



The N -Clock Model: Variational Analysis for Fast and Slow Divergence Rates of N

MARCO CICALESSE, GIANLUCA ORLANDO & MATTHIAS RUF 

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Abstract

We study a nearest neighbors ferromagnetic classical spin system on the square lattice in which the spin field is constrained to take values in a discretization of the unit circle consisting of N equi-spaced vectors, also known as the N -clock model. We find a fast rate of divergence of N with respect to the lattice spacing for which the N -clock model has the same discrete-to-continuum variational limit as the classical XY model (also known as planar rotator model), in particular concentrating energy on topological defects of dimension 0. We prove the existence of a slow rate of divergence of N at which the coarse-grain limit does not detect topological defects, but it is instead a BV -total variation. Finally, the two different types of limit behaviors are coupled in a critical regime for N , whose analysis requires the aid of Cartesian currents.

1. Introduction

The emergence of phase transitions mediated by the formation of topological singularities has been proposed in pioneering works on the ferromagnetic XY model (also planar rotator model) [15, 34, 35]. The latter describes a system of \mathbb{S}^1 vectors sitting on a square lattice, in which only the nearest neighbors interact in a ferromagnetic way. If the spin field is allowed to attain only finitely many, say N , equi-spaced values on \mathbb{S}^1 (as in the N -clock model) this topological concentration is ruled out. Instead, the phase transitions are characterized by a typical domain structure as in Ising systems. These two different behaviors lead to the natural question whether the N -clock model approximates the XY model as $N \rightarrow +\infty$. Fröhlich and Spencer give a positive answer to this question, showing in [29] that the N -clock model (for N large enough) presents phase transitions mediated by the formation and interaction of topological singularities.

The results in this paper and in [25, 26] concern a related problem regarding the behavior of low-energy states of the two systems in the discrete-to-continuum

variational analysis as the lattice spacing vanishes and N diverges simultaneously. With the help of fine concepts in geometric measure theory and in the theory of cartesian currents, these results show to which extent the coarse-grain limit of the classical N -clock model resembles the one of the classical XY model obtained in [4, 7]. To state precisely the results, we set the mathematical framework for the problem.

We consider a bounded, open set with Lipschitz boundary $\Omega \subset \mathbb{R}^2$. Given a small parameter $\varepsilon > 0$, we consider the square lattice $\varepsilon\mathbb{Z}^2$ and we define $\Omega_\varepsilon := \Omega \cap \varepsilon\mathbb{Z}^2$. The classical XY energy (relative to its minimum) is defined on spin fields $u: \Omega_\varepsilon \rightarrow \mathbb{S}^1$ by

$$XY_\varepsilon(u) = \frac{1}{2} \sum_{\langle i, j \rangle} \varepsilon^2 |u(\varepsilon i) - u(\varepsilon j)|^2,$$

where the sum is taken over ordered pairs of nearest neighbors $\langle i, j \rangle$, i.e., $(i, j) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ such that $|i - j| = 1$ and $\varepsilon i, \varepsilon j \in \Omega_\varepsilon$. We consider the additional parameter $N_\varepsilon \in \mathbb{N}$ or, equivalently, $\theta_\varepsilon := \frac{2\pi}{N_\varepsilon}$, and we set

$$\mathcal{S}_\varepsilon := \{\exp(\iota k \theta_\varepsilon) : k = 0, \dots, N_\varepsilon - 1\},$$

where ι is the imaginary unit. The admissible spin fields we consider here are only those taking values in the discrete set \mathcal{S}_ε . We study the energy defined for every $u: \Omega_\varepsilon \rightarrow \mathbb{S}^1$ by

$$E_\varepsilon(u) := \begin{cases} XY_\varepsilon(u) & \text{if } u: \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1}$$

We are interested in the behavior of low-energy states u_ε such that $E_\varepsilon(u_\varepsilon) \leq C\kappa_\varepsilon$ with $\kappa_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ to be determined. To this end, given $\theta_\varepsilon \rightarrow 0$, we find the relevant scaling κ_ε and we study the Γ -limit of $\frac{1}{\kappa_\varepsilon} E_\varepsilon$. The limit strongly depends on the rate of convergence $\theta_\varepsilon \rightarrow 0$ and can be characterized by interfacial-type singularities [2, 8, 12, 17, 20, 21, 23, 24, 27] (see also [5, 18]) or vortex-like singularities [3, 4, 6, 7, 14, 19, 38], possibly coexisting. In this paper we are interested in the following three regimes: $\varepsilon |\log \varepsilon| \ll \theta_\varepsilon$, $\theta_\varepsilon \sim \varepsilon |\log \varepsilon|$, and $\theta_\varepsilon \ll \varepsilon$. The intermediate case $\varepsilon \ll \theta_\varepsilon \ll \varepsilon |\log \varepsilon|$ has been covered in [25].

To understand how the limit is affected by the choice of $\theta_\varepsilon \rightarrow 0$, we start by considering the following example. Let $\Omega = B_{1/2}(0)$ be the ball of radius $\frac{1}{2}$ centered at 0, let $\varphi_1, \varphi_2 \in [0, 2\pi)$ and $v_1 = \exp(\iota\varphi_1), v_2 = \exp(\iota\varphi_2) \in \mathbb{S}^1$, and for $x = (x_1, x_2) \in \mathbb{R}^2$ define

$$u(x) := \begin{cases} v_1 & \text{if } x_1 \leq 0, \\ v_2 & \text{if } x_1 > 0. \end{cases} \tag{1.2}$$

For $\varepsilon i = (\varepsilon i_1, \varepsilon i_2) \in \Omega_\varepsilon$ and $\eta_\varepsilon > 0$ we define

$$u_\varepsilon(\varepsilon i) := \begin{cases} v_1 & \text{if } \varepsilon i_1 \leq 0, \\ \exp\left(\iota\left((\varphi_1 - \varphi_2)\left(1 - \frac{\varepsilon i_1}{\eta_\varepsilon}\right) + \varphi_2\right)\right) & \text{if } 0 < \varepsilon i_1 \leq \eta_\varepsilon, \\ v_2 & \text{if } \varepsilon i_1 > \eta_\varepsilon. \end{cases} \tag{1.3}$$

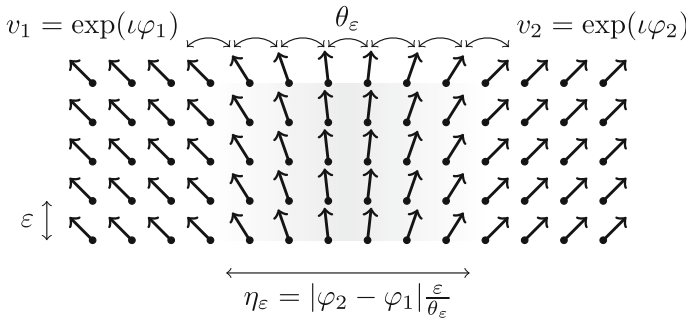


Fig. 1. Construction which shows that $\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon$ approximates the geodesic distance between the two values v_1 and v_2 of a pure-jump function. During the transition between v_1 and v_2 in the strip of size $\eta_\varepsilon = |\varphi_1 - \varphi_2| \frac{\varepsilon}{\theta_\varepsilon}$ the minimal angle between two adjacent vectors is θ_ε

If u_ε satisfies the pointwise constraint $u_\varepsilon(\varepsilon i) \in \mathcal{S}_\varepsilon$, then $|\varphi_1 - \varphi_2| \frac{\varepsilon}{\eta_\varepsilon} \sim \theta_\varepsilon$, i.e., $\eta_\varepsilon \sim |\varphi_1 - \varphi_2| \frac{\varepsilon}{\theta_\varepsilon}$, see Fig. 1. As a result,

$$\begin{aligned} \frac{1}{\kappa_\varepsilon} E_\varepsilon(u_\varepsilon) &\sim \left(1 - \cos\left(\frac{\varepsilon}{\eta_\varepsilon}(\varphi_1 - \varphi_2)\right)\right) \frac{\eta_\varepsilon}{\kappa_\varepsilon} \\ &\sim (1 - \cos(\theta_\varepsilon)) \frac{\varepsilon}{\theta_\varepsilon \kappa_\varepsilon} |\varphi_1 - \varphi_2| \sim \frac{\varepsilon \theta_\varepsilon}{\kappa_\varepsilon} |\varphi_1 - \varphi_2|. \end{aligned}$$

This suggests that the nontrivial scaling $\kappa_\varepsilon = \varepsilon\theta_\varepsilon$ leads to a finite energy proportional to $|\varphi_1 - \varphi_2|$. The construction can be optimized by choosing the angles φ_1 and φ_2 in such a way that $|\varphi_1 - \varphi_2|$ equals the geodesic distance on \mathbb{S}^1 between v_1 and v_2 , namely $d_{\mathbb{S}^1}(v_1, v_2)$. This back-of-the-envelope calculation shows that the presence of θ_ε allows us to detect energy concentration on interfaces. Such a behavior is ruled out in the classical XY model, see [4, Example 1].

The fact that $d_{\mathbb{S}^1}(v_1, v_2)$ is the total variation (in the sense of [10, Formula (2.11)]) of the \mathbb{S}^1 -valued pure-jump function u defined in (1.2) suggests that at the scaling $\kappa_\varepsilon = \varepsilon\theta_\varepsilon$ the Γ -limit of $\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon$ might be finite on the class $BV(\Omega; \mathbb{S}^1)$ of \mathbb{S}^1 -valued functions of bounded variation. This is confirmed by the following theorem, for which we introduce some notation, cf. [11]. Given a function $u \in BV(\Omega; \mathbb{S}^1)$, its distributional derivative Du can be decomposed as $Du = \nabla u \mathcal{L}^2 + D^{(c)}u + (u^+ - u^-) \otimes \nu_u \mathcal{H}^1 \llcorner J_u$, where ∇u denotes the approximate gradient, \mathcal{L}^2 is the Lebesgue measure in \mathbb{R}^2 , $D^{(c)}u$ is the Cantor part of Du , \mathcal{H}^1 is the 1-dimensional Hausdorff measure, J_u is the \mathcal{H}^1 -countably rectifiable jump set of u with normal vector ν_u , and u^+ and u^- are the traces of u on J_u . By $|\cdot|_1$ we denote the 1-norm on vectors and by $|\cdot|_{2,1}$ the anisotropic norm on matrices given by the sum of the Euclidean norms of the columns.

Theorem 1.1. (Regime $\varepsilon |\log \varepsilon| \ll \theta_\varepsilon \ll 1$) Assume that $\varepsilon |\log \varepsilon| \ll \theta_\varepsilon \ll 1$. Then the following results hold:

- i) (Compactness) Let $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ be such that $\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$. Then there exists a subsequence (not relabeled) and a function $u \in BV(\Omega; \mathbb{S}^1)$ such that $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$.

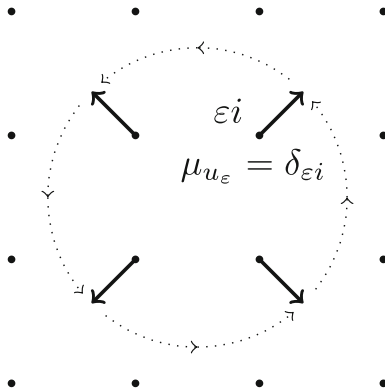


Fig. 2. Example of discrete vorticity measure equal to a Dirac delta on the point $\varepsilon i \in \varepsilon \mathbb{Z}^2$. By following a closed path on the square of the lattice with the top-right corner in εi , the spin field covers the whole \mathbb{S}^1 . The discrete vorticity measure can only have weights in $\{-1, 0, 1\}$

ii) (Γ -liminf inequality) Assume that $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ and $u \in BV(\Omega; \mathbb{S}^1)$ satisfy $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$. Then

$$\int_\Omega |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \int_{J_u} d_{\mathbb{S}^1}(u^-, u^+) |v_u|_1 \, d\mathcal{H}^1 \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon).$$

iii) (Γ -limsup inequality) Let $u \in BV(\Omega; \mathbb{S}^1)$. Then there exists a sequence $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ such that $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$ and

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq \int_\Omega |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \int_{J_u} d_{\mathbb{S}^1}(u^-, u^+) |v_u|_1 \, d\mathcal{H}^1.$$

The previous theorem does not hold true if $\theta_\varepsilon \lesssim \varepsilon |\log \varepsilon|$, i.e., $\frac{\theta_\varepsilon}{\varepsilon |\log \varepsilon|} \rightarrow C \in [0, +\infty)$. In this regime, an additional object plays a role, namely the discrete vorticity measures μ_{u_ε} associated to the spin field u_ε (see Fig. 2 and cf. (2.6) for the precise definition). By (1.1), we have

$$\frac{1}{\varepsilon^2 |\log \varepsilon|} XY_\varepsilon(u_\varepsilon) = \frac{\varepsilon \theta_\varepsilon}{\varepsilon^2 |\log \varepsilon| \varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \sim \frac{\theta_\varepsilon}{\varepsilon |\log \varepsilon|} \sim C. \tag{1.4}$$

The bound $\frac{1}{\varepsilon^2 |\log \varepsilon|} XY_\varepsilon(u_\varepsilon) \leq C$ yields compactness for the discrete vorticity measure μ_{u_ε} . More precisely, in [4] it is proven that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ up to a subsequence in the flat convergence (i.e., in the norm of the dual of Lipschitz functions with compact support, see (2.7)), where $\mu = \sum_{h=1}^N d_h \delta_{x_h}$, $x_h \in \Omega$, $d_h \in \mathbb{Z}$, is a measure that represents the vortex-like singularities of the spin field u_ε as ε goes to zero. The limit of $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon)$ is, in general, strictly greater than the anisotropic total variation in $BV(\Omega; \mathbb{S}^1)$ obtained in Theorem 1.1, since u_ε must satisfy the topological constraint $\mu_{u_\varepsilon} \xrightarrow{f} \mu$. To describe the limit, we associate to u_ε with $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$ the current G_{u_ε} given by the extended graph in $\Omega \times \mathbb{S}^1$ of its piecewise constant interpolation, see Section 3.5. In Section 4 we prove a compactness result for G_{u_ε} to deduce

that $G_{u_\varepsilon} \rightharpoonup T$ in the sense of currents. In Proposition 3.11 we show that $\partial G_{u_\varepsilon} = -\mu_{u_\varepsilon} \times \llbracket \mathbb{S}^1 \rrbracket$, where $\llbracket \mathbb{S}^1 \rrbracket$ is the current given by the integration over \mathbb{S}^1 oriented counterclockwise. Since $\mu_{u_\varepsilon} \xrightarrow{f} \mu$, the limit $T \in \text{Adm}(\mu, u; \Omega)$ of the currents G_{u_ε} satisfies, among other properties that characterize the class $\text{Adm}(\mu, u; \Omega)$ given in (4.1), $\partial T = -\mu \times \llbracket \mathbb{S}^1 \rrbracket$. For this reason, in general, the current T is different from the graph G_u of the limit map u , which may have a boundary different from $-\mu \times \llbracket \mathbb{S}^1 \rrbracket$. Nevertheless, T can be represented as

$$T = G_u + L \times \llbracket \mathbb{S}^1 \rrbracket, \tag{1.5}$$

where L is an integer multiplicity 1-rectifiable current, which keeps track of the possible concentration of $|Du_\varepsilon|$ on 1-dimensional sets. In [25] we have proved that the energy concentration on vortex-like singularities and the BV -type concentration on 1-dimensional sets occur at two separate energy scalings if $\varepsilon \ll \theta_\varepsilon \ll \varepsilon |\log \varepsilon|$. In this paper, we investigate the critical regime $\theta_\varepsilon \sim \varepsilon |\log \varepsilon|$ and prove that the two concentration effects appear simultaneously and are coupled in the limit energy by

$$\mathcal{J}(\mu, u; \Omega) := \inf \left\{ \int_{J_T} \ell_T(x) |v_T(x)|_1 \, d\mathcal{H}^1(x) : T \in \text{Adm}(\mu, u; \Omega) \right\}.$$

Here J_T is the 1-dimensional jump-concentration set of T oriented by the normal v_T , accounting for both the jump set of u and the support of the concentration part L in the decomposition (1.5). At each point $x \in J_T$, the current T has a vertical part, given by a curve in \mathbb{S}^1 which connects the traces of u on the two sides of J_T ; $\ell_T(x)$ is its length. See also Fig. 3.

Theorem 1.2. (Regime $\theta_\varepsilon \sim \varepsilon |\log \varepsilon|$) *Assume that $\theta_\varepsilon = \varepsilon |\log \varepsilon|$.¹ Then the following results hold:*

- (i) (Compactness) *Let $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ be such that $\frac{1}{\varepsilon^2 |\log \varepsilon|} E_\varepsilon(u_\varepsilon) \leq C$. Then there exists a measure $\mu = \sum_{h=1}^M d_h \delta_{x_h}$, $M \in \mathbb{N}$, $x_h \in \Omega$, $d_h \in \mathbb{Z}$, such that (up to a subsequence) $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ and there exists a function $u \in BV(\Omega; \mathbb{S}^1)$ such that (up to a subsequence) $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$.*
- (ii) (Γ -liminf inequality) *Let $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$, let $\mu = \sum_{h=1}^M d_h \delta_{x_h}$, $M \in \mathbb{N}$, $x_h \in \Omega$, $d_h \in \mathbb{Z}$, and let $u \in BV(\Omega; \mathbb{S}^1)$. Assume that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ and $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$. Then*

$$\begin{aligned} & \int_\Omega |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(\mu, u; \Omega) + 2\pi |\mu|(\Omega) \\ & \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 |\log \varepsilon|} E_\varepsilon(u_\varepsilon). \end{aligned}$$

¹ If $\theta_\varepsilon = \theta_0 \varepsilon |\log \varepsilon|$, with $\theta_0 \in (0, +\infty)$, the limit functional needs to be modified accordingly by multiplying by θ_0 the first three terms, while the term $2\pi |\mu|(\Omega)$ is unaffected.

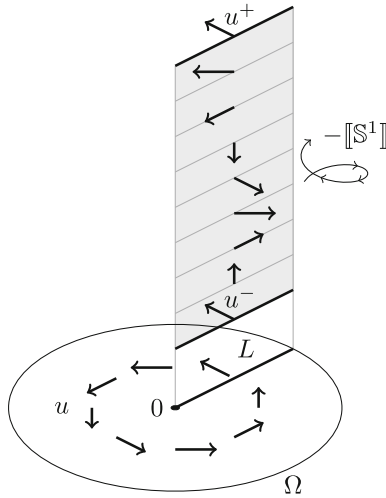


Fig. 3. Depiction of a current of the form $T = G_u - L \times \llbracket \mathbb{S}^1 \rrbracket \in \text{Adm}(0, u; \Omega)$. In the picture, Ω is the unit disc centered at the origin. The function u has no jumps and presents a vortex-like singularity, turning once counterclockwise around the origin. In particular, the graph G_u has a hole, namely, $\partial G_u = -\delta_0 \times \llbracket \mathbb{S}^1 \rrbracket$. The current T features a concentration part $-L \times \llbracket \mathbb{S}^1 \rrbracket$. It is supported on a radius of the ball and is characterized by a vertical part (in gray) that connects clockwise in \mathbb{S}^1 the (equal) traces u^- and u^+ of u on the two sides of the radius. Note that the vertical part is not given by the geodesic connecting u^- and u^+ . The concentration part is needed to compensate the boundary of the graph G_u , so that $\partial T = 0$. Indeed, $-\partial L \times \llbracket \mathbb{S}^1 \rrbracket = \delta_0 \times \llbracket \mathbb{S}^1 \rrbracket = -\partial G_u$ inside Ω . In conclusion, the current T does not turn around the origin. In this figure, for \mathcal{H}^1 -a.e. x in the support of L , the length $\ell_T(x)$ is 2π

(iii) (Γ -limsup inequality) Let $\mu = \sum_{h=1}^M d_h \delta_{x_h}$, $M \in \mathbb{N}$, $x_h \in \Omega$, $d_h \in \mathbb{Z}$ and let $u \in BV(\Omega; \mathbb{S}^1)$. Then there exists a sequence $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ such that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$, $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$, and

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 |\log \varepsilon|} E_\varepsilon(u_\varepsilon) \leq \int_{\Omega} |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(\mu, u; \Omega) + 2\pi |\mu|(\Omega).$$

Theorems 1.1, 1.2, and [25, Theorem 1.1] lead, in particular, to the conclusion that for $\varepsilon \ll \theta_\varepsilon$ the N_ε -clock model does not share the same asymptotic behavior of the classical XY model. For the latter model, the following asymptotic expansion is known to hold true in the sense of Γ -convergence (see [7, Section 4] and also [1, 9, 16, 39] for the Ginzburg-Landau model)

$$\frac{1}{\varepsilon^2} XY_\varepsilon(u_\varepsilon) \sim 2\pi M |\log \varepsilon| + \mathbb{W}(\mu) + M\gamma, \tag{1.6}$$

where \mathbb{W} is a Coulomb-type interaction potential referred to as renormalized energy, while γ is the core energy carried by each vortex, cf. (7.3) and (7.5) for the precise definitions. The next theorem shows that the E_ε energy has the same asymptotic

expansion if $\theta_\varepsilon \ll \varepsilon$, finally providing a precise range for θ_ε for which the N_ε -clock model approximates the classical XY model.

Theorem 1.3. (Regime $\theta_\varepsilon \ll \varepsilon$) *Assume that $\theta_\varepsilon \ll \varepsilon$ and $M \in \mathbb{N}$. Then the following results hold:*

- (i) (Compactness) *Assume that $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ satisfies $\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon) - 2\pi M |\log \varepsilon| \leq C$. Then there exists a measure $\mu = \sum_{h=1}^N d_h \delta_{x_h}$, $x_h \in \Omega$, $d_h \in \mathbb{Z}$, with $|\mu|(\Omega) \leq M$ such that (up to a subsequence) $\mu_{u_\varepsilon} \xrightarrow{f} \mu$. Moreover, if $|\mu|(\Omega) = M$, then $|d_h| = 1$.*
- (ii) (Γ -liminf inequality) *Let $\mu = \sum_{h=1}^M d_h \delta_{x_h}$, $x_h \in \Omega$, $|d_h| = 1$, and let $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ be such that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$. Then*

$$\mathbb{W}(\mu) + M\gamma \leq \liminf_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon) - 2\pi M |\log \varepsilon| \right).$$

- (iii) (Γ -limsup inequality) *Let $\mu = \sum_{h=1}^M d_h \delta_{x_h}$, $x_h \in \Omega$, $|d_h| = 1$. Then there exists $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ such that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ and*

$$\limsup_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon) - 2\pi M |\log \varepsilon| \right) \leq \mathbb{W}(\mu) + M\gamma.$$

We summarize all the results obtained in this paper, in [25], and in [26] in Table 1.

It is worth mentioning that the paper also contains an extension result for Cartesian currents, Lemma 3.4, that we reckon to be of independent interest.

2. Notation and preliminaries

In order to avoid confusion with lattice points we denote the imaginary unit by i . We shall often identify \mathbb{R}^2 with the complex plane \mathbb{C} . By $x \odot y$ we mean the product of $x, y \in \mathbb{R}^2$ seen as complex numbers. Given a vector $a = (a_1, a_2) \in \mathbb{R}^2$, its 1-norm is $|a|_1 = |a_1| + |a_2|$. We define the $(2, 1)$ -norm of a matrix $A = (a_{ij}) \in \mathbb{R}^{2 \times 2}$ by

$$|A|_{2,1} := (a_{11}^2 + a_{21}^2)^{1/2} + (a_{12}^2 + a_{22}^2)^{1/2}.$$

If $u, v \in \mathbb{S}^1$, their geodesic distance on \mathbb{S}^1 is denoted by $d_{\mathbb{S}^1}(u, v)$. It is given by the angle in $[0, \pi]$ between the vectors u and v , i.e., $d_{\mathbb{S}^1}(u, v) = \arccos(u \cdot v)$. Observe that

$$\frac{1}{2}|u - v| = \sin\left(\frac{1}{2}d_{\mathbb{S}^1}(u, v)\right) \quad \text{and} \quad |u - v| \leq d_{\mathbb{S}^1}(u, v) \leq \frac{\pi}{2}|u - v|. \tag{2.1}$$

Given two sequences α_ε and β_ε , we write $\alpha_\varepsilon \ll \beta_\varepsilon$ if $\lim_{\varepsilon \rightarrow 0} \frac{\alpha_\varepsilon}{\beta_\varepsilon} = 0$ and $\alpha_\varepsilon \sim \beta_\varepsilon$ if $\lim_{\varepsilon \rightarrow 0} \frac{\alpha_\varepsilon}{\beta_\varepsilon} \in (0, +\infty)$. We will use the notation $\text{deg}(u)(x_0)$ to denote the topological degree of a continuous map $u \in C(B_\rho(x_0) \setminus \{x_0\}; \mathbb{S}^1)$, i.e., the topological degree of its restriction $u|_{\partial B_r(x_0)}$, independent of $r < \rho$.

We denote by $I_\lambda(x)$ the half-open squares given by

$$I_\lambda(x) = x + [0, \lambda)^2. \tag{2.2}$$

Table 1. In this table we summarize our results

Regime	Energy bound	Limit of μ_{u_ε}	Energy Behavior	Theorem
θ_ε finite	$\frac{1}{\varepsilon} E_\varepsilon \leq C$	Not relevant	Interfaces	[26, Thm 1.2]
$\varepsilon \log \varepsilon \ll \theta_\varepsilon$	$\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon \leq C$	Not relevant	BV vortices	Thm 1.1
$\theta_\varepsilon \sim \varepsilon \log \varepsilon $	$\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon \leq C$	$\mu_{u_\varepsilon} \xrightarrow{f} \mu$	+ BV + concentration vortices	Thm 1.2
$\varepsilon \ll \theta_\varepsilon \ll \varepsilon \log \varepsilon $	$\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon - 2\pi M \log \varepsilon \frac{\varepsilon}{\theta_\varepsilon} \leq C$	$\mu_{u_\varepsilon} \xrightarrow{f} \mu$	+ BV + concentration	[25, Thm 1.1]
$\theta_\varepsilon \ll \varepsilon$	$\frac{1}{\varepsilon^2} E_\varepsilon - 2\pi M \log \varepsilon \leq C$	$\mu_{u_\varepsilon} \xrightarrow{f} \mu$	XY	Thm 1.3

By “interfaces” we mean that the energy concentrates on 1-dimensional domain walls that separate the different phases [26], while “ BV ” denotes a BV -type total variation, Theorem 1.1. The expression “ BV +concentration” indicates the presence in a BV -type energy of a surface term of the form $\mathcal{J}(\mu, u; \Omega)$ which accounts for concentration effects on 1-dimensional surfaces, as in [25] and Theorem 1.2. By “vortices” we mean that a logarithmic energy is carried by the system for the creation of vortex-like singularities in the limit. Finally, “ XY ” expresses the fact that the energy is a good approximation (at first order) of the classical XY model, Theorem 1.3. The missing case $\theta_\varepsilon \sim \varepsilon$ is still mainly open

2.1. BV -functions

In this section we recall basic facts about functions of bounded variation. For more details we refer to the monograph [11].

Let $O \subset \mathbb{R}^d$ be an open set. A function $u \in L^1(O; \mathbb{R}^n)$ is a function of bounded variation if its distributional derivative Du is given by a finite matrix-valued Radon measure on O . We write $u \in BV(O; \mathbb{R}^n)$.

The space $BV_{loc}(O; \mathbb{R}^n)$ is defined as usual. The space $BV(O; \mathbb{R}^n)$ becomes a Banach space when endowed with the norm $\|u\|_{BV(O)} = \|u\|_{L^1(O)} + |Du|(O)$, where $|Du|$ denotes the total variation measure of Du . The total variation with respect to the anisotropic norm $|\cdot|_{2,1}$ is denoted by $|Du|_{2,1}$. When O is a bounded Lipschitz domain, then $BV(O; \mathbb{R}^n)$ is compactly embedded in $L^1(O; \mathbb{R}^n)$. We say that a sequence u_n converges weakly* in $BV(O; \mathbb{R}^n)$ to u if $u_n \rightarrow u$ in $L^1(O; \mathbb{R}^n)$ and $Du_n \xrightarrow{*} Du$ in the sense of measures.

Given $x \in O$ and $v \in \mathbb{S}^{d-1}$ we set

$$B_\rho^\pm(x, v) = \{y \in B_\rho(x) : \pm(y - x) \cdot v > 0\}.$$

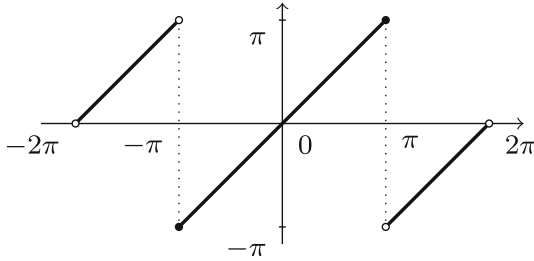


Fig. 4. Graph of the function Ψ for $t \in (-2\pi, 2\pi)$. Observe that Ψ is an odd function

We say that $x \in O$ is an approximate jump point of u if there exist $a \neq b \in \mathbb{R}^n$ and $v \in \mathbb{S}^{d-1}$ such that

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho^d} \int_{B_\rho^+(x,v)} |u(y) - a| \, dy = \lim_{\rho \rightarrow 0} \frac{1}{\rho^d} \int_{B_\rho^-(x,v)} |u(y) - b| \, dy = 0.$$

The triplet (a, b, v) is determined uniquely up to the change to $(b, a, -v)$. We denote it by $(u^+(x), u^-(x), v_u(x))$ and let J_u be the set of approximate jump points of u . The triplet (u^+, u^-, v_u) can be chosen as a Borel function on the Borel set J_u . Denoting by ∇u the approximate gradient of u , we can decompose the measure Du as

$$Du(B) = \int_B \nabla u \, dx + \int_{J_u \cap B} (u^+(x) - u^-(x)) \otimes v_u(x) \, d\mathcal{H}^{d-1} + D^{(c)}u(B),$$

where $D^{(c)}u$ is the so-called Cantor part and $D^{(j)}u = (u^+ - u^-) \otimes v_u \mathcal{H}^{d-1} \llcorner J_u$ is the so-called jump part.

2.2. Results for the classical XY model

We recall here some results about the classical XY model, namely when the spin field $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathbb{S}^1$ is not constrained to take values in a discrete set.

