\end{aligned}$$

which concludes the proof, as the inequality holds for any ε. □

7.2. *Proof of Theorem 1.3 (T6)*

The claim (1.8) follows from Proposition 7.2. Thus, we only need to prove (1.9). First, notice that, if $|\nabla d_b|(p_\infty) = 1$, then (1.9) follows from (1.8) and the fact that u_m is 1-Lipschitz. Let now $|\nabla d_b|(p_\infty) < 1$. Suppose by contradiction that there are a subsequence $m_k \underset{k\to+\infty}{\longrightarrow} +\infty$ and constants $\varepsilon > 0$ and $\eta_0 > 0$ such that

$$|\nabla d_b|(p_\infty) + \varepsilon \leq |\nabla u_{m_k}|(p) \quad \text{for every} \quad p \in B_{\eta_0}(p_\infty) \quad \text{and every} \quad k \geq 0.$$

We now fix $\eta \leq \eta_0$, which will be chosen later in the proof. Let $(q_t)_{t\geqq 0}$ be the curve defined by

$$q_0 = p_\infty \quad \text{and} \quad \frac{\mathrm{d}q_t}{\mathrm{d}t} = \nabla u_{m_k}(q_t).$$

Let $T > 0$ be such that for any $t \in [0, T]$, $d(q_t, p_\infty) \leqq \eta$, and in particular

$$|\nabla d_b|(p_\infty) + \varepsilon \leq |\nabla u_{m_k}|(q_t) \quad \text{for every} \quad t \in [0, T].$$

We have

$$u_{m_k}(q_T) - u_{m_k}(p_\infty) = \int_0^T \left|\nabla u_{m_k}(q_t)\right|^2 \mathrm{d}t \geqq \int_0^T \left(|\nabla d_b|(p_\infty) + \varepsilon\right)^2 \mathrm{d}t$$
$$= T\left(|\nabla d_b|(p_\infty) + \varepsilon\right)^2.$$

As u_{m_k} is bounded by the diameter of M, this estimate implies that there exists a finite biggest time $T > 0$ such that for any $t \in [0, T]$, $d(q_t, p_\infty) \leqq \eta$. In particular, $d(p_\infty, q_T) = \eta$. Let γ be a unit speed minimizing geodesic between p_∞ and q_T. By Proposition 2.7, there is a constant $C_d > 0$ such that d_b is C_d-semiconcave in $B_{\eta_0}(p_\infty)$. In particular, by Lemma 7.1, we have that

$$d_b(q_T) - d_b(p_\infty) \leqq |\nabla d_b(p_\infty)|\, d(p_\infty, q_T) + C_d(d(p_\infty, q_T))^2 \tag{7.2}$$
$$= |\nabla d_b(p_\infty)|\, \eta + C_d\eta^2 \,. \tag{7.3}$$

On the other hand,

$$u_{m_k}(q_T) - u_{m_k}(p_\infty) \tag{7.4}$$
$$= \int_0^T \left|\nabla u_{m_k}(q_t)\right| \left|\frac{\mathrm{d}q_t}{\mathrm{d}t}\right| \mathrm{d}t \geqq \int_0^T \left(|\nabla d_b|(p_\infty) + \varepsilon\right) \left|\frac{\mathrm{d}q_t}{\mathrm{d}t}\right| \mathrm{d}t$$
$$= \left(|\nabla d_b|(p_\infty) + \varepsilon\right) \int_0^T \left|\frac{\mathrm{d}q_t}{\mathrm{d}t}\right| \mathrm{d}t \geqq \left(|\nabla d_b|(p_\infty) + \varepsilon\right) d(q_0, q_T)$$
$$= \left(|\nabla d_b|(p_\infty) + \varepsilon\right)\eta. \tag{7.5}$$

Combining (7.3) and (7.5), we get that

$$\varepsilon\eta - C_d\eta^2 \leq \left(u_{m_k}(q_T) - u_{m_k}(p_\infty)\right)$$
$$-\left(d_b(q_T) - d_b(p_\infty)\right) \leq 2\|u_{m_k} - d_b\|_{L^\infty(M)}.$$

Now, taking η small enough, we get that

$$\frac{1}{2}\varepsilon\eta \leq 2\|u_{m_k} - d_b\|_{L^\infty(M)} \quad \text{for every} \quad k \geq 0,$$

but this is in contradiction with the uniform convergence of u_m to d_b. □

Acknowledgements. The three authors were partially supported by Agence Nationale de la Recherche (ANR) with the projects GeoSpec (LabEx PERSYVAL-Lab, ANR-11-LABX-0025-01), CoMeDiC (ANR-15-CE40-0006) and ShapO (ANR-18-CE40-0013). The third author is supported by the European Research Council (ERC), under the European Union's Horizon 2020 research and innovation programme, through the project ERC VAREG *Variational approach to the regularity of the free boundaries* (grant agreement No. 853404).

