



Complete Entropic Inequalities for Quantum Markov Chains

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Abstract

We prove that every GNS-symmetric quantum Markov semigroup on a finite dimensional matrix algebra satisfies a modified log-Sobolev inequality. In the discrete time setting, we prove that every finite dimensional GNS-symmetric quantum channel satisfies a strong data processing inequality with respect to its decoherence free part. Moreover, we establish the first general approximate tensorization property of the relative entropy. This extends the famous strong subadditivity of the quantum entropy (SSA) of two subsystems to the general setting of two subalgebras. All three results are independent of the size of the environment and hence satisfy the tensorization property. They are obtained via a common, conceptually simple method for proving entropic inequalities via spectral or L_2 -estimates. As an application, we combine our results on the modified log-Sobolev inequality and approximate tensorization to derive tight bounds for local generators.

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1. Introduction and Main Results

Relative entropy is a fundamental information measure that has been widely used in probability, statistics and information theory. It was first introduced by Kullback and Leibler [52] for probability distributions (also called KL-divergence), and later extended by Umegaki [79] to the noncommutative setting for quantum states. For two quantum states with density matrices ρ and σ , the relative entropy of ρ with respect to σ is defined as

$$D(\rho\|\sigma) = \text{tr}(\rho \ln \rho - \rho \ln \sigma) , \tag{1}$$

where tr is the matrix trace. When ρ and σ share a same eigenbasis, (1) recovers the KL-divergence for two (discrete) probability densities. In both classical and quantum cases, $D(\rho\|\sigma)$ measures how well the classical or quantum state ρ can be distinguished from σ by statistical or quantum-mechanical experiments [13, 43, 65]. In this work, we study several related inequalities of the quantum relative entropy which have direct applications in quantum information theory and quantum many-body systems. Some of our results also yield new insights in the classical cases for probability distributions.

1.1. Modified Logarithmic Sobolev Inequality

The logarithmic Sobolev inequality is a functional inequality that was first introduced by Gross in his study of quantum field theory [40] as an equivalent formulation of hypercontractivity [64]. Over the past several decades, logarithmic Sobolev inequalities have been intensively studied for their applications in analysis, probability and information theory (see e.g. the [41, 54] and the references therein). Let (Ω, μ) be a probability space and $(\mathcal{T}_t : L_\infty(\Omega, \mu) \rightarrow L_\infty(\Omega, \mu))_{t \geq 0}$ be a Markov semigroup with the unique invariant measure μ . The semigroup $(\mathcal{T}_t)_{t \geq 0}$ is said to satisfy the α -logarithmic Sobolev inequality (in short, α -LSI) for $\alpha > 0$ if

$$\int f^2 \ln f^2 d\mu - \left(\int f^2 d\mu \right) \ln \left(\int f^2 d\mu \right) \leq -\frac{1}{\alpha} \int f(\mathcal{L}f) d\mu \tag{2}$$

for any real function f in the domain of \mathcal{L} , where \mathcal{L} is the generator of the semigroup, i.e. $\mathcal{T}_t = e^{t\mathcal{L}}$. It is well-known that the logarithmic Sobolev inequality admits a (weaker) variant formulation, called *modified logarithmic Sobolev inequality*, which is directly related to the relative entropy. The semigroup $(\mathcal{T}_t)_{t \geq 0}$ is said to satisfy the α -modified logarithmic Sobolev inequality (in short, α -MLSI) for $\alpha > 0$ if, for any probability density $f \geq 0$ with $\int f d\mu = 1$,

$$\alpha \int f \ln f d\mu \leq - \int \mathcal{L}(f) \ln f d\mu. \tag{3}$$

The left hand side is the (classical) entropy functional $\text{Ent}(f) := \int f \ln f d\mu$. The α -MLSI (3) is equivalent to

$$\text{Ent}(\mathcal{T}_t f) \leq e^{-\alpha t} \text{Ent}(f),$$

which means that the entropy of the system decays exponentially. This entropic convergence property is a powerful tool to derive mixing times for the semigroup.

The main purpose of this work is to study modified logarithmic Sobolev inequalities for quantum Markov semigroups. Quantum Markov semigroups are non-commutative generalizations of Markov semigroups where the underlying function spaces are replaced by matrix algebras or operator algebras. Let \mathcal{H} be a finite dimensional Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the bounded operators on \mathcal{H} . A quantum Markov semigroup (QMS) $(\mathcal{P}_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}))_{t \geq 0}$ is a continuous semigroup of completely positive trace preserving maps. Such continuous-time families of quantum channels model the Markovian evolution of dissipative open quantum systems. In recent years, the connection between logarithmic Sobolev inequalities and other functional inequalities, such as hypercontractivity, Poincaré inequality and transport cost inequalities, have been largely extended to quantum Markov semigroup (see [21, 22, 29, 50, 66, 74]). Some of these works found direct applications in quantum information and quantum computational complexity (see e.g. [11, 16, 63]).

Despite the rich connections to many aspects of quantum Markov processes, logarithmic Sobolev inequalities in the quantum framework are missing one key property—the tensorization property. For two classical Markov semigroups $(\mathcal{S}_t)_{t \geq 0}$ and $(\mathcal{T}_t)_{t \geq 0}$, if each semigroup satisfies α -MLSI, then $(\mathcal{S}_t \otimes \mathcal{T}_t)_{t \geq 0}$ also satisfies α -MLSI [14] with the same constant α . Tensorization is a powerful property that allow us to obtain MLSI for large, composite systems in terms of the dynamics on smaller subsystems, which is a technique that was already used by Gross in his very first work on the logarithmic Sobolev inequality. Nevertheless, the tensor stability of MLSI fails for general (non-ergodic) quantum Markov semigroups [17, Proposition 4.21]. The lack of tensorization property is a common difficulty in quantum information (see e.g. the super-additivity of the channel capacity [42, 75]). On the other hand, it was discovered in [37] that the tensorization property is satisfied with a stronger definition of MLSI: a quantum Markov semigroup $(\mathcal{P}_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}))_{t \geq 0}$ is said to satisfy the α -complete modified logarithmic Sobolev inequality (in short, α -CMLSI) if for any $n \geq 1$, the amplification $\mathcal{P}_t \otimes \text{id}_n$ satisfies α -MLSI, where id_n is the identity map on an n -dimensional quantum system. Our first main result shows that such tensor stable modified log-Sobolev inequalities generically hold in finite dimensions.