Following [6], in order to define the discrete vorticity of the spin variable, it is convenient to introduce the projection $Q : \mathbb{R} \rightarrow 2\pi\mathbb{Z}$ defined by

$$Q(t) := \operatorname{argmin}\{|t - s| : s \in 2\pi\mathbb{Z}\}, \tag{2.3}$$

with the convention that, if the argmin is not unique, then we choose the one with minimal modulus. Then, for every $t \in \mathbb{R}$, we define (see Fig. 4)

$$\Psi(t) := t - Q(t) \in [-\pi, \pi]. \tag{2.4}$$

Let $u : \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$ and let $\varphi : \varepsilon\mathbb{Z}^2 \rightarrow [0, 2\pi)$ be the phase of u defined by the relation $u = \exp(i\varphi)$. The discrete vorticity of u is defined for every $\varepsilon i \in \varepsilon\mathbb{Z}^2$ by

$$d_u(\varepsilon i) := \frac{1}{2\pi} \left[\Psi(\varphi(\varepsilon i + \varepsilon e_1) - \varphi(\varepsilon i)) + \Psi(\varphi(\varepsilon i + \varepsilon e_1 + \varepsilon e_2) - \varphi(\varepsilon i + \varepsilon e_1)) + \Psi(\varphi(\varepsilon i + \varepsilon e_2) - \varphi(\varepsilon i + \varepsilon e_1 + \varepsilon e_2)) + \Psi(\varphi(\varepsilon i) - \varphi(\varepsilon i + \varepsilon e_2)) \right]. \tag{2.5}$$

As already noted in [6], the discrete vorticity d_u only takes values in $\{-1, 0, 1\}$, i.e., only vortices of degree ± 1 can be present in the discrete setting. We introduce the discrete measure representing all vortices of the discrete spin field defined by

$$\mu_u := \sum_{\varepsilon i \in \varepsilon \mathbb{Z}^2} d_u(\varepsilon i) \delta_{\varepsilon i + \varepsilon(e_1 + e_2)}. \tag{2.6}$$

Remark 2.1. In [6,7] the vorticity measure $\dot{\mu}_u$ is supported in the centers of the squares completely contained in Ω , i.e.,

$$\dot{\mu}_u = \sum_{\substack{\varepsilon i \in \varepsilon \mathbb{Z}^2 \\ \varepsilon i + [0, \varepsilon]^2 \subset \Omega}} d_u(\varepsilon i) \delta_{\varepsilon i + \varepsilon/2(e_1 + e_2)}.$$

In this paper we prefer definition (2.6) since it fits well with our definition of discrete currents in Section 3.5 on the whole set Ω . However, as we will borrow some results from [6,7], we have to ensure that these definitions are asymptotically equivalent with respect to the flat convergence defined below.

Definition 2.2. (Flat convergence) Let $O \subset \mathbb{R}^2$ be an open set. A sequence of finite Radon measures $\mu_j \in \mathcal{M}_b(O)$ converges flat to $\mu \in \mathcal{M}_b(O)$, denoted by $\mu_j \xrightarrow{f} \mu$, if

$$\sup_{\substack{\psi \in C_c^{0,1}(O) \\ \|\psi\|_{C^{0,1}} \leq 1}} \left| \int_O \psi \, d\mu_j - \int_O \psi \, d\mu \right| \rightarrow 0. \tag{2.7}$$

Observe that the flat convergence is weaker than the weak* convergence. The two notions are equivalent when the measures μ_j have equibounded total variations.

The two vorticity measures μ_u and $\dot{\mu}_u$ are then close as explained in Lemma 2.3 below. For $A \subset \mathbb{R}^2$ we shall use the localized energy given by

$$E_\varepsilon(u; A) := \frac{1}{2} \sum_{\substack{(i,j) \\ \varepsilon i, \varepsilon j \in A}} \varepsilon^2 |u(\varepsilon i) - u(\varepsilon j)|^2.$$

We shall adopt the same notation for the classical XY_ε energy. We work with spin fields $u_\varepsilon : \varepsilon \mathbb{Z}^2 \rightarrow \mathbb{S}^1$ defined on the whole lattice $\varepsilon \mathbb{Z}^2$. We can always assume that $XY_\varepsilon(u_\varepsilon; \overline{\Omega}^\varepsilon) \leq CXY_\varepsilon(u_\varepsilon; \Omega)$, where $\overline{\Omega}^\varepsilon$ is the union of the squares $\varepsilon i + [0, \varepsilon]^2$ that intersect Ω . (If not, thanks to the Lipschitz regularity of Ω , we modify u_ε outside Ω in such a way that the energy estimate is satisfied, see [4, Remark 2].)

Lemma 2.3. Assume that $u_\varepsilon : \varepsilon \mathbb{Z}^2 \rightarrow \mathbb{S}^1$ is a sequence such that $\frac{1}{\varepsilon^2} XY_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$. Then $\mu_{u_\varepsilon} \llcorner \Omega - \dot{\mu}_{u_\varepsilon} \xrightarrow{f} 0$.

Proof. Note that, for any $\psi \in C_c^{0,1}(\Omega)$ with $\|\psi\|_{C^{0,1}} \leq 1$, we have

$$|\langle \mu_{u_\varepsilon} \llcorner \Omega - \dot{\mu}_{u_\varepsilon}, \psi \rangle| \leq \sum_{\varepsilon i \in \varepsilon \mathbb{Z}^2 \cap \overline{\Omega}^\varepsilon} |d_{u_\varepsilon}(\varepsilon i)| \frac{\varepsilon}{\sqrt{2}} \leq \frac{\varepsilon}{\sqrt{2}} |\dot{\mu}_{u_\varepsilon}|(\overline{\Omega}^\varepsilon)$$

$$\leq C\varepsilon \frac{1}{\varepsilon^2} XY_\varepsilon(u; \overline{\Omega}^\varepsilon) \leq C\varepsilon |\log \varepsilon|,$$

where in the last but one inequality we used [6, Remark 3.4]. This proves the claim.

We recall the following compactness and lower bound for the classical XY model:

Proposition 2.4. *Let $u_\varepsilon : \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$ and assume that there is a constant $C > 0$ such that $\frac{1}{\varepsilon^2 |\log \varepsilon|} XY_\varepsilon(u_\varepsilon) \leq C$. Then there exists a measure $\mu \in \mathcal{M}_b(\Omega)$ of the form $\mu = \sum_{h=1}^M d_h \delta_{x_h}$ with $d_h \in \mathbb{Z}$ and $x_h \in \Omega$, and a subsequence (not relabeled) such that $\mu_{u_\varepsilon} \llcorner \Omega \xrightarrow{f} \mu$. Moreover*

$$2\pi |\mu|(\Omega) \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 |\log \varepsilon|} XY_\varepsilon(u_\varepsilon).$$

Proof. In [7, Theorem 3.1-(i)] it is proven that (up to a subsequence) the discrete vorticity measures $\dot{\mu}_{u_\varepsilon}$ converge flat to a measure of the claimed form satisfying also the lower bound. The claim thus follows from Lemma 2.3.

3. Currents

For the theory of currents and cartesian currents we refer to the books [30, 31]. We recall here some notation, definitions, and basic facts about currents. We additionally prove some technical lemmata that we need in this paper and that were also used in [25].

3.1. Definitions and basic facts

Given an open set $O \subset \mathbb{R}^d$, we denote by $\mathcal{D}^k(O)$ the space of k -forms $\omega : O \mapsto \Lambda^k \mathbb{R}^d$ that are C^∞ with compact support in O . A k -current $T \in \mathcal{D}_k(O)$ is an element of the dual of $\mathcal{D}^k(O)$. The duality between a k -current and a k -form ω will be denoted by $T(\omega)$. The *boundary* of a k -current T is the $(k-1)$ -current $\partial T \in \mathcal{D}_{k-1}(O)$ defined by $\partial T(\omega) := T(d\omega)$ for every $\omega \in \mathcal{D}^{k-1}(O)$ (or $\partial T := 0$ if $k = 0$). As for distributions, the *support* of a current T is the smallest relatively closed set K in O such that $T(\omega) = 0$ if ω is supported outside K . Given a smooth map $f : O \rightarrow O' \subset \mathbb{R}^{N'}$ such that f is proper², $f^\# \omega \in \mathcal{D}^k(O)$ denotes the pull-back of a k -form $\omega \in \mathcal{D}^k(O')$ through f . The *push-forward* of a k -current $T \in \mathcal{D}_k(O)$ is the k -current $f\#T \in \mathcal{D}_k(O')$ defined by $f\#T(\omega) := T(f^\# \omega)$. Given a k -form $\omega \in \mathcal{D}^k(O)$, we can write it via its components

$$\omega = \sum_{|\alpha|=k} \omega_\alpha dx^\alpha, \quad \omega_\alpha \in C_c^\infty(O),$$

² which means $f^{-1}(K)$ is compact in O for all compact sets $K \subset O'$.

where the expression $|\alpha| = k$ denotes all multi-indices $\alpha = (\alpha_1, \dots, \alpha_k)$ with $1 \leq \alpha_i \leq d$, and $dx^\alpha = dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k}$. The norm of $\omega(x)$ is denoted by $|\omega(x)|$ and it is the Euclidean norm of the vector with components $(\omega_\alpha(x))_{|\alpha|=k}$. The *total variation* of a k -current $T \in \mathcal{D}_k(O)$ is defined by

$$|T|(O) := \sup\{T(\omega) : \omega \in \mathcal{D}^k(O), |\omega(x)| \leq 1\}.$$

If $T \in \mathcal{D}_k(O)$ with $|T|(O) < \infty$, then we can define the measure $|T| \in \mathcal{M}_b(O)$

$$|T|(\psi) := \sup\{T(\omega) : \omega \in \mathcal{D}^k(O), |\omega(x)| \leq \psi(x)\}$$

for every $\psi \in C_0(O)$, $\psi \geq 0$. As a consequence of Riesz’s Representation Theorem (see [30, 2.2.3, Theorem 1]) there exists a $|T|$ -measurable function $\mathbf{T} : O \mapsto \Lambda_k \mathbb{R}^d$ with $|\mathbf{T}(x)| = 1$ for $|T|$ -a.e. $x \in O$ such that

$$T(\omega) = \int_O \langle \omega(x), \mathbf{T}(x) \rangle d|T|(x) \tag{3.1}$$

for every $\omega \in \mathcal{D}^k(O)$. We note that if T has finite total variation, then it can be extended to a linear functional acting on all forms with bounded, Borel-measurable coefficients via the dominated convergence theorem. In particular, in this case the push-forward $f_\#T$ can be defined also for $f \in C^1(O, O')$ with bounded derivatives, cf. the discussion in [30, p. 132].

A set $\mathcal{M} \subset O$ is a countably \mathcal{H}^k -rectifiable set if it can be covered, up to an \mathcal{H}^k -negligible subset, by countably many k -manifolds of class C^1 . As such, it admits at \mathcal{H}^k -a.e. $x \in \mathcal{M}$ a tangent space $\text{Tan}(\mathcal{M}, x)$ in a measure theoretic sense. A current $T \in \mathcal{D}_k(O)$ is an *integer multiplicity (i.m.) rectifiable current* if it is representable as

$$T(\omega) = \int_{\mathcal{M}} \langle \omega(x), \xi(x) \rangle \theta(x) d\mathcal{H}^k(x), \quad \text{for } \omega \in \mathcal{D}^k(O), \tag{3.2}$$

where $\mathcal{M} \subset O$ is a \mathcal{H}^k -measurable and countably \mathcal{H}^k -rectifiable set, $\theta : \mathcal{M} \rightarrow \mathbb{Z}$ is locally $\mathcal{H}^k \llcorner \mathcal{M}$ -summable, and $\xi : \mathcal{M} \rightarrow \Lambda_k \mathbb{R}^d$ is a \mathcal{H}^k -measurable map such that $\xi(x)$ spans $\text{Tan}(\mathcal{M}, x)$ and $|\xi(x)| = 1$ for \mathcal{H}^k -a.e. $x \in \mathcal{M}$. We use the short-hand notation $T = \tau(\mathcal{M}, \theta, \xi)$. One can always replace \mathcal{M} by the set $\mathcal{M} \cap \theta^{-1}(\{0\})$, so that we may always assume that $\theta \neq 0$. Then the triple $(\mathcal{M}, \theta, \xi)$ is uniquely determined up to \mathcal{H}^k -negligible modifications. Moreover, one can show, according to the Riesz’s representation in (3.1), that $\mathbf{T} = \xi$ and the total variation³ is given by $|T| = |\theta| \mathcal{H}^k \llcorner \mathcal{M}$.

If T_j are i.m. rectifiable currents and $T_j \rightarrow T$ in $\mathcal{D}_k(O)$ with $\sup_j (|T_j|(V) + |\partial T_j|(V)) < \infty$ for every $V \subset\subset O$, then by the Closure Theorem [30, 2.2.4, Theorem 1] T is an i.m. rectifiable current, too. By $[[\mathcal{M}]]$ we denote the current defined by integration over \mathcal{M} .

³ For i.m. rectifiable currents, the total variation coincides with the so-called mass. Hence, we will not distinguish between these two concepts.

3.2. Currents in product spaces

Let us introduce some notation for currents defined on the product space $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. We will denote by (x, y) the points in this space. The standard basis of the first space \mathbb{R}^{d_1} is $\{e_1, \dots, e_{d_1}\}$, while $\{\bar{e}_1, \dots, \bar{e}_{d_2}\}$ is the standard basis of the second space \mathbb{R}^{d_2} . Given $O_1 \subset \mathbb{R}^{d_1}$, $O_2 \subset \mathbb{R}^{d_2}$ open sets, $T_1 \in \mathcal{D}_{k_1}(O_1)$, $T_2 \in \mathcal{D}_{k_2}(O_2)$ and a $(k_1 + k_2)$ -form $\omega \in \mathcal{D}^{k_1+k_2}(O_1 \times O_2)$ of the type

$$\omega(x, y) = \sum_{\substack{|\alpha|=k_1 \\ |\beta|=k_2}} \omega_{\alpha\beta}(x, y) dx^\alpha \wedge dy^\beta,$$

the product current $T_1 \times T_2 \in \mathcal{D}_{k_1+k_2}(O_1 \times O_2)$ is defined by

$$T_1 \times T_2(\omega) := T_1\left(\sum_{|\alpha|=k_1} T_2\left(\sum_{|\beta|=k_2} \omega_{\alpha\beta}(x, y) dy^\beta\right) dx^\alpha\right),$$

while $T_1 \times T_2(\phi dx^\alpha \wedge dy^\beta) = 0$ if $|\alpha| + |\beta| = k_1 + k_2$ but $|\alpha| \neq k_1, |\beta| \neq k_2$.

3.3. Graphs

Let $O \subset \mathbb{R}^d$ be an open set and $u: \Omega \rightarrow \mathbb{R}^2$ a Lipschitz map. Then we can consider the d -current associated to the graph of u given by $G_u := (\text{id} \times u) \# \llbracket O \rrbracket \in \mathcal{D}_2(O \times \mathbb{R}^2)$, where $\text{id} \times u: O \rightarrow O \times \mathbb{R}^2$ is the map $(\text{id} \times u)(x) = (x, u(x))$. Note that by definition we have

$$G_u(\omega) = \int_O \langle \omega(x, u(x)), M(\nabla u(x)) \rangle dx$$

for all $\omega \in \mathcal{D}^d(O \times \mathbb{R}^2)$, with the d -vector

$$M(\nabla u) = (e_1 + \partial_{x^1} u^1 \bar{e}_1 + \partial_{x^1} u^2 \bar{e}_2) \wedge \dots \wedge (e_d + \partial_{x^d} u^1 \bar{e}_1 + \partial_{x^d} u^2 \bar{e}_2). \quad (3.3)$$

The above formula can be extended to the class $\mathcal{A}^1(O; \mathbb{R}^2)$ defined by

$$\mathcal{A}^1(O; \mathbb{R}^2) := \{u \in L^1(O; \mathbb{R}^2) : u \text{ approx. diff. almost everywhere and all minors of } \nabla u \text{ are in } L^1(O)\}.$$

Remark 3.1. We recall that $\partial G_u|_{\Omega \times \mathbb{R}^2} = 0$ when $u \in W^{1,2}(O; \mathbb{R}^2) \subset \mathcal{A}^1(O; \mathbb{R}^2)$, see [30, 3.2.1, Proposition 3]. This property however fails for general functions $u \in \mathcal{A}^1(O; \mathbb{R}^2)$.

In Lemma 3.4 we need to interpret the graphs of $W^{1,1}(O; \mathbb{S}^1)$ as currents. This can be done because of the following observation:

Lemma 3.2. *Let $O \subset \mathbb{R}^d$ be an open, bounded set. Then $W^{1,1}(O; \mathbb{S}^1) \subset \mathcal{A}^1(O; \mathbb{R}^2)$.*

Proof. It is well-known that Sobolev functions are approximately differentiable a.e. Moreover, all 1-minors of ∇u are in $L^1(O)$. We argue that all 2-minors vanish at a.e. point. To this end, denote by $P: \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^2$ the smooth mapping $P(x) = x/|x|$. Since for $u \in W^{1,1}(O; \mathbb{S}^1)$ we have $u = P \circ u$ almost everywhere, for a.e. $x \in O$ the chain rule for approximate differentials yields

$$\nabla u(x) = \nabla P(u(x))\nabla u(x).$$

Since $\nabla P(u(x))$ has at most rank 1, also $\nabla u(x)$ has at most rank 1 and therefore all 2-minors have to vanish as claimed.

We will use the orientation of the graph of a smooth function $u: O \subset \mathbb{R}^2 \rightarrow \mathbb{S}^1$ (cf. [30, 2.2.4]). For such maps we have $|G_u| = \mathcal{H}^2 \llcorner \mathcal{M}$, where $\mathcal{M} = (\text{id} \times u)(\Omega)$, and, for every $(x, y) \in \mathcal{M}$,

$$\begin{aligned} \sqrt{1 + |\nabla u(x)|^2} G_u(x, y) &= e_1 \wedge e_2 \\ &\quad + \partial_{x_2} u^1(x) e_1 \wedge \bar{e}_1 + \partial_{x_2} u^2(x) e_1 \wedge \bar{e}_2 \\ &\quad - \partial_{x_1} u^1(x) e_2 \wedge \bar{e}_1 - \partial_{x_1} u^2(x) e_2 \wedge \bar{e}_2. \end{aligned} \tag{3.4}$$

3.4. Cartesian currents

Let $O \subset \mathbb{R}^d$ be a bounded, open set. We recall that the class of *cartesian currents* in $O \times \mathbb{R}^2$ is defined by

$$\begin{aligned} \text{cart}(O \times \mathbb{R}^2) &:= \{T \in \mathcal{D}_d(O \times \mathbb{R}^2) : T \text{ is i.m. rectifiable, } \partial T|_{O \times \mathbb{R}^2} = 0, \\ &\quad \pi_{\#}^O T = \llbracket O \rrbracket, T|_{dx} \geq 0, |T| < \infty, \|T\|_1 < \infty\}, \end{aligned}$$

where $\pi^O: O \times \mathbb{R}^2 \rightarrow O$ denotes the projection on the first component, $T|_{dx} \geq 0$ means that $T(\phi(x, y) dx) \geq 0$ for every $\phi \in C_c^\infty(O \times \mathbb{R}^2)$ with $\phi \geq 0$, and

$$\|T\|_1 = \sup\{T(\phi(x, y)|y| dx) : \phi \in C_c^\infty(O \times \mathbb{R}^2), |\phi| \leq 1\}.$$

Note that, if for some function u

$$T(\phi(x, y) dx) = \int_O \phi(x, u(x)) dx \quad \text{then} \quad \|T\|_1 = \int_O |u| dx. \tag{3.5}$$

The class of *cartesian currents* in $O \times \mathbb{S}^1$ is

$$\text{cart}(O \times \mathbb{S}^1) := \{T \in \text{cart}(O \times \mathbb{R}^2) : \text{supp}(T) \subset \bar{O} \times \mathbb{S}^1\},$$

(cf. [31, 6.2.2] for this definition). We recall the following approximation theorem which explains that cartesian currents in $O \times \mathbb{S}^1$ are precisely those currents that arise as limits of graphs of \mathbb{S}^1 -valued smooth maps. The proof, based on a regularization argument on the lifting of T , can be found in [32, Theorem 7].⁴

⁴ Notice that some results in [32] require O to have smooth boundary. This is not the case for this theorem, which is based on a local construction.

Theorem 3.3. (Approximation Theorem) *Let $T \in \text{cart}(O \times \mathbb{S}^1)$. Then there exists a sequence of smooth maps $u_h \in C^\infty(O; \mathbb{S}^1)$ such that*

$$G_{u_h} \rightharpoonup T \text{ in } \mathcal{D}_d(O \times \mathbb{R}^2) \text{ and } |G_{u_h}|(O \times \mathbb{R}^2) \rightarrow |T|(O \times \mathbb{R}^2).$$

Using the above approximation result, we now prove an extension result for cartesian currents, which we could not find in the literature.

Lemma 3.4. (Extension of cartesian currents) *Let $O \subset \mathbb{R}^d$ be a bounded, open set with Lipschitz boundary and let $T \in \text{cart}(O \times \mathbb{S}^1)$. Then there exist an open set $\tilde{O} \supseteq O$ and a current $T \in \text{cart}(\tilde{O} \times \mathbb{S}^1)$ such that $\tilde{T}|_{O \times \mathbb{R}^2} = T$ and $|\tilde{T}|(\partial O \times \mathbb{R}^2) = 0$.*

Proof. Applying Theorem 3.3 we find a sequence $u_k \in C^\infty(O; \mathbb{S}^1)$ such that $G_{u_k} \rightharpoonup T$ in $\mathcal{D}_d(O \times \mathbb{R}^2)$ and $|G_{u_k}|(O \times \mathbb{R}^2) \rightarrow |T|(O \times \mathbb{R}^2)$. In particular, the sequence $|G_{u_k}|(O \times \mathbb{R}^2)$ is bounded, which implies that

$$\sup_k \int_O |\nabla u_k| dx < C. \tag{3.6}$$

Next we extend the functions u_k . To this end, note that there exists $t > 0$ and a bi-Lipschitz map $\Gamma : (\partial O \times (-t, t)) \rightarrow \Gamma(\partial O \times (-t, t))$ such that $\Gamma(x, 0) = x$ for all $x \in \partial O$, $\Gamma(\partial O \times (-t, t))$ is an open neighborhood of ∂O and

$$\Gamma(\partial O \times (-t, 0)) \subset O, \quad \Gamma(\partial O \times (0, t)) \subset \mathbb{R}^2 \setminus \bar{O}. \tag{3.7}$$

This result is a consequence of [37, Theorem 7.4 & Corollary 7.5]; details can be found for instance in [36, Theorem 2.3]. The extension of u_k is then achieved via reflection. More precisely, for a sufficiently small $t' > 0$ we define it on O' with $O' = O + B_{t'}(0)$ by

$$\tilde{u}_k(x) = \begin{cases} u_k(\Gamma(P(\Gamma^{-1}(x)))) & \text{if } x \notin O, \\ u_k(x) & \text{otherwise,} \end{cases} \tag{3.8}$$

where $P(x, \tau) = (x, -\tau)$. Since Γ is bi-Lipschitz, we have that $\tilde{u}_k \in W^{1,1}(O'; \mathbb{S}^1)$ and by a change of variables we can bound the L^1 -norm of its gradient via

$$\int_{O'} |\nabla \tilde{u}_k| dx \leq \int_O |\nabla u_k| dx + C_\Gamma \int_{O' \setminus O} |(\nabla u_k) \circ \Gamma \circ P \circ \Gamma^{-1}| dx \leq C_\Gamma \int_O |\nabla u_k| dx, \tag{3.9}$$

where the constant C_Γ depends only on the bi-Lipschitz properties of Γ and the dimension. Lemma 3.2 implies that $\tilde{u}_k \in \mathcal{A}^1(O'; \mathbb{R}^2)$. In particular, the current $G_{\tilde{u}_k} \in \mathcal{D}_d(O' \times \mathbb{R}^2)$ is well-defined in the sense of

$$G_{\tilde{u}_k}(\omega) = \int_{O'} \langle \omega(x, \tilde{u}_k(x)), M(\nabla \tilde{u}_k(x)) \rangle dx,$$

with $M(\nabla \tilde{u}_k)$ given by (3.3). We next prove that $G_{\tilde{u}_k} \in \text{cart}(O' \times \mathbb{S}^1)$. First note that whenever $\omega \in \mathcal{D}^d(O' \times \mathbb{R}^2)$ is a form with $\text{supp}(\omega) \subset\subset O' \times \mathbb{R}^2 \setminus (O' \times \mathbb{S}^1)$, then the definition yields $G_{\tilde{u}_k}(\omega) = 0$. It then suffices to prove that $\partial G_{\tilde{u}_k}|_{O' \times \mathbb{R}^2} = 0$. We will argue locally. For each $x \in O'$ we choose a rotation Q_x , radii $r_x > 0$, and heights $h_x > 0$ such that the cylinders $C_x := x + Q_x((-r_x, r_x)^{d-1} \times (-h_x, h_x))$ satisfy

- (i) $C_x \subset\subset O$ if $x \in O$;
- (ii) $C_x \subset\subset O' \setminus \overline{O}$ if $x \in O' \setminus \overline{O}$;
- (iii) $C_x \subset\subset O'$ if $x \in \partial O$ and

$$C_x \cap O = C_x \cap \left(x + Q_x \{ (x', x_d) \in \mathbb{R}^d : x' \in (-r_x, r_x)^{d-1} : -h_x < x_d < \psi(x') \} \right)$$

for some $\psi \in \text{Lip}(\mathbb{R}^{d-1})$.

For $x \in O$ we have $\partial G_{\tilde{u}_k}|_{C_x \times \mathbb{R}^2} = \partial G_{u_k}|_{C_x \times \mathbb{R}^2} = 0$ since $u_k \in C^\infty(C_x; \mathbb{S}^1)$. Next consider the second case, namely $x \in O' \setminus \overline{O}$. Since $C_x \subset\subset O' \setminus \overline{O}$, the properties in (3.7) imply that $\Gamma \circ P \circ \Gamma^{-1}(C_x) \subset\subset O$. In particular, by the smoothness of u_k on O we have that $\tilde{u}_k \in W^{1,\infty}(C_x)$, so that by Remark 3.1 again $\partial G_{\tilde{u}_k}|_{C_x \times \mathbb{R}^2} = 0$. Finally, we consider $x \in \partial O$. Since $C_x \cap O$ is (up to a rigid motion) the subgraph of a Lipschitz function, it is in particular simply connected. By classical lifting theory, we find a sequence of scalar functions $\varphi_k \in C^\infty(C_x \cap O)$ such that $u_k(x) = \exp(i\varphi_k(x))$. In particular, using the chain rule we see that $\varphi_k \in W^{1,1}(C_x \cap O)$. Now fix $0 < \delta_x < r_x$ small enough such that $B_{\delta_x}(x) \subset C_x$ and

$$(\Gamma \circ P \circ \Gamma^{-1})(B_{\delta_x}(x) \cap (O' \setminus \overline{O})) \subset C_x \cap O,$$

which is possible due to (3.7). We then extend the lifting φ_k to a function $\tilde{\varphi}_k \in W^{1,1}(B_{\delta_x}(x))$ via the same reflection construction as in (3.8), which is well-defined due to the above inclusion. Observe that this definition guarantees that $\tilde{u}_k(y) = \exp(i\tilde{\varphi}_k(y))$ for almost every $y \in B_{\delta_x}(x)$. Expressed in terms of currents this means that

$$G_{\tilde{u}_k}|_{B_{\delta_x}(x) \times \mathbb{R}^2} = \chi \# G_{\tilde{\varphi}_k}|_{B_{\delta_x}(x) \times \mathbb{R}^2},$$

where $G_{\tilde{\varphi}_k} \in \mathcal{D}_d(B_{\delta_x}(x) \times \mathbb{R})$ is the current associated to the graph of $\tilde{\varphi}_k$ and $\chi: \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d \times \mathbb{S}^1$ is the covering map defined by $\chi(x, \vartheta) := (x, \cos(\vartheta), \sin(\vartheta))$. In particular, by [32, Theorem 2, p. 97 & Proposition 1 (i), p. 100] we have $G_{\tilde{u}_k}|_{B_{\delta_x}(x) \times \mathbb{R}^2} \in \text{cart}(B_{\delta_x}(x) \times \mathbb{S}^1)$, so that by the definition of cartesian currents we have $\partial G_{\tilde{u}_k}|_{B_{\delta_x}(x) \times \mathbb{R}^2} = 0$.

Thus we have shown that, for every $x \in O'$, there exists a ball $B_{\delta_x}(x) \subset O'$ such that $\partial G_{\tilde{u}_k}|_{B_{\delta_x}(x) \times \mathbb{R}^2} = 0$. Using a partition of unity to localize the support of any form $\omega \in \mathcal{D}^{d-1}(O' \times \mathbb{R}^2)$ with respect to the x -variable, we conclude that $\partial G_{\tilde{u}_k}|_{O' \times \mathbb{R}^2} = 0$ and therefore $G_{\tilde{u}_k} \in \text{cart}(O' \times \mathbb{S}^1)$. As seen in the proof of Lemma 3.2, all 2-minors of Du vanish, so that the bounds (3.6) and (3.9) yield

$$|G_{\tilde{u}_k}|(O' \times \mathbb{R}^2) = \int_{O'} |M(\nabla \tilde{u}_k)| \, dx \leq \int_{O'} \left(1 + |\nabla \tilde{u}_k|^2 \right)^{\frac{1}{2}} \, dx \leq C.$$

Hence, up to a subsequence, we can assume that $G_{\tilde{u}_k} \rightarrow \tilde{T}$ in $\mathcal{D}_d(O' \times \mathbb{R}^2)$, see [30, 2.2.4 Theorem 2]. From [30, 4.2.2. Theorem 1] it follows that $\tilde{T} \in \text{cart}(O' \times \mathbb{S}^1)$. Since $\tilde{u}_k = u_k$ on O , we find that $\tilde{T}|_{O \times \mathbb{R}^2} = T$. It remains to show that $|\tilde{T}|(\partial O \times \mathbb{R}^2) = 0$. To this end, note that for $0 < \eta < \eta' < 1$ (and η small enough), by the bi-Lipschitz continuity of Γ and (3.7) we have that

$$(\Gamma \circ P \circ \Gamma^{-1})(O_\eta^{\text{out}}) \subset O_{\eta'}^{\text{in}},$$

where the sets $O_{\eta'}^{\text{out}}$ and $O_{\eta'}^{\text{in}}$ are defined as

$$O_{\eta'}^{\text{out}} := \{x \in O' \setminus \overline{O} : \text{dist}(x, \partial O) < \eta\}, \quad O_{\eta'}^{\text{in}} := \{x \in O : \text{dist}(x, \partial O) < \eta'\}.$$

Hence, similar to (3.9) we obtain that

$$|G_{\tilde{u}_k}|((\partial O + B_{\eta}(0)) \times \mathbb{R}^2) \leq C_{\Gamma} \int_{O_{\eta'}^{\text{in}}} \left(1 + |\nabla u_k|^2\right)^{\frac{1}{2}} dx = C_{\Gamma} |G_{u_k}|(O_{\eta'}^{\text{in}} \times \mathbb{R}^2). \tag{3.10}$$

Since $|G_{u_k}|(O \times \mathbb{R}^2) \rightarrow |T|(O \times \mathbb{R}^2)$ and $|T|$ is a finite measure, for a.e. $\eta' \in (0, 1)$ we have $|G_{u_k}|(O_{\eta'}^{\text{in}} \times \mathbb{R}^2) \rightarrow |T|(O_{\eta'}^{\text{in}} \times \mathbb{R}^2)$. Applying the lower semicontinuity of the mass with respect to weak convergence of currents in (3.10), we infer that

$$|\tilde{T}|((\partial O + B_{\eta}(0)) \times \mathbb{R}^2) \leq C_{\Gamma} |T|(O_{\eta'}^{\text{in}} \times \mathbb{R}^2).$$

Sending $\eta \rightarrow 0$ first and then $\eta' \rightarrow 0$ we conclude that $|\tilde{T}|(\partial O \times \mathbb{R}^2) = 0$, as claimed.