Funding Open access funding provided by Università di Pisa within the CRUI-CARE Agreement.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A. Appendix About Semiconcavity

In this section we prove that defining local semiconcavity through charts (as in [20]), or through geodesics, is the same (see Proposition 2.3). We recall the notation $\lambda_{ab} = (1-\lambda)a + \lambda b$, for $a, b \in \mathbb{R}$ and $\lambda \in [0, 1]$ and we notice that the C-semiconcavity of $u : M \to \mathbb{R}$ (in the sense of Definition 1.9) can be rewritten as

$$\lambda_{u(\gamma(a))u(\gamma(b))} - u(\gamma(\lambda_{ab})) \leqq C\lambda(1-\lambda)(b-a)^2,$$

for every unit speed geodesic $\gamma : [a, b] \to M$ and any $\lambda \in [0, 1]$.
In order to prove Proposition 2.3, we need the following lemma, which shows how to estimate the difference between two geodesics linking a pair of given points, for two different metrics.

Lemma A.1. *Let g be a metric on the unit ball $B_1(0) \subset \mathbb{R}^n$. There exists a constant $B > 0$ such that for any unit speed geodesic $\gamma : [a, b] \to (B_1(0), g)$ and $\lambda \in [0, 1]$, we have*

$$\left|\gamma(\lambda_{ab}) - \lambda_{\gamma(a)\gamma(b)}\right| \leqq B\lambda(1-\lambda)(b-a)^2.$$

Proof. It suffices to prove that the estimate holds for $\lambda \leqq \frac{1}{2}$, as the case $\lambda \geqq \frac{1}{2}$ can be deduced by considering $\widetilde{\gamma} : t \mapsto \gamma(b-t)$ instead of γ. A unit speed geodesic $\gamma : [a, b] \to (B_1(0), g)$ satisfies the geodesic equation

$$\ddot{\gamma}^l + \Gamma^l_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0,$$

where Γ^l_{ij} are the Christoffel symbols of the metric g. As γ is unit speed, the $(\dot{\gamma}^i)$ are bounded, uniformly in γ. Therefore, there exists a constant $\alpha > 0$ independent of γ such that $|\ddot{\gamma}| \leqq \alpha$. By integration, we find

$$|\gamma(t) - \gamma(a) - \dot{\gamma}(a)(t-a)| \leqq \alpha(t-a)^2.$$

Evaluating this expression at b yields

$$|\gamma(b) - \gamma(a) - \dot{\gamma}(a)(b-a)| \leqq \alpha(b-a)^2.$$

From these two estimates, we deduce

$$\left|\gamma(t) - \gamma(a) - \frac{\gamma(b)-\gamma(a)}{b-a}(t-a)\right| \leqq \alpha(t-a)^2 + \alpha(b-a)(t-a).$$

Taking $t = (1-\lambda)a + \lambda b$ in this estimate yields

$$
\begin{aligned}
&|\gamma((1-\lambda)a + \lambda b) - ((1-\lambda)\gamma(a) + \lambda\gamma(b))| \\
&\quad \leqq \alpha\lambda(1+\lambda)(b-a)^2 = \frac{\alpha(1+\lambda)}{1-\lambda}\lambda(1-\lambda)(b-a)^2.
\end{aligned}
$$

Taking $B := \frac{\alpha(1+1/2)}{1-1/2}$, this proves the desired estimate when $\lambda \leqq 1/2$. This concludes the proof. □