Theorem 1.1. *Let $(\mathcal{P}_t)_{t \geq 0}$ be a quantum Markov semigroup and assume $(\mathcal{P}_t)_{t \geq 0}$ is GNS-symmetric to some full-rank invariant state σ . Denote $E_* = \lim_{t \rightarrow \infty} \mathcal{P}_{t*}$ as the limit of the pre-adjoint map, which gives the projection onto the fixed point space of \mathcal{P}_{t*} . Then for all $n \in \mathbb{N}$ and all states $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$,*

$$D(\mathcal{P}_{t*} \otimes \text{id}_n(\rho) \| E_* \otimes \text{id}_n(\rho)) \leq e^{-\alpha t} D(\rho \| E_* \otimes \text{id}_n(\rho)) , \tag{CMLSI}$$

where $D(\cdot \| \cdot)$ denotes the relative entropy and the constant α satisfies

$$\frac{\lambda}{C_{\text{cb}}(E_*)} \leq \alpha \leq 2\lambda .$$

Here λ is the spectral gap of the generator \mathcal{L} of the QMS and $C_{\text{cb}}(E_*)$ is the complete Pimsner–Popa index of the map E_* .

We refer to Section 3 for details concerning the spectral gap and GNS symmetry, and to Section 2 for the index $C_{\text{cb}}(E_*)$. We emphasise that Theorem 1.1 asserts that there exists an exponential decay rate α for the relative entropy independent of $n \in \mathbb{N}$, and which holds not only for the semigroup $(\mathcal{P}_t)_{t \geq 0}$ itself but also for its amplifications $(\mathcal{P}_t \otimes \text{id}_n)_{t \geq 0}$ coupling it to an environment system \mathbb{C}^n . This definition was introduced in [37], and proved to satisfy the tensorization property: whenever two quantum Markov semigroups satisfy α -CMLSI, their tensor product satisfies α -CMLSI. Later, Li, Junge and LaRacuente [56] proved that the heat semigroup of Riemannian manifolds of positive curvature and all classical (continuous-time) finite Markov chains satisfy CMLSI. Using the noncommutative curvature lower bound introduced in [21, 29], CMLSI was obtained for heat semigroup on all compact Riemannian manifolds and some examples from operator algebras [17, 18]. Despite the constant progress on this topic in the recent years, the problem of the positivity of the CMLSI constant for finite dimensional QMS has been left open. Here, Theorem 1.1 finally provides a positive answer to the question via a relatively simple proof.

1.2. Strong Data Processing Inequality

One key property behind the widespread applications of the quantum relative entropy is the data processing inequality. It states that the relative entropy is non-increasing under the action of a quantum channel Φ (complete positive trace perserving map). Namely, for all states ρ and σ ,

$$D(\Phi(\rho) \parallel \Phi(\sigma)) \leq D(\rho \parallel \sigma). \quad (4)$$

As the relative entropy is a measure of distinguishability, the data processing inequality asserts that two states can not become more distinguishable after applying a same channel to them. First proved by Lindblad [60] and Uhlmann [78], the data processing inequality for the relative entropy has been largely refined and improved in recent years (e.g. [23, 48, 62]). As discussed in [12, 44, 55, 63], one natural refinement of the inequality consists in asking when the contraction of the relative entropy observed in (4) can be strict, i.e. whether there exists a constant $c < 1$ such that

$$D(\Phi(\rho) \parallel \Phi(\sigma)) \leq c D(\rho \parallel \sigma). \quad (5)$$

This question has been intensively studied for classical channels and more general entropies (see e.g. [3, 26, 32, 33, 59, 71, 73] and the references therein) under the name *strong data processing inequality* (SDPI). In the quantum setting, despite progresses on some special cases [44, 63], the existence of a contractive coefficient for general channels in (5) remains open. Our second main result is the following strong data processing inequality as a discrete time analog of Theorem 1.1:

Theorem 1.2. (c.f. Corollary 4.3) *Let $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a quantum channel. Suppose Φ is GNS-symmetric to a full-rank invariant state $\sigma = \Phi(\sigma)$. Then there exists an explicit constant $c < 1$ such that for any $n \in \mathbb{N}$ and all bipartite states $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathbb{C}^n)$,*

$$D((\Phi \otimes \text{id}_n)(\rho) \| (\Phi \circ E_* \otimes \text{id}_n)(\rho)) \leq c D(\rho \| (E_* \otimes \text{id}_n)(\rho)), \quad (\text{CSDPI})$$

where E_* is the projection onto the decoherence-free space of Φ .

We refer to Section 4 for the definition of E_* and remark that the constant c explicitly depends on the index $C_{\text{cb}}(E_*)$ and an L_2 -condition $\lambda := \|\Phi - E_* : L_2 \rightarrow L_2\|$. The above inequality (CSDPI) implies a discrete time entropy decay. Moreover, the inequality (CSDPI) again gives a uniform control for all amplifications $\Phi \otimes \text{id}_n$, which is the reason that we call it *complete strong data processing inequality* (CSDPI). These improvements over the standard data processing inequality have applications to quantum state preparation and quantum channel capacities [9, 19]. For instance, similarly to CMLSI, CSDPI admits tensorization: if two quantum channels Φ and Ψ satisfy CSDPI with contraction coefficient $c < 1$, so does $\Phi \otimes \Psi$. Also, thanks to its “completeness”, Theorem 1.2 implies a concrete estimate on the convergence $\Phi^n \rightarrow \Phi^n \circ E_*$ in terms of the diamond norm.