We will also use the structure theorem for cartesian currents in $O \times \mathbb{S}^1$ that has been proven in [32, Section 3, Theorems 1, 5, 6].⁵ However, to simplify notation, from now on we focus on dimension two. Recall that $\Omega \subset \mathbb{R}^2$ is a bounded, open set with Lipschitz boundary. To state the theorem, we recall the following decomposition for a current $T \in \text{cart}(\Omega \times \mathbb{S}^1)$. Letting \mathcal{M} be the countably \mathcal{H}^2 -rectifiable set where T is concentrated, we denote by $\mathcal{M}^{(a)}$ the set of points $(x, y) \in \mathcal{M}$ at which the tangent plane $\text{Tan}(\mathcal{M}, (x, y))$ does not contain vertical vectors (namely, the Jacobian of the projection π^{Ω} restricted to $\text{Tan}(\mathcal{M}, (x, y))$ has maximal rank), by $\mathcal{M}^{(j,c)} := (\mathcal{M} \setminus \mathcal{M}^{(a)}) \cap (J_T \times \mathbb{S}^1)$, where $J_T := \{x \in \Omega : \frac{d\pi_{\#}^{\Omega}|T|}{d\mathcal{H}^1}(x) > 0\}$, and by $\mathcal{M}^{(c)} := \mathcal{M} \setminus (\mathcal{M}^{(a)} \cup \mathcal{M}^{(j,c)})$. Then we can split the current as

$$T = T^{(a)} + T^{(c)} + T^{(j,c)},$$

where $T^{(a)} := T \llcorner \mathcal{M}^{(a)}$, $T^{(c)} := T \llcorner \mathcal{M}^{(c)}$, $T^{(j,c)} := T \llcorner \mathcal{M}^{(j,c)}$ are mutually singular measures, and we denote by \llcorner the restriction of the Radon measure T . Hereafter we use the notation $\widehat{x}^1 = x^2$ and $\widehat{x}^2 = x^1$.

Theorem 3.5. (Structure Theorem for $\text{cart}(\Omega \times \mathbb{S}^1)$) *Let $T \in \text{cart}(\Omega \times \mathbb{S}^1)$. Then there exists a unique map $u_T \in BV(\Omega; \mathbb{S}^1)$ and an (not unique) i.m. rectifiable 1-current $L_T = \tau(\mathcal{L}, k, \mathbf{L}_T) \in \mathcal{D}_1(\Omega)$ such that $T^{(j,c)} = T^{(j)} + L_T \times \llbracket \mathbb{S}^1 \rrbracket$ and*

$$T(\phi(x, y) dx) = T^{(a)}(\phi(x, y) dx) = \int_{\Omega} \phi(x, u_T(x)) dx, \tag{3.11}$$

$$T^{(a)}(\phi(x, y) d\widehat{x}^l \wedge dy^m) = (-1)^{2-l} \int_{\Omega} \phi(x, u_T(x)) \partial_{x^l}^{(a)} u_T^m(x) dx, \tag{3.12}$$

$$T^{(c)}(\phi(x, y) d\widehat{x}^l \wedge dy^m) = (-1)^{2-l} \int_{\Omega} \phi(x, \tilde{u}_T(x)) d\partial_{x^l}^{(c)} u_T^m(x), \tag{3.13}$$

⁵ As for the Approximation Theorem, no boundary regularity is required for this result.

$$T^{(j)}(\phi(x, y) d\tilde{x}^j \wedge dy^m) = (-1)^{2-l} \int_{J_{u_T}} \left\{ \int_{\gamma_x} \phi(x, y) dy^m \right\} v_{u_T}^l(x) d\mathcal{H}^1(x) \tag{3.14}$$

for every $\phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$, γ_x being the (oriented) geodesic arc in \mathbb{S}^1 that connects $u_T^-(x)$ to $u_T^+(x)$ and \tilde{u}_T being the precise representative of u_T .

Remark 3.6. In [32, Theorem 6] the structure of $T^{(j)}$ is formulated in a slightly different way, using the counter-clockwise arc $\gamma_{\varphi^-, \varphi^+}$ between $(\cos(\varphi^-), \sin(\varphi^-))$ and $(\cos(\varphi^+), \sin(\varphi^+))$ and replacing J_{u_T} by J_φ , where $\varphi \in BV(\Omega)$ is a local lifting of T . More precisely, the notion of lifting is understood in the sense that $T = \chi\#G_\varphi$, where $\chi : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2 \times \mathbb{S}^1$ is the covering map $(x, \vartheta) \mapsto (x, \cos(\vartheta), \sin(\vartheta))$ and $G_\varphi \in \text{cart}(\Omega \times \mathbb{R})$ is the cartesian current given by the boundary of the subgraph of φ (hence the push-forward via χ is well-defined as G_φ has finite mass, see Section 3.1). To explain how to deduce (3.14), we recall the local construction in [32]: for every $x \in J_\varphi$ one chooses $p^+(x) \geq 0$ and $k'(x) \in \mathbb{N} \cup \{0\}$ such that

$$\varphi^+(x) = p^+(x) + 2\pi k'(x), \quad 0 \leq p^+(x) - \varphi^-(x) < 2\pi,$$

where we recall that in the scalar case the traces (and the normal to the jump set) are arranged to satisfy $\varphi^- < \varphi^+$ on J_φ . Then, locally, the 1-current L'_T in [32, Theorem 6] is given by $L'_T = \tau(\mathcal{L}', k'(x), L'_T)$, where $\mathcal{L}' \subset J_\varphi$ denotes the set of points with $k'(x) \geq 1$ and L'_T is the orientation of \mathcal{L}' defined via $L'_T = v_\varphi^2 e_1 - v_\varphi^1 e_2$. To obtain the representation via geodesics, we let

$$q^+(x), k(x) = \begin{cases} (p^+(x), k'(x)) & \text{if } p^+(x) - \varphi^-(x) < \pi, \\ (p^+(x) - 2\pi, k'(x) + 1) & \text{if } p^+(x) - \varphi^-(x) > \pi, \end{cases}$$

The case $p^+(x) - \varphi^-(x) = \pi$, i.e. antipodal points, needs special care. In this case we define $q^+(x)$ and $k(x)$ according to the following rule: let $\tilde{\varphi}^\pm(x) := \varphi^\pm(x) \bmod 2\pi \in [0, 2\pi)$. Then

$$q^+(x), k(x) = \begin{cases} (p^+(x), k'(x)) & \text{if } \Psi(\tilde{\varphi}^+(x) - \tilde{\varphi}^-(x)) = \pi, \\ (p^+(x) - 2\pi, k'(x) + 1) & \text{if } \Psi(\tilde{\varphi}^+(x) - \tilde{\varphi}^-(x)) = -\pi, \end{cases}$$

with the function Ψ defined in (2.4). Replacing $(p^+(x), k'(x))$ by $(q^+(x), k(x))$, the modified structure of $T^{(j)}$ can be proven following exactly the lines of [32, p.107-108], noting that by the chain rule in BV [11, Theorem 3.96] we have $J_\varphi = J_{u_T} \cup \{x \in J_\varphi : q^+(x) = \varphi^-(x)\}$. In particular,

$$L_T = \tau(\mathcal{L}, k, L_T), \quad \mathcal{L} = \{x \in J_\varphi : k(x) \geq 1\}, \quad L_T = v_\varphi^2 e_1 - v_\varphi^1 e_2 \tag{3.15}$$

still depend on the local lifting φ , but in (3.14) the curves $\gamma_{\varphi^-, \varphi^+}$ are replaced by the more intrinsic geodesic arcs γ_x connecting $u_T^-(x) = (\cos(\varphi^-(x)), \sin(\varphi^-(x)))$ to $u_T^+(x) = (\cos(\varphi^+(x)), \sin(\varphi^+(x)))$ (these formulas are consistent with the choice $v_{u_T}(x) = v_\varphi(x)$). In particular, exchanging $u_T^-(x)$ and $u_T^+(x)$ will change the

orientation of the arc (also in the case of antipodal points) and of the normal $\nu_{u_T}(x)$,⁶ so that the formula for $T^{(j)}$ is invariant, hence well-defined without the use of local liftings.

It is convenient to recast the jump-concentration part of $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ in the following way. Let $L_T = \tau(\mathcal{L}, k, \mathbf{L}_T)$ as in Theorem 3.5. We introduce for \mathcal{H}^1 -a.e. $x \in J_T$ the normal $\nu_T(x)$ to the 1-rectifiable set $J_T = J_{u_T} \cup \mathcal{L}$ as

$$\nu_T(x) = \begin{cases} \nu_{u_T}(x) & \text{if } x \in J_{u_T}, \\ (-\mathbf{L}_T^2(x), \mathbf{L}_T^1(x)) & \text{if } x \in \mathcal{L} \setminus J_{u_T}, \end{cases} \tag{3.16}$$

where we choose $\nu_{u_T}(x) = (-\mathbf{L}_T^2(x), \mathbf{L}_T^1(x))$ if $x \in \mathcal{L} \cap J_{u_T}$. For \mathcal{H}^1 -a.e. $x \in J_T$ we consider the curve γ_x^T given by: the (oriented) geodesic arc γ_x which connects $u_T^-(x)$ to $u_T^+(x)$ if $x \in J_{u_T} \setminus \mathcal{L}$ (in the sense of Remark 3.6 in case of antipodal points); the whole \mathbb{S}^1 turning $k(x)$ times if $x \in \mathcal{L} \setminus J_{u_T}$, $k(x)$ being the integer multiplicity of L_T ; the sum (in the sense of currents)⁷ of the oriented geodesic arc γ_x and of \mathbb{S}^1 with multiplicity $k(x)$ if $x \in J_{u_T} \cap \mathcal{L}$. Then

$$T^{(j^c)}(\phi(x, y) \, d\mathbf{x}^l \wedge dy^m) = (-1)^{2-l} \int_{J_T} \left\{ \int_{\gamma_x^T} \phi(x, y) \, dy^m \right\} \nu_T^l(x) \, d\mathcal{H}^1(x). \tag{3.17}$$

The integration over γ_x^T with respect to the form dy^m in the formula above is intended with the correct multiplicity of the curve γ_x^T defined for \mathcal{H}^1 -a.e. $x \in J_T$ by the integer number

$$m(x, y) := \begin{cases} \pm 1, & \text{if } x \in J_{u_T} \setminus \mathcal{L}, \, y \in \text{supp}(\gamma_x), \\ k(x), & \text{if } x \in \mathcal{L} \setminus J_{u_T}, \, y \in \mathbb{S}^1, \\ k(x) \pm 1, & \text{if } x \in \mathcal{L} \cap J_{u_T}, \, y \in \text{supp}(\gamma_x), \\ k(x), & \text{if } x \in \mathcal{L} \cap J_{u_T}, \, y \in \text{supp}(\gamma_x^T) \setminus \text{supp}(\gamma_x), \end{cases} \tag{3.18}$$

where $\pm = +/-$ if the geodesic arc γ_x is oriented counterclockwise/clockwise, respectively. More precisely,

$$\int_{\gamma_x^T} \phi(y) \, dy^m = (-1)^m \int_{\text{supp}(\gamma_x^T)} \phi(y) \widehat{y}^m m(x, y) \, d\mathcal{H}^1(y). \tag{3.19}$$

⁶ More precisely, assume that $u_1, u_2, \nu \in \mathbb{S}^1$ and assume that the geodesic arc from u_1 to u_2 is counterclockwise. If $(u_T^+(x), u_T^-(x), \nu_{u_T}(x)) = (u_2, u_1, \nu)$, then γ_x is oriented counterclockwise. If, instead, $(u_T^+(x), u_T^-(x), \nu_{u_T}(x)) = (u_1, u_2, -\nu)$ (equivalent to the first choice, according to the definition of jump point), then γ_x is oriented clockwise.

⁷ In this case, a more elementary way of defining γ_x^T is the following: let $\gamma_x : [0, 1] \rightarrow \mathbb{S}^1$ be the geodesic arc, and let $\varphi_x : [0, 1] \rightarrow \mathbb{R}$ be a continuous function (unique up to translations of an integer multiple of 2π) such that $\gamma_x(t) = \exp(i\varphi_x(t))$. Then $\gamma_x^T(t) = \exp(i(1-t)\varphi_x(0) + t(\varphi_x(1) + 2\pi k(x)))$.

Remark 3.7. Note that we constructed $m(x, y)$ based on the orientation (3.16) of ν_T . As discussed in Remark 3.6, changing the orientation of ν_{u_T} changes the orientation of the geodesic γ_x , while a change of the orientation of L_T switches the sign of $k(x)$. Hence changing the orientation of $\nu_T(x)$ changes $m(x, y)$ into $-m(x, y)$. If we choose locally $\nu_T = \nu_\varphi$ as in Remark 3.6, our construction above yields $m(x, y) \geq 0$.

In the proposition below, we derive an explicit formula for the 2-vector T of a cartesian current.

Proposition 3.8. *Let $T \in \text{cart}(\Omega \times \mathbb{S}^1)$, let u_T be the BV function associated to T . Then $|T^{(a)}| = \mathcal{H}^2 \llcorner \mathcal{M}^{(a)}$, $|T^{(c)}| = \mathcal{H}^2 \llcorner \mathcal{M}^{(c)}$, $|T^{(j c)}| = |m| \mathcal{H}^2 \llcorner \mathcal{M}^{(j c)}$, and*

$$\begin{aligned} \sqrt{1 + |\nabla u_T(x)|^2} T(x, y) &= e_1 \wedge e_2 \\ &+ \partial_{x_2}^{(a)} u_T^1(x) e_1 \wedge \bar{e}_1 + \partial_{x_2}^{(a)} u_T^2(x) e_1 \wedge \bar{e}_2 \\ &- \partial_{x_1}^{(a)} u_T^1(x) e_2 \wedge \bar{e}_1 - \partial_{x_1}^{(a)} u_T^2(x) e_2 \wedge \bar{e}_2, \end{aligned} \tag{3.20}$$

for \mathcal{H}^2 -a.e. $(x, y) \in \mathcal{M}^{(a)}$,

$$\begin{aligned} T(x, y) &= \frac{d\partial_{x_2}^{(c)} u_T^1}{d|D^{(c)} u_T|}(x) e_1 \wedge \bar{e}_1 + \frac{d\partial_{x_2}^{(c)} u_T^2}{d|D^{(c)} u_T|}(x) e_1 \wedge \bar{e}_2 \\ &- \frac{d\partial_{x_1}^{(c)} u_T^1}{d|D^{(c)} u_T|}(x) e_2 \wedge \bar{e}_1 - \frac{d\partial_{x_1}^{(c)} u_T^2}{d|D^{(c)} u_T|}(x) e_2 \wedge \bar{e}_2, \end{aligned} \tag{3.21}$$

for \mathcal{H}^2 -a.e. $(x, y) \in \mathcal{M}^{(c)}$, and

$$\begin{aligned} \text{sign}(m(x, y)) T(x, y) &= -v_T^2(x) y^2 e_1 \wedge \bar{e}_1 + v_T^2(x) y^1 e_1 \wedge \bar{e}_2 \\ &+ v_T^1(x) y^2 e_2 \wedge \bar{e}_1 - v_T^1(x) y^1 e_2 \wedge \bar{e}_2, \end{aligned} \tag{3.22}$$

for \mathcal{H}^2 -a.e. $(x, y) \in \mathcal{M}^{(j c)}$, where $m(x, y)$ is the integer defined in (3.18).

Proof. Assume Ω simply connected (if not, the following arguments can be repeated locally). Let us consider the covering map $\chi: \Omega \times \mathbb{R} \rightarrow \Omega \times \mathbb{S}^1$ defined by $\chi(x, \vartheta) := (x, \cos(\vartheta), \sin(\vartheta))$. By [32, Corollary 1, p. 105] there exists a lifting of T , i.e., there is a function $\varphi \in BV(\Omega; \mathbb{R})$ such that $T = \chi\#G_\varphi$, where $G_\varphi \in \text{cart}(\Omega \times \mathbb{R})$ is the cartesian current given by the boundary of the subgraph of φ . The fine structure of such currents is well known, compare [28, Theorem 4.5.9], [30, 4.1.5 & 4.2.4]. We recall here that, if we consider the subgraph $SG_\varphi := \{(x, y) \in \Omega \times \mathbb{R} : y < \varphi(x)\}$, then SG_φ is a set of finite perimeter; G_φ is the current $G_\varphi = \partial \llbracket SG_\varphi \rrbracket$. The interior normal to SG_φ is given by

$$\begin{aligned} n(x, \varphi(x)) &= \frac{d(D\varphi, -\mathcal{L}^2)}{d|(D\varphi, -\mathcal{L}^2)|}(x), \quad \text{for } x \in \Omega \setminus J_\varphi, \\ n(x, \vartheta) &= (\nu_\varphi(x), 0), \quad \text{for } x \in J_\varphi, \vartheta \in [\varphi^-(x), \varphi^+(x)], \end{aligned} \tag{3.23}$$

where ν_φ is the normal to the jump set J_φ and \mathcal{L}^2 denotes the two-dimensional Lebesgue measure. Moreover, the current G_φ can be represented as $G_\varphi = \mathbf{G}_\varphi |G_\varphi|$

where $|G_\varphi|$ is concentrated on the reduced boundary $\partial^- SG_u$, $|G_\varphi| = \mathcal{H}^2 \llcorner \partial^- SG_u$, and \mathbf{G}_φ is the 2-vector in \mathbb{R}^3 such that $-\mathbf{G}_\varphi(x, \vartheta) \wedge n(x, \vartheta) = e_1 \wedge e_2 \wedge e_3$, i.e.,

$$\mathbf{G}_\varphi = -n^3 e_1 \wedge e_2 + n^2 e_1 \wedge e_3 - n^1 e_2 \wedge e_3.$$

Finally, letting that

$$\begin{aligned} \Sigma^{(a)} &:= \{(x, \tilde{\varphi}(x)) : x \in \Omega \setminus J_\varphi, n^3(x, u(x)) \neq 0\}, \\ \Sigma^{(c)} &:= \{(x, \tilde{\varphi}(x)) : x \in \Omega \setminus J_\varphi, n^3(x, u(x)) = 0\}, \\ \Sigma^{(j)} &:= \{(x, \vartheta) : x \in J_\varphi, \vartheta \in [\varphi^-(x), \varphi^+(x)], n^3(x, \vartheta) = 0\}, \end{aligned}$$

we have that $\partial^- SG_\varphi = \Sigma^{(a)} \cup \Sigma^{(c)} \cup \Sigma^{(j)}$ and, denoting $G_\varphi^{(a)} = G_\varphi \llcorner \Sigma^{(a)}$, $G_\varphi^{(c)} = G_\varphi \llcorner \Sigma^{(c)}$, $G_\varphi^{(j)} = G_\varphi \llcorner \Sigma^{(j)}$, and by [32, formulas (2) and (16)] we have on the one hand that $u_T = (\cos(\varphi), \sin(\varphi))$ a.e. and

$$\chi_\# G_\varphi^{(a)} = T^{(a)}, \quad \chi_\# G_\varphi^{(c)} = T^{(c)}, \quad \chi_\# G_\varphi^{(j)} = T^{(j)}. \tag{3.24}$$

On the other hand, observe that the Jacobian of $d\chi : \text{Tan}(\partial^- SG_\varphi, x) \mapsto \mathbb{R}^4$ equals 1 (indeed $d\chi$ maps any pair of orthonormal vectors of \mathbb{R}^3 to a pair of orthonormal vectors in \mathbb{R}^4). Hence, by the area formula, for $\sigma \in \{a, c, j\}$ we obtain

$$\begin{aligned} \chi_\# G_\varphi^{(\sigma)}(\omega) &= \int_{\Sigma^{(\sigma)}} \langle \chi^\# \omega, \mathbf{G}_\varphi \rangle d\mathcal{H}^2(x, \vartheta) \\ &= \int_{\chi(\Sigma^{(\sigma)})} \langle \omega(x, y), \sum_{(x, \vartheta) \in \chi^{-1}(x, y)} d\chi(x, \vartheta) \mathbf{G}_\varphi(x, \vartheta) \rangle d\mathcal{H}^2(x, y). \end{aligned} \tag{3.25}$$

Next, note that for $\sigma \in \{a, c\}$ the map $\tilde{\chi} : \Sigma^{(\sigma)} \rightarrow \chi(\Sigma^{(\sigma)})$ is one-to-one and for any $(x, \tilde{\varphi}(x)) \in \Sigma^{(a)} \cup \Sigma^{(c)}$ we have

$$\begin{aligned} d\chi(x, \tilde{\varphi}(x)) \mathbf{G}_\varphi(x, \tilde{\varphi}(x)) &= -n^3(x, \tilde{\varphi}(x)) e_1 \wedge e_2 \\ &\quad - n^2(x, \tilde{\varphi}(x)) \sin(\tilde{\varphi}(x)) e_1 \wedge \bar{e}_1 + n^2(x, \tilde{\varphi}(x)) \cos(\tilde{\varphi}(x)) e_1 \wedge \bar{e}_2 \\ &\quad + n^1(x, \tilde{\varphi}(x)) \sin(\tilde{\varphi}(x)) e_2 \wedge \bar{e}_1 - n^1(x, \tilde{\varphi}(x)) \cos(\tilde{\varphi}(x)) e_2 \wedge \bar{e}_2. \end{aligned} \tag{3.26}$$

Since $|n| = 1$ we see that $|d\chi(x, \tilde{\varphi}(x)) \mathbf{G}_\varphi(x, \tilde{\varphi}(x))| = 1$, too. Moreover, for \mathcal{H}^2 -a.e. $(x, y) \in \chi(\Sigma^{(\sigma)})$ the vector $d\chi(x, \tilde{\varphi}(x)) \mathbf{G}_\varphi(x, \tilde{\varphi}(x))$ orients the tangent space at (x, y) . Hence (3.24) and the uniqueness of the representation of i.m. rectifiable currents (cf. Section 3.1) implies $\chi(\Sigma^{(\sigma)}) = \mathcal{M}^{(\sigma)}$ up to a \mathcal{H}^2 -negligible set, $|T^{(\sigma)}| = \mathcal{H}^2 \llcorner \mathcal{M}^{(\sigma)}$, and

$$T(\chi(x, \tilde{\varphi}(x))) = d\chi(x, \tilde{u}(x)) \mathbf{G}_\varphi(x, \tilde{\varphi}(x))$$

for \mathcal{H}^2 -almost every $(x, y) = \chi(x, \tilde{\varphi}(x)) \in \Sigma^{(\sigma)}$. By the chain rule in BV [11, Theorem 3.96] we deduce that

$$\nabla u_T = \begin{pmatrix} -u_T^2 \\ u_T^1 \end{pmatrix} \otimes \nabla \varphi, \quad D^{(c)} u_T = \begin{pmatrix} -\tilde{u}_T^2 \\ \tilde{u}_T^1 \end{pmatrix} \otimes D^{(c)} \varphi.$$

Combined with the formula for n given by (3.23), the formulas (3.20) and (3.21) then follow from (3.26) by a straightforward calculation.

In order to treat the case $\sigma = j$, note that due to (3.23) we have for any $(x, y) = \chi(x, \vartheta) \in \chi(\Sigma^{(j)})$

$$\begin{aligned} d\chi(x, \vartheta)\mathbf{G}_\varphi(x, \vartheta) &= -v_\varphi^2(x)y^2e_1 \wedge \bar{e}_1 + v_\varphi^2(x)y^1e_1 \wedge \bar{e}_2 \\ &\quad + v_\varphi^1(x)y^2e_2 \wedge \bar{e}_1 - v_\varphi^1(x)y^1e_2 \wedge \bar{e}_2 \\ &=: \xi(x, y). \end{aligned}$$

Again $|\xi(x, y)| = 1$ and $\xi(x, y)$ orients the tangent space at \mathcal{H}^2 -a.e. $(x, y) \in \chi(\Sigma^{(j)})$. Thus (3.25) and the uniqueness of the representation of i.m. rectifiable currents imply (up to \mathcal{H}^2 -negligible sets) that $\mathcal{M}^{(jc)} = \chi(\Sigma^{(j)})$, $T = \xi$ on $\mathcal{M}^{(jc)}$, and $|T^{(jc)}| = N(x, y)\mathcal{H}^2 \llcorner \mathcal{M}^{(jc)}$, with

$$N(x, y) = \#\{\vartheta \in [\varphi^-(x), \varphi^+(x)] : (\cos(\vartheta), \sin(\vartheta)) = y\}.$$

To conclude, we have to relate $m(x, y)$ to $N(x, y)$ and $v_T(x)$ to $v_\varphi(x)$. First note that the proof of the structure theorem (sketched in Remark 3.6) yields $J_{u_T} \cup \mathcal{L} = J_\varphi$ and, combined with the definition of the curves γ_x^T (cf. (3.17)), implies that $\chi[\varphi^-(x), \varphi^+(x)] = \text{supp}(\gamma_x^T)$ for $x \in J_\varphi$. Hence

$$\mathcal{M}^{(jc)} = \chi(\Sigma^{(j)}) = \{(x, y) \in \Omega \times \mathbb{R}^2 : x \in J_{u_T} \cup \mathcal{L}, y \in \text{supp}(\gamma_x^T)\}. \tag{3.27}$$

Moreover, provided we orient J_{u_T} the same way as J_φ and \mathcal{L} according to (3.15), equation (3.16) also yields $v_T = v_\varphi$ and $m(x, y) = N(x, y)$ (a detailed proof of the latter requires to distinguish different cases, which we omit here).⁸ Inserting this equality in (3.27) concludes the proof of (3.22).

Finally, we recall the following result, proven in [32, Section 4].

Proposition 3.9. *If $u \in BV(\Omega; \mathbb{S}^1)$, then there exists a $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ such that $u_T = u$ a.e. in Ω .*

3.5. Currents associated to discrete spin fields

We introduce the piecewise constant interpolations of spin fields. For a set S , we define

$$\mathcal{PC}_\varepsilon(S) := \{u: \mathbb{R}^2 \rightarrow S : u(x) = u(\varepsilon i) \text{ if } x \in \varepsilon i + [0, \varepsilon)^2 \text{ for some } i \in \varepsilon\mathbb{Z}^2\}$$

Given $u: \Omega_\varepsilon \rightarrow \mathbb{S}^1$, we can always identify it with its piecewise constant interpolation belonging to $\mathcal{PC}_\varepsilon(\mathbb{S}^1)$, arbitrarily extended to \mathbb{R}^2 . Note that the piecewise constant interpolation of u coincides with u on the bottom-left corners of the squares of the lattice $\varepsilon\mathbb{Z}^2$.

⁸ As noted in Remark 3.7, the choice $v_T(x) = v_\varphi(x)$ always yields $m(x, y) \geq 0$. The factor $\text{sign}(m(x, y))$ in (3.22) makes the formula invariant under the change of $v_T(x)$.

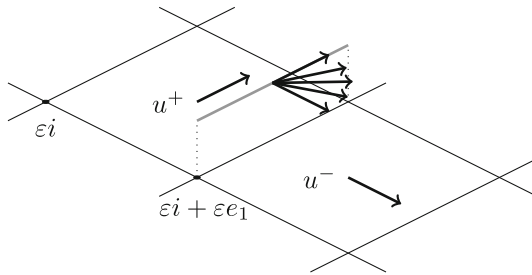


Fig. 5. The current G_u has vertical parts concentrated on the jump set J_u , where a transition from u^- to u^+ occurs

We associate to $u \in \mathcal{PC}_\varepsilon(\mathbb{S}^1)$ the current $G_u \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ defined by

$$G_u(\phi(x, y) dx^1 \wedge dx^2) := \int_\Omega \phi(x, u(x)) dx, \tag{3.28}$$

$$G_u(\phi(x, y) d\tilde{x}^l \wedge dy^m) := (-1)^{2-l} \int_{J_u} \left\{ \int_{\gamma_x} \phi(x, y) dy^m \right\} v_u^l(x) d\mathcal{H}^1(x), \tag{3.29}$$

$$G_u(\phi(x, y) dy^1 \wedge dy^2) := 0, \tag{3.30}$$

for every $\phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$, where J_u is the jump set of u , $v_u(x)$ is the normal to J_u at x , and $\gamma_x \subset \mathbb{S}^1$ is the (oriented) geodesic arc which connects the two traces $u^-(x)$ and $u^+(x)$ (Fig. 5). If $u^+(x)$ and $u^-(x)$ are opposite vectors, the choice of the geodesic arc $\gamma_x \subset \mathbb{S}^1$ is done consistently with the choice made in (2.3) for the values $\Psi(\pi)$ and $\Psi(-\pi)$ as follows: let $\varphi^\pm(x) \in [0, 2\pi)$ be the phase of $u^\pm(x)$; if $\Psi(\varphi^+(x) - \varphi^-(x)) = \pi$, then γ_x is the arc that connects $u^-(x)$ to $u^+(x)$ counterclockwise; if $\Psi(\varphi^+(x) - \varphi^-(x)) = -\pi$, then γ_x is the arc that connects $u^-(x)$ to $u^+(x)$ clockwise. Note that the choice of the arc γ_x is independent of the orientation of the normal $v_u(x)$.

We define for \mathcal{H}^1 -a.e. $x \in J_u$ the integer number $m(x) = \pm 1$, where $\pm = +/ -$ if the geodesic arc γ_x is oriented counterclockwise/clockwise, respectively. Then

$$\int_{\gamma_x} \phi(x, y) dy^m = (-1)^m m(x) \int_{\text{supp}(\gamma_x)} \phi(x, y) \tilde{y}^m d\mathcal{H}^1(y). \tag{3.31}$$

In the proposition below we characterize the current G_u associated to a discrete spin field in terms of the decomposition $G_u = \mathbf{G}_u |G_u|$.

Proposition 3.10. *Let $u \in \mathcal{PC}_\varepsilon(\mathbb{S}^1)$ and let $G_u \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ be the current defined in (3.28)–(3.30). Then G_u is an i.m. rectifiable current and, according to the representation formula (3.1), $G_u = \mathbf{G}_u |G_u|$, where $|G_u| = \mathcal{H}^2 \llcorner \mathcal{M}$,*

$$\mathcal{M} = \mathcal{M}^{(a)} \cup \mathcal{M}^{(j)} = \{(x, u(x)) : x \in \Omega \setminus J_u\} \cup \{(x, y) : x \in J_u, y \in \gamma_x\},$$

and

$$\mathbf{G}_u(x, y) = e_1 \wedge e_2 \tag{3.32}$$

for \mathcal{H}^2 -a.e. $(x, y) \in \mathcal{M}^{(a)}$ and

$$\begin{aligned} \mathbf{G}_u(x, y) = \text{sign}(m(x)) & \left[-v_u^2(x)y^2e_1 \wedge \bar{e}_1 + v_u^2(x)y^1e_1 \wedge \bar{e}_2 \right. \\ & \left. + v_u^1(x)y^2e_2 \wedge \bar{e}_1 - v_u^1(x)y^1e_2 \wedge \bar{e}_2 \right] \end{aligned} \tag{3.33}$$

for \mathcal{H}^2 -a.e. $(x, y) \in \mathcal{M}^{(j)}$.