Proof of Proposition 2.3. Let us assume that u is locally semiconcave. Let $\psi : U \to V$ be a chart from an open set U of M to on open set V of $\mathbb{R}^n$, and $y \in V$. Let $f := u \circ \psi^{-1}$. We want to show that f is semiconcave in a neighborhood of y, as a function of $\mathbb{R}^n$. We first observe that f is locally semiconcave on the manifold $(V, \psi_\star g)$. Let $V' \subset V$ be a neighborhood of y that is geodesically convex for the metric $\psi_\star g$, and such that there exists a constant $C > 0$ such that f is C-semiconcave on $(V', \psi_\star g)$. Let d denote the distance function on $(V', \psi_\star g)$. Up to taking V' smaller, we may assume that the metric $\psi_\star g$ is bounded on V', and so there exists a constant $\beta > 0$ such that

$$
\forall x, y \in V', \quad d(x, y) \leqq \beta\, |x - y|\,.
$$

Let $x, y \in V'$ be such that $[x, y] \subset V'$, and $\lambda \in [0, 1]$. Let $\gamma : [a, b] \to V'$ be a unit speed geodesic of $(V', \psi_\star g)$ from x to y. By the C-semiconcavity of f on $(V', \psi_\star g)$, we have that

$$
\begin{aligned}
\lambda_{f(x)f(y)} - f(\lambda_{xy}) &= \lambda_{f(\gamma(a))f(\gamma(b))} - f(\lambda_{\gamma(a)\gamma(b)}) \\
&\leqq C\lambda(1-\lambda)(b-a)^2 + f(\gamma(\lambda_{ab})) - f(\lambda_{\gamma(a)\gamma(b)}) \\
&\leqq C\lambda(1-\lambda)(b-a)^2 + \mathrm{Lip}(f)\left|\gamma(\lambda_{ab}) - \lambda_{\gamma(a)\gamma(b)}\right|.
\end{aligned}
$$

Applying Lemma A.1 above, we get a constant $B > 0$ such that

$$
\begin{aligned}
\lambda_{f(x)f(y)} - f(\lambda_{xy}) &\leqq (C + \mathrm{Lip}(f)B)\lambda(1-\lambda)(b-a)^2 \\
&= (C + \mathrm{Lip}(f)B)\lambda(1-\lambda)(d(x, y))^2 \\
&\leqq (C + \mathrm{Lip}(f)B)\beta^2\lambda(1-\lambda)\,|x-y|^2\,,
\end{aligned}
$$

and so f is semiconcave on V', as a function of $\mathbb{R}^n$.
Reciprocally, let us assume that $u \circ \psi^{-1}$ is locally semiconcave as a function of $\mathbb{R}^n$ for any chart ψ. Then, we can show that $u \circ \psi^{-1}$ is locally semiconcave for the metric $\psi_\star g$, for any chart ψ, by using the same technique. From there we deduce that u is locally semiconcave. This concludes the proof. □

Appendix B. A Counter-Example to the Equivalence of (1.3) and (1.15) for Small m

Theorem B.1. *There exist a surface of revolution M and a parameter $m > 0$ such that $u_m \neq u_m^d$.*

Proof of Theorem B.1. Let r_θ denote the rotation of $\mathbb{R}^3$ of angle $\theta \in [0, 2\pi)$ around the z-axis.

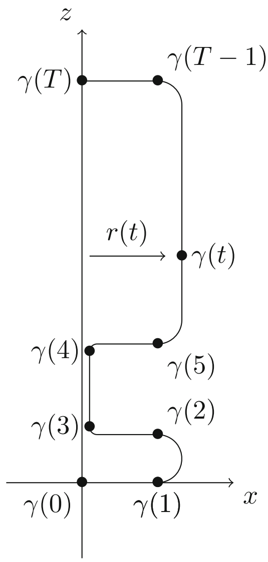

Fig. 3. The curve γ

Let $T := 10^{10}$ and $r, h : [0, T] \to \mathbb{R}$ be two smooth functions such that

$$\begin{aligned}
\gamma : t \mapsto (r(t), 0, h(t)) &\text{ is a unit speed curve.} \\
M := \{r_\theta(\gamma(t)) : (t, \theta) \in [0, T] \times [0, 2\pi)]\} &\text{ is a smooth surface,} \\
r(0) = r(T) = 0, & \\
r \leqq 2, & \\
r([1, 2]) \subset [1, 2], & \\
r([3, 4]) \subset (0, 10^{-10}), & \\
r([5, T-1]) \subset [1, 2]. &
\end{aligned}$$

This information is pictured in Fig. 3. We chose $b = (0, 0, 0)$ as the base point on M, and $m = 10^{-10}$. Let us assume that $u_m^d = u_m$ and build a better competitor in (1.15) to contradict the minimality of u_m^d. We will first reduce (1.15) to a one-dimensional problem. Note that the functional we are minimizing is rotation-invariant. More precisely, for any $\theta \in (0, 2\pi)$ and $u \in H^1(M)$, we have that

$$\int_M |\nabla(u \circ r_\theta)|^2 - m(u \circ r_\theta) = \int_M |\nabla u|^2 - mu. \tag{B.1}$$