1.3. Approximate Tensorization of Relative Entropy

The data processing inequality is closely related to another celebrated inequality in quantum information theory, namely the *strong subadditivity* (SSA). SSA can be equivalently stated in terms of relative entropies as follows: for any tripartite state ρ^{ABC} ,

$$D\left(\rho^{ABC} \left\| \frac{\mathbb{1}_{AB}}{d_{AB}} \otimes \rho^C\right.\right) \leq D\left(\rho^{ABC} \left\| \frac{\mathbb{1}_A}{d_A} \otimes \rho^{BC}\right.\right) + D\left(\rho^{ABC} \left\| \frac{\mathbb{1}_B}{d_B} \otimes \rho^{AC}\right.\right).$$

here $\frac{\mathbb{1}_{AB}}{d_{AB}}$ is the completely mixed state on AB whereas ρ^C denotes the reduced density on C (and similarly for the other terms). SSA was long known in classical information theory, and proved by Lieb and Ruskai [58] for the quantum entropy. Later Petz [68] proved SSA in a very general setting: given any four matrix subalgebras $\mathcal{N} \subset \mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{M}$, and corresponding projections $E_1, E_2, E_{\mathcal{N}}$ from \mathcal{M} onto $\mathcal{N}_1, \mathcal{N}_2$ and \mathcal{N} , for all states ρ on \mathcal{M} , it holds that

$$D(\rho \| E_{\mathcal{N}*}(\rho)) \leq D(\rho \| E_{1*}(\rho)) + D(\rho \| E_{2*}(\rho)), \quad (6)$$

as long as $E_1 \circ E_2 = E_2 \circ E_1 = E_{\mathcal{N}}$. This last commutation relation is usually referred to as a “*commuting square*” condition and was introduced by Popa [72].

Although the commuting square gives a nice characterization of SSA, SSA-type inequalities are also desired when the “commuting square” condition is not fully satisfied. For instance, in the context of classical lattice spin systems, where the projections are conditional expectations onto different regions of the lattice with respect to a given Gibbs measure, the commuting square condition corresponds to the infinite temperature regime [8]. To assess the finite temperature regime, (6) has

to be modified in the following way [24,27]: there exists a constant $c > 1$ such that, for all states ρ ,

$$D(\rho \| E_{\mathcal{N}^*}(\rho)) \leq c (D(\rho \| E_{1^*}(\rho)) + D(\rho \| E_{2^*}(\rho))), \quad (7)$$

where the constant c is some measure of the violation of the commutation relation $\|E_1 \circ E_2 - E_{\mathcal{N}}\|$ in some appropriate norm. This inequality, called *approximate tensorization of the relative entropy*, was used in the classical case (i.e. when all algebras are commutative) in the study of logarithmic Sobolev inequalities for lattice spin system [24]. In the quantum setting, a weaker bound to (7) was derived in [8] with a further additive error term vanishing on classical states. However, the question of finding general bounds like (7) without the additive error term was left unresolved. Our third main theorem answers this question.

Theorem 1.3. *Let $\mathcal{N} \subset \mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{M}$ be four finite dimensional von Neumann algebras. Let $E_1, E_2, E_{\mathcal{N}}$ be the corresponding projections from \mathcal{M} onto $\mathcal{N}_1, \mathcal{N}_2$ and \mathcal{N} such that $E_{\mathcal{N}} \circ E_1 = E_{\mathcal{N}} \circ E_2 = E_{\mathcal{N}}$. Then there exists an explicit constant c_{cb} such that, for any $n \in \mathbb{N}$ and all states $\rho \in \mathcal{M} \otimes \mathcal{B}(\mathbb{C}^n)$, we have*

$$D(\rho \| (E_{\mathcal{N}^*} \otimes \text{id})(\rho)) \leq c_{\text{cb}} (D(\rho \| (E_{1^*} \otimes \text{id})(\rho)) + D(\rho \| (E_{2^*} \otimes \text{id})(\rho))). \quad (8)$$

We refer to Theorems 5.1 and Corollary 5.4 for concrete estimates on the constant c_{cb} . All of the three results above rely on a common conceptually simple tool, namely a two-sided estimate of the relative entropy via the so-called *Bogoliubov-Kubo-Mori Fisher information* (see Lemma 2.2 in Section 2 for more details). The Bogoliubov-Kubo-Mori Fisher information is closely related to a special case of monotone Riemannian metric on the state space studied in [55,69] and a quantum χ_2 -divergence studied in [82]. This allows us to approach each of the three above entropic inequalities via corresponding spectral gap conditions. Given the simplicity of our approach, we believe it will also prove useful in the study of other entropic inequalities.

1.4. Applications and Examples

Based on the above results, we exploit the approximate tensorization estimate from Theorem 1.3 to get tighter bounds on the optimal CMLSI constant for quantum Markov semigroups (QMS) relevant to the communities of mathematical physics and quantum information theory. For a QMS $(\mathcal{P}_t = e^{t\mathcal{L}})_{t \geq 0}$ with the generator \mathcal{L} , we denote by $\alpha_{\text{CMLSI}}(\mathcal{L})$ the largest constant α satisfying (CMLSI) in Theorem 1.1. In Section 6, we restrict our analysis to the class of symmetric QMS, that is QMS symmetric to the trace inner product or equivalently the maximally mixed state. The generators of these semigroups admit a simple form as a sum of double commutators with self-adjoint operators $\{a_k\}$:

$$\mathcal{L}(\rho) = - \sum_{k=1}^l [a_k, [a_k, \rho]]. \quad (9)$$

Using approximate tensorization, we obtain the following improved CMLSI constant for symmetric QMS:

Theorem 1.4. (c.f. Corollary 6.3) *For a symmetric generator \mathcal{L} given as above,*

$$\alpha_{\text{CMLS}}(\mathcal{L}) \geq \Omega(\lambda^2 \text{polylog}(d_{\mathcal{H}})^{-1}),$$

where $d_{\mathcal{H}}$ is the dimension of the underlying Hilbert space, and $\lambda := \min_k \lambda(\mathcal{L}_{a_k})$ is the minimum spectral gap of any of the generators $\mathcal{L}_{a_k}(\rho) = [a_k, [a_k, \rho]]$.