Proof. First note that the set \mathcal{M} is countably \mathcal{H}^2 -rectifiable. Since u is piecewise constant, for horizontal forms we have

$$G_u(\phi(x, y) dx) = \int_{\Omega} \phi(x, u(x)) dx = \int_{\Omega \times \mathbb{R}^2} \phi(x, y) d\mathcal{H}^2 \llcorner \mathcal{M}^{(a)}(x, y).$$

By (3.31) we deduce that for $l, m = 1, 2$

$$\begin{aligned} & G_u(\phi(x, y) d\widehat{x}^l \wedge dy^m) \\ &= (-1)^{2-l} \int_{J_u} \left\{ \int_{\gamma_x} \phi(x, y) dy^m \right\} v_u^l(x) d\mathcal{H}^1(x) \\ &= (-1)^{2-l+m} \int_{J_u} \left\{ \int_{\text{supp}(\gamma_x)} \phi(x, y) \widehat{y}^m d\mathcal{H}^1(y) \right\} v_u^l(x) m(x) d\mathcal{H}^1(x) \\ &= (-1)^{2-l+m} \int_{\Omega \times \mathbb{R}^2} \phi(x, y) \widehat{y}^m v_u^l(x) m(x) d\mathcal{H}^2 \llcorner \mathcal{M}^{(j)}(x, y). \end{aligned}$$

Then for every $\omega \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ we have

$$G_u(\omega) = \int_{\Omega \times \mathbb{R}^2} \langle \omega, \mathbf{G}_u \rangle d\mathcal{H}^2 \llcorner \mathcal{M}$$

for \mathbf{G}_u defined as in (3.32)–(3.33) and moreover $\mathbf{G}_u(x, y)$ is associated to the tangent space at $(x, y) \in \mathcal{M}$. Since also $|\mathbf{G}_u(x, y)| = 1$ for $|G_u|$ -a.e. $(x, y) \in \Omega \times \mathbb{R}^2$, we conclude the proof.

The next proposition is crucial since it relates the boundary of the current G_u associated to a discrete spin field to the vorticity measure μ_u .

Proposition 3.11. *Let $u \in \mathcal{PC}_\varepsilon(\mathbb{S}^1)$ and let $G_u \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ be the current defined in (3.28)–(3.30). Then*

$$\partial G_u|_{\Omega \times \mathbb{R}^2} = -\mu_u \times \llbracket \mathbb{S}^1 \rrbracket,$$

where μ_u is the discrete vorticity measure defined in (2.6) for $u|_{\varepsilon\mathbb{Z}^2} : \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$.

Proof. Let us fix $0 < \rho < \min\{\varepsilon/4, \text{dist}(\Omega_\varepsilon, \partial\Omega)\}$ and $\eta \in \mathcal{D}^1(\Omega \times \mathbb{R}^2)$. With a partition of unity we can split η into the sum of 1-forms depending on their supports. We discuss here all the possibilities for the supports.

Case 1 $\text{supp}(\eta) \subset (\varepsilon i + (0, \varepsilon)^2) \times \mathbb{R}^2$ for some $i \in \mathbb{Z}^2$. Since u is constant in $(\varepsilon i + (0, \varepsilon)^2)$, we get automatically $\partial G_u(\eta) = 0$ by Remark 3.1.

Case 2 Let H be the side of the square $\varepsilon i + [0, \varepsilon]^2$ connecting two vertices $p, q \in \varepsilon\mathbb{Z}^2$ and let U be the $\rho/2$ -neighborhood of $H \setminus (B_\rho(p) \cup B_\rho(q))$. Assume that $\text{supp}(\eta) \subset U \times \mathbb{R}^2$. We claim that

$$\partial G_u(\eta) = 0. \tag{3.34}$$

To prove this, we approximate the pure-jump function u by means of a sequence of Lipschitz functions u_j . Let u^\pm be the traces of u on the two sides of H and let ν_H be the normal to H oriented as ν_u . We let $\widehat{\varphi}^\pm \in [0, 2\pi)$ be the phases of u^\pm defined by $u^\pm = \exp(i\widehat{\varphi}^\pm)$. We set $\varphi^- := \widehat{\varphi}^-$ and $\varphi^+ := \widehat{\varphi}^- + \Psi(\widehat{\varphi}^+ - \widehat{\varphi}^-) \in (-\pi, 3\pi)$, where Ψ is the function given by (2.4). We then define

$$\varphi(t) := \begin{cases} \varphi^-, & \text{if } t \leq -\frac{1}{2} \\ \varphi^- + (\varphi^+ - \varphi^-)(t + \frac{1}{2}) & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ \varphi^+ & \text{if } t \geq \frac{1}{2}, \end{cases}$$

and $\varphi_k(s) := \varphi(ks)$ for k large enough. Note that the curve $t \in (-1/2, 1/2) \mapsto \exp(i\varphi(t))$ parametrizes the geodesic arc $\gamma_\pm \subset \mathbb{S}^1$ which connects u^- to u^+ , consistently with the choice done in formula (3.29). Then we put

$$u_k(x) := \exp(i\varphi_k(\nu_H \cdot (x - p))) \quad \text{for } x \in U.$$

We prove that $G_{u_k} \rightarrow G_u$ in $\mathcal{D}_2(U \times \mathbb{R}^2)$. Let us fix $\phi \in C_c^\infty(U \times \mathbb{R}^2)$. Since $u_k \rightarrow u$ in measure, we have that

$$G_{u_k}(\phi(x, y) dx) = \int_U \phi(x, u_k(x)) dx \rightarrow \int_U \phi(x, u(x)) dx = G_u(\phi(x, y) dx).$$

Writing $x \in U$ as $x = x' + s\nu_H$ with $x' \in H, s \in \mathbb{R}$, for $l = 1, 2$ we further obtain that

$$\begin{aligned} G_{u_k}(\phi(x, y) d\widehat{x}^l \wedge dy^1) &= (-1)^{2-l} \int_U \phi(x, u_k(x)) \partial_{x_l} u_k^1(x) dx \\ &= (-1)^{3-l} \int_U \phi(x, u_k(x)) \sin(\varphi_k(\nu_H \cdot (x - p))) \varphi_k'(\nu_H \cdot (x - p)) v_H^l dx \\ &= (-1)^{3-l} \int_H \left\{ \int_{-1/2k}^{1/2k} \phi(x' + s\nu_H, \exp(i\varphi_k(s))) \sin(\varphi_k(s)) \varphi_k'(s) ds \right\} v_H^l d\mathcal{H}^1(x') \\ &= (-1)^{3-l} \int_H \left\{ \int_{-1/2}^{1/2} \phi(x' + \frac{t}{k}\nu_H, \exp(i\varphi(t))) \sin(\varphi(t)) \varphi'(t) dt \right\} v_H^l d\mathcal{H}^1(x') \\ &= (-1)^{2-l} \int_H \left\{ \int_{\gamma_\pm} \phi(x' + \frac{t}{k}\nu_H, y) dy^1 \right\} v_H^l d\mathcal{H}^1(x') \\ &\rightarrow (-1)^{2-l} \int_H \left\{ \int_{\gamma_\pm} \phi(x', y) dy^1 \right\} v_H^l d\mathcal{H}^1(x') = G_u(\phi(x, y) d\widehat{x}^l \wedge dy^1), \end{aligned}$$

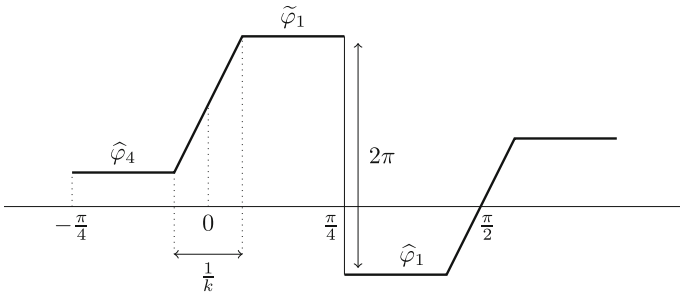


Fig. 6. Example for the definition of φ_k for $h = 0$

where $\gamma_{\pm} \subset \mathbb{S}^1$ is the geodesic arc connecting u^- to u^+ . With analogous computations one proves $G_{u_k}(\phi(x, y) \, d\hat{x}^1 \wedge dy^2) \rightarrow G_u(\phi(x, y) \, d\hat{x}^1 \wedge dy^2)$.

Hence, due to Stokes' Theorem we have that

$$0 = \partial G_{u_k}(\eta) = G_{u_k}(d\eta) \rightarrow G_u(d\eta) = \partial G_u(\eta),$$

which proves (3.34).

Case 3 $\text{supp}(\eta) \subset B_\rho(p) \times \mathbb{R}^2$, where $p = \varepsilon i + \varepsilon e_1 + \varepsilon e_2$ for some $i \in \mathbb{Z}^2$. In this case we will approximate the current G_u with graphs of a sequence of functions u_k which are Lipschitz outside the point p . For notation simplicity we let $\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3, \widehat{\varphi}_4 \in [0, 2\pi)$ be the phases defined by the relations

$$\begin{aligned} u(\varepsilon i + \varepsilon e_1 + \varepsilon e_2) &=: u_1 = \exp(i\widehat{\varphi}_1), & u(\varepsilon i + \varepsilon e_2) &=: u_2 = \exp(i\widehat{\varphi}_2), \\ u(\varepsilon i) &=: u_3 = \exp(i\widehat{\varphi}_3), & u(\varepsilon i + \varepsilon e_1) &=: u_4 = \exp(i\widehat{\varphi}_4). \end{aligned} \tag{3.35}$$

We define the auxiliary angles

$$\widetilde{\varphi}_{\sigma(h+1)} := \widehat{\varphi}_{\sigma(h)} + \Psi(\widehat{\varphi}_{\sigma(h+1)} - \widehat{\varphi}_{\sigma(h)}), \tag{3.36}$$

for $h = 1, 2, 3, 4$, where $\sigma(h) \in \{1, 2, 3, 4\}$ is such that $\sigma(h) \equiv h \pmod{4}$ (the term $\Psi(\widehat{\varphi}_{\sigma(h+1)} - \widehat{\varphi}_{\sigma(h)})$ is the oriented angle in $[-\pi, \pi]$ between the two vectors $u_{\sigma(h)}$ and $u_{\sigma(h+1)}$). We introduce the 2π -periodic function $\varphi_k : \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{aligned} \varphi_k(\vartheta) & \\ := \begin{cases} \widehat{\varphi}_{\sigma(h)}, & \text{if } -\frac{\pi}{4} + h\frac{\pi}{2} < \vartheta \leq h\frac{\pi}{2} - \frac{1}{2k}, \\ \widehat{\varphi}_{\sigma(h)} + k(\widetilde{\varphi}_{\sigma(h+1)} - \widehat{\varphi}_{\sigma(h)})(\vartheta - h\frac{\pi}{2} + \frac{1}{2k}), & \text{if } h\frac{\pi}{2} - \frac{1}{2k} < \vartheta < h\frac{\pi}{2} + \frac{1}{2k} \\ \widetilde{\varphi}_{\sigma(h+1)}, & \text{if } h\frac{\pi}{2} + \frac{1}{2k} \leq \vartheta < \frac{\pi}{4} + h\frac{\pi}{2}. \end{cases} \end{aligned}$$

for $\vartheta \in (-\frac{\pi}{4} + h\frac{\pi}{2}, \frac{\pi}{4} + h\frac{\pi}{2})$, $h \in \mathbb{Z}$ (see also Fig. 6). The function φ_k might have jumps at the points $\frac{\pi}{4} + h\frac{\pi}{2}$, $h \in \mathbb{Z}$; note, however, that according to (2.3) the amplitude of the jump is given by

$$\begin{aligned} \widehat{\varphi}_{\sigma(h+1)} - \widetilde{\varphi}_{\sigma(h+1)} &= \widehat{\varphi}_{\sigma(h+1)} - \widehat{\varphi}_{\sigma(h)} - \Psi(\widehat{\varphi}_{\sigma(h+1)} - \widehat{\varphi}_{\sigma(h)}) \\ &= Q(\widehat{\varphi}_{\sigma(h+1)} - \widehat{\varphi}_{\sigma(h)}) \in 2\pi\mathbb{Z}. \end{aligned}$$

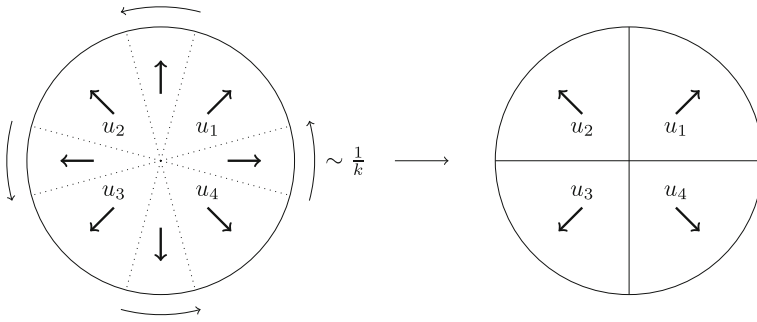


Fig. 7. Example of the approximation u_k (on the left) of the function u (on the right). The jump set of the function u is expanded and a transition between the jumps of u is constructed using the geodesic arcs in \mathbb{S}^1 between the traces. If u has a nontrivial discrete vorticity as in the picture, then the graph G_{u_k} of the function u_k has a hole in the center, as it happens for the graph of the map $x \mapsto \frac{x}{|x|}$. The hole is then preserved in the passage to limit to G_u , see formula (3.37)

We now define a map $v_k: \mathbb{S}^1 \rightarrow \mathbb{S}^1$. Given $y \in \mathbb{S}^1$, let $\vartheta(y) \in [0, 2\pi)$ be the angle such that $y = \exp(i\vartheta(y))$ and set

$$v_k(y) := \exp(i\varphi_k(\vartheta(y))).$$

The definition actually does not depend on the choice of the phase $\vartheta(y)$, due to the 2π -periodicity of φ_k . Thus we could also choose $\vartheta(y) \in [2\pi h, 2\pi(h + 1))$ for any $h \in \mathbb{Z}$. Note that v_k is continuous: indeed the possible jumps of φ_k have amplitude in $2\pi\mathbb{Z}$, and thus are not seen by v_k . In particular, we can compute the degree of the map v_k via the formula

$$\begin{aligned} \deg(v_k)2\pi &= \deg(v_k) \int_{\mathbb{S}^1} \omega_{\mathbb{S}^1} = \int_{\mathbb{S}^1} v_k^\# \omega_{\mathbb{S}^1} = \sum_{h=0}^3 \tilde{\varphi}_\sigma(h+1) - \widehat{\varphi}_\sigma(h) \\ &= \sum_{h=0}^3 \Psi(\widehat{\varphi}_\sigma(h+1) - \widehat{\varphi}_\sigma(h)) = d_u(\varepsilon i)2\pi, \end{aligned}$$

where $\omega_{\mathbb{S}^1}$ is the volume form on \mathbb{S}^1 and $d_u(\varepsilon i)$ is the discrete vorticity defined in (2.5).

We now define the map $u_k: B_\rho(p) \rightarrow \mathbb{S}^1$ by

$$u_k(x) := v_k\left(\frac{x-p}{|x-p|}\right).$$

Note that, if (r, ϑ) are polar coordinates for the point $x - p$, then the polar coordinates of $u_k(x)$ are $(1, \varphi_k(\vartheta))$ (see also Fig. 7).

By [30, 3.2.2, Example 2] we get that

$$\begin{aligned} \partial G_{u_k}|_{B_\rho(p) \times \mathbb{R}^2} &= -\deg(v_k)\delta_p \times \llbracket \mathbb{S}^1 \rrbracket \\ &= -d_u(\varepsilon i)\delta_p \times \llbracket \mathbb{S}^1 \rrbracket = -\mu_u \times \llbracket \mathbb{S}^1 \rrbracket|_{B_\rho(p) \times \mathbb{R}^2}. \end{aligned} \tag{3.37}$$

Therefore, to conclude the proof it suffices to show the convergence $G_{u_k} \rightarrow G_u$ in $\mathcal{D}_2(\Omega \times \mathbb{R}^2)$, so that

$$-\mu_u \times \llbracket \mathbb{S}^1 \rrbracket(\eta) = \partial G_{u_k}(\eta) \rightarrow \partial G_u(\eta).$$

To do so, let us fix $\phi \in C_c^\infty(B_\rho(p) \times \mathbb{R}^2)$. Since $u_k \rightarrow u$ in measure, we have that

$$\begin{aligned} G_{u_k}(\phi(x, y) \, dx) &= \int_{B_\rho(p)} \phi(x, u_k(x)) \, dx \rightarrow \int_{B_\rho(p)} \phi(x, u(x)) \, dx \\ &= G_u(\phi(x, y) \, dx). \end{aligned}$$

To compute the limit on forms of the type $\phi(x, y) \, d\widehat{x}^l \wedge dy^m$, observe that u_k is not constant only in the 4 sectors of $B_\rho(p)$ given in polar coordinates by

$$A_k^h - p := \left\{ (r, \vartheta) : r \in (0, \rho), \vartheta \in \left(h\frac{\pi}{2} - \frac{1}{2k}, h\frac{\pi}{2} + \frac{1}{2k} \right) \right\}, \quad h \in \{0, 1, 2, 3\},$$

thus, for $l, m = 1, 2$,

$$\begin{aligned} (-1)^{2-l} G_{u_k}(\phi(x, y) \, d\widehat{x}^l \wedge dy^m) &= \int_{B_\rho(p)} \phi(x, u_k(x)) \partial_{x^l} u_k^m(x) \, dx \\ &= \sum_{h=0}^3 \int_{A_k^h} \phi(x, u_k(x)) \partial_{x^l} u_k^m(x) \, dx. \end{aligned}$$

The integrals on the sets A_k^h can be computed in polar coordinates. We show the computations for $h = 0$ and $m = 1$, the other cases being analogous. Changing variables in the integral on the interval $(-1/2k, 1/2k)$ we obtain

$$\begin{aligned} &\int_{A_k^0} \phi(x, u_k(x)) \partial_{x^2} u_k^1(x) \, dx \\ &= - \int_0^\rho \int_{-1/2k}^{1/2k} \phi(p + r \exp(\iota\vartheta), \exp(\iota\varphi_k(\vartheta))) \sin(\varphi_k(\vartheta)) \varphi_k'(\vartheta) \cos(\vartheta) \, d\vartheta \, dr \\ &= - \int_0^\rho \int_{-1/2}^{1/2} \phi(p + r \exp(\iota\frac{t}{k}), \exp(\iota\varphi_1(t))) \sin(\varphi_1(t)) \varphi_1'(t) \cos(\frac{t}{k}) \, dt \, dr \\ &\rightarrow - \int_0^\rho \int_{-1/2}^{1/2} \phi(p + (r, 0), \exp(\iota\varphi_1(t))) \sin(\varphi_1(t)) \varphi_1'(t) \, dt \, dr \\ &= \int_{J_{41}} \left\{ \int_{\gamma_{41}} \phi(x, y) \, dy^1 \right\} v^2 \, d\mathcal{H}^1(x), \end{aligned}$$

where $t \in (-1/2, 1/2) \mapsto \gamma_{41}(t) := \exp(\iota(\widehat{\varphi}_4 + (\widetilde{\varphi}_1 - \widehat{\varphi}_4)(t + \frac{1}{2}))$ is a parametrization of the geodesic arc $\gamma_{41} \in \mathbb{S}^1$ which connects u_4 to u_1 (cf. the definition of $\widehat{\varphi}_4$

in (3.35) and of $\tilde{\varphi}_1$ in (3.36)) and J_{41} is the subset of $J_u \cap B_\rho(p)$ where u jumps from u_4 to u_1 , oriented with normal $\nu = (0, 1)$. Moreover,

$$\begin{aligned} & \int_{A_k^0} \phi(x, u_k(x)) \partial_{x^1} u_k^1(x) \, dx \\ &= \int_0^\rho \int_{-1/2k}^{1/2k} \phi(p + r \exp(\iota \vartheta), \exp(\iota \varphi_k(\vartheta))) \sin(\varphi_k(\vartheta)) \varphi_k'(\vartheta) \sin(\vartheta) \, d\vartheta \, dr \\ &= \int_0^\rho \int_{-1/2}^{1/2} \phi(p + r \exp(\iota \frac{t}{k}), \exp(\iota \varphi_1(t))) \sin(\varphi_1(t)) \varphi_1'(t) \sin(\frac{t}{k}) \, dt \, dr \rightarrow 0. \end{aligned}$$

This concludes the proof.

In the elementary lemma below we show that the flat convergence of the vorticity measure implies convergence of the boundaries of the graphs associated to the corresponding spin field.

Lemma 3.12. *Let $\mu_\varepsilon, \mu \in \mathcal{M}_b(\Omega)$ and assume that $\mu_\varepsilon \xrightarrow{f} \mu$ in Ω . Then $\mu_\varepsilon \times \llbracket \mathbb{S}^1 \rrbracket \rightarrow \mu \times \llbracket \mathbb{S}^1 \rrbracket$ in $\mathcal{D}_1(\Omega \times \mathbb{R}^2)$.*

Proof. Let us fix $\phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$. For $l = 1, 2$, by the very definition of the product of a 0-current and a 1-current we infer

$$\mu_\varepsilon \times \llbracket \mathbb{S}^1 \rrbracket (\phi(x, y) \, dx^l) = 0 = \mu \times \llbracket \mathbb{S}^1 \rrbracket (\phi(x, y) \, dx^l).$$

Next note that $\psi^m(x) := \int_{\mathbb{S}^1} \phi(x, y) \, dy^m$ belongs to $C_c^{0,1}(\Omega)$ for $m = 1, 2$. Hence

$$\begin{aligned} \mu_\varepsilon \times \llbracket \mathbb{S}^1 \rrbracket (\phi(x, y) \, dy^m) &= \int_\Omega \llbracket \mathbb{S}^1 \rrbracket (\phi(x, y) \, dy^m) \, d\mu_\varepsilon(x) \\ &= \int_\Omega \left\{ \int_{\mathbb{S}^1} \phi(x, y) \, dy^m \right\} \, d\mu_\varepsilon(x) \\ &= \int_\Omega \psi^m(x) \, d\mu_\varepsilon(x) \rightarrow \int_\Omega \psi^m(x) \, d\mu(x). \end{aligned}$$

4. A compactness result

In this section we prove a general compactness result that includes the statement in Theorem 1.2-(i) but can be also applied in other regimes, as in [25]. For this reason, in each result we give precisely the assumptions on θ_ε for which the statements hold true. The notation $\theta_\varepsilon \lesssim \varepsilon |\log \varepsilon|$ stands for $\lim_{\varepsilon \rightarrow 0} \frac{\theta_\varepsilon}{\varepsilon |\log \varepsilon|} \in [0, +\infty)$. Given a measure $\mu = \sum_{h=1}^M d_h \delta_{x_h}$ and an open set A , we adopt the notation

$$A_\mu := A \setminus \text{supp}(\mu) = A \setminus \{x_1, \dots, x_M\}$$

and $A_\mu^\rho := A \setminus \bigcup_{h=1}^M B_\rho(x_h)$.

Our first goal is to prove a compactness result for the graphs G_{u_ε} in the class of i.m. rectifiable currents. To state the result, given $\mu = \sum_{h=1}^M d_h \delta_{x_h}$ with $d_h \in \mathbb{Z}$ and $u \in BV(\Omega; \mathbb{S}^1)$, we introduce the set of admissible currents

$$\text{Adm}(\mu, u; \Omega) := \left\{ T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2) : T \in \text{cart}(\Omega_\mu \times \mathbb{S}^1), \right. \\ \left. \partial T|_{\Omega \times \mathbb{R}^2} = -\mu \times \llbracket \mathbb{S}^1 \rrbracket, u_T = u \text{ a.e. in } \Omega \right\}. \tag{4.1}$$

This is the main result in this section.

Proposition 4.1. (Compactness in the sense of currents) *Assume that $u_\varepsilon: \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$ satisfies $\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$ with $\theta_\varepsilon \lesssim \varepsilon |\log \varepsilon|$. Let $G_{u_\varepsilon} \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ be the current associated to u_ε as in (3.28)–(3.30) and let μ_{u_ε} the discrete vorticity measure associated to u_ε as in (2.6). Then there exists a subsequence (not relabeled) and*

- (i) $\mu = \sum_{h=1}^M d_h \delta_{x_h}$ with $d_h \in \mathbb{Z}$ such that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$;
- (ii) $u \in BV(\Omega; \mathbb{S}^1)$ such that $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$ and $u_\varepsilon \xrightarrow{*} u$ in $BV_{\text{loc}}(\Omega; \mathbb{R}^2)$;
- (iii) $T \in \text{Adm}(\mu, u; \Omega)$ such that $G_{u_\varepsilon} \rightarrow T$ in $\mathcal{D}_2(\Omega \times \mathbb{R}^2)$.

In particular, if $\theta_\varepsilon \ll \varepsilon |\log \varepsilon|$, then $\mu = 0$ and $T \in \text{cart}(\Omega \times \mathbb{S}^1)$.

We postpone the proof, since we need some preliminary results. To deduce a bound on the mass $|G_{u_\varepsilon}|$ (and thus compactness in $\mathcal{D}_2(\Omega \times \mathbb{R}^2)$), we rewrite the energy as a parametric integral of the currents G_{u_ε} . Specifically, defining the convex and positively 1-homogeneous function $\Phi: \Lambda_2(\mathbb{R}^2 \times \mathbb{R}^2) \mapsto \mathbb{R}$ by

$$\Phi(\xi) := \sqrt{(\xi^{21})^2 + (\xi^{22})^2} + \sqrt{(\xi^{11})^2 + (\xi^{12})^2} \tag{4.2}$$

for every

$$\xi = \xi^{00} e_1 \wedge e_2 + \xi^{21} e_1 \wedge \bar{e}_1 + \xi^{22} e_1 \wedge \bar{e}_2 + \xi^{11} e_2 \wedge \bar{e}_1 + \xi^{12} e_2 \wedge \bar{e}_2 \\ + \xi^{00} \bar{e}_1 \wedge \bar{e}_2 \in \Lambda_2(\mathbb{R}^2 \times \mathbb{R}^2),$$

we have the following representation proven in [25, Lemma 4.3].

Lemma 4.2. *Assume that $\theta_\varepsilon \ll 1$. Let $\sigma \in (0, 1)$ and $A \subset\subset \Omega$. Then for ε small enough*

$$\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \geq (1-\sigma) \int_{J_{u_\varepsilon} \cap A} d_{\mathbb{S}^1}(u_\varepsilon^-, u_\varepsilon^+) |v_{u_\varepsilon}|_1 d\mathcal{H}^1 = (1-\sigma) \int_{A \times \mathbb{R}^2} \Phi(G_{u_\varepsilon}) d|G_{u_\varepsilon}|.$$

One of the features of the limit current T is that it is an i.m. rectifiable current. This will follow from the Closure Theorem [30, 2.2.4, Theorem 1]. However, we first need a technical lemma to circumvent the fact that, in general, the masses $|\partial G_{u_\varepsilon}|$ are not equibounded. By Proposition 3.11 the boundaries $\partial G_{u_\varepsilon}$ are indeed related to the vorticity μ_{u_ε} , which thanks to the well-known ball construction is equivalent to a sequence of measures with equibounded masses. The precise statement suited for our purposes is the following.

Lemma 4.3. *Assume that $u_\varepsilon: \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$ satisfies $\frac{1}{\varepsilon^2|\log \varepsilon|}XY_\varepsilon(u_\varepsilon; \Omega) \leq C$. Let $G_{u_\varepsilon} \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ be the current associated to u_ε defined as in (3.28)–(3.30). Let $\Omega' \subset\subset \Omega$. Then there exist a subsequence (not relabeled), finitely many points $z_1, \dots, z_J \in \Omega'$, and $\bar{u}_\varepsilon: \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$ such that*

- (i) $G_{\bar{u}_\varepsilon} - G_{u_\varepsilon} \rightharpoonup 0$ in $\mathcal{D}_2(\Omega' \times \mathbb{R}^2)$;
- (ii) $\sup_\varepsilon |G_{\bar{u}_\varepsilon}|(\Omega' \times \mathbb{R}^2) \leq \sup_\varepsilon |G_{u_\varepsilon}|(\Omega' \times \mathbb{R}^2) + 1$ for ε small enough;
- (iii) $\partial G_{\bar{u}_\varepsilon}|_{(\Omega' \setminus \{z_1, \dots, z_J\}) \times \mathbb{R}^2} = 0$ for ε small enough.

Proof. The proof relies on some arguments for the discrete vorticity measure that can be adapted in this from [13]. We provide some details for the sake of completeness.

We consider an auxiliary discrete vorticity measure $\mu_{u_\varepsilon}^\Delta$ defined through a triangulation with respect to the lattice $\varepsilon\mathbb{Z}^2$. (We do this since the results in [13] are stated on the triangular lattice.) More precisely, let $\varphi_\varepsilon: \varepsilon\mathbb{Z}^2 \rightarrow [0, 2\pi)$ be such that $u_\varepsilon(x) = \exp(i\varphi_\varepsilon(x))$. As in (2.5), for $\pm \in \{+, -\}$ in the triangle $\text{conv}\{\varepsilon i, \varepsilon i \pm \varepsilon e_1, \varepsilon i \pm \varepsilon e_2\}$ we set

$$d_{u_\varepsilon}^\pm(\varepsilon i) := \frac{1}{2\pi} \left[\Psi(\varphi(\varepsilon i \pm \varepsilon e_1) - \varphi(\varepsilon i)) + \Psi(\varphi(\varepsilon i \pm \varepsilon e_2) - \varphi(\varepsilon i \pm \varepsilon e_1)) + \Psi(\varphi(\varepsilon i) - \varphi(\varepsilon i \pm \varepsilon e_2)) \right]$$

and

$$\mu_{u_\varepsilon}^\Delta \llcorner \text{conv}\{\varepsilon i, \varepsilon i \pm \varepsilon e_1, \varepsilon i \pm \varepsilon e_2\} := d_{u_\varepsilon}^\pm(\varepsilon i) \delta_{\varepsilon i \pm (1 - \frac{\sqrt{2}}{2})\varepsilon e_1 \pm (1 - \frac{\sqrt{2}}{2})\varepsilon e_2},$$

where $\varepsilon i \pm (1 - \frac{\sqrt{2}}{2})\varepsilon e_1 \pm (1 - \frac{\sqrt{2}}{2})\varepsilon e_2$ is the incenter of the triangle $\text{conv}\{\varepsilon i, \varepsilon i \pm \varepsilon e_1, \varepsilon i \pm \varepsilon e_2\}$. Note that, if $d_{u_\varepsilon}^\pm(\varepsilon i) \neq 0$, then $\frac{1}{\varepsilon^2}XY_\varepsilon(u_\varepsilon; \text{conv}\{\varepsilon i, \varepsilon i \pm \varepsilon e_1, \varepsilon i \pm \varepsilon e_2\}) \geq c_0$ for some universal constant c_0 . Thus $|\mu_{u_\varepsilon}^\Delta|(\Omega') \leq \frac{C}{\varepsilon^2}XY_\varepsilon(u_\varepsilon; \Omega) \leq C|\log \varepsilon|$ for every $\Omega' \subset\subset \Omega$. Moreover,

$$|\mu_{u_\varepsilon}^\Delta|(A') = 0 \implies |\mu_{u_\varepsilon}|(A) = 0 \text{ for } A \subset\subset A' \text{ and } \varepsilon \text{ small enough.} \tag{4.3}$$

Indeed, if $\mu_{u_\varepsilon}^\Delta \llcorner \text{conv}\{\varepsilon i, \varepsilon i + \varepsilon e_1, \varepsilon i + \varepsilon e_2\} = 0$ and $\mu_{u_\varepsilon}^\Delta \llcorner \text{conv}\{\varepsilon i + \varepsilon e_1 + \varepsilon e_2, \varepsilon i + \varepsilon e_1, \varepsilon i + \varepsilon e_2\} = 0$, then $\mu_{u_\varepsilon} \llcorner (\varepsilon i + (0, \varepsilon]^2) = 0$.