By the uniqueness of the minimizer u_m^d, we deduce that u_m^d is rotation-invariant, i.e. there exists a function $\rho_m : [0, T] \to \mathbb{R}$ such that for any $\theta \in [0, 2\pi)$ and $t \in [0, T]$, $u_m^d(r_\theta(\gamma(t))) = \rho_m(t)$. Thus u_m^d is a minimizer of (1.15) among rotation-invariant functions. Let $u : M \to \mathbb{R}$ be any rotation-invariant function, and $\rho : [0, T] \to \mathbb{R}$ be such that for any $\theta \in [0, 2\pi)$, $u(r_\theta(\gamma(t))) = \rho(t)$. We will translate the minimization problem (1.15) on u into a problem on ρ.

First, because M is a surface of revolution, all the geodesics starting from $b = (0, 0, 0)$ have a constant angle θ. Thus, they are of the form $t \mapsto r_\theta(\gamma(t))$ for some $\theta \in [0, 2\pi)$. These are actually unit speed geodesics as γ is unit speed. Hence, $d_b(r_\theta(\gamma(t))) = t$, and the constraint $u \leqq d_b$ in (1.15) is equivalent to $\rho(t) \leqq t$.

Secondly, we translate the H^1 constraint. To this end, let us define some coordinates (t, θ) on M via the map

$$\phi : (0, T) \times (0, 2\pi) \to M\ , \quad \phi(t, \theta) = r_\theta(\gamma(t)).$$

We have that

$$\begin{aligned}\int_M |\nabla u|^2 &= \int_0^{2\pi} \int_0^T (|\nabla u|^2 \circ \phi) J\phi \, \mathrm{d}t \, \mathrm{d}\theta \\ &= \int_0^{2\pi} \int_0^T |\nabla u|^2 (r_\theta(\gamma(t)))r(t) \, \mathrm{d}t \, \mathrm{d}\theta = 2\pi \int_0^T |\nabla u|^2 (\gamma(t))r(t) \, \mathrm{d}t, \quad \text{(B.2)}\end{aligned}$$

because u is rotation-invariant. Moreover, as u is rotation-invariant, its gradient at the point $\gamma(t)$ is parallel to $\gamma'(t)$, and so

$$\left|\rho'(t)\right| = \left|\nabla u(\gamma(t)) \cdot \gamma'(t)\right| = |\nabla u(\gamma(t))| \left|\gamma'(t)\right| = |\nabla u(\gamma(t))| \,.$$

Hence (B.2) gives

$$\int_M |\nabla u|^2 = 2\pi \int_0^T \rho'(t)^2 r(t) \, dt$$

Thus, the constraint $u \in H^1(M)$ in (1.15) is equivalent to $v \in H^1((0, T), r(t)\mathrm{d}t)$. Thirdly, we may compute the functional likewise:

$$\int_M |\nabla u|^2 - mu = 2\pi \int_0^T \left(\rho'(t)^2 - m\rho(t)\right) r(t)\mathrm{d}t.$$

All in all, as u_m^d is a minimizer in (1.15), ρ_m is a minimizer of

$$\inf \left\{ \int_0^T \left(\rho'(t)^2 - m\rho(t)\right) r(t)\mathrm{d}t \ : \ \rho \in H^1\big((0, T), r(t)\mathrm{d}t\big), \ \rho(t) \leqq t \right\}. \tag{B.3}$$

The idea of the rest of the proof is as follows: first, we recall the assumption $u_m^d = u_m$, which means that $\left|\nabla u_m^d\right| \leqq 1$, and so $\left|\rho_m'\right| \leqq 1$. Now, if $\rho_m(4)$ is close to 4, then $\rho_m'(t)$ is close to 1 for $t \leqq 4$, so a competitor v such that $\rho'(t)$ is small for $t \leqq 4$ will contradict the minimality of ρ_m in (B.3). If on the contrary $\rho_m(4)$ is significantly smaller than 4, then for $t \geqq 4$, $\rho_m(t)$ will be significantly smaller than t, so a competitor ρ such that $\rho(t)$ is closer to t for $t \geqq 4$ will contradict the minimality of ρ_m in (B.3). Because we chose r very small on the interval $[3, 4]$ (see Fig. 3), we can define a competitor ρ independently on $[0, 3]$ and $[4, T]$, without paying much for the behavior of ρ on $[3, 4]$.