Note that the above bound is asymptotically better than Theorem 1.1 because the index is $C_{\text{cb}}(E_*) = d_{\mathcal{H}}^2$ for primitive semigroups.

Example 1.5. Consider the quantum Markov semigroups induced by sub-Laplacians of the special unitary group $\text{SU}(2)$ on its irreducible representations:

$$\mathcal{L}_m^H(\rho) := -[X_m, [X_m, \rho]] - [Y_m, [Y_m, \rho]].$$

Here X_m (resp. Y_m) is the spin- $\frac{m-1}{2}$ representation of the Pauli X matrix (resp. Y -matrix). In contrast to the induced semigroup of the standard Laplace-Beltrami operator $\Delta = X^2 + Y^2 + Z^2$ the CMLS constant of \mathcal{L}_m^H is not accessible from the corresponding classical Markov semigroup due to the lack of curvature lower bound in the sub-Riemannian setting. With help of numerics, we obtain that

$$\alpha_{\text{CMLS}}(\mathcal{L}_m^H) > 0.18$$

uniformly for all $m \geq 2$.

In Section 6.2, we focus on symmetric semigroups which bare a locality structure inherited from a graph. More precisely, given a finite graph $G = (V, E)$, we consider the n -fold tensor product $\mathcal{H}_V := \bigotimes_{v \in V} \mathcal{H}_v$ of a finite dimensional local Hilbert space \mathcal{H} , namely, an n -qudit system for $d = \dim(\mathcal{H})$. The Lindblad operators are supported on the edges $e \in E$ of the graph

$$\mathcal{L}_G := \sum_{e \in E} \mathcal{L}_e, \quad \text{where} \quad \mathcal{L}_e(\rho) := \sum_{j \in J^{(e)}} L_j^{(e)} \rho L_j^{(e)} - \frac{1}{2} \{L_j^{(e)} L_j^{(e)}, \rho\}, \quad (10)$$

where for any edge $e \in (v, w) \in E$ and any $j \in J^{(e)}$, the local Lindblad operator $L_j^{(e)}$ acts trivially on subsystems other than $\mathcal{H}_v \otimes \mathcal{H}_w$. We call (10) a *subsystem Lindbladian*, which means that the global dynamics consists of local interactions on subsystems of adjacent vertices. This gives a general model of 2-local interacting quantum lattice spin systems. Using approximate tensorization again, we provide a lower bounds on the CMLS constant for the global Lindbladian \mathcal{L}_G based on the local Lindbladians \mathcal{L}_e .

Theorem 1.6. (c.f. Theorem 6.6) *Let $G = (V, E)$ be a finite, connected graph of maximum degree γ and let \mathcal{L}_G be a symmetric subsystem Lindbladian of the form (10). Denote by E_e the projection onto the kernel of the local Lindbladian \mathcal{L}_e . Then*

$$\alpha_{\text{CMLS}}(\mathcal{L}_G) \geq \Omega \left(\frac{\ln \left(\frac{\lambda(\tilde{\mathcal{L}}_G)}{4(\gamma-1)^2} + 1 \right)}{\ln(C) + 1} \right) \min_{e \in E} \alpha_{\text{CMLS}}(\mathcal{L}_e),$$

where $\alpha_{\text{CMLS}}(\mathcal{L}_e)$ is the CMLS constant of \mathcal{L}_e , and $\lambda(\tilde{\mathcal{L}}_G)$ is the spectral gap of the generator $\tilde{\mathcal{L}}_G := \sum_{e \in E} E_e - \text{id}$.

Here the index C can be chosen as either the complete Pimsner–Popa index [70] of the algebra \mathcal{N} of fixed points of the evolution, or the inverse minimal eigenvalue of the Choi state of the projection map $E_G := \lim_{t \rightarrow \infty} e^{t\mathcal{L}_G}$. The index C can be thought of as what replaces the size of the graph in the case of classical graph Laplacians. In particular, for expander graphs, our bound gives

$$\alpha_{\text{CMSLI}}(\tilde{\mathcal{L}}_G) \geq \Omega\left(\frac{1}{\ln(C)}\right).$$

We exemplify our bound on the random transposition model.

Example 1.7. (Random transposition) Motivated by the classical random transposition model in [14,35], we introduce the quantum nearest neighbor random transposition via the local Lindbladian on an edge $(i, j) \in E$ given by

$$\mathcal{L}_{(i,j)}(\rho) := \frac{1}{2}(S_{i,j}\rho S_{i,j} - \rho), \quad S_{i,j}(|\psi\rangle \otimes |\varphi\rangle) = |\varphi\rangle \otimes |\psi\rangle, \quad (11)$$

where $S_{i,j} : \mathcal{H}_i \otimes \mathcal{H}_j \rightarrow \mathcal{H}_i \otimes \mathcal{H}_j$ is the swap unitary gate between vertex i and j . Then the global Lindbladian $\mathcal{L}_G^{\text{NNRT}} := \sum_{e \in E} \mathcal{L}_e$ is generated by local random swaps on $|V| = n$ qudits. In this case, we find that

$$\alpha_{\text{CMSLI}}(\mathcal{L}_G^{\text{NNRT}}) \geq \lambda(\mathcal{L}_G^{\text{NNRT}}) \Omega((\ln n!)^{-1}),$$

where $\lambda(\mathcal{L}_G^{\text{NNRT}})$ is the spectral gap and the factorial $n!$ is the size of the permutation group \mathcal{S}_n . This presents an exponential improvement over the bounds from Theorem 1.1, where the constant was controlled by the inverse size of the group $(n!)^{-1}$.