Let us fix $\Omega' \subset\subset \Omega$. We define the family of balls

$$\mathcal{B}_\varepsilon := \{B_{(1 - \frac{\sqrt{2}}{2})\varepsilon}(x) : x \in \text{supp}(\mu_{u_\varepsilon}^\Delta) \cap \Omega'\}$$

and we let $\mathcal{R}(\mathcal{B}_\varepsilon) := \sum_{B_r(x) \in \mathcal{B}_\varepsilon} r$. Each ball in \mathcal{B}_ε is contained in a triangle of the lattice $\varepsilon\mathbb{Z}^2$. Since $|\mu_{u_\varepsilon}^\Delta|(\Omega') \leq C|\log \varepsilon|$, we have that $\#\mathcal{B}_\varepsilon \leq C|\log \varepsilon|$. For every $0 < r < R$ and for every $x \in \mathbb{R}^2$ we set $A_{r,R}(x) := B_R(x) \setminus \bar{B}_r(x)$. If $A_{r,R}(x) \cap \bigcup_{B \in \mathcal{B}_\varepsilon} B = \emptyset$ we set

$$\mathcal{E}_\varepsilon(A_{r,R}(x)) := |\mu_{u_\varepsilon}^\Delta(B_r(x))| \log \frac{R}{r},$$

and we extend \mathcal{E}_ε to every open set A by

$$\mathcal{E}_\varepsilon(A) := \sup \left\{ \sum_{j=1}^N \mathcal{E}_\varepsilon(A^j) : N \in \mathbb{N}, A^j = A_{r_j, R_j}(x_j), A^j \cap \bigcup_{B \in \mathcal{B}_\varepsilon} B = \emptyset, \right. \\ \left. A^j \cap A^k = \emptyset \text{ for } j \neq k, A^j \subset A \text{ for all } j \right\}. \tag{4.4}$$

The set function \mathcal{E}_ε is increasing, superadditive and equals $-\infty$ iff $A \subset \bigcup_{B \in \mathcal{B}_\varepsilon} B$. Let Ω'' be such that $\Omega' \subset \subset \Omega'' \subset \subset \Omega$. As in [13, Lemma 7.1] one can prove that

$$\mathcal{E}_\varepsilon(\Omega'') \leq \frac{C}{\varepsilon^2} XY_\varepsilon(u_\varepsilon; \Omega) \leq C |\log \varepsilon|. \tag{4.5}$$

We apply the ball construction to the triplet $(\mathcal{E}_\varepsilon, \mu_{u_\varepsilon}^\Delta, \mathcal{B}_\varepsilon)$. The form which suits most the arguments here is the one stated in [13, Lemma 6.1]. To keep track of the constants, we let \bar{C} be such that $XY_\varepsilon(u_\varepsilon; \Omega) \leq \bar{C} \varepsilon^2 |\log \varepsilon|$. We fix $p \in (\frac{3}{4}, 1)$ and we set $\alpha_\varepsilon := \bar{C} \varepsilon^p |\log \varepsilon|$. Then there exists a family $\{\mathcal{B}_\varepsilon(t)\}_{t \geq 0}$ which satisfies that

- (1) $\bigcup_{B \in \mathcal{B}_\varepsilon} B \subset \bigcup_{B \in \mathcal{B}_\varepsilon(t_1)} B \subset \bigcup_{B \in \mathcal{B}_\varepsilon(t_2)} B$, for every $0 \leq t_1 \leq t_2$;
- (2) $\bar{B} \cap \bar{B}' = \emptyset$ for every $B, B' \in \mathcal{B}_\varepsilon(t)$, $B \neq B'$, and $t \geq 0$;
- (3) for every $0 \leq t_1 \leq t_2$ and every open set U we have that

$$\mathcal{E}_\varepsilon\left(U \cap \left(\bigcup_{B \in \mathcal{B}_\varepsilon(t_2)} B \setminus \bigcup_{B \in \mathcal{B}_\varepsilon(t_1)} \bar{B} \right)\right) \geq \sum_{\substack{B \in \mathcal{B}_\varepsilon(t_2) \\ B \subset U}} |\mu_{u_\varepsilon}^\Delta(B)| \log \frac{1+t_2}{1+t_1};$$

- (4) for $B = B_r(x) \in \mathcal{B}_\varepsilon(t)$ and $t \geq 0$, we have that $r > \alpha_\varepsilon$ and $|\mu_{u_\varepsilon}^\Delta(B_{r+\alpha_\varepsilon}(x) \setminus \bar{B}_{r-\alpha_\varepsilon}(x))| = 0$;
- (5) for every $t \geq 0$ we have that $\mathcal{R}(\mathcal{B}_\varepsilon(t)) \leq (1+t)(\mathcal{R}(\mathcal{B}_\varepsilon) + \#\mathcal{B}_\varepsilon \alpha_\varepsilon)$, where $\mathcal{R}(\mathcal{B}_\varepsilon(t)) := \sum_{B_r(x) \in \mathcal{B}_\varepsilon(t)} r$;

Note that in general $\mathcal{B}_\varepsilon(0)$ is not \mathcal{B}_ε . We let $t_\varepsilon := \varepsilon^{p-1} - 1$ and we define the measures

$$\tilde{\mu}_\varepsilon := \sum_{B_r(x) \in \mathcal{B}_\varepsilon(t_\varepsilon)} \mu_{u_\varepsilon}^\Delta(B_r(x)) \delta_x.$$

Since $\#\mathcal{B}_\varepsilon \leq C |\log \varepsilon|$, by property (5) above we have that

$$\mathcal{R}(\mathcal{B}_\varepsilon(t_\varepsilon)) \leq (1+t_\varepsilon)(\mathcal{R}(\mathcal{B}_\varepsilon) + \#\mathcal{B}_\varepsilon \alpha_\varepsilon) \leq C \varepsilon^{2p-1} |\log \varepsilon|^2. \tag{4.6}$$

Moreover, since \mathcal{E}_ε is an increasing set function, by (4.5), and property (3) in the ball construction, for ε small enough we have that

$$C |\log \varepsilon| \geq \mathcal{E}_\varepsilon(\Omega'') \geq \sum_{\substack{B \in \mathcal{B}_\varepsilon(t_\varepsilon) \\ B \subset \Omega''}} |\mu_{u_\varepsilon}^\Delta(B)| \log(1+t_\varepsilon) \geq |\tilde{\mu}_\varepsilon|(\Omega')(1-p) |\log \varepsilon|$$

and thus

$$|\tilde{\mu}_\varepsilon|(\Omega') \leq C. \tag{4.7}$$

We consider the following two subclasses of $\mathcal{B}_\varepsilon(t_\varepsilon)$:

$$\begin{aligned}\mathcal{B}_\varepsilon^{=0} &:= \{B_r(x) \in \mathcal{B}_\varepsilon(t_\varepsilon) : \mu_{u_\varepsilon}^\Delta(B_r(x)) = 0, x \in \Omega'\}, \\ \mathcal{B}_\varepsilon^{\neq 0} &:= \{B_r(x) \in \mathcal{B}_\varepsilon(t_\varepsilon) : \mu_{u_\varepsilon}^\Delta(B) \neq 0, x \in \Omega'\}.\end{aligned}\quad (4.8)$$

Let $B_{r_\varepsilon}(x_\varepsilon) \in \mathcal{B}_\varepsilon^{=0}$. Thanks to property (4) in the ball construction, $|\mu_{u_\varepsilon}^\Delta|(B_{r_\varepsilon+\alpha_\varepsilon}(x_\varepsilon) \setminus \overline{B_{r_\varepsilon-\alpha_\varepsilon}(x_\varepsilon)}) = 0$. We set $K_\varepsilon := \lfloor \frac{\alpha_\varepsilon}{4\varepsilon^p} \rfloor \geq 1$ and $r_k := r_\varepsilon - \frac{\alpha_\varepsilon}{2} + k\varepsilon^p$ for $k = 0, \dots, K_\varepsilon$. Note that $r_k \leq r_\varepsilon - \frac{\alpha_\varepsilon}{4}$. For every ε there exists $k_\varepsilon \in \{1, \dots, K_\varepsilon\}$ such that

$$\begin{aligned}\overline{C\varepsilon^2} |\log \varepsilon| &\geq XY_\varepsilon(u_\varepsilon; B_{r_\varepsilon+\alpha_\varepsilon}(x_\varepsilon) \setminus \overline{B_{r_\varepsilon-\alpha_\varepsilon}(x_\varepsilon)}) \geq \sum_{k=1}^{K_\varepsilon} XY_\varepsilon(u_\varepsilon; A_{r_{k-1}, r_k}(x_\varepsilon)) \\ &\geq K_\varepsilon XY_\varepsilon(u_\varepsilon; A_{r_{k_\varepsilon-1}, r_{k_\varepsilon}}(x_\varepsilon)) \geq \frac{\alpha_\varepsilon}{8\varepsilon^p} XY_\varepsilon(u_\varepsilon; A_{r_{k_\varepsilon-1}, r_{k_\varepsilon}}(x_\varepsilon)).\end{aligned}$$

Rearranging terms the definition of α_ε yields the bound $XY_\varepsilon(u_\varepsilon; A_{r_{k_\varepsilon-1}, r_{k_\varepsilon}}(x_\varepsilon)) \leq C_1\varepsilon^2$, with $C_1 := 8$. Hence we can apply Lemma 4.4 below to find $\bar{u}_\varepsilon : \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$ such that, for $\varepsilon < (r_{k_\varepsilon} - r_{k_\varepsilon-1})\frac{1}{C_0C_1}$, we have

$$\bar{u}_\varepsilon = u_\varepsilon \text{ on } \varepsilon\mathbb{Z}^2 \setminus \overline{B_{\frac{r_{k_\varepsilon-1}+r_{k_\varepsilon}}{2}}(x_\varepsilon)} \quad \text{and} \quad |\mu_{\bar{u}_\varepsilon}^\Delta|(B_{r_{k_\varepsilon}}(x_\varepsilon)) = 0.$$

The condition $\varepsilon < (r_{k_\varepsilon} - r_{k_\varepsilon-1})\frac{1}{C_0C_1}$ is satisfied as $\varepsilon < \varepsilon^p\frac{1}{8C_0}$ for ε small enough. Since $r_{k_\varepsilon} \leq r_\varepsilon - \frac{\alpha_\varepsilon}{4} \ll r_\varepsilon - \sqrt{2}\varepsilon$, we also have that the piecewise constant functions \bar{u}_ε and u_ε coincide outside $B_{r_\varepsilon}(x_\varepsilon)$.

We apply the modification described above to every $B_{r_\varepsilon}(x_\varepsilon) \in \mathcal{B}_\varepsilon^{=0}$. In this way, for every ε we construct $\bar{u}_\varepsilon : \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$ such that $\bar{u}_\varepsilon = u_\varepsilon$ (as piecewise constant functions) in $\mathbb{R}^2 \setminus \bigcup_{B \in \mathcal{B}_\varepsilon^{=0}} B$ and $|\mu_{\bar{u}_\varepsilon}^\Delta|(A) = 0$ for every open set A such that $A \subset \subset \Omega' \setminus \bigcup_{B \in \mathcal{B}_\varepsilon^{\neq 0}} B$. By (4.7)–(4.8) and by the definition of \tilde{u}_ε , we have that $\#\mathcal{B}_\varepsilon^{\neq 0}$ is equibounded and, up to a subsequence, we can assume $\mathcal{B}_\varepsilon^{\neq 0} = \{B_{r_1^\varepsilon}(x_1^\varepsilon), \dots, B_{r_M^\varepsilon}(x_M^\varepsilon)\}$. There exists a set of points $\{z_1, \dots, z_J\} \subset \overline{\Omega}'$ with $J \leq M$ such that, up to a subsequence, each sequence x_m^ε converges to a point z_h as $\varepsilon \rightarrow 0$. (The points belonging to $\partial\Omega'$ are actually not relevant for the following discussion.) From now on, we work with this subsequence.

Let us show that $G_{\bar{u}_\varepsilon} - G_{u_\varepsilon} \rightarrow 0$ in $\mathcal{D}_2(\Omega' \times \mathbb{R}^2)$. Let $\phi \in C_c^\infty(\Omega' \times \mathbb{R}^2)$. By (4.6) we have

$$\begin{aligned} |(G_{\bar{u}_\varepsilon} - G_{u_\varepsilon})(\phi(x, y) \, dx) | &\leq \int_{\Omega'} |\phi(x, \bar{u}_\varepsilon(x)) - \phi(x, u_\varepsilon(x))| \, dx \\ &\leq \sum_{B \in \mathcal{B}_\varepsilon^{=0}} \int_B |\phi(x, \bar{u}_\varepsilon(x)) - \phi(x, u_\varepsilon(x))| \, dx \leq 2\|\phi\|_{L^\infty} \pi \mathcal{R}(\mathcal{B}_\varepsilon(t_\varepsilon))^2 \\ &\leq C\|\phi\|_{L^\infty} \varepsilon^{4p-2} |\log \varepsilon|^4.\end{aligned}$$

For the next estimate we observe that, given a ball $B_r(x)$, we have $\mathcal{H}^1(J_{u_\varepsilon} \cap B_r(x)) \leq C\frac{r^2}{\varepsilon}$. Indeed, since u_ε is piecewise constant on the squares of $\varepsilon\mathbb{Z}^2$, the

measure of its jump set in $B_r(x)$ can be roughly estimated by 4ε times the number of squares that intersect $B_r(x)$, which is of the order of $\frac{r^2}{\varepsilon^2}$, at least for $\varepsilon \lesssim r$. The same holds true for \bar{u}_ε . For $x \in J_{\bar{u}_\varepsilon}$ (resp., $x \in J_{u_\varepsilon}$), let $\bar{\gamma}_x$ (resp., γ_x) be the (oriented) geodesic arc that connects $\bar{u}_\varepsilon^+(x)$ and $\bar{u}_\varepsilon^-(x)$ (or $u_\varepsilon^+(x)$ and $u_\varepsilon^-(x)$). Then, using that $\bar{u}_\varepsilon = u_\varepsilon$ on $\varepsilon\mathbb{Z}^2 \setminus \bigcup_{B \in \mathcal{B}_\varepsilon^0} B$,

$$\begin{aligned} & |(G_{\bar{u}_\varepsilon} - G_{u_\varepsilon})(\phi(x, y) \, d\hat{x}^l \wedge dy^m)| \\ & \leq \left| \sum_{B \in \mathcal{B}_\varepsilon^0} \int_{J_{\bar{u}_\varepsilon} \cap B} \left\{ \int_{\bar{\gamma}_x} \phi(x, y) \, dy^m \right\} v_{\bar{u}_\varepsilon}^l(x) \, d\mathcal{H}^1(x) \right. \\ & \quad \left. - \int_{J_{u_\varepsilon} \cap B} \left\{ \int_{\gamma_x} \phi(x, y) \, dy^m \right\} v_{u_\varepsilon}^l(x) \, d\mathcal{H}^1(x) \right| \\ & \leq 2\pi \|\phi\|_{L^\infty} \sum_{B \in \mathcal{B}_\varepsilon^0} (\mathcal{H}^1(J_{\bar{u}_\varepsilon} \cap B) + \mathcal{H}^1(J_{u_\varepsilon} \cap B)) \leq C \|\phi\|_{L^\infty} \frac{2}{\varepsilon} \mathcal{R}(\mathcal{B}_\varepsilon(t_\varepsilon))^2 \\ & \leq C \|\phi\|_{L^\infty} \varepsilon^{4p-3} |\log \varepsilon|^4. \end{aligned}$$

The two previous inequalities and the fact that $p \in (\frac{3}{4}, 1)$ imply that $G_{\bar{u}_\varepsilon} - G_{u_\varepsilon} \rightarrow 0$ in $\mathcal{D}_2(\Omega' \times \mathbb{R}^2)$. Moreover, taking the supremum over 2-forms with L^∞ -norm less than 1, they also imply that $|G_{\bar{u}_\varepsilon}|(\Omega' \times \mathbb{R}^2) \leq |G_{u_\varepsilon}|(\Omega' \times \mathbb{R}^2) + 1$ for ε small enough.

Let $A \subset\subset A' \subset\subset \Omega' \setminus \{z_1, \dots, z_J\}$. By (4.6), for ε small enough it follows that $A' \subset\subset \Omega' \setminus \bigcup_{B \in \mathcal{B}_\varepsilon^0} B$ and thus $|\mu_{\bar{u}_\varepsilon}^\Delta|(A') = 0$. By (4.3), we obtain that $|\mu_{\bar{u}_\varepsilon}|(A) = 0$, i.e., $\partial G_{\bar{u}_\varepsilon}|_{A \times \mathbb{R}^2} = 0$. By the arbitrariness of A and A' we get $\partial G_{\bar{u}_\varepsilon}|_{(\Omega' \setminus \{z_1, \dots, z_J\}) \times \mathbb{R}^2} = 0$.

In the proof we applied the following extension lemma proven in [13, Lemma 3.5 and Remark 3.6].

Lemma 4.4. *There exists a constant $C_0 > 0$ such that the following holds true. Let $\varepsilon > 0$, $x_0 \in \mathbb{R}^2$, and $R > r > \varepsilon$, let $C_1 > 1$ and $u_\varepsilon : \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$ with $XY_\varepsilon(u_\varepsilon; B_R(x_0) \setminus \bar{B}_r(x_0)) \leq C_1 \varepsilon^2$, $\mu_{u_\varepsilon}^\Delta(B_r(x_0)) = 0$, and $|\mu_{u_\varepsilon}^\Delta|(\varepsilon i + (0, \varepsilon]^2) = 0$ whenever $(\varepsilon i + (0, \varepsilon]^2) \cap (B_R(x_0) \setminus \bar{B}_r(x_0)) \neq \emptyset$. Then there exists $\bar{u}_\varepsilon : \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$ such that, for $\varepsilon < \frac{R-r}{C_0 C_1}$,*

- $\bar{u}_\varepsilon = u_\varepsilon$ on $\varepsilon\mathbb{Z}^2 \setminus \bar{B}_{\frac{r+R}{2}}(x_0)$;
- $|\mu_{\bar{u}_\varepsilon}^\Delta|(B_R(x_0)) = 0$.

We are finally in a position to prove Proposition 4.1.

Proof of Proposition 4.1. From the assumptions $\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$ and $\theta_\varepsilon \lesssim \varepsilon |\log \varepsilon|$ it follows that

$$\frac{1}{\varepsilon^2 |\log \varepsilon|} E_\varepsilon(u_\varepsilon) = \frac{\theta_\varepsilon}{\varepsilon |\log \varepsilon|} \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C,$$

so that by Proposition 2.4 we get that (up to a subsequence) $\mu_{u_\varepsilon} \xrightarrow{f} \mu = \sum_{h=1}^M d_h \delta_{x_h}$, proving the first point.

Applying Lemma 4.2 with $\sigma = \frac{1}{2}$ we deduce that for every $A \subset\subset \Omega$

$$C \geq \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \geq \frac{1}{2} \int_{J_{u_\varepsilon} \cap A} d_{\mathbb{S}^1}(u_\varepsilon^-, u_\varepsilon^+) |v_{u_\varepsilon}|_1 d\mathcal{H}^1 \geq \frac{1}{2} |Du_\varepsilon|(A).$$

Hence u_ε is bounded in $BV(A; \mathbb{S}^1)$ and we conclude that (up to a subsequence) $u_\varepsilon \rightarrow u$ in $L^1(A)$ and $u_\varepsilon \xrightarrow{*} u$ in $BV(A; \mathbb{R}^2)$ for some $u \in BV(A; \mathbb{S}^1)$ with $|Du|(A) \leq C$. Since $A \subset\subset \Omega$ was arbitrary and the constant C does not depend on A , the second point follows from a diagonal argument and the equiintegrability of u_ε .

Applying Lemma 4.2 with $\sigma = \frac{1}{2}$ and since $\Phi(\xi) \geq \frac{\sqrt{(\xi^{21})^2 + (\xi^{22})^2 + (\xi^{11})^2 + (\xi^{12})^2}}{2}$, we obtain, by Proposition 3.10, that for every $A \subset\subset \Omega$,

$$\begin{aligned} |G_{u_\varepsilon}|(A \times \mathbb{R}^2) &= |G_{u_\varepsilon}|(\mathcal{M}^{(a)} \cap A \times \mathbb{R}^2) + |G_{u_\varepsilon}|(\mathcal{M}^{(j)} \cap A \times \mathbb{R}^2) \\ &\leq |A| + \int_{A \times \mathbb{R}^2} \Phi(G_{u_\varepsilon}) d|G_{u_\varepsilon}| \leq |\Omega| + \frac{2}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C. \end{aligned}$$

By the Compactness Theorem for currents [30, 2.2.3, Proposition 2 and Theorem 1-(i)] we deduce that there exists a subsequence (not relabeled) and a current $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ with $|T| < \infty$ such that $G_{u_\varepsilon} \rightarrow T$ in $\mathcal{D}_2(\Omega \times \mathbb{R}^2)$.

It remains to prove that $T \in \text{Adm}(\mu, u; \Omega)$:

- T is an i.m. rectifiable current: Let $\Omega' \subset\subset \Omega$. We consider the subsequence (not relabeled), the points $z_1, \dots, z_J \in \Omega'$, and the spin field $\bar{u}_\varepsilon: \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$ given by Lemma 4.3. By Lemma 4.3-(i) we have that $G_{\bar{u}_\varepsilon} \rightarrow T$ in $\mathcal{D}_2(\Omega' \times \mathbb{R}^2)$. Let us fix $A \subset\subset \Omega' \setminus \{x_1, \dots, x_M, z_1, \dots, z_J\}$. We have that $\sup_\varepsilon |G_{u_\varepsilon}|(A \times \mathbb{R}^2) < +\infty$ and $\partial G_{u_\varepsilon}|_{A \times \mathbb{R}^2} = 0$. By the Closure Theorem [30, 2.2.4, Theorem 1], the limit $T|_{A \times \mathbb{R}^2}$ is an i.m. rectifiable current. By the arbitrariness of A and Ω' and since $\{x_h\} \times \mathbb{S}^1$ and $\{z_j\} \times \mathbb{S}^1$ are \mathcal{H}^2 -negligible sets, this proves that T is an i.m. rectifiable current in $\Omega \times \mathbb{R}^2$.
- $\partial T|_{\Omega \times \mathbb{R}^2} = -\mu \times \llbracket \mathbb{S}^1 \rrbracket$: by Proposition 3.11 we have $\partial G_{u_\varepsilon}|_{\Omega \times \mathbb{R}^2} = -\mu_{u_\varepsilon} \times \llbracket \mathbb{S}^1 \rrbracket$. Since $\mu_{u_\varepsilon} \xrightarrow{f} \mu$, by Lemma 3.12, and since $\partial G_{u_\varepsilon} \rightarrow \partial T$ in $\mathcal{D}_1(\Omega \times \mathbb{R}^2)$, we conclude that $\partial T|_{\Omega \times \mathbb{R}^2} = -\mu \times \llbracket \mathbb{S}^1 \rrbracket$. In particular, $\partial T|_{\Omega_\mu \times \mathbb{R}^2} = 0$.
- $T|_{dx} \geq 0$: let $\omega \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ be of the form $\omega(x, y) = \phi(x, y) dx$ with $\phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$ and $\phi \geq 0$. Then $G_{u_\varepsilon}(\omega) = \int_\Omega \phi(x, u_\varepsilon(x)) dx \geq 0$. Passing to the limit as $\varepsilon \rightarrow 0$ we get $T(\omega) \geq 0$.
- $|T| < \infty$: this is a consequence of the Compactness Theorem for currents (see above).
- $\|T\|_1 < \infty$: note that, by (3.5), $\|G_{u_\varepsilon}\|_1 = \int_\Omega |u_\varepsilon| dx = |\Omega|$. By the lower semicontinuity of $\|\cdot\|_1$ with respect to the convergence in $\mathcal{D}_2(\Omega \times \mathbb{R}^2)$ we deduce that $\|T\|_1 \leq |\Omega|$.
- $\pi_\#^\Omega T = \llbracket \Omega \rrbracket$: let us fix $\omega \in \mathcal{D}^2(\Omega)$, i.e., a 2-form of the type $\omega(x) = \phi(x) dx$ with $\phi \in C_c^\infty(\Omega)$. Then $G_{u_\varepsilon}(\omega) = \int_\Omega \phi(x) dx$. Thus $\pi_\#^\Omega G_{u_\varepsilon} = \llbracket \Omega \rrbracket$. Passing to the limit as $\varepsilon \rightarrow 0$ we get the desired condition (cf. also [30, 4.2.1, Proposition 3]).

- $\text{supp}(T) \subset \overline{\Omega} \times \mathbb{S}^1$: let us fix $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^2)$ with $\text{supp}(\omega) \subset\subset (\Omega \times \mathbb{R}^2) \setminus (\Omega \times \mathbb{S}^1)$. Then $G_{u_\varepsilon}(\omega) = 0$. Passing to the limit as $\varepsilon \rightarrow 0$, we conclude that $T(\omega) = 0$.

To prove that $u_T = u$ a.e. we observe that $u_\varepsilon \rightarrow u$ implies

$$G_{u_\varepsilon}(\phi(x, y) \, dx) = \int_{\Omega} \phi(x, u_\varepsilon(x)) \, dx \rightarrow \int_{\Omega} \phi(x, u(x)) \, dx$$

for every $\phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$. On the other hand, due to Theorem 3.5

$$G_{u_\varepsilon}(\phi(x, y) \, dx) \rightarrow T(\phi(x, y) \, dx) = \int_{\Omega} \phi(x, u_T(x)) \, dx.$$

By the arbitrariness of ϕ , we get $u_T = u$ a.e. in Ω .

Finally, if $\theta_\varepsilon \ll \varepsilon |\log \varepsilon|$, then Proposition 2.4 and the assumed energy bound yield that $\mu = 0$, whence $T \in \text{cart}(\Omega \times \mathbb{S}^1)$.

5. Proofs in the regime $\theta_\varepsilon \sim \varepsilon |\log \varepsilon|$

In this section we prove Theorem 1.2. We remark that the compactness result Theorem 1.2-(i) is already covered by Proposition 4.1. Thus, we only need to prove Theorem 1.2-(ii) and (iii).

From the lower semicontinuity of parametric integrals with respect to the mass bounded weak convergence of currents, [31, 1.3.1, Theorem 1], we obtain the following asymptotic lower bound:

Proposition 5.1. (Lower bound for the parametric integral) *Assume that $\theta_\varepsilon \lesssim \varepsilon |\log \varepsilon|$ and that $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$. Let $G_{u_\varepsilon} \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ be the currents associated to u_ε defined as in (3.28)–(3.30) and assume that $G_{u_\varepsilon} \rightharpoonup T$ with $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ represented as $T = \mathbf{T}|T|$. Then*

$$\int_{A \times \mathbb{R}^2} \Phi(\mathbf{T}) \, d|T| \leq \liminf_{\varepsilon \rightarrow 0} \int_{A \times \mathbb{R}^2} \Phi(G_{u_\varepsilon}) \, d|G_{u_\varepsilon}|. \tag{5.1}$$

for every open set $A \subset\subset \Omega$.

We can write explicitly the parametric integral in the left-hand side of (5.1) in terms of the BV function u , limit of the sequence u_ε . We recall that by (3.17) the jump-concentration part of T is given by

$$T^{(j_c)}(\phi(x, y) \, d\widehat{x}^l \wedge dy^m) = (-1)^{2-l} \int_{J_T} \left\{ \int_{\gamma_x^T} \phi(x, y) \, dy^m \right\} \nu_T^l(x) \, d\mathcal{H}^1(x).$$

For \mathcal{H}^1 -a.e. $x \in J_T$ we define the number

$$\ell_T(x) := \text{length}(\gamma_x^T) = \int_{\text{supp}(\gamma_x^T)} |m(x, y)| \, d\mathcal{H}^1(y), \tag{5.2}$$

where $m(x, y)$ is the integer defined in (3.18). Notice that by $\text{length}(\gamma_x^T)$ we mean the length of the curve γ_x^T counted with its multiplicity and not the \mathcal{H}^1 Hausdorff measure of its support. Observe that, in particular, $\ell_T(x) = d_{\mathbb{S}^1}(u^-(x), u^+(x))$ if $x \in J_u \setminus \mathcal{L}$, whilst $\ell_T(x) = 2\pi|k(x)|$ if $x \in \mathcal{L} \setminus J_u$. The full form of the parametric integral is contained in the lemma below.

Lemma 5.2. *Let Φ be the parametric integrand defined in (4.2). Let $\mu = \sum_{h=1}^M d_h \delta_{x_h}$ with $d_h \in \mathbb{Z}$ and $u \in BV(\Omega; \mathbb{S}^1)$. Let $T \in \text{Adm}(\mu, u; \Omega)$. Then*

$$\int_{\Omega \times \mathbb{R}^2} \Phi(T) d|T| = \int_{\Omega} |\nabla u|_{2,1} dx + |D^{(c)}u|_{2,1}(\Omega) + \int_{J_T \cap \Omega} \ell_T(x) |v_T(x)|_1 d\mathcal{H}^1(x).$$

Proof. We first prove the statement in the case $\mu = 0$, namely $T \in \text{cart}(\Omega \times \mathbb{S}^1)$. We employ the mutually singular decomposition given by Theorem 3.5 and Proposition 3.8, so that $|T| = \mathcal{H}^2 \llcorner \mathcal{M}^{(a)} + \mathcal{H}^2 \llcorner \mathcal{M}^{(c)} + |T^{(j.c)}|$. First of all note that by (3.20) and (3.11) for every $\psi \in C_c^\infty(\Omega)$ we have

$$\int_{\mathcal{M}^{(a)}} \psi(x) \frac{1}{\sqrt{1 + |\nabla u(x)|^2}} d\mathcal{H}^2(x, y) = \int_{\Omega} \psi(x) dx, \tag{5.3}$$

since both integrals are equal to $T^{(a)}(\psi(x) dx)$. By approximation, the above equality is true for every $\psi : \Omega \rightarrow \mathbb{R}$ such that $(x, y) \mapsto \psi(x)$ is $\mathcal{H}^2 \llcorner \mathcal{M}^{(a)}$ -measurable and $x \mapsto \psi(x)$ is \mathcal{L}^2 -measurable. Note that $x \mapsto |\nabla u(x)|_{2,1}$ satisfies these measurability properties thanks to (3.20). In particular, we deduce that

$$\begin{aligned} & \int_{\Omega \times \mathbb{R}^2} \Phi(T(x, y)) d\mathcal{H}^2 \llcorner \mathcal{M}^{(a)}(x, y) \\ &= \int_{\mathcal{M}^{(a)}} |\nabla u(x)|_{2,1} \frac{1}{\sqrt{1 + |\nabla u(x)|^2}} d\mathcal{H}^2(x, y) \\ &= \int_{\Omega} |\nabla u(x)|_{2,1} dx. \end{aligned}$$

Next note that, by (3.13), for every function $\psi \in C_c(\Omega)$ it holds true

$$\sup_{\substack{\tilde{\omega}(x,y) \leq \psi(x) \\ \tilde{\omega} \in \mathcal{D}^2(\Omega \times \mathbb{R}^2)}} T^{(c)}(\tilde{\omega}) = \sup_{\substack{|\omega(x)| \leq \psi(x) \\ \omega \text{ } |D^{(c)}u| \text{-measurable}}} T^{(c)}(\omega). \tag{5.4}$$

Indeed given $\tilde{\omega} \in \mathcal{D}^2(\Omega \times \mathbb{R}^2)$ such that $|\tilde{\omega}(x, y)| \leq \psi(x)$, one can define the $|D^{(c)}u|$ -measurable 2-form $\omega(x) := \tilde{\omega}(x, \tilde{u}_T(x))$ to prove that the left-hand side is greater than or equal to the right-hand side. For the reverse inequality, given a $|D^{(c)}u|$ -measurable 2-form ω such that $|\omega(x)| \leq \psi(x)$, one can regularize it and then define the 2-form $\tilde{\omega}(x, y) := \omega(x)\zeta(y)$, where $\zeta \in C_c^\infty(B_2)$ is such that $\zeta(y) = 1$ for $|y| \leq 1$ (note that ζ does not affect the value of $T^{(c)}(\omega)$ thanks to (3.13)).