Case one: $\rho_m(4) \in [3.5, 4]$. Let us define a competitor ρ for (B.3):

$$\rho : [0, T] \to \mathbb{R}\ , \quad \rho(t) = \begin{cases} 0 & \text{if} \quad t \in [0, 3] \\ 4(t-3) & \text{if} \quad t \in [3, 4] \\ \rho_m(t) + 4 - \rho_m(4) & \text{if} \quad t \geqq 4 \end{cases}.$$

Let us call $\mathcal{F}(\rho)$ the functional appearing in (B.3). We have, from the definition of r and ρ,

$$\begin{aligned}\mathcal{F}(\rho) = &\int_3^4 \big(16 - 4m(t-3)\big) r(t)\mathrm{d}t + \int_4^T \left(\rho_m'^2(t) - m\rho_m(t)\right) r(t)\mathrm{d}t \\ &- m(4 - \rho_m(4)) \int_4^T r(t)\mathrm{d}t \\ &\leqq (16 - 0) \cdot 10^{-10} + \int_4^T \left(\rho_m'^2(t) - m\rho_m(t)\right) r(t)\mathrm{d}t - 0,\end{aligned} \tag{B.4}$$

so

$$\begin{aligned}\mathcal{F}(\rho)-\mathcal{F}(\rho_m) &\leqq 16\cdot 10^{-10}-\int_0^4\Big(\rho_m'^2(t)-m\rho_m(t)\Big)\,r(t)\,\mathrm{d}t\\ &\leqq 16\cdot 10^{-10}-\int_1^2\rho_m'^2(t)r(t)\,\mathrm{d}t+m\int_0^4\rho_m(t)r(t)\,\mathrm{d}t\\ &\leqq 16\cdot 10^{-10}-\int_1^2\rho_m'^2(t)r(t)\,\mathrm{d}t+m\int_0^4 2t\,\mathrm{d}t\\ &= 16\cdot 10^{-10}-\int_1^2\rho_m'^2(t)r(t)\,\mathrm{d}t+16m.\end{aligned}\tag{B.5}$$

We are left to bound from below the integral term in (B.5). By, Hölder inequality we have that

$$\int_1^2\rho_m'\leqq\left(\int_1^2\frac{1}{r}\right)^{1/2}\left(\int_1^2\rho_m'^2 r\right)^{1/2},$$

and so

$$\int_1^2\rho_m'^2 r\geqq\frac{(\rho_m(2)-\rho_m(1))^2}{\int_1^2\frac{1}{r}}\geqq(\rho_m(2)-\rho_m(1))^2,\tag{B.6}$$

by the construction of r. Now we use the fact $u_m^d=u_m$, which means that $\left|\nabla u_m^d\right|\leqq 1$, and so $\left|\rho_m'\right|\leqq 1$. With the running assumption $\rho_m(4)\geqq 3.5$, this implies $\rho_m(2)\geqq 1.5$. As $\rho_m(1)\leqq 1$, we get $\rho_m(2)-\rho_m(1)\geqq 0.5$. Then, (B.6) and (B.5) yield

$$\mathcal{F}(\rho)-\mathcal{F}(\rho_m)\leqq 16\cdot 10^{-10}-0.25+16m.\tag{B.7}$$

Recalling that we have chosen $m=10^{-10}$, it contradicts the minimality of ρ_m in (B.3).
Case two: $\rho_m(4)\leqq 3.5$. We use the same competitor ρ as in case one. We even perform similar estimates, the only difference being that we don't estimate the term $-m(4-\rho_m(4))\int_{(4,T)}r(t)\mathrm{d}t$ by 0 as in (B.4). Thus (B.5) becomes instead

$$\begin{aligned}&\mathcal{F}(\rho)-\mathcal{F}(\rho_m)\\ &\leqq 16\cdot 10^{-10}-\int_1^2\rho_m'^2(t)r(t)\mathrm{d}t+16m-m(4-\rho_m(4))\int_4^T r(t)\mathrm{d}t.\\ &\leqq 16\cdot 10^{-10}+16m-0.5m\int_4^T r(t)\mathrm{d}t\\ &\leqq 16\cdot 10^{-10}+16m-0.5m\int_5^{T-1} r(t)\mathrm{d}t.\end{aligned}$$

Recalling that we have chosen $m=10^{-10}$, $T=10^{10}$ and $r\geqq 1$ between 5 and $T-1$, it contradicts the minimality of ρ_m in (B.3). This concludes the proof. □

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(*Received April 6, 2021 / Accepted September 15, 2022*)
Published online October 1, 2022