The rest of the paper is organized as follows: in the next section, we review some preliminary definitions and prove our key lemma. Section 3 is devoted to the proof of Theorem 1.1, which is our first main result on the complete modified log-Sobolev inequality. In Section 4, we prove the complete strong data processing inequality of Theorem 1.2. The approximate tensorization results are discussed in Section 5. Section 6 provides the improved CMSLI constant of Theorem 1.4 for symmetric quantum Markov semigroups. In Section 6.2, we discuss examples from subsystem Lindbladians. We end the paper with some discussion on questions that remain open. We remark that although we restrict our discussion to finite dimensions, the general results in Sections 3, 4, and 5 can be extended to (trace) symmetric maps in the setting of finite von Neumann algebras, as long as the index $C_{\text{cb}}(E)$ is finite and the corresponding spectral gap condition is satisfied. For more examples and applications, we invite the interested reader to consult a longer version of the present article available at [39].

Notations. We denote \mathcal{H} as a Hilbert space, $\mathcal{B}(\mathcal{H})$ as the bounded operators on \mathcal{H} , and $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ as a von Neumann subalgebra. We write “tr” for the standard matrix trace, $\langle \cdot, \cdot \rangle_{\text{HS}}$ for the trace inner product and $\| \cdot \|_2$ for the Hilbert–Schmidt norm. The corresponding Hilbert–Schmidt space (resp. trace class operators) is denoted by $\mathcal{T}_2(\mathcal{H})$ (resp. $\mathcal{T}_1(\mathcal{H})$). Operators will be denoted by capital letters A, X, Y, \dots , and states or density operators are denoted by Greek letters $\rho, \sigma, \omega, \dots$. Sometimes

we will also use lowercase letters to emphasize their belonging to a subalgebra. We write A^\dagger for the adjoint of an operator $A \in \mathcal{B}(\mathcal{H})$, and Φ^* (or Φ_*) for the adjoint (or pre-adjoint) of a map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$. The identity operator on \mathcal{H} is denoted as $\mathbb{1}_{\mathcal{H}}$ and the identity map on a von Neumann subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is $\text{id}_{\mathcal{M}}$. We also denote the dimension of \mathcal{H} by $d_{\mathcal{H}} = \dim(\mathcal{H})$. Given two maps $\Phi, \Psi : \mathcal{M} \rightarrow \mathcal{M}$ on a von Neumann subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, we write $\Phi \leq_{\text{cp}} \Psi$ if $\Psi - \Phi$ is completely positive. Given a subalgebra $\mathcal{N} \subset \mathcal{M}$, we will use $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ for a conditional expectation onto \mathcal{N} and write $C(E_{\mathcal{N}})$ for the corresponding Pimsner–Popa index (see Section 2). We will also write $C(E_{\mathcal{N}}) = C_{\tau}(\mathcal{M} : \mathcal{N})$ where τ is the special operator (17) uniquely determined by $E_{\mathcal{N}}$. When $E_{\mathcal{N}}$ is trace preserving and $\tau = \frac{1}{d_{\mathcal{H}}}$, we omit τ and write the original Pimsner–Popa index as $C(\mathcal{M} : \mathcal{N})$.

2. Preliminaries

2.1. Relative Entropy and Conditional Expectation

Throughout the paper, we will consider \mathcal{H} to be a finite dimensional Hilbert space. We say that an operator $\rho \in \mathcal{B}(\mathcal{H})$ is a state (or density operator) if $\rho \geq 0$ and $\text{tr}(\rho) = 1$. We denote by $\mathcal{D}(\mathcal{H})$ the set of states on \mathcal{H} . A quantum channel $\Phi : \mathcal{T}_1(\mathcal{H}) \rightarrow \mathcal{T}_1(\mathcal{H})$ (or more generally, $\Phi : \mathcal{M}_* \rightarrow \mathcal{M}_*$) is a completely positive trace preserving map. With slight abuse of notation, we will often write $\Psi(\rho) := (\Psi \otimes \text{id})(\rho)$ for a bipartite state $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathbb{C}^n)$ and a quantum channel $\Psi : \mathcal{T}_1(\mathcal{H}) \rightarrow \mathcal{T}_1(\mathcal{H})$. For two states ρ and σ , their relative entropy is defined as

$$D(\rho \parallel \sigma) = \begin{cases} \text{tr}(\rho \ln \rho - \rho \ln \sigma), & \text{if } \text{supp}(\rho) \leq \text{supp}(\sigma) \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\text{supp}(\rho)$ (resp. $\text{supp}(\sigma)$) is the support projection of ρ (resp. σ).

Let $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be two von Neumann subalgebras. Recall that a conditional expectation onto \mathcal{N} is a completely positive unital map $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ satisfying that

- (i) for all $a \in \mathcal{N}$, $E_{\mathcal{N}}(a) = a$
- (ii) for all $a, b \in \mathcal{N}$, $X \in \mathcal{B}(\mathcal{H})$, $E_{\mathcal{N}}(aXb) = aE_{\mathcal{N}}(X)b$.

We denote by $E_{\mathcal{N}*}$ its adjoint map with respect to the trace inner product, i.e.

$$\text{tr}(E_{\mathcal{N}*}(X)Y) = \text{tr}(XE_{\mathcal{N}}(Y)) .$$

For a state ρ , the relative entropy with respect to the conditional expectation $E_{\mathcal{N}*}$ is given by

$$D(\rho \parallel E_{\mathcal{N}*}(\rho)) = \inf_{E_{\mathcal{N}*}(\sigma) = \rho} D(\rho \parallel \sigma) ,$$

where the infimum is always attained by $E_{\mathcal{N}*}(\rho)$. We note that $D(\rho \parallel E_{\mathcal{N}*}(\rho))$ depends not only on the subalgebra \mathcal{N} , but also on the conditional expectation $E_{\mathcal{N}}$,

which is not unique for a subalgebra \mathcal{N} . Indeed, for any σ satisfying $E_{\mathcal{N}^*}(\sigma) = \sigma$, we have the chain rule (see [46, Lemma 3.4])

$$D(\rho\|\sigma) = D(\rho\|E_{\mathcal{N}^*}(\rho)) + D(E_{\mathcal{N}^*}(\rho)\|\sigma) . \tag{12}$$