Since $|T^{(c)}| = \mathcal{H}^2 \llcorner \mathcal{M}^{(c)}$ and by (3.13), equality (5.4) implies that

$$\int_{\mathcal{M}^{(c)}} \psi(x) d\mathcal{H}^2(x, y) = \int_{\Omega} \psi(x) d|D^{(c)}u|(x) \tag{5.5}$$

for every function $\psi \in C_c(\Omega)$. By approximation, (5.5) holds true for every $\psi : \Omega \rightarrow \mathbb{R}$ such that ψ is $\mathcal{H}^2 \llcorner \mathcal{M}^{(c)}$ -measurable and $|D^{(c)}u|$ -measurable. The function $x \mapsto \left| \frac{dD^{(c)}u}{d|D^{(c)}u|}(x) \right|_{2,1}$ satisfies these measurability properties, cf. (3.21), thus (3.21) implies

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^2} \Phi(\mathbf{T}(x, y)) \, d\mathcal{H}^2 \llcorner \mathcal{M}^{(c)}(x, y) &= \int_{\mathcal{M}^{(c)}} \left| \frac{dD^{(c)}u}{d|D^{(c)}u|}(x) \right|_{2,1} \, d\mathcal{H}^2(x, y) \\ &= |D^{(c)}u|_{2,1}(\Omega). \end{aligned}$$

Finally, by (3.22) we get that

$$\begin{aligned} &\int_{\Omega \times \mathbb{R}^2} \Phi(\mathbf{T}(x, y)) \, d|T^{(j_c)}|(x, y) \\ &= \int_{\Omega \times \mathbb{R}^2} |\mathbf{m}(x, y)| |v_T(x)|_1 \, d\mathcal{H}^2 \llcorner \mathcal{M}^{(j_c)}(x, y) \\ &= \int_{J_T} \left\{ \int_{\mathbb{S}^1} \mathbb{1}_{\mathcal{M}^{(j_c)}}(x, y) |\mathbf{m}(x, y)| |v_T(x)|_1 \, d\mathcal{H}^1(y) \right\} \, d\mathcal{H}^1(x) \\ &= \int_{J_T} \left\{ \int_{\text{supp}(\gamma_x^T)} |\mathbf{m}(x, y)| \, d\mathcal{H}^1(y) \right\} |v_T(x)|_1 \, d\mathcal{H}^1(x) \\ &= \int_{J_T} \ell_T(x) |v_T(x)|_1 \, d\mathcal{H}^1(x). \end{aligned}$$

In the second equality we employed the coarea formula for rectifiable sets [28, Theorem 3.2.22] (applied with $W = J_T \times \mathbb{S}^1$, $Z = J_T$, f given by the projection $J_T \times \mathbb{S}^1 \rightarrow J_T$, and $g = \mathbb{1}_{\mathcal{M}^{(j_c)}} |\mathbf{m}| |v_T|_1$) and in the third equality we used (3.27).

Let us prove the general case $T \in \text{Adm}(\mu, u; \Omega)$. We observe that a current $T \in \text{cart}(\Omega_\mu \times \mathbb{S}^1)$ can be extended to a current $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$. Indeed, since $T \in \text{cart}(\Omega_\mu \times \mathbb{S}^1)$, it can be represented as

$$T(\omega) = \int_{\Omega_\mu \times \mathbb{R}^2} \langle \omega, \xi \rangle \theta \, d\mathcal{H}^2 \llcorner \mathcal{M}, \quad \text{for } \omega \in \mathcal{D}^2(\Omega_\mu \times \mathbb{R}^2),$$

according to the notation in (3.2), where $\mathcal{M} \subset \Omega_\mu \times \mathbb{S}^1$ \mathcal{H}^2 -a.e. (cf. the proof of Proposition 3.8 for the last fact). The integral above can be extended to a linear functional on forms $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^2)$, namely,

$$T(\omega) = \int_{\Omega \times \mathbb{R}^2} \langle \omega, \xi \rangle \theta \, d\mathcal{H}^2 \llcorner \mathcal{M}, \quad \text{for } \omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^2).$$

To prove the continuity of this functional, let us fix a form $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^2)$ with $\sup_x |\omega(x)| \leq 1$. We have the bound

$$\begin{aligned} |T(\omega)| &\leq |T((1 - \zeta)\omega)| + \left| \int_{\Omega \times \mathbb{R}^2} \zeta \langle \omega, \xi \rangle \theta \, d\mathcal{H}^2 \llcorner \mathcal{M} \right| \\ &\leq |T|(\Omega_\mu \times \mathbb{R}^2) + \sum_{h=1}^N \int_{B_\rho(x_h) \times \mathbb{R}^2} |\theta| \, d\mathcal{H}^2 \llcorner \mathcal{M}, \end{aligned} \tag{5.6}$$

where $\zeta \in C_c^\infty(\Omega)$ is such that $0 \leq \zeta \leq 1$, $\text{supp}(\zeta) \subset \bigcup_{h=1}^N B_\rho(x_h)$, and $\zeta \equiv 1$ on $B_{\rho/2}(x_h)$ for every $h = 1, \dots, N$. Letting $\rho \rightarrow 0$ in the inequality above, we get $|T(\omega)| \leq |T|(\Omega_\mu \times \mathbb{R}^2)$ since $\mathcal{H}^2(\mathcal{M} \cap (\{x_h\} \times \mathbb{R}^2)) \leq \mathcal{H}^2(\{x_h\} \times \mathbb{S}^1) = 0$ for $h = 1, \dots, N$ and θ is $\mathcal{H}^2 \llcorner \mathcal{M}$ -summable. This shows that $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$.

Moreover, by the arbitrariness of ω in (5.6) we deduce that $|T|(\Omega \times \mathbb{R}^2) = |T|(\Omega_\mu \times \mathbb{R}^2)$ and, in particular, from the first step of the proof applied to Ω_μ we infer that

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^2} \Phi(T) \, d|T| &= \int_{\Omega_\mu \times \mathbb{R}^2} \Phi(T) \, d|T| \\ &= \int_{\Omega} |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \int_{J_T} \ell_T(x) |v_T(x)|_1 \, d\mathcal{H}^1(x). \end{aligned}$$

We are now in a position to prove the lower bound in the regime $\theta_\varepsilon \sim \varepsilon |\log \varepsilon|$. We recall that the asymptotic lower bound is written in terms of the energy

$$\mathcal{J}(\mu, u; \Omega) := \inf \left\{ \int_{J_T} \ell_T(x) |v_T(x)|_1 \, d\mathcal{H}^1(x) : T \in \text{Adm}(\mu, u; \Omega) \right\} \quad (5.7)$$

with $\ell_T(x)$ defined in (5.2) and $\text{Adm}(\mu, u; \Omega)$ in (4.1). We remark that $\text{Adm}(\mu, u; \Omega)$ is non-empty.⁹ Moreover,

$$\mathcal{J}(\mu, u; \Omega) \geq \int_{J_u \cap \Omega} d_{\mathbb{S}^1}(u^-, u^+) |v_u|_1 \, d\mathcal{H}^1. \quad (5.8)$$

Indeed, for \mathcal{H}^1 -a.e. $x \in J_T$ we have $d_{\mathbb{S}^1}(u^-(x), u^+(x)) \leq \text{length}(\gamma_x^T) = \ell_T(x)$, since γ_x^T is a curve connecting $u^-(x)$ and $u^+(x)$ in \mathbb{S}^1 .

Using the previous results, we obtain the Γ -liminf inequality.

Proof of Theorem 1.2-ii. Let $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ (extended arbitrarily to $\varepsilon\mathbb{Z}^2$), $u \in BV(\Omega; \mathbb{S}^1)$, and $\mu = \sum_{h=1}^M d_h \delta_{x_h}$ be as in the statement of the theorem. Let $T \in \text{Adm}(\mu, u; \Omega)$ be given by Proposition 4.1 and fix a set $A \subset\subset \Omega_\mu$. Let $\rho > 0$ be such that the balls $\{B_\rho(x_h)\}_{h=1}^M$ are pairwise disjoint and $A \subset\subset \Omega_\mu^\rho$. Let $\sigma \in (0, 1)$. Then, by Lemma 4.2,

$$\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \geq \sum_{h=1}^M \frac{1}{\varepsilon^2 |\log \varepsilon|} E_\varepsilon(u_\varepsilon; B_\rho(x_h)) + (1 - \sigma) \int_{A \times \mathbb{R}^2} \Phi(G_{u_\varepsilon}) \, d|G_{u_\varepsilon}|. \quad (5.9)$$

To estimate the first term, we exploit the localized lower bound for the XY -model [7, Theorem 3.1], which yields the existence of a constant $\tilde{C} \in \mathbb{R}$ such that

$$\liminf_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon; B_\rho(x_h)) - 2\pi |d_h| \log \frac{\rho}{\varepsilon} \right] \geq \tilde{C}$$

⁹ By Proposition 3.9 there exists a current $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ such that $u_T = u$. Let $\gamma_1, \dots, \gamma_M$ be pairwise disjoint unit speed Lipschitz curves such that γ_h connects x_h to $\partial\Omega$. Define L_h to be the 1-current $\tau(\text{supp}(\gamma_h), -d_h, \dot{\gamma}_h)$, so that $\partial L_h = d_h \delta_{x_h}$. Then $T + \sum_{h=1}^M L_h \times \llbracket \mathbb{S}^1 \rrbracket \in \text{Adm}(\mu, u; \Omega)$.

and, in particular, that

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 |\log \varepsilon|} E_\varepsilon(u_\varepsilon; B_\rho(x_h)) \geq 2\pi |d_h|. \tag{5.10}$$

By (5.9), Proposition 5.1, letting $\sigma \rightarrow 0$ and $A \nearrow \Omega_\mu$, and by Lemma 5.2 we infer that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) &\geq \sum_{h=1}^M 2\pi |d_h| + \int_{\Omega_\mu \times \mathbb{R}^2} \Phi(T) \, d|T| \\ &= 2\pi |\mu|(\Omega) + \int_\Omega |\nabla u|_{2,1} \, dx \\ &\quad + |D^{(c)}u|_{2,1}(\Omega) + \int_{J_T \cap \Omega} \ell_T(x) |v_T(x)|_1 \, d\mathcal{H}^1(x). \end{aligned}$$

Taking the infimum over all $T \in \text{Adm}(\mu, u; \Omega)$ we deduce the claim.

Let us prove the Γ -limsup inequality. In the definition of the recovery sequence we use a map that projects vectors of \mathbb{S}^1 on \mathcal{S}_ε . Given $u \in \mathbb{S}^1$ we let $\varphi_u \in [0, 2\pi)$ be the unique angle such that $u = \exp(i\varphi_u)$. We define $\mathfrak{P}_\varepsilon : \mathbb{S}^1 \rightarrow \mathcal{S}_\varepsilon$ by

$$\mathfrak{P}_\varepsilon(u) = \exp\left(i\theta_\varepsilon \left\lfloor \frac{\varphi_u}{\theta_\varepsilon} \right\rfloor\right). \tag{5.11}$$

Proof of Theorem 1.2-iii). To construct the recovery sequence, we closely follow the proof of [25, Proposition 4.22] done for the regime $\varepsilon \ll \theta_\varepsilon \ll \varepsilon |\log \varepsilon|$. Most of the arguments hold true also when $\theta_\varepsilon = \varepsilon |\log \varepsilon|$, see [25, Remark 4.23]. Here we will sketch the proof and provide more details for the steps that need to be adapted.

Let us fix $\mu = \sum_{h=1}^M d_h \delta_{x_h}$ and $u \in BV(\Omega; \mathbb{S}^1)$ as in the statement. The function u is gradually approximated as explained in the following.

Step 1 (Approximation with currents) Fix $\sigma > 0$. By the definition (5.7) of \mathcal{J} there exists $T \in \text{Adm}(\mu, u; \Omega)$ such that

$$\int_{\Omega \times \mathbb{R}^2} \Phi(T) \, d|T| \leq \int_\Omega |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(\mu, u; \Omega) + \sigma. \tag{5.12}$$

Step 2 (Approximation with \mathbb{S}^1 -valued maps with finitely many singularities) Exploiting the extension Lemma 3.4 and the approximation Theorem 3.3, we find an open set $\tilde{\Omega} \supset \supset \Omega$ and a sequence of maps $u_k \in C^\infty(\tilde{\Omega}_\mu; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ such that $u_k \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$, $|G_{u_k}|(\Omega \times \mathbb{R}^2) \rightarrow |T|(\Omega \times \mathbb{R}^2)$, and $\deg(u_k)(x_h) = d_h$ for $h = 1, \dots, N$. We refer to [25, Lemma 4.17] for a detailed proof. Reshetnyak’s Continuity Theorem implies that

$$\int_\Omega |\nabla u_k|_{2,1} \, dx = \int_{\Omega \times \mathbb{R}^2} \Phi(G_{u_k}) \, d|G_{u_k}| \leq \int_{\Omega \times \mathbb{R}^2} \Phi(T) \, d|T| + \sigma, \tag{5.13}$$

for k large enough. In the first equality we applied Lemma 5.2. Thanks to this step (and via a diagonal argument as $\sigma \rightarrow 0$), it is enough to prove the Γ -limsup inequality assuming the stronger regularity $u \in C^\infty(\tilde{\Omega}_\mu; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$.

Step 3 (Splitting of the degree) Without loss of generality, hereafter we shall assume that $|\deg(u)(x_h)| = 1$ for $h = 1, \dots, N$. If this is not the case, we split each singularity x_h with degree d_h into $|d_h|$ singularities of degree with modulus 1, without increasing the energy asymptotically. More precisely, by [25, Lemma 4.18], for $0 < \tau \ll 1$ there exist measures μ^τ and $u^\tau \in C^\infty(\tilde{\Omega}_{\mu^\tau}; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ with $\mu^\tau = \sum_{h=1}^{N^\tau} \deg(u^\tau)(x_h^\tau) \delta_{x_h^\tau}$ and $|\deg(u^\tau)(x_h^\tau)| = 1$ such that $u^\tau \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$, $\mu^\tau \xrightarrow{f} \mu$, and $\int_\Omega |\nabla u^\tau|_{2,1} dx \rightarrow \int_\Omega |\nabla u|_{2,1} dx$, as $\tau \rightarrow 0$.

Step 4 (Moving singularities on a lattice) We introduce the additional parameter $\lambda_n := 2^{-n}$, $n \in \mathbb{N}$, which will be used later to obtain a piecewise constant approximation. Without loss of generality, we shall assume that $x_h \in \lambda_n \mathbb{Z}^2$ for $h = 1, \dots, N$. If this is not the case, we find an approximation of u in the $W^{1,1}(\tilde{\Omega})$ -norm satisfying that property as follows. For every n and $h = 1, \dots, N$ we choose $x_h^n \in \lambda_n \mathbb{Z}^2 \cap \Omega$ such that $x_h^n \rightarrow x_h$ as $n \rightarrow +\infty$. For every n there exists a diffeomorphism $\psi_n: \tilde{\Omega} \rightarrow \tilde{\Omega}$ such that $\psi_n(x_h^n) = x_h$ for $h = 1, \dots, N$ (see, e.g., [33, p. 210] for an explicit construction). We remark that it is possible to construct ψ_n in such a way that $\|\psi_n - \text{id}\|_{C^1}$ and $\|\psi_n^{-1} - \text{id}\|_{C^1}$ are controlled by $\max_h |x_h^n - x_h|$ for every n . In particular, $\|\psi_n - \text{id}\|_{C^1}, \|\psi_n^{-1} - \text{id}\|_{C^1} \rightarrow 0$ for $n \rightarrow +\infty$. We define $u^n := u \circ \psi_n \in C^\infty(\tilde{\Omega} \setminus \{x_1^n, \dots, x_N^n\}; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$. Then $u^n \rightarrow u$ strongly in $W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ as $n \rightarrow +\infty$. Let us fix $\rho > 0$ such that the balls $\overline{B}_\rho(x_h)$ are pairwise disjoint and contained in $\tilde{\Omega}$. For n large enough, we have that $x_h^n \in B_{\rho/4}(x_h)$ for $h = 1, \dots, N$. Let $\zeta \in C_c^\infty(B_\rho(x_h))$ such that $\zeta \equiv 1$ on $B_{\rho/2}(x_h)$. By [22, Theorem B.1] we have that

$$\begin{aligned} 2\pi \deg(u^n)(x_h^n) &= \int_{B_\rho(x_h)} (u^n \times \nabla u^n)^\perp \cdot \nabla \zeta \, dx \xrightarrow{n \rightarrow +\infty} \int_{B_\rho(x_h)} (u \times \nabla u)^\perp \cdot \nabla \zeta \, dx \\ &= 2\pi \deg(u)(x_h) \end{aligned}$$

where $(u \times \nabla u)^\perp = (u_1 \partial_2 u_2 - u_2 \partial_2 u_1, u_2 \partial_1 u_1 - u_1 \partial_1 u_2)$.

Step 5 (Modification near singularities) Let us fix $\sigma > 0$. Then there exists $\eta_0 > 0$ (small enough) and $u^\sigma \in C^\infty(\tilde{\Omega}_\mu; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ such that $\int_\Omega |\nabla u^\sigma|_{2,1} dx \leq \int_\Omega |\nabla u|_{2,1} dx + \sigma$, $u^\sigma(x) = u(x)$ in $\tilde{\Omega} \setminus \bigcup_{h=1}^M \overline{B}_{\sqrt{\eta_0}}(x_h)$, and $u^\sigma(x) = \left(\frac{x-x_h}{|x-x_h|}\right)^{d_h}$ in $B_{\eta_0}(x_h) \setminus \{x_h\}$ (where the power is meant in the sense of complex functions). We refer to [25, Lemma 4.21] for a proof of this modification result. Thus, up to a diagonal argument as $\sigma \rightarrow 0$, we assume that u has the structure of u^σ with singularities $x_h \in \lambda_n \mathbb{Z}^2$.

Step 6 (Recovery sequence near singularities) By the assumption in Step 5, $u(x) := \left(\frac{x-x_h}{|x-x_h|}\right)^{d_h}$ in $B_{\eta_0}(x_h)$, where $d_h = \pm 1$. In [25, Formula (4.75)] we showed that the projection $\mathfrak{P}_\varepsilon(u)$ is concentrating the energy of a vortex near the singularity. More precisely, for every $\eta \in (0, \eta_0)$ we have that

$$\limsup_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(\mathfrak{P}_\varepsilon(u); B_\eta(x_h)) - 2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right) \leq C \eta, \tag{5.14}$$

for some universal constant C . In the regime $\theta_\varepsilon = \varepsilon |\log \varepsilon|$, this yields

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon \left(\mathfrak{P}_\varepsilon(u); \bigcup_{h=1}^M B_\eta(x_h) \right) \leq 2\pi M + C \eta = 2\pi |\mu|(\Omega) + C \eta.$$

Step 7 (Recovery sequence far from singularities) Fix $\eta \in (0, \eta_0)$. We consider a suitable square centered at the singularities x_h and with corners on $\lambda_n \mathbb{Z}^2$. More precisely, let $m(\lambda_n) \in \mathbb{N}$ be the maximal integer such that $Q(\lambda_n, x_h) := x_h + [-2^{m(\lambda_n)}\lambda_n, 2^{m(\lambda_n)}\lambda_n]^2 \subset B_{\eta/2}(x_h)$, so that the estimate in Step 5 holds true in $\bigcup_{h=1}^M Q(\lambda_n, x_h)$. We also consider the square $Q_0(\lambda_n, x_h) := x_h + [(-2^{m(\lambda_n)} + 1)\lambda_n, (2^{m(\lambda_n)} - 1)\lambda_n]^2$. The squares $Q(\lambda_n, x_h)$ and $Q_0(\lambda_n, x_h)$ differ by a frame made by 1 layer of squares of $\lambda_n \mathbb{Z}^2$. By the choice of $m(\lambda_n)$, one can prove that $B_{\eta/16}(x_h) \subset\subset Q_0(\lambda_n, x_h)$. Far from the singularities, we discretize u on the lattice $\lambda_n \mathbb{Z}^2$. Specifically, we exploit the fact that $u \in C^\infty(\tilde{\Omega} \setminus \bigcup_{h=1}^M \bar{B}_{\eta/32}(x_h); \mathbb{S}^1)$ to find a sequence of piecewise constant functions $u_n \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$ such that

$$u_n \rightarrow u \text{ strongly in } L^1\left(\Omega \setminus \bigcup_{h=1}^M \bar{B}_{\eta/16}(x_h)\right), \tag{5.15}$$

$$\limsup_{n \rightarrow +\infty} \int_{J_{u_n} \cap O^{\lambda_n}} d_{\mathbb{S}^1}(u_n^-, u_n^+) |v_{u_n}|_1 d\mathcal{H}^1 \leq \int_{\Omega} |\nabla u|_{2,1} dx, \tag{5.16}$$

where O^{λ_n} is the union of half-open squares $I_{\lambda_n}(\lambda_n z)$, with $z \in \mathbb{Z}^2$, that intersect $\Omega \setminus \bigcup_{h=1}^M \bar{B}_{\eta/16}(x_h)$. Note that, since $B_{\eta/16}(x_h) \subset Q_0(\lambda_n, x_h)$,

$$\Omega \setminus \bigcup_{h=1}^M Q_0(\lambda_n, x_h) \subset \Omega \setminus \bigcup_{h=1}^M B_{\eta/16}(x_h) \subset O^{\lambda_n} \subset \tilde{\Omega} \setminus \bigcup_{h=1}^M \bar{B}_{\eta/32}(x_h).$$

For a detailed proof of this discretization result see [25, Lemma 4.13].

We consider a recovery sequence $u'_\varepsilon \in \mathcal{PC}_\varepsilon(\mathcal{S}_\varepsilon)$ for u_n satisfying $u'_\varepsilon \rightarrow u_n$ strongly in $L^1(O^{\lambda_n})$ and

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon \left(u'_\varepsilon; \Omega \setminus \bigcup_{h=1}^M Q_0(\lambda_n, x_h) \right) \leq \int_{J_{u_n} \cap O^{\lambda_n}} d_{\mathbb{S}^1}(u_n^-, u_n^+) |v_{u_n}|_1 d\mathcal{H}^1. \tag{5.17}$$

The recovery sequence u'_ε is defined as in the regime $\varepsilon \ll \theta_\varepsilon \ll \varepsilon |\log \varepsilon|$ in the case of no vortex-like singularities, exploiting the piecewise constant structure of u_n . The details of this construction can be found in [25, Proposition 4.16].

Step 8 (Joining the two constructions) A careful dyadic decomposition of the square $Q(\lambda_n, x_h)$ leads to the construction of a spin field $u_\varepsilon: \varepsilon \mathbb{Z}^2 \cap Q(\lambda_n, x_h) \rightarrow \mathcal{S}_\varepsilon$ such that $u_\varepsilon = \mathfrak{P}_\varepsilon(u)$ on $B_{\eta/16}(x_h) \subset Q_0(\lambda_n, x_h)$, while $u_\varepsilon(\varepsilon i) = u'_\varepsilon(\varepsilon i)$ if $\varepsilon i \in \varepsilon \mathbb{Z}^2 \cap Q(\lambda_n, x_h)$ satisfies $\text{dist}(\varepsilon i, \partial Q(\lambda_n, x_h)) \leq \varepsilon$, and

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon \left(u_\varepsilon; \bigcup_{h=1}^M Q(\lambda_n, x_h) \right) \leq 2\pi |\mu|(\Omega) + C\eta. \tag{5.18}$$

Since u_ε and u'_ε agree close to $\partial Q(\lambda, x_h)$, we define a global spin field $u_\varepsilon \in \mathcal{PC}_\varepsilon(\mathcal{S}_\varepsilon)$ (equal to u'_ε outside $\bigcup_{h=1}^M Q(\lambda_n, x_h)$) satisfying, thanks to (5.17) and (5.18),

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) &\leq \limsup_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon \left(u'_\varepsilon; \Omega \setminus \bigcup_{h=1}^M Q_0(\lambda_n, x_h) \right) \right. \\ &\quad \left. + \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon \left(u_\varepsilon; \bigcup_{h=1}^M Q(\lambda_n, x_h) \right) \right] \\ &\leq \int_{J_{u_n} \cap O^{\lambda_n}} d_{\mathbb{S}^1}(u_n^-, u_n^+) |v_{u_n}|_1 d\mathcal{H}^1 + 2\pi |\mu|(\Omega) + C\eta. \end{aligned} \tag{5.19}$$

We refer to [25, Steps 2–4 in the proof of Proposition 4.22] for the details about this construction.

Step 9 (Identifying the L^1 -limit of u_ε) As $\varepsilon \rightarrow 0$, the spin fields u_ε converge in $L^1(\Omega; \mathbb{R}^2)$ to the map $\hat{u}_n \in L^1(\Omega; \mathbb{S}^1)$ given by

$$\hat{u}_n(x) := \begin{cases} u_n(x), & \text{if } x \in \Omega \setminus \bigcup_{h=1}^M Q(\lambda_n, x_h), \\ \left(\frac{x-x_h}{|x-x_h|} \right)^{d_h}, & \text{if } x \in Q_\infty(\lambda_n, x_h) \text{ for } h = 1, \dots, M, \\ u_0^{\lambda_n}(x), & \text{if } x \in Q(\lambda_n, x_h) \setminus Q_\infty(\lambda_n, x_h) \text{ for } h = 1, \dots, M, \end{cases}$$

where $u_0^{\lambda_n}$ is an \mathbb{S}^1 -valued map whose value is not relevant here and $Q_\infty(\lambda_n, x_h) := x_h + [(-2^{m(\lambda_n)} + 2)\lambda_n, (2^{m(\lambda_n)} - 2)\lambda_n] \subset Q(\lambda_n, x_h)$ (This notation is used in agreement to [25, Proposition 4.22].) Since $Q(\lambda_n, x_h) \setminus Q_\infty(\lambda_n, x_h)$ is a frame of width $2\lambda_n$ and $Q(\lambda_n, x_h) \subset B_{\eta/2}(x_h)$, where $u(x) = \left(\frac{x-x_h}{|x-x_h|} \right)^{d_h}$ by our assumptions on u in Step 5, by (5.15) we get that

$$\hat{u}_n \rightarrow u \text{ in } L^1(\Omega; \mathbb{R}^2). \tag{5.20}$$

Hence, via a diagonal argument as $n \rightarrow +\infty$ we find a subsequence such that $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$ and, by (5.16) and (5.19), that

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq \int_\Omega |\nabla u|_{2,1} dx + 2\pi |\mu|(\Omega) + C\eta,$$

which will give the claim after an additional diagonal argument as $\eta \rightarrow 0$.

Step 10 (Identifying the flat limit of μ_{u_ε}) In order to implement the diagonal arguments proven in Step 9, we need to identify the flat limit of μ_{u_ε} for n fixed. After the diagonal argument we will obtain the desired convergence $\mu_{u_\varepsilon} \xrightarrow{f} \mu$. As this is the major difference in the regime $\theta_\varepsilon = \varepsilon |\log \varepsilon|$, we provide all the details. Since $\theta_\varepsilon = \varepsilon |\log \varepsilon|$, the energy bound (5.19) reads

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 |\log \varepsilon|} E_\varepsilon(u_\varepsilon; \Omega) \leq \int_{J_{u_n} \cap O^{\lambda_n}} d_{\mathbb{S}^1}(u_n^-, u_n^+) |v_{u_n}|_1 d\mathcal{H}^1 + 2\pi |\mu|(\Omega) + C\eta. \tag{5.21}$$

Note that the left hand side agrees with the unconstrained scaled XY -model. In particular, by Proposition 2.4 we deduce that there exists a measure $\mu_n \in \mathcal{M}_b(\Omega)$

of the form $\mu_n = \sum_{k=1}^{K_n} d_{k,n} \delta_{x_{k,n}}$ with $d_{k,n} \in \mathbb{Z} \setminus \{0\}$ such that, up to a subsequence, $\mu_{u_\varepsilon} \xrightarrow{f} \mu_n$. Thus the (already proven) lower bound in Theorem 1.2-(ii), the convergence $u_\varepsilon \rightarrow \hat{u}_n$, and (5.21) yield

$$\begin{aligned} & \int_{\Omega} |\nabla \hat{u}_n|_{2,1} \, dx + |D^{(c)} \hat{u}_n|_{2,1}(\Omega) + \mathcal{J}(\mu_n, \hat{u}_n; \Omega) + 2\pi |\mu_n|(\Omega) \\ & \leq \int_{J_{u_n} \cap O^{\lambda_n}} d_{\mathbb{S}^1}(u_n^-, u_n^+) |v_{u_n}|_1 d\mathcal{H}^1 + 2\pi |\mu|(\Omega) + C\eta. \end{aligned}$$

Since the last term in the above estimate is controlled via (5.16), we deduce that $|\mu_n|(\Omega)$ is equibounded in n . In particular, up to a subsequence, there exists a measure $\mu_0 \in \mathcal{M}_b(\Omega)$ that has the structure $\mu_0 = \sum_{k=1}^K d_k \delta_{x_k}$ for some $d_k \in \mathbb{Z} \setminus \{0\}$ such that $\mu_n \xrightarrow{f} \mu_0$ (and weakly* in the sense of measures). We next want to use a lower semicontinuity property of the left hand side. However, due to the mixed term $\mathcal{J}(\mu, u; \Omega)$, this is not straightforward, so we estimate the left hand side from below with a negligible error when $\lambda_n \rightarrow 0$. Indeed, by (5.8) we have that

$$\mathcal{J}(\mu_n, \hat{u}_n; \Omega) \geq \int_{J_{\hat{u}_n} \cap \Omega} d_{\mathbb{S}^1}(\hat{u}_n^-, \hat{u}_n^+) |v_{\hat{u}_n}|_1 d\mathcal{H}^1.$$

Inserting this lower bound in the previous estimate we obtain that

$$\begin{aligned} & \int_{\Omega} |\nabla \hat{u}_n|_{2,1} \, dx + |D^{(c)} \hat{u}_n|_{2,1}(\Omega) + \int_{J_{\hat{u}_n} \cap \Omega} d_{\mathbb{S}^1}(\hat{u}_n^-, \hat{u}_n^+) |v_{\hat{u}_n}|_1 d\mathcal{H}^1 + 2\pi |\mu_n|(\Omega) \\ & \leq C\eta + 2\pi |\mu|(\Omega) + \int_{J_{u_n} \cap O^{\lambda_n}} d_{\mathbb{S}^1}(u_n^-, u_n^+) |v_{u_n}|_1 d\mathcal{H}^1. \end{aligned}$$

The left hand side is lower semicontinuous with respect to the $L^1(\Omega; \mathbb{R}^2)$ -convergence of \hat{u}_n (see Proposition 6.1 below) and the weak*-convergence of μ_n . The limit of the right hand side is given by (5.16). Due to (5.20) and the fact that $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ (recall the standing assumption in Step 5) we conclude that

$$\int_{\Omega} |\nabla u|_{2,1} \, dx + 2\pi |\mu_0| \leq C\eta + \int_{\Omega} |\nabla u|_{2,1} \, dx + 2\pi |\mu|(\Omega),$$

which implies that $|\mu_0|(\Omega) \leq |\mu|(\Omega)$ (recall that $\eta < \eta_0$ can be chosen arbitrary small, while the constant C is bounded uniformly).