Hence the infimum is attained if and only if $D(E_{\mathcal{N}^*}(\rho)\|\sigma) = 0$. More explicitly, a finite dimensional von Neumann (sub)algebra can always be expressed as a direct sum of matrix algebras with multiplicity, i.e.

$$\mathcal{N} = \bigoplus_{i=1}^n \mathcal{B}(\mathcal{H}_i) \otimes \mathbb{C}\mathbb{1}_{\mathcal{K}_i} , \quad \mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i \otimes \mathcal{K}_i .$$

Denote P_i as the projection onto $\mathcal{H}_i \otimes \mathcal{K}_i$. There exists a family of density operators $\tau_i \in \mathcal{D}(\mathcal{K}_i)$ such that

$$\begin{aligned} E_{\mathcal{N}}(X) &= \bigoplus_{i=1}^n \text{tr}_{\mathcal{K}_i}(P_i X P_i (\mathbb{1}_{\mathcal{K}_i} \otimes \tau_i)) \otimes \mathbb{1}_{\mathcal{K}_i} , \\ E_{\mathcal{N}^*}(\rho) &= \bigoplus_{i=1}^n \text{tr}_{\mathcal{K}_i}(P_i \rho P_i) \otimes \tau_i , \end{aligned} \tag{13}$$

where $\text{tr}_{\mathcal{K}_i}$ is the partial trace with respect to \mathcal{K}_i . A state σ satisfies $E_{\mathcal{N}^*}(\sigma) = \sigma$ if and only if

$$\sigma = \bigoplus_{i=1}^n p_i \sigma_i \otimes \tau_i$$

for some density operators $\sigma_i \in \mathcal{D}(\mathcal{H}_i)$ and a probability distribution $\{p_i\}_{i=1}^n$. We denote $\mathcal{D}(E_{\mathcal{N}}) := \{\sigma \in \mathcal{D}(\mathcal{H}) \mid \sigma = E_{\mathcal{N}^*}(\sigma)\}$ as the subset of states that are invariant under $E_{\mathcal{N}^*}$. For any $\sigma \in \mathcal{D}(E_{\mathcal{N}})$ and all $X \in \mathcal{M}$,

$$E_{\mathcal{N}^*}(\sigma^{\frac{1}{2}} X \sigma^{\frac{1}{2}}) = \sigma^{\frac{1}{2}} E_{\mathcal{N}}(X) \sigma^{\frac{1}{2}} .$$

2.2. Subalgebra Index and Max-relative Entropy

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a finite dimensional von Neumann algebra and let $\mathcal{N} \subset \mathcal{M}$ be a subalgebra of \mathcal{M} . The trace preserving conditional expectation $E_{\mathcal{N},\text{tr}} : \mathcal{M} \rightarrow \mathcal{N}$ is defined so that, for any $X \in \mathcal{M}$ and $Y \in \mathcal{N}$,

$$\text{tr}(XY) = \text{tr}(E_{\mathcal{N},\text{tr}}(X)Y) .$$

$E_{\mathcal{N},\text{tr}}$ is self-adjoint and corresponds to taking $\tau_i = d_{\mathcal{K}_i}^{-1} \mathbb{1}_{\mathcal{K}_i}$ in (13). We recall the definition of the index associated to the algebra inclusion $\mathcal{N} \subset \mathcal{M}$,

$$\begin{aligned} C(\mathcal{M} : \mathcal{N}) &= \inf\{c > 0 \mid \rho \leq c E_{\mathcal{N},\text{tr}}(\rho) \text{ for all states } \rho \in \mathcal{M}\} , \\ C_{\text{cb}}(\mathcal{M} : \mathcal{N}) &= \sup_{n \in \mathbb{N}} C(\mathcal{M} \otimes \mathbb{M}_n : \mathcal{N} \otimes \mathbb{M}_n) , \end{aligned}$$

where the supremum in $C_{\text{cb}}(\mathcal{M} : \mathcal{N})$ is taken over all finite dimensional matrix algebras \mathbb{M}_n . The index $C(\mathcal{M} : \mathcal{N})$ was first introduced by Pimsner and Popa in [70] for the connection to subfactor index and Connes entropy, and the completely bounded version $C_{\text{cb}}(\mathcal{M} : \mathcal{N})$ was studied in [38]. In particular, it was proved in [38, Theorem 3.9] that $C_{\text{cb}}(\mathcal{M} : \mathcal{N})$ is indeed some completely bounded norm (see (69) in “Appendix B”).

These indices are closely related to the notion of maximal relative entropy. Recall that for two states, ρ, ω , their maximal relative entropy is [28]

$$D_{\max}(\rho \|\omega) = \ln \inf \{ c > 0 \mid \rho \leq c \omega \} .$$

Indeed,

$$\ln C(\mathcal{M} : \mathcal{N}) = \sup_{\rho \in \mathcal{D}(E_{\mathcal{M}, \text{tr}})} D_{\max}(\rho \|\| E_{\mathcal{N}, \text{tr}}(\rho)) .$$

For all finite dimensional inclusion $\mathcal{N} \subset \mathcal{M}$, the index $C(\mathcal{M} : \mathcal{N})$ is explicitly calculated in [70, Theorem 6.1] (hence also for $C_{\text{cb}}(\mathcal{M} : \mathcal{N})$). In particular, for $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{N} = \bigoplus_{i=1}^n \mathcal{B}(\mathcal{H}_i) \otimes \mathbb{C} \mathbb{1}_{\mathcal{K}_i}$,

$$C(\mathcal{B}(\mathcal{H}) : \mathcal{N}) = \sum_{i=1}^n \min \{ d_{\mathcal{H}_i}, d_{\mathcal{K}_i} \} d_{\mathcal{K}_i}, \quad C_{\text{cb}}(\mathcal{B}(\mathcal{H}) : \mathcal{N}) = \sum_{i=1}^n d_{\mathcal{K}_i}^2 . \quad (14)$$