We finish the proof by showing that all measures μ_n have mass 1 in a uniform neighborhood of each of the points x_h given by the target measure $\mu = \sum_{h=1}^M \deg(u)(x_h) \delta_{x_h}$. Indeed, by Step 6, $u_\varepsilon = \mathfrak{P}_\varepsilon(u)$ on each $B_{\eta/16}(x_h)$. Due to (5.14) we have for ε small enough

$$\frac{1}{\varepsilon^2} E_\varepsilon(\mathfrak{P}_\varepsilon(u); B_\eta(x_h)) \leq 2C\eta \frac{\theta_\varepsilon}{\varepsilon} + 2\pi |\log \varepsilon| \leq C |\log \varepsilon|.$$

This allows us to apply [6, Proposition 5.2], so that the flat convergence of discrete vorticities is equivalent to the flat convergence of the (normalized) Jacobians of the piecewise affine interpolations. Denote by \widehat{v}_ε and $\widehat{u}(\varepsilon)$ the piecewise affine

interpolations associated to $\mathfrak{P}_\varepsilon(u)$ and u on $B_{\eta/16}(x_h)$, respectively. (We adopt the notation $\widehat{u}(\varepsilon)$ to stress that the interpolated function is independent of ε .) We have that

$$\begin{aligned} & \|\widehat{v}_\varepsilon - \widehat{u}(\varepsilon)\|_{L^2(B_{\eta/16}(x_h))} \left(\|\nabla \widehat{v}_\varepsilon\|_{L^2(B_{\eta/16}(x_h))} + \|\nabla \widehat{u}(\varepsilon)\|_{L^2(B_{\eta/16}(x_h))} \right) \\ & \leq C\eta\theta_\varepsilon \left(\frac{1}{\varepsilon^2} E_\varepsilon(\mathfrak{P}_\varepsilon(u); B_\eta(x_h)) + \frac{1}{\varepsilon^2} E_\varepsilon\left(\left(\frac{x-x_h}{|x-x_h|}\right)^{d_h}; B_\eta(x_h)\right) \right)^{\frac{1}{2}} \\ & \leq C\theta_\varepsilon |\log \varepsilon|^{\frac{1}{2}}. \end{aligned}$$

As $\theta_\varepsilon = \varepsilon |\log \varepsilon|$, the right hand side vanishes when $\varepsilon \rightarrow 0$. Hence [6, Lemma 3.1] yields that $\widehat{Jv}_\varepsilon - \widehat{Ju}(\varepsilon) \xrightarrow{f} 0$ on $B_{\eta/16}(x_h)$. Since $u = \left(\frac{x-x_h}{|x-x_h|}\right)^{\pm 1}$ on $B_{\eta/16}(x_h)$, Step 1 of the proof of [4, Theorem 5.1 (ii)] implies that $\frac{1}{\pi} \widehat{Ju}(\varepsilon) \xrightarrow{f} \text{deg}(u)(x_h) \delta_{x_h}$. Choosing an arbitrary $\varphi \in C_c^{0,1}(B_{\eta/16}(x_h))$, the above arguments imply

$$\begin{aligned} \langle \mu_n \llcorner B_{\eta/16}(x_h), \varphi \rangle &= \lim_{\varepsilon \rightarrow 0} \langle \mu_{u_\varepsilon}, \varphi \rangle \\ &= \lim_{\varepsilon \rightarrow 0} \langle \frac{1}{\pi} \widehat{Jv}_\varepsilon, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \langle \frac{1}{\pi} \widehat{Ju}(\varepsilon), \varphi \rangle = \text{deg}(u)(x_h) \varphi(x_h). \end{aligned}$$

Letting $n \rightarrow +\infty$ in the above equality we infer that $\mu_0 \llcorner B_{\eta/16}(x_h) = \text{deg}(u)(x_h) \delta_{x_h}$. Now consider the decomposition of μ_0 into the mutually singular measures

$$\mu_0 = \sum_{h=1}^M \text{deg}(u)(x_h) \delta_{x_h} + \mu_0 \llcorner \left(\Omega \setminus \bigcup_{h=1}^M B_{\eta/16}(x_h) \right).$$

From mutual singularity we deduce that

$$\begin{aligned} |\mu|(\Omega) &\geq |\mu_0|(\Omega) = \sum_{h=1}^M |\text{deg}(u)(x_h)| + \left| \mu_0 \llcorner \left(\Omega \setminus \bigcup_{h=1}^M B_{\eta/16}(x_h) \right) \right| \\ &\geq |\mu|(\Omega) + \left| \mu_0 \llcorner \left(\Omega \setminus \bigcup_{h=1}^M B_{\eta/16}(x_h) \right) \right|. \end{aligned}$$

Hence $\mu_0 \llcorner \left(\Omega \setminus \bigcup_{h=1}^M B_{\eta/16}(x_h) \right) = 0$ and therefore $\mu_0 = \sum_{h=1}^M \text{deg}(u)(x_h) \delta_{x_h} = \mu$. In conclusion, we have proved that $\mu_{u_\varepsilon} \xrightarrow{f} \mu_n$ as $\varepsilon \rightarrow 0$ and $\mu_n \xrightarrow{f} \mu$ as $n \rightarrow +\infty$ which justifies the diagonal arguments in Step 9.

6. Proofs in the regime $\varepsilon |\log \varepsilon| \ll \theta_\varepsilon \ll 1$

In the present scaling regime the discrete vorticity measures μ_{u_ε} for sequences with bounded energy are not necessarily compact. Hence we cannot use the parametric integral as a comparison, but we will work directly with the spin variable u_ε . We recall the following lower-semicontinuity result proven in the d -dimensional case in [26, Lemma 3.2] via a slicing argument.

Proposition 6.1. *For every open set $A \subset \Omega$ define the functional $E(\cdot; A) : L^1(A; \mathbb{R}^2) \rightarrow [0, +\infty]$ with domain $BV(A; \mathbb{S}^1)$, on which it is given by*

$$E(u; A) := \int_A |\nabla w|_{2,1} \, dx + |D^{(c)}w|_{2,1}(A) + \int_{J_u \cap A} d_{\mathbb{S}^1}(u^+, u^-) |v_u|_1 d\mathcal{H}^1.$$

Then $E(\cdot; A)$ is lower semicontinuous with respect to strong convergence in $L^1(A; \mathbb{R}^2)$.

Proof of Theorem 1.1. The compactness result in (i) follows by Lemma 4.2 as in the proof of Proposition 4.1. Indeed, Lemma 4.2 with $\sigma = \frac{1}{2}$ yields that u_ε is bounded in $BV(A; \mathbb{S}^1)$ for every $A \subset\subset \Omega$.

To prove (ii), it suffices to consider $u \in BV(\Omega; \mathbb{S}^1)$ and a sequence $u_\varepsilon \in \mathcal{PC}_\varepsilon(\mathcal{S}_\varepsilon)$ such that $u_\varepsilon \rightarrow u$ strongly in $L^1(\Omega; \mathbb{R}^2)$ and

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C.$$

Let us fix an open set $A \subset\subset \Omega$ and $\sigma > 0$. By Lemma 4.2, for ε small enough we have that

$$\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \geq (1 - \sigma) \int_{J_{u_\varepsilon} \cap A} d_{\mathbb{S}^1}(u_\varepsilon^+, u_\varepsilon^-) |v_{u_\varepsilon}|_1 d\mathcal{H}^1 = (1 - \sigma) E(u_\varepsilon; A).$$

Hence Proposition 6.1 yields that

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \geq (1 - \sigma) E(u; A).$$

We conclude the proof of the lower bound letting $\sigma \rightarrow 0$ and $A \nearrow \Omega$.

In order to show the Γ -limsup inequality in (iii), we first remark that, following the approximation procedure in [26, Proof of Proposition 4.3 (Step 1-3)], it suffices to prove the upper bound for a target function $u \in C^\infty(\tilde{\Omega} \setminus V; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$, where $\tilde{\Omega} \supset\supset \Omega$ has Lipschitz-boundary and $V = \{x_1, \dots, x_M\} \subset \tilde{\Omega}$ is a finite set. Moreover, following Steps 3–5 in the proof of Theorem 1.2-(iii) above, we can assume, without loss of generality, that $V \subset \lambda_n \mathbb{Z}^2 \cap \tilde{\Omega}$ with $\lambda_n = 2^{-n}$, that there exists $\Omega \subset\subset \Omega' \subset\subset \tilde{\Omega}$ such that $V \cap (\Omega' \setminus \Omega) = \emptyset$, and that there exists $\eta_0 > 0$ such that for every $x_i \in V$ we have $u(x) = \left(\frac{x-x_i}{|x-x_i|}\right)^{\pm 1}$ for $x \in \bar{B}_{\eta_0}(x_i)$.

We are finally in a position to apply [25, Proposition 4.22] (which is valid also in the case $\varepsilon \ll \theta_\varepsilon$) to the modified field $u \in C^\infty(\Omega' \setminus V; \mathbb{S}^1) \cap W^{1,1}(\Omega'; \mathbb{S}^1)$. It gives the existence of a recovery sequence $u_\varepsilon \in \mathcal{PC}_\varepsilon(\mathcal{S}_\varepsilon)$ such that $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$ and

$$\limsup_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) - 2\pi |\mu|(\Omega) |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right) \leq \int_\Omega |\nabla u|_{2,1} \, dx.$$

Since $|\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \rightarrow 0$, we obtain that

$$\Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u) \leq \int_\Omega |\nabla u|_{2,1} \, dx.$$

Note that the right hand side coincides with the functional claimed to be the Γ -limit in Theorem 1.1 since $u \in W^{1,1}(\Omega; \mathbb{S}^1)$.

7. Proofs in the regime $\theta_\varepsilon \ll \varepsilon$

We now come to the scaling regime which yields a discretization of \mathbb{S}^1 that is fine enough to commit asymptotically no error compared to the XY -model up to the first order development. Throughout this section we shall always assume that

$$\theta_\varepsilon \ll \varepsilon. \tag{7.1}$$

Moreover, we will use the following elementary estimate: for any $x, y \in \mathbb{R}^2 \setminus \{0\}$ it holds that

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right| \leq 2 \frac{|x - y|}{|y|}. \tag{7.2}$$

7.1. Renormalized and core energy

Following [16], we define the *renormalized energy* corresponding to the configuration of vortices $\mu = \sum_{h=1}^M d_h \delta_{x_h}$ by

$$\mathbb{W}(\mu) = -2\pi \sum_{h \neq k} d_h d_k \log |x_h - x_k| - 2\pi \sum_h d_h R_0(x_h),$$

where R_0 is harmonic in Ω and $R_0(x) = -\sum_{h=1}^M d_h \log |x - x_h|$ for $x \in \partial\Omega$ (see [16]). The renormalized energy can also be recast as

$$\mathbb{W}(\mu) = \lim_{\eta \rightarrow 0} [\tilde{m}(\eta, \mu) - 2\pi |\mu|(\Omega) |\log \eta|], \tag{7.3}$$

where

$$\tilde{m}(\eta, \mu) := \min \left\{ \int_{\Omega_\mu^\eta} |\nabla w|^2 dx : w(x) = \alpha_h \odot \left(\frac{x - x_h}{|x - x_h|} \right)^{d_h} \text{ for } x \in \partial B_\eta(x_h), |\alpha_h| = 1 \right\}. \tag{7.4}$$

To define the *core energy* (see also [16, Section IX]), we introduce the discrete minimization problem in a ball B_r

$$\gamma(\varepsilon, r) := \min \left\{ \frac{1}{\varepsilon^2} E_\varepsilon(v; B_r) : v : \varepsilon\mathbb{Z}^2 \cap B_r \rightarrow \mathbb{S}^1, v(x) = \frac{x}{|x|} \text{ for } x \in \partial_\varepsilon B_r \right\}, \tag{7.5}$$

where $\partial_\varepsilon B_r$ is the discrete boundary of B_r , defined for a general open set by

$$\partial_\varepsilon A = \{\varepsilon i \in \varepsilon\mathbb{Z}^2 \cap A : \text{dist}(\varepsilon i, \partial A) \leq \varepsilon\}.$$

Note that $\partial_\varepsilon B_r \subset \varepsilon\mathbb{Z}^2 \cap B_r \setminus B_{r-\varepsilon}$. Then the *core energy* of a vortex is the number γ given by the following lemma. The result is analogous to [7, Theorem 4.1] with some differences: here we consider $r_\varepsilon \rightarrow 0$ depending on ε and we use a different notion of discrete boundary of a set. The modifications in the proof are minor, but we give the details for the convenience of the reader.

Lemma 7.1. *Let r_ε be a family of radii such that $\varepsilon \ll r_\varepsilon \leq C$. Then there exists*

$$\lim_{\varepsilon \rightarrow 0} \left[\gamma(\varepsilon, r_\varepsilon) - 2\pi \left| \log \frac{\varepsilon}{r_\varepsilon} \right| \right] =: \gamma \in \mathbb{R}, \tag{7.6}$$

where γ is independent of the sequence r_ε .

Proof. We introduce the function

$$I(t) = \min \left\{ E_1(v; B_{\frac{1}{t}}) : v : \mathbb{Z}^2 \cap B_{\frac{1}{t}} \rightarrow \mathbb{S}^1, v(x) = \frac{x}{|x|} \text{ for } x \in \partial_1 B_{\frac{1}{t}} \right\}.$$

Let us show that

$$I(t_1) \leq I(t_2) + 2\pi \log \frac{t_2}{t_1} + \varrho_{t_2} \quad \text{for } 0 < t_1 \leq t_2, \tag{7.7}$$

where ϱ_{t_2} is a generic sequence (which may change from line to line) satisfying $\varrho_{t_2} \rightarrow 0$ as $t_2 \rightarrow 0$. To this end, let $v_2 : \mathbb{Z}^2 \cap B_{\frac{1}{t_2}} \rightarrow \mathbb{S}^1$ be such that $v_2(x) = \frac{x}{|x|}$ for $x \in \partial_1 B_{\frac{1}{t_2}}$ and $E_1(v_2; B_{\frac{1}{t_2}}) = I(t_2)$. We extend v_2 to $B_{\frac{1}{t_1}}$ setting

$$v_1(i) := \begin{cases} v_2(i), & \text{if } i \in \mathbb{Z}^2 \cap B_{\frac{1}{t_2}}, \\ \frac{i}{|i|}, & \text{if } i \in \mathbb{Z}^2 \cap B_{\frac{1}{t_1}} \setminus B_{\frac{1}{t_2}}. \end{cases}$$

To reduce notation, we define $A(t_1, t_2) := B_{\frac{1}{t_1}} \setminus \overline{B_{\frac{1}{t_2}}}$. Next, note that if $i \in B_{\frac{1}{t_2}}$ and $j \notin B_{\frac{1}{t_2}}$ with $|i - j| = 1$, then $|i| \geq \frac{1}{t_2} - 1 > \frac{1}{t_2} - 2$. Hence

$$\begin{aligned} I(t_1) &\leq E_1(v_1; B_{\frac{1}{t_1}}) \leq E_1(v_2; B_{\frac{1}{t_2}}) + E_1(v_1; A(t_1, t_2)) \\ &= I(t_2) + \frac{1}{2} \sum_{\substack{(i,j) \\ i,j \in A(t_1,t_2)}} \left| \frac{i}{|i|} - \frac{j}{|j|} \right|^2. \end{aligned}$$

To control the last sum, we derive an estimate for the finite differences away from the singularity. Set $u(x) = \frac{x}{|x|}$. For any $t \in [0, 1]$ and $i, j \in \mathbb{Z}^2$ with $|i - j| = 1$ we have

$$|(1 - t)\varepsilon i + t\varepsilon j| \geq |\varepsilon i| - \varepsilon.$$

Hence, by the regularity of u in $\mathbb{R}^2 \setminus \{0\}$, for any $\varepsilon i, \varepsilon j \in \varepsilon \mathbb{Z}^2 \setminus B_{2\varepsilon}$ with $|i - j| = 1$,

$$|u(\varepsilon i) - u(\varepsilon j)| \leq \int_0^1 |\nabla u(t\varepsilon i + (1 - t)\varepsilon j)(\varepsilon i - \varepsilon j)| dt.$$

Since $i - j \in \{\pm e_1, \pm e_2\}$, a direct computation yields the two cases

$$|u(\varepsilon i) - u(\varepsilon j)| \leq \begin{cases} \int_0^1 \frac{|i \cdot e_2|}{|ti + (1 - t)j|^2} dt & \text{if } (i - j) \parallel e_1, \\ \int_0^1 \frac{|i \cdot e_1|}{|ti + (1 - t)j|^2} dt & \text{if } (i - j) \parallel e_2, \end{cases} \tag{7.8}$$

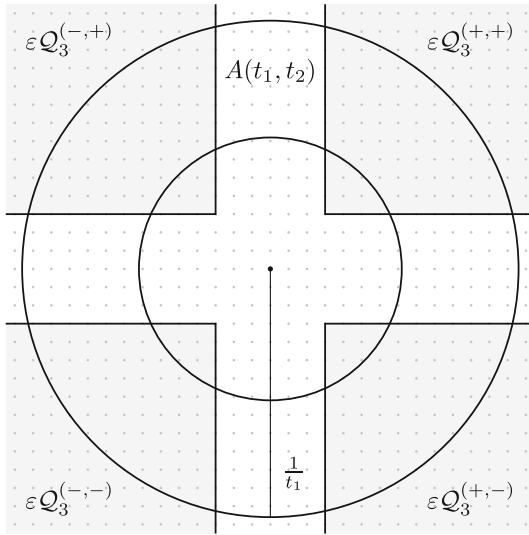


Fig. 8. Schematic illustration of the trimmed quadrants εQ_3^s

Next note that $|ti + (1 - t)j| \geq |i| - 1 \geq \max\{|i \cdot e_1|, |i \cdot e_2|\} - 1$. The right-hand side is non-negative if $i \neq 0$, so that we can take the square of this inequality. Using Jensen’s inequality, we infer from (7.8) that

$$|u(\varepsilon i) - u(\varepsilon j)|^2 \leq \frac{k^2}{(k - 1)^4} \Big|_{k=\max\{|i \cdot e_1|, |i \cdot e_2|\}}. \tag{7.9}$$

We shall use both (7.8) and (7.9) to estimate the sum of interactions in $A(t_1, t_2)$. To do so, we split the annulus $A(t_1, t_2)$ using trimmed quadrants defined as follows: given a tuple of signs $s = (s_1, s_2) \in \{(+, +), (-, +), (-, -), (+, -)\}$ and $n \in \mathbb{N}$ we define the trimmed quadrants Q_n^s as

$$Q_n^s := \{x \in \mathbb{R}^2 : s_1 x \cdot e_1 \geq n, s_2 x \cdot e_2 \geq n\}. \tag{7.10}$$

Fix $n = 3$. We then consider interactions $\langle i, j \rangle$ where both points belong to one trimmed quadrant and the remaining interactions. For the latter we use the estimate (7.9), noting that $\max\{|i \cdot e_1|, |i \cdot e_2|\} \geq \frac{1}{\sqrt{2}}|i| \geq \frac{1}{\sqrt{2}}t_2$ and that for t_2 small enough, i.e., the inner circle of the annulus large enough, the interactions outside the trimmed quadrants can be counted along 20 lines parallel to one of the coordinate axes in a way that the maximal component strictly increases along the line (cf. Fig. 8). Summing over all pairs of signs $s \in \{(+, +), (-, +), (+, -), (-, -)\}$ then yields that

$$\frac{1}{2} \sum_{\substack{\langle i, j \rangle \\ i, j \in A(t_1, t_2)}} \left| \frac{i}{|i|} - \frac{j}{|j|} \right|^2 \leq \frac{1}{2} \sum_s \sum_{i, j \in Q_3^s \cap A(t_1, t_2)} \left| \frac{i}{|i|} - \frac{j}{|j|} \right|^2 + C \sum_{k=\lfloor \frac{1}{\sqrt{2}t_2} \rfloor}^{\lceil \frac{1}{t_1} \rceil} \frac{k^2}{(k - 1)^4}$$

$$\leq \frac{1}{2} \sum_s \sum_{\substack{\langle i, j \rangle \\ i, j \in \mathbb{Q}_3^s \cap A(t_1, t_2)}} \left| \frac{i}{|i|} - \frac{j}{|j|} \right|^2 + \rho_{t_2},$$

where we used that the series $\sum_{k=2}^\infty \frac{k^2}{(k-1)^4}$ converges. The contributions on the trimmed cubes have to be treated more carefully since we need the precise prefactor 2π in (7.7). The idea is to switch from the discrete lattice \mathbb{Z}^2 to a continuum environment that leads to an integral. We have

$$I_s := \frac{1}{2} \sum_{\substack{\langle i, j \rangle \\ i, j \in \mathbb{Q}_3^s \cap A(t_1, t_2)}} \left| \frac{i}{|i|} - \frac{j}{|j|} \right|^2 = \sum_{i \in \mathbb{Z}^2} \left| \frac{i+s_1 e_1}{|i+s_1 e_1|} - \frac{i}{|i|} \right|^2 + \left| \frac{i+s_2 e_2}{|i+s_2 e_2|} - \frac{i}{|i|} \right|^2.$$

For each term on the right hand side we apply (7.8) noting that $|t(i + s_r e_r) + (1-t)i| = |i + t s_r e_r| \geq |i|$ for $i \in \mathbb{Q}_3^s$, $s = (s_1, s_2)$, and $r \in \{1, 2\}$, so that by Jensen’s inequality

$$\left| \frac{i+s_1 e_1}{|i+s_1 e_1|} - \frac{i}{|i|} \right|^2 + \left| \frac{i+s_2 e_2}{|i+s_2 e_2|} - \frac{i}{|i|} \right|^2 \leq \frac{|i \cdot e_1|^2 + |i \cdot e_2|^2}{|i|^4} = \frac{1}{|i|^2}$$

Note that for $i \in \mathbb{Z}^2 \cap \mathbb{Q}_3^s$ it holds that $\frac{1}{|x|^2} \geq \frac{1}{|i|^2}$ for all $x \in i - s_1 e_1 - s_2 e_2 + [-\frac{1}{2}, \frac{1}{2}]^2$. Since $i - s_1 e_1 - s_2 e_2 + [-\frac{1}{2}, \frac{1}{2}]^2 \in \mathbb{Q}_2^s$, we can control I_s by

$$I_s \leq \sum_{i \in \mathbb{Z}^2} \frac{1}{|i|^2} \leq \int_{\mathbb{Q}_2^s \cap B_{\frac{1}{4}} \setminus B_{\frac{1}{2}-4}} \frac{1}{|x|^2} dx. \tag{7.11}$$

Summing this estimate over all couples of signs s , we infer that

$$\sum_s I_s \leq \int_{B_{\frac{1}{4}} \setminus \bar{B}_{\frac{1}{2}-4}} \frac{1}{|x|^2} dx \leq 2\pi \log \frac{t_2}{t_1} + Q_{t_2}$$

where we also used that by the mean value theorem $|\log \frac{1}{t_2} - \log(\frac{1}{t_2}-4)| \leq 4|\frac{t_2}{1-4t_2}|$. This proves (7.7). As a consequence, the limit $\lim_{t \rightarrow 0} [I(t) - 2\pi |\log t|] =: \gamma$ exists. Since $\gamma(\varepsilon, r_\varepsilon) = I(\frac{\varepsilon}{r_\varepsilon})$, it only remains to show that $\gamma \neq -\infty$. To this end, we show that the boundary conditions in the definition of $\gamma(\varepsilon, 1)$ force concentration of the Jacobians, so that we can use localized lower bounds. Let $v_\varepsilon : \varepsilon \mathbb{Z}^2 \cap B_1 \rightarrow \mathbb{S}^1$ be an admissible minimizer for the problem defining $\gamma(\varepsilon, 1)$ and extend it to $\varepsilon \mathbb{Z}^2 \setminus B_1$ via $v_\varepsilon(\varepsilon i) = \frac{\varepsilon i}{|\varepsilon i|}$. Then, using the boundary conditions imposed on v_ε and (7.2), we deduce that

$$\begin{aligned} \frac{1}{\varepsilon^2} E_\varepsilon(v_\varepsilon; B_3) &\leq \frac{1}{\varepsilon^2} E_\varepsilon(v_\varepsilon; B_1) + \frac{1}{2\varepsilon^2} \sum_{\substack{\langle i, j \rangle \\ \varepsilon i, \varepsilon j \in B_3 \setminus B_{1/2}}} \varepsilon^2 \left| \frac{\varepsilon i}{|\varepsilon i|} - \frac{\varepsilon j}{|\varepsilon j|} \right|^2 \\ &\leq \gamma(\varepsilon, 1) + \frac{C}{\varepsilon^2} \sum_{\substack{\langle i, j \rangle \\ \varepsilon i, \varepsilon j \in B_3 \setminus B_{1/2}}} \varepsilon^4 \leq \gamma(\varepsilon, 1) + C. \end{aligned} \tag{7.12}$$

Since we already proved that $\gamma(\varepsilon, 1) - 2\pi|\log \varepsilon|$ remains bounded from above when $\varepsilon \rightarrow 0$, Proposition 2.4 implies that (up to a subsequence) $\mu_{v_\varepsilon} \llcorner B_3 \xrightarrow{f} \mu$ for some $\mu = d_1 \delta_{x_1}$ with $d_1 \in \mathbb{Z}$ and $x_1 \in B_3$. We claim that $\mu \neq 0$. Indeed, let us denote by \widehat{v}_ε the piecewise affine interpolation of v_ε and let $\eta: [0, 3] \rightarrow \mathbb{R}$ be the piecewise affine function such that $\eta = 1$ on $[0, 1]$, $\eta = 0$ on $[2, 3]$ and η is affine on $[1, 2]$. Then define the Lipschitz function $\varphi \in C_c^{0,1}(B_3)$ via $\varphi(x) = \eta(|x|)$. Using the flat convergence of μ_{v_ε} , which transfers to the scaled Jacobian $\pi^{-1} \mathbb{J} \widehat{v}_\varepsilon$ due to [6, Proposition 5.2], we infer that

$$\begin{aligned} \langle \mu, \varphi \rangle &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{B_3} \mathbb{J} \widehat{v}_\varepsilon \varphi \, dx \\ &= - \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{B_2 \setminus B_1} \begin{pmatrix} (v_\varepsilon)_1 \partial_2 (v_\varepsilon)_2 - (v_\varepsilon)_2 \partial_2 (v_\varepsilon)_1 \\ -(v_\varepsilon)_1 \partial_1 (v_\varepsilon)_2 + (v_\varepsilon)_2 \partial_1 (v_\varepsilon)_1 \end{pmatrix} \cdot \nabla \varphi \, dx, \end{aligned}$$

where in the last equality we integrated by parts due to the fact that in dimension two the Jacobian can be written as a divergence (in two ways). Note that on $B_2 \setminus B_1$ the function v_ε agrees with the discrete version of $x/|x|$. Hence one can pass to the limit in ε as \widehat{v}_ε converges to $x/|x|$ weakly in $H^1(B_2 \setminus B_1)$. Moreover, it holds that $\nabla \varphi(x) = -x/|x|$ a.e. in $B_2 \setminus B_1$. An explicit computation shows that

$$\langle \mu, \varphi \rangle = \frac{1}{2\pi} \int_{B_2 \setminus B_1} |x|^{-1} \, dx = 1. \tag{7.13}$$

Consequently $\mu \llcorner B_3 = \delta_{x_1}$. Let $\sigma < \frac{1}{2} \text{dist}(x_1, \partial B_3)$. Then [7, Theorem 3.1(ii)] yields

$$\begin{aligned} &\liminf_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon^2} E_\varepsilon(v_\varepsilon; B_3) - 2\pi|\log \varepsilon| \right) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon^2} E_\varepsilon(v_\varepsilon; B_\sigma(x_1)) - 2\pi \log \frac{\sigma}{\varepsilon} \right) + 2\pi \log \sigma \\ &\geq -C + 2\pi \log \sigma \end{aligned}$$

for some constant C . Combining this lower bound with (7.12) yields that

$$-C + 2\pi \log \sigma \leq \lim_{\varepsilon \rightarrow 0} (\gamma(\varepsilon, 1) - 2\pi|\log \varepsilon|) + C.$$

This shows that $\gamma > -\infty$ and concludes the proof.

Below we will use a shifted version of $\gamma(\varepsilon, r_\varepsilon)$. More precisely, given $x_0 \in \Omega$, set

$$\begin{aligned} \gamma_{x_0}(\varepsilon, r) &:= \min \left\{ \frac{1}{\varepsilon^2} E_\varepsilon(v; B_r(x_0)) : v: \varepsilon \mathbb{Z}^2 \cap B_r(x_0) \rightarrow \mathbb{S}^1, \right. \\ &\quad \left. v(x) = \frac{x-x_0}{|x-x_0|} \text{ on } \partial_\varepsilon B_r(x_0) \right\}. \end{aligned}$$

As shown in the lemma below the asymptotic behavior does not depend on x_0 .

Lemma 7.2. *Let $\gamma \in \mathbb{R}$ be given by Lemma 7.1 and let $\varepsilon \ll r_\varepsilon \leq C$. Then it holds that*

$$\lim_{\varepsilon \rightarrow 0} \left(\gamma_{x_0}(\varepsilon, r_\varepsilon) - 2\pi \left| \log \frac{\varepsilon}{r_\varepsilon} \right| \right) = \gamma.$$

Proof. Consider a point $x_\varepsilon \in \varepsilon\mathbb{Z}^2$ such that $|x_0 - x_\varepsilon| \leq 2\varepsilon$. Then, given a minimizer $v : \varepsilon\mathbb{Z}^2 \cap B_{r_\varepsilon - 4\varepsilon} \rightarrow \mathbb{S}^1$ for the problem defining $\gamma(\varepsilon, r_\varepsilon - 4\varepsilon)$ (extended via the boundary conditions on $\varepsilon\mathbb{Z}^2 \setminus B_{r_\varepsilon - 4\varepsilon}$) we define $\tilde{v}(\varepsilon i) = v(\varepsilon i - x_\varepsilon)$. This function is admissible in the definition of $\gamma_{x_0}(\varepsilon, r_\varepsilon)$. Hence

$$\begin{aligned} & \gamma_{x_0}(\varepsilon, r_\varepsilon) - 2\pi \left| \log \frac{\varepsilon}{r_\varepsilon} \right| \\ & \leq \gamma(\varepsilon, r_\varepsilon - 4\varepsilon) - 2\pi \left| \log \frac{\varepsilon}{r_\varepsilon} \right| + \frac{1}{2\varepsilon^2} \sum_{\substack{\langle i, j \rangle \\ \varepsilon i, \varepsilon j \in B_{r_\varepsilon + 2\varepsilon} \\ \varepsilon i, \varepsilon j \notin B_{r_\varepsilon - 5\varepsilon}}} \varepsilon^2 \left| \frac{\varepsilon i}{|\varepsilon i|} - \frac{\varepsilon j}{|\varepsilon j|} \right|^2. \end{aligned}$$

The last sum can be bounded applying (7.2) which leads to

$$\begin{aligned} & \frac{1}{2\varepsilon^2} \sum_{\substack{\langle i, j \rangle \\ \varepsilon i, \varepsilon j \in B_{r_\varepsilon + 2\varepsilon} \\ \varepsilon i, \varepsilon j \notin B_{r_\varepsilon - 5\varepsilon}}} \varepsilon^2 \left| \frac{\varepsilon i}{|\varepsilon i|} - \frac{\varepsilon j}{|\varepsilon j|} \right|^2 \leq \frac{C}{r_\varepsilon^2} \sum_{\substack{\varepsilon i \in \varepsilon\mathbb{Z}^2 \\ \varepsilon i \in B_{r_\varepsilon + 2\varepsilon} \\ \varepsilon i \notin B_{r_\varepsilon - 5\varepsilon}}} \\ & \varepsilon^2 \leq \frac{C}{r_\varepsilon^2} |B_{r_\varepsilon + 4\varepsilon} \setminus B_{r_\varepsilon - 7\varepsilon}| \leq C \frac{r_\varepsilon \varepsilon + \varepsilon^2}{r_\varepsilon^2}, \end{aligned}$$

which vanishes due to the assumption that $\varepsilon \ll r_\varepsilon$. Thus we proved that

$$\limsup_{\varepsilon \rightarrow 0} \left(\gamma_{x_0}(\varepsilon, r_\varepsilon) - 2\pi \left| \log \frac{\varepsilon}{r_\varepsilon} \right| \right) \leq \gamma.$$

The reverse inequality for the lim inf can be proven by a similar argument.