For example, if we take $\mathcal{D} \subset \mathcal{B}(\mathcal{H})$ to be the subalgebra of diagonal matrices and \mathbb{C} as the multiple of identity

$$\begin{aligned} C(\mathcal{B}(\mathcal{H}) : \mathcal{D}) &= C_{\text{cb}}(\mathcal{B}(\mathcal{H}) : \mathcal{D}) = d_{\mathcal{H}} , \\ C(\mathcal{B}(\mathcal{H}) : \mathbb{C}) &= d_{\mathcal{H}}, \quad C_{\text{cb}}(\mathcal{B}(\mathcal{H}) : \mathbb{C}) = d_{\mathcal{H}}^2 . \end{aligned} \quad (15)$$

In this paper, we will also consider the index for a general conditional expectation $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ (see e.g [51] for more information). For a conditional expectation $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ onto \mathcal{N} , we define

$$\begin{aligned} C(E_{\mathcal{N}}) &= \inf \{ c > 0 \mid \rho \leq c E_{\mathcal{N}*}(\rho) \text{ for all states } \rho \in \mathcal{M} \} , \\ C_{\text{cb}}(E_{\mathcal{N}}) &= \sup_{n \in \mathbb{N}} C(E_{\mathcal{N}} \otimes \text{id}_{\mathbb{M}_n}) . \end{aligned} \quad (16)$$

Here, we recall that \mathbb{M}_n is the n -dimensional matrix algebra and $E_{\mathcal{N}} \otimes \text{id}_{\mathbb{M}_n}$ is a conditional expectation from $\mathcal{M} \otimes \mathbb{M}_n \rightarrow \mathcal{N} \otimes \mathbb{M}_n$. Note that given the subalgebra \mathcal{N} , $E_{\mathcal{N}}$ and $E_{\mathcal{N}*}$ are uniquely determined by any invariant state $\sigma \in \mathcal{D}(E_{\mathcal{N}})$, or equivalently the densities $\{\tau_i\}$ in (13). Indeed, denoting

$$\tau = \bigoplus_{i=1}^n \mathbb{1}_{\mathcal{H}_i} \otimes \tau_i, \quad (17)$$

we have

$$E_{\mathcal{N}}(X) = E_{\mathcal{N}, \text{tr}}(\tau^{\frac{1}{2}} X \tau^{\frac{1}{2}}), \quad E_{\mathcal{N}*}(\rho) = \tau^{\frac{1}{2}} E_{\mathcal{N}, \text{tr}}(\rho) \tau^{\frac{1}{2}}. \quad (18)$$

In particular, $E_{\mathcal{N}}$ is faithful if and only if τ is full-rank. By definition, the Pimsner–Popa index $C(\mathcal{M} : \mathcal{N})$ is the special case for the trace preserving condition expectation $C(E_{\mathcal{N},\text{tr}})$. In the later discussion, we will often use the alternative notation

$$C_{\tau}(\mathcal{M} : \mathcal{N}) := C(E_{\mathcal{N}}) , \quad C_{\tau,\text{cb}}(\mathcal{M} : \mathcal{N}) := C_{\text{cb}}(E_{\mathcal{N}}).$$

Since τ commutes with \mathcal{N} ,

$$\begin{aligned} C_{\tau}(\mathcal{M} : \mathcal{N}) &\leq \mu_{\min}(\tau)^{-1} C(\mathcal{M} : \mathcal{N}), \\ C_{\tau,\text{cb}}(\mathcal{M} : \mathcal{N}) &\leq \mu_{\min}(\tau)^{-1} C_{\text{cb}}(\mathcal{M} : \mathcal{N}), \end{aligned} \tag{19}$$

where $\mu_{\min}(\tau) = \min_i \mu_{\min}(\tau_i)$ is the minimal eigenvalue of τ . Hence in finite dimensions, both $C(E_{\mathcal{N}})$ and $C_{\text{cb}}(E_{\mathcal{N}})$ are finite if and only if $E_{\mathcal{N}}$ is faithful. Moreover, for any invariant state $\sigma \in \mathcal{D}(E_{\mathcal{N}})$, by the obvious bound $\sigma \leq \tau$, we also have

$$\begin{aligned} C_{\tau}(\mathcal{M} : \mathcal{N}) &\leq \mu_{\min}(\sigma)^{-1} C(\mathcal{M} : \mathcal{N}), \\ C_{\tau,\text{cb}}(\mathcal{M} : \mathcal{N}) &\leq \mu_{\min}(\sigma)^{-1} C_{\text{cb}}(\mathcal{M} : \mathcal{N}). \end{aligned} \tag{20}$$

2.3. A Key Lemma

We shall now discuss the key lemma that will be repeatedly used in the later sections. Given a density operator $\rho \in \mathcal{D}(\mathcal{H})$, we define the multiplication operator

$$\Gamma_{\rho}(X) := \int_0^1 \rho^s X \rho^{1-s} ds .$$

Γ_{ρ} is a positive operator on the Hilbert–Schmidt space $\mathcal{T}_2(\mathcal{H}) := L_2(\mathcal{B}(\mathcal{H}), \text{tr})$ and hence induces a weighted L_2 -norm (semi-norm if ρ is not full-rank) defined for $X \in \mathcal{B}(\mathcal{H})$ as

$$\|X\|_{\rho}^2 := \langle X, \Gamma_{\rho}(X) \rangle_{\text{HS}} = \int_0^1 \text{tr}(X^{\dagger} \rho^s X \rho^{1-s}) ds . \tag{21}$$

We denote by $L_2(\rho)$ the corresponding L_2 -space. For a full-rank density ρ , the inverse operator of Γ_{ρ} is given by

$$\Gamma_{\rho}^{-1}(X) := \int_0^{\infty} (\rho + r)^{-1} X (\rho + r)^{-1} dr ,$$

which is the double operator integral for the difference quotient of $f(t) = \ln t$ and operator ρ (see e.g. [21]). We denote by slight abuse of notations the corresponding weighted L_2 -norm as

$$\|X\|_{\rho^{-1}}^2 := \langle X, \Gamma_{\rho}^{-1}(X) \rangle_{\text{HS}} = \int_0^{\infty} \text{tr}(X^{\dagger} (\rho + r)^{-1} X (\rho + r)^{-1}) dr ,$$

and the corresponding L_2 space as $L_2(\rho^{-1})$. Note that the definition $\|\cdot\|_{\rho^{-1}}$ does not amount to plugging in the inverse operator $\sigma = \rho^{-1}$ into the definition of $\|\cdot\|_{\sigma}$

in (21). Namely, in our notations $\|X\|_{\rho^{-1}} \neq \|X\|_{\sigma}$ for $\sigma = \rho^{-1}$. The inverse weighted norm $\|\cdot\|_{\rho^{-1}}$ is closely related to the quantum χ^2 -divergence introduced in [77, Definition 1] for the logarithmic function. It is easy to see that

$$\|\Gamma_{\rho}(X)\|_{\rho^{-1}} = \|X\|_{\rho} \quad \text{and} \quad \|\Gamma_{\rho^{-1}}(X)\|_{\rho} = \|X\|_{\rho^{-1}} .$$

Lemma 2.1. *If $\rho \leq c\sigma$ for any two states ρ, σ and some $c > 0$, then, for any $X \in \mathcal{B}(\mathcal{H})$ and all $\mu_1, \mu_2 > 0$,*

$$\begin{aligned} & \int_0^{\infty} \text{tr}(X^{\dagger}(\mu_1\sigma + r)^{-1}X(\mu_2\sigma + r)^{-1}) \, dr \\ & \leq c \int_0^{\infty} \text{tr}(X^{\dagger}(\mu_1\rho + r)^{-1}X(\mu_2\rho + r)^{-1}) \, dr . \end{aligned}$$

In particular, $\|X\|_{\sigma^{-1}} \leq c \|X\|_{\rho^{-1}}$.

Proof. This is a standard comparison. Using cyclicity of the trace and the fact that $t \mapsto t^{-1}$ is operator anti-monotone,

$$\begin{aligned} & \int_0^{\infty} \text{tr}(X^{\dagger}(\mu_1\rho + r)^{-1}X(\mu_2\rho + r)^{-1}) \, dr \\ & \geq \int_0^{\infty} \text{tr}(X^{\dagger}(c\mu_1\sigma + r)^{-1}X(\mu_2\rho + r)^{-1}) \, dr \\ & \geq \int_0^{\infty} \text{tr}(X^{\dagger}(c\mu_1\sigma + r)^{-1}X(c\mu_2\sigma + r)^{-1}) \, dr \\ & = \int_0^{\infty} \frac{1}{c^2} \text{tr}\left(X^{\dagger}\left(\mu_1\sigma + \frac{r}{c}\right)^{-1}X\left(\mu_2\sigma + \frac{r}{c}\right)^{-1}\right) \, dr \\ & = \frac{1}{c} \int_0^{\infty} \text{tr}(X^{\dagger}(\mu_1\sigma + r)^{-1}X(\mu_2\sigma + r)^{-1}) \, dr . \end{aligned}$$

In the last equality, we used the change of variable $r \rightarrow \frac{r}{c}$. □

Our key lemma is a two-sided estimate of $D(\rho\|\sigma)$ via the inverse weighted norm.

Lemma 2.2. *Let ρ and σ be two full-rank density operators and suppose $\rho \leq c\sigma$ for some $c > 0$. Then*

$$k(c) \|\rho - \sigma\|_{\sigma^{-1}}^2 \leq D(\rho\|\sigma) \leq \|\rho - \sigma\|_{\sigma^{-1}}^2 \tag{22}$$

where $k(c) = \frac{c \ln c - c + 1}{(c - 1)^2}$. Note that $k(c) \leq 1/2$ for $c \geq 1$.

Proof. For the lower bound, we consider $\rho_t := (1 - t)\sigma + t\rho$, $t \in [0, 1]$ and the function $f(t) = D(\rho_t\|\sigma)$. We have $f(0) = 0$, $f(1) = D(\rho\|\sigma)$ and the derivatives

$$\begin{aligned} f'(t) &= \text{tr}((\rho - \sigma) \ln \rho_t - (\rho - \sigma) \ln \sigma) , \\ f''(t) &= \int_0^{\infty} \text{tr}\left((\rho - \sigma) \frac{1}{\rho_t + r} (\rho - \sigma) \frac{1}{\rho_t + r}\right) \, dr = \|\rho - \sigma\|_{\rho_t^{-1}}^2 . \end{aligned}$$

Note that $f'(0) = 0$ and $\rho_t \leq (ct + (1 - t))\sigma$. We have for the lower bound

$$\begin{aligned} D(\rho\|\sigma) &= \int_0^1 \left(\int_0^s f''(t) dt \right) ds \\ &= \int_0^1 \int_0^s \|\rho - \sigma\|_{\rho_t^{-1}}^2 dt ds \\ &\geq \int_0^1 \int_0^s \frac{1}{1 + (c - 1)t} dt ds \|\rho - \sigma\|_{\sigma^{-1}}^2 \\ &\geq k(c) \|\rho - \sigma\|_{\sigma^{-1}}^2, \end{aligned}$$

where we used Lemma 2.1 and

$$k(c) = \int_0^1 \int_0^s \frac{1}{1 + (c - 1)t} dt ds = \frac{c \ln c - c + 1}{(c - 1)^2}.$$

The upper bound is a special case of [77, Proposition 6]. Here we present a different proof using a method similar to our lower bound. Note that $\rho_t = (1 - t)\sigma + t\rho \geq (1 - t)\sigma$. Then,

$$\begin{aligned