7.2. Compactness and Γ -convergence

We recall the compactness result and the Γ -liminf inequality obtained in [7, Theorem 4.2]. We emphasize that these results also hold in our setting, regarding $u_\varepsilon \in \mathcal{PC}_\varepsilon(\mathcal{S}_\varepsilon)$ as a spin field $u_\varepsilon : \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$, that means, neglecting the \mathcal{S}_ε constraint.

Theorem 7.3. (Theorem 4.2 in [7]) *The following results hold:*

- (i) (Compactness) *Let $M \in \mathbb{N}$ and let $u_\varepsilon : \varepsilon\mathbb{Z}^2 \cap \Omega \rightarrow \mathbb{S}^1$ be such that $\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon) - 2\pi M \left| \log \varepsilon \right| \leq C$. Then there exists a subsequence (which we do not relabel) such that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ for some $\mu = \sum_{h=1}^{M'} d_h \delta_{x_h}$ with $|\mu|(\Omega) \leq M$. Moreover, if $|\mu|(\Omega) = M$, then $|d_h| = 1$.*
- (ii) (Γ -liminf inequality) *Let $u_\varepsilon : \varepsilon\mathbb{Z}^2 \cap \Omega \rightarrow \mathbb{S}^1$ be such that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ with $\mu = \sum_{h=1}^M d_h \delta_{x_h}$, $|d_h| = 1$. Then*

$$\liminf_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon) - 2\pi M \left| \log \varepsilon \right| \right] \geq \mathbb{W}(\mu) + M\gamma.$$

For the construction of the recovery sequence our arguments slightly differ from the proof of [7, Theorem 4.2]. For the reader’s convenience we give here the detailed proof, which together with Theorem 7.3 establishes Theorem 1.3.

Proposition 7.4. (Γ -limsup inequality) *Let $\mu = \sum_{h=1}^M d_h \delta_{x_h}$ with $|d_h| = 1$. Then there exists a sequence $u_\varepsilon : \varepsilon\mathbb{Z}^2 \cap \Omega \rightarrow \mathcal{S}_\varepsilon$ with $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ such that*

$$\limsup_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon) - 2\pi M |\log \varepsilon| \right] \leq \mathbb{W}(\mu) + M\gamma. \tag{7.14}$$

Proof. To avoid confusion among infinitesimal sequences, in this proof we denote by ϱ_ε a sequence, which may change from line to line, such that $\varrho_\varepsilon \rightarrow 0$ when $\varepsilon \rightarrow 0$.

Step 1 (Construction of the recovery sequence)

Let us fix $0 < \eta' < \eta < 1$ with η small enough such that the balls $\overline{B}_\eta(x_h)$ are pairwise disjoint and their union is contained in Ω . We denote by w^η a solution to the minimum problem (7.4) in Ω_μ^η . Then for $h = 1, \dots, M$ there exists $\alpha_h^\eta \in \mathbb{C}$ with $|\alpha_h^\eta| = 1$ such that $w^\eta(x) = \alpha_h^\eta \odot \left(\frac{x-x_h}{|x-x_h|}\right)^{d_h}$ for $x \in \partial B_\eta(x_h)$. Extend w^η to $\Omega_\mu^{\eta'/2}$ by $w^\eta(x) := \alpha_h^\eta \odot \left(\frac{x-x_h}{|x-x_h|}\right)^{d_h}$ for $x \in B_\eta(x_h) \setminus \overline{B}_{\eta'/2}(x_h)$. To reduce notation, we set $A_{\eta'}^\eta(x_h) := B_\eta(x_h) \setminus \overline{B}_{\eta'}(x_h)$. The extension w^η then belongs to $W^{1,2}(\Omega_\mu^{\eta'}; \mathbb{S}^1)$ and its Dirichlet energy is given by

$$\begin{aligned} \int_{\Omega_\mu^{\eta'}} |\nabla w^\eta|^2 dx &= \int_{\Omega_\mu^\eta} |\nabla w^\eta|^2 dx + \sum_{h=1}^M \int_{A_{\eta'}^\eta(x_h)} \frac{1}{|x-x_h|^2} dx \\ &= \tilde{m}(\eta, \mu) + 2\pi M \log \frac{\eta}{\eta'}. \end{aligned} \tag{7.15}$$

Set $r_\varepsilon = |\log \varepsilon|^{-\frac{1}{2}} \gg \varepsilon$ and let $\tilde{u}_\varepsilon : \varepsilon\mathbb{Z}^2 \cap B_{r_\varepsilon}(x_h) \rightarrow \mathbb{S}^1$ be a function that agrees with $x \mapsto \alpha_h^\eta \odot \left(\frac{x-x_h}{|x-x_h|}\right)^{d_h}$ on $\partial_\varepsilon B_{r_\varepsilon}(x_h)$ and such that, cf. Lemmata 7.1 and 7.2,

$$\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; B_{r_\varepsilon}(x_h)) = \gamma_{x_h}(\varepsilon, r_\varepsilon) = 2\pi \log \frac{r_\varepsilon}{\varepsilon} + \gamma + \varrho_\varepsilon \leq 2\pi |\log \varepsilon| + \gamma + \varrho_\varepsilon. \tag{7.16}$$

We now extend \tilde{u}_ε to $\varepsilon\mathbb{Z}^2 \cap \Omega$ distinguishing two cases: we set that

$$\tilde{u}_\varepsilon(\varepsilon i) := \alpha_h^\eta \odot \left(\frac{\varepsilon i - x_h}{|\varepsilon i - x_h|}\right)^{d_h} \quad \text{if } \varepsilon i \in \overline{B}_{\eta'}(x_h) \setminus B_{r_\varepsilon}(x_h); \tag{7.17}$$

on $\varepsilon\mathbb{Z}^2 \cap \Omega_\mu^{\eta'}$ the definition is more involved since w^η has only Sobolev regularity up to the (Lipschitz)-boundary and we are not aware of any density results preserving the traces on part of the boundary and the \mathbb{S}^1 -constraint. First we need to extend w^η to $\tilde{\Omega}_\mu^\eta$ for some open set $\tilde{\Omega} \supset \Omega$ with Lipschitz boundary. This can be achieved via a local reflection as in (3.8), so that we may assume from now on that $w^\eta \in W^{1,2}(\tilde{\Omega}_\mu^{\eta'}; \mathbb{S}^1)$. We further extend it to \mathbb{R}^2 with compact support (neglecting the \mathbb{S}^1 constraint outside $\tilde{\Omega}_\mu^{\eta'/2}$). Now let us define the discrete approximation of this extended w^η . Consider the shifted lattice $\mathbb{Z}_\varepsilon^x = x + \varepsilon\mathbb{Z}^2$ with $x \in B_\varepsilon$ and denote by $\widehat{w}_{\varepsilon,x}^\eta \in W^{1,2}(\mathbb{R}^2; \mathbb{R}^2)$ the piecewise affine interpolation of (the quasicontinuous

representative of) w^η on a standard triangulation associated to \mathbb{Z}_ε^x . As shown in the proof of [40, Theorem 1] there exists $x_\varepsilon \in B_\varepsilon$ such that $\widehat{w}_{\varepsilon, x_\varepsilon}^\eta \rightarrow w^\eta$ strongly in $W^{1,2}(\mathbb{R}^2; \mathbb{R}^2)$ (the proof is given in the scalar-case, but the argument also works component-wise; see also [40, Section 3.1]). Thus it is natural to define $\tilde{u}_\varepsilon : \varepsilon\mathbb{Z}^2 \cap \Omega_\mu^{\eta'} \rightarrow \mathbb{S}^1$ by

$$\tilde{u}_\varepsilon(\varepsilon i) = w_{\varepsilon, x_\varepsilon}^\eta(\varepsilon i + x_\varepsilon). \tag{7.18}$$

Observe that since w^η is defined on $\tilde{\Omega}_\mu^{\eta'}/2$ with values in \mathbb{S}^1 , for ε small enough \tilde{u}_ε is indeed \mathbb{S}^1 -valued. Moreover, the strong convergence of the affine interpolations ensures that

$$\begin{aligned} & \frac{1}{2} \sum_{\substack{\langle i, j \rangle \\ \varepsilon i, \varepsilon j \in \Omega_\mu^{\eta'}}} \varepsilon^2 \left| \frac{\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)}{\varepsilon} \right|^2 \\ & \leq \sum_{\substack{T \text{ triangle} \\ \varepsilon T \cap \Omega_\mu^{\eta'} \neq \emptyset}} \int_{\varepsilon T + x_\varepsilon} |\nabla \widehat{w}_{\varepsilon, x_\varepsilon}^\eta|^2 dx \leq \int_{\Omega_\mu^{\eta'}} |\nabla w^\eta|^2 dx + \varrho_\varepsilon. \end{aligned} \tag{7.19}$$

Finally, we define the global sequence $u_\varepsilon := \mathfrak{P}_\varepsilon(\tilde{u}_\varepsilon) : \varepsilon\mathbb{Z}^2 \cap \Omega \rightarrow \mathcal{S}_\varepsilon$ with the function \mathfrak{P}_ε given by (5.11). Note that the piecewise constant maps u_ε and \tilde{u}_ε actually depend on η' and η . For the computations to following however, we drop the dependence on these parameters to simplify notation.

We start estimating the error in energy due to the projection \mathfrak{P}_ε . To this end, we use the elementary inequality

$$|a|^2 - |b|^2 \leq |a - b|(|a| + |b|) \leq 2|b||a - b| + |a - b|^2, \tag{7.20}$$

which yields that

$$\begin{aligned} \frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon) &= \frac{1}{2} \sum_{\langle i, j \rangle} |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 \\ &\leq \frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon) + C|\Omega| \frac{\theta_\varepsilon^2}{\varepsilon^2} + 2 \sum_{\langle i, j \rangle} \theta_\varepsilon |\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)|. \end{aligned} \tag{7.21}$$

We shall prove that $\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon)$ carries the whole energy, that means,

$$\limsup_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon) - 2\pi M |\log \varepsilon| \right] \leq \mathbb{W}(\mu) + M\gamma, \tag{7.22}$$

whereas the remainder satisfies

$$\lim_{\varepsilon \rightarrow 0} \sum_{\langle i, j \rangle} \theta_\varepsilon |\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)| = 0. \tag{7.23}$$

Inequalities (7.21), (7.22), and (7.23) then yield (7.14) thanks to the assumption (7.1).

In order to prove both (7.22) and (7.23), we estimate separately the contribution of the energy and that of the remainder in the regions $B_{r_\varepsilon}(x_h)$, $\Omega_{\mu'}^{\eta'}$, and $(B_{\eta'+\varepsilon}(x_h) \setminus B_{r_\varepsilon-\varepsilon}(x_h))$. We remark that this decomposition of $\varepsilon\mathbb{Z}^2 \cap \Omega$ takes into account all the nearest-neighbors interactions.

Step 2 (Estimates close to the singularities)

Let us start with the estimates inside $B_{r_\varepsilon}(x_h)$. Notice that (7.16) already gives explicitly the value of $\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; B_{r_\varepsilon})$, so we only have to estimate the remainder in $B_{r_\varepsilon}(x_h)$. Combining the Cauchy–Schwarz inequality with (7.16), (7.1), and taking into account that $r_\varepsilon = |\log \varepsilon|^{-\frac{1}{2}}$, we obtain that

$$\begin{aligned} \sum_{\substack{\langle i,j \rangle \\ \varepsilon i, \varepsilon j \in B_{r_\varepsilon}(x_h)}} \theta_\varepsilon |\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)| &\leq \left(\sum_{\substack{\langle i,j \rangle \\ \varepsilon i, \varepsilon j \in B_{r_\varepsilon}(x_h)}} \theta_\varepsilon^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{\langle i,j \rangle \\ \varepsilon i, \varepsilon j \in B_{r_\varepsilon}(x_h)}} |\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)|^2 \right)^{\frac{1}{2}} \\ &\leq Cr_\varepsilon \frac{\theta_\varepsilon}{\varepsilon} \left(\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; B_{r_\varepsilon}(x_h)) \right)^{\frac{1}{2}} \\ &\leq Cr_\varepsilon \frac{\theta_\varepsilon}{\varepsilon} (2\pi |\log \varepsilon| + \gamma + \varrho_\varepsilon)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \tag{7.24}$$

Step 3 (Estimates in the perforated domain)

We go on with the estimates inside $\Omega_{\mu'}^{\eta'}$. In this set the function \tilde{u}_ε is given by (7.18). In particular, by (7.15) and (7.19),

$$\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; \Omega_{\mu'}^{\eta'}) \leq \int_{\Omega_{\mu'}^{\eta'}} |\nabla w^\eta|^2 dx + \varrho_\varepsilon = \tilde{m}(\eta, \mu) + 2\pi M \log \frac{\eta}{\eta'} + \varrho_\varepsilon. \tag{7.25}$$

Concerning the remainder, the Cauchy–Schwarz inequality, (7.25), and (7.1) imply that

$$\begin{aligned} \sum_{\substack{\langle i,j \rangle \\ \varepsilon i, \varepsilon j \in \Omega_{\mu'}^{\eta'}}} \theta_\varepsilon |\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)| &\leq \left(\sum_{\substack{\langle i,j \rangle \\ \varepsilon i, \varepsilon j \in \Omega_{\mu'}^{\eta'}}} \theta_\varepsilon^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{\langle i,j \rangle \\ \varepsilon i, \varepsilon j \in \Omega_{\mu'}^{\eta'}}} |\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)|^2 \right)^{\frac{1}{2}} \\ &\leq C|\Omega|^{\frac{1}{2}} \frac{\theta_\varepsilon}{\varepsilon} \left(\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; \Omega_{\mu'}^{\eta'}) \right)^{\frac{1}{2}} \\ &\leq C|\Omega|^{\frac{1}{2}} \frac{\theta_\varepsilon}{\varepsilon} (\tilde{m}(\eta, \mu) + 2\pi M \log \frac{\eta}{\eta'} + \varrho_\varepsilon)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \tag{7.26}$$

Step 4 (Estimates in the annulus)

Finally, we need to estimate the energy $\frac{1}{\varepsilon^2} E_\varepsilon$ in the set $(B_{\eta'+\varepsilon}(x_h) \setminus B_{r_\varepsilon-\varepsilon}(x_h))$, where $\tilde{u}_\varepsilon(x) = \alpha_h^\eta \odot \left(\frac{x-x_h}{|x-x_h|} \right)^{dh}$ (or slightly shifted in $B_{\eta'+\varepsilon}(x_h) \setminus \overline{B_{\eta'}(x_h)}$ due to

(7.18), for which we recall that $w^\eta = \alpha_h^\eta \odot \left(\frac{x-x_h}{|x-x_h|}\right)^{d_h}$ in $B_\eta(x_h) \setminus B_{\eta'/2}(x_h)$. To simplify notation, for $R > r > 0$ we denote in this step $A_r^R(x) := B_R(x) \setminus \overline{B}_r(x)$.

We first estimate the contribution involving the shifted function. For any $\varepsilon i, \varepsilon j$ with $|i - j| = 1$ and $\varepsilon i \in A_{\eta'}^{\eta'+\varepsilon}(x_h)$ the condition $|x_\varepsilon| \leq \varepsilon$ and (7.2) imply that

$$|\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)| \leq \frac{4\varepsilon}{\eta' - \varepsilon}.$$

Summing these estimate we deduce that

$$\begin{aligned} \frac{1}{\varepsilon^2} \sum_{\substack{\langle i, j \rangle \\ \varepsilon i \in A_{\eta'}^{\eta'+\varepsilon}(x_h)}} \varepsilon^2 |\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)|^2 &\leq C \frac{(\eta' + 3\varepsilon)^2 - (\eta' - 2\varepsilon)^2}{(\eta' - \varepsilon)^2} \\ &\leq C \frac{\varepsilon\eta' + \varepsilon^2}{(\eta' - \varepsilon)^2} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Hence we can write

$$\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; A_{r_\varepsilon-\varepsilon}^{\eta'+\varepsilon}(x_h)) \leq \frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; \overline{A}_{r_\varepsilon-\varepsilon}^{\eta'}(x_h)) + \varrho_\varepsilon. \tag{7.27}$$

Note that on the set $\varepsilon\mathbb{Z}^2 \cap \overline{A}_{r_\varepsilon-\varepsilon}^{\eta'}(x_h)$ the function \tilde{u}_ε coincides with $x \mapsto \alpha_h^\eta \odot \left(\frac{x-x_h}{|x-x_h|}\right)^{d_h}$, so that the invariance of the discrete energy under orthogonal transformations implies that

$$\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; \overline{A}_{r_\varepsilon-\varepsilon}^{\eta'}(x_h)) = \frac{1}{\varepsilon^2} \sum_{\substack{\langle i, j \rangle \\ \varepsilon i, \varepsilon j \in \overline{A}_{r_\varepsilon-\varepsilon}^{\eta'}(x_h)}} \varepsilon^2 \left| \frac{\varepsilon i - x_h}{|\varepsilon i - x_h|} - \frac{\varepsilon j - x_h}{|\varepsilon j - x_h|} \right|^2.$$

Using a shifted version of the trimmed quadrants defined in (7.10) and summing over all possible pairs of signs $s \in \{(+, +), (-, +), (+, -), (-, -)\}$, we can split the energy as

$$\begin{aligned} &\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; \overline{A}_{r_\varepsilon-\varepsilon}^{\eta'}(x_h)) \\ &\leq \sum_s \frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; (\varepsilon\mathcal{Q}_3^s + x_h) \cap \overline{A}_{r_\varepsilon-\varepsilon}^{\eta'}(x_h)) + C \sum_{k=\lfloor \frac{r_\varepsilon}{\varepsilon} \rfloor - 2}^\infty \frac{1}{k^2}, \end{aligned} \tag{7.28}$$

where we used the bound (7.2) to estimate the contributions not fully contained in one of the trimmed quadrants by the last sum. Since the sum $\sum_{k=1}^{+\infty} k^{-2}$ is finite and $\frac{r_\varepsilon}{\varepsilon} \rightarrow +\infty$, the second term in the right-hand side is infinitesimal as $\varepsilon \rightarrow 0$. On each trimmed quadrant we use a shifted version of (7.8) and a monotonicity argument as in (7.11) to deduce that

$$\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; (\varepsilon\mathcal{Q}_3^s + x_h) \cap \overline{A}_{r_\varepsilon-\varepsilon}^{\eta'}(x_h))$$

$$\leq \sum_{\substack{\varepsilon i \in \overline{A}_{r_\varepsilon - \varepsilon}^{\eta'}(x_h) \\ \varepsilon i \in \varepsilon \mathbb{Z}^2 \cap (\mathcal{Q}_3^s + x_h)}} \varepsilon^2 \frac{1}{|\varepsilon i - x_h|^2} \leq \int_{\varepsilon \mathcal{Q}_2^s \cap A_{r_\varepsilon - 3\varepsilon}^{\eta'}} \frac{1}{|x|^2} dx,$$

with the annulus $A_{r_\varepsilon - 3\varepsilon}^{\eta'} = B_{\eta'} \setminus \overline{B}_{r_\varepsilon - 3\varepsilon}$ centered at 0. Summing over all four quadrants, since $\varepsilon \ll r_\varepsilon$, we get that

$$\begin{aligned} & \sum_s \frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; (\varepsilon \mathcal{Q}_3^s + x_h) \cap \overline{A}_{r_\varepsilon - \varepsilon}^{\eta'}(x_h)) \\ & \leq \int_{A_{r_\varepsilon - 3\varepsilon}^{\eta'}} \frac{1}{|x|^2} dx = 2\pi \log \frac{\eta'}{r_\varepsilon - 3\varepsilon} = 2\pi \log \frac{\eta'}{r_\varepsilon} + \varrho_\varepsilon. \end{aligned}$$

In combination with (7.27) and (7.28) we conclude that

$$\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; A_{r_\varepsilon - \varepsilon}^{\eta' + \varepsilon}(x_h)) \leq 2\pi \log \frac{\eta'}{r_\varepsilon} + \varrho_\varepsilon. \tag{7.29}$$

We now estimate the remainder term in $A_{r_\varepsilon - \varepsilon}^{\eta' + \varepsilon}(x_h)$, for which applying the Cauchy–Schwarz inequality as in (7.24) is too rough. However, note that for any $i \in \mathbb{Z}^2$ and $x \in \varepsilon i + [0, \varepsilon)^2$ with $|\varepsilon i - x_h| \gg \varepsilon$, we have for ε small enough that

$$|\varepsilon i - x_h| \geq |x - x_h| - \sqrt{2}\varepsilon \geq \frac{1}{2}|x - x_h|.$$

Hence, using (7.2) and a change of variables we obtain that

$$\begin{aligned} \sum_{\substack{(i,j) \\ \varepsilon i, \varepsilon j \in A_{r_\varepsilon - \varepsilon}^{\eta' + \varepsilon}(x_h)}} \theta_\varepsilon |\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)| & \leq C \sum_{\varepsilon i \in \varepsilon \mathbb{Z}^2 \cap A_{r_\varepsilon - \varepsilon}^{\eta' + \varepsilon}(x_h)} \theta_\varepsilon \frac{\varepsilon}{|\varepsilon i - x_h|} \leq C \frac{\theta_\varepsilon}{\varepsilon} \int_{A_{r_\varepsilon - 3\varepsilon}^{\eta' + 3\varepsilon}} \frac{1}{|x|} dx. \end{aligned} \tag{7.30}$$

Since the last integral is proportional to η' , inserting assumption (7.1) shows that the right-hand side of (7.30) is infinitesimal as $\varepsilon \rightarrow 0$.

Step 5 (Proof of (7.22) and (7.23) and conclusion) To prove (7.22), we employ (7.16), (7.25), and (7.29) to split the energy as follows:

$$\begin{aligned} \frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon) - 2\pi M |\log \varepsilon| & \leq \frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; \Omega_\mu^{\eta'}) - 2\pi M \log \frac{\eta}{\eta'} + 2\pi M \log \eta \\ & \quad + \sum_{h=1}^M \left[\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; B_{r_\varepsilon}(x_h)) - 2\pi \log \frac{r_\varepsilon}{\varepsilon} \right] \\ & \quad + \sum_{h=1}^M \left[\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_\varepsilon; A_{r_\varepsilon - \varepsilon}^{\eta' + \varepsilon}(x_h)) - 2\pi \log \frac{\eta'}{r_\varepsilon} \right] \\ & \leq \tilde{m}(\eta, \mu) + 2\pi M \log \eta + M\gamma + \varrho_\varepsilon. \end{aligned}$$

Now we stress the dependence of \tilde{u}_ε on η' and η , denoting the sequence by $\tilde{u}_{\varepsilon,\eta}$ (set for instance $\eta' = \eta/2$). Letting $\varepsilon \rightarrow 0$, for $\eta < 1$ we deduce that

$$\limsup_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_{\varepsilon,\eta}) - 2\pi M |\log \varepsilon| \right] \leq \tilde{m}(\eta, \mu) - 2\pi M |\log \eta| + M\gamma.$$

Moreover, (7.23) follows from (7.24), (7.26), and (7.30) splitting the remainder in the same way. Hence, for each $\eta < 1$ we found a sequence $u_{\varepsilon,\eta} \in \mathcal{PC}_\varepsilon(\mathcal{S}_\varepsilon)$ such that

$$\limsup_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon^2} E_\varepsilon(u_{\varepsilon,\eta}) - 2\pi M |\log \varepsilon| \right] \leq \tilde{m}(\eta, \mu) - 2\pi M |\log \eta| + M\gamma. \tag{7.31}$$

Before we conclude via a diagonal argument, we have to identify the flat limit of the vorticity measure $\mu_{u_{\varepsilon,\eta}}$. From the above energy estimate and Proposition 2.4 we deduce that, passing to a subsequence, $\mu_{u_{\varepsilon,\eta}} \xrightarrow{f} \mu_\eta$ for some $\mu_\eta = \sum_{k=1}^M d_k^\eta \delta_{x_k^\eta}$ with $|\mu_\eta|(\Omega) \leq M$ (we allow for $d_k^\eta = 0$ to sum up to M). Fix a Lipschitz set $A = A_\eta \subset\subset \Omega$ such that $\text{supp}(\mu_\eta) \subset A$. Due the logarithmic energy bound we can apply [6, Proposition 5.2] and deduce that on the set A it holds that $\pi^{-1} J\widehat{u}_{\varepsilon,\eta} - \mu_{u_{\varepsilon,\eta}} \xrightarrow{f} 0$, where $\widehat{u}_{\varepsilon,\eta}$ denotes the piecewise affine interpolation associated to $u_{\varepsilon,\eta}$ (which is at least defined on A for ε small enough). Let now $\widehat{v}_{\varepsilon,\eta}$ be the function defined via piecewise affine interpolation of the values $\tilde{u}_{\varepsilon,\eta}(\varepsilon i)$, $\varepsilon i \in \varepsilon\mathbb{Z}^2 \cap \Omega$. We argue that on A the Jacobian of $\widehat{u}_{\varepsilon,\eta}$ is close with respect to the flat convergence to the Jacobian of $\widehat{v}_{\varepsilon,\eta}$, i.e.,

$$J(\widehat{u}_{\varepsilon,\eta}) - J(\widehat{v}_{\varepsilon,\eta}) \xrightarrow{f} 0 \quad \text{on } A. \tag{7.32}$$

To this end, we apply [6, Lemma 3.1] which states that the Jacobians of two functions u and w are close if $\|u - w\|_{L^2} (\|\nabla u\|_{L^2} + \|\nabla w\|_{L^2})$ is small. Since by definition $u_{\varepsilon,\eta} = \mathfrak{P}_\varepsilon(\tilde{u}_{\varepsilon,\eta})$, we know that

$$|u_{\varepsilon,\eta}(\varepsilon i) - \tilde{u}_{\varepsilon,\eta}(\varepsilon i)| \leq \theta_\varepsilon.$$

Inserting this estimate in the definition of the piecewise affine interpolation one can show that

$$|\widehat{u}_{\varepsilon,\eta}(x) - \widehat{v}_{\varepsilon,\eta}(x)| \leq C\theta_\varepsilon \quad \text{for all } x \in A. \tag{7.33}$$

Taking into account the energy bounds (7.31) and (7.22), we conclude that

$$\begin{aligned} & \|\widehat{u}_{\varepsilon,\eta} - \widehat{v}_{\varepsilon,\eta}\|_{L^2(A)} \left(\|\nabla \widehat{u}_{\varepsilon,\eta}\|_{L^2(A)} + \|\nabla \widehat{v}_{\varepsilon,\eta}\|_{L^2(A)} \right) \\ & \leq C\sqrt{|A|}\theta_\varepsilon \left(\frac{1}{\varepsilon^2} E_\varepsilon(u_{\varepsilon,\eta}; \Omega) + \frac{1}{\varepsilon^2} E_\varepsilon(\tilde{u}_{\varepsilon,\eta}; \Omega) \right)^{\frac{1}{2}} \leq C\sqrt{|A|}\theta_\varepsilon |\log \varepsilon|^{\frac{1}{2}}. \end{aligned} \tag{7.34}$$

The above right hand side vanishes when $\varepsilon \rightarrow 0$, so that [6, Lemma 3.1] implies (7.32). Hence it suffices to study the limit of the Jacobians of $\widehat{v}_{\varepsilon,\eta}$. We show that the limit carries mass in each ball $B_\rho(x_h)$ for all $\rho > 0$ small enough such that

$B_\rho(x_h) \subset\subset A$. To this end, we test the flat convergence against the Lipschitz cut-off function

$$\varphi_\rho^h(x) := \min\{\max\{\frac{2}{\rho}(\rho - |x - x_h|), 0\}, 1\}.$$

Using the distributional divergence form of the Jacobian and the fact that $\widehat{v}_{\varepsilon,\eta}$ agrees with the piecewise affine interpolation of the map $x \mapsto \alpha_h^\eta \odot \left(\frac{x-x_h}{|x-x_h|}\right)^{d_h}$ on the support of the gradient of φ_ρ^h provided $\rho < \eta/4$ and $r_\varepsilon \ll \rho/2$, we infer that

$$\begin{aligned} \langle \mu_\eta, \varphi_\rho^h \rangle &= \lim_{\varepsilon \rightarrow 0} \langle \pi^{-1} \widehat{J} \widehat{v}_{\varepsilon,\eta}, \varphi_\rho^h \rangle \\ &= - \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{A_{\rho/2}^\rho(x_h)} \begin{pmatrix} (\widehat{v}_{\varepsilon,\eta})_1 \partial_2 (\widehat{v}_{\varepsilon,\eta})_2 - (\widehat{v}_{\varepsilon,\eta})_2 \partial_2 (\widehat{v}_{\varepsilon,\eta})_1 \\ -(\widehat{v}_{\varepsilon,\eta})_1 \partial_1 (\widehat{v}_{\varepsilon,\eta})_2 + (\widehat{v}_{\varepsilon,\eta})_2 \partial_1 (\widehat{v}_{\varepsilon,\eta})_1 \end{pmatrix} \nabla \varphi_\rho^h \, dx \\ &= d_h, \end{aligned}$$

where the limit can be calculated similarly to (7.13). From this equality and the arbitrariness of $\rho > 0$, we deduce that $\{x_1, \dots, x_M\} \subset \text{supp}(\mu_\eta)$ and $\mu \llcorner \{x_h\} = d_h \delta_{x_h}$. Since $|\mu_\eta|(\Omega) \leq M$, it follows that $\mu_\eta = \mu$ independently of η and the subsequence of ε . Since the flat convergence is given by a metric, we can thus use a diagonal argument with $\eta = \eta_\varepsilon$ to find a sequence $u_\varepsilon := u_{\varepsilon,\eta_\varepsilon}$ satisfying $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ and, due to (7.31), also the claimed inequality (7.14).

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Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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M. CICALEASE
Technische Universität München,
Munich
Germany.
e-mail: cicaless@ma.tum.de

and

G. ORLANDO
Politecnico di Bari,
Bari
Italy.
e-mail: gianluca.orlando@poliba.it

and

M. RUF
Ecole Polytechnique Fédérale de Lausanne,
Lausanne
Switzerland.
e-mail: matthias.ruf@epfl.ch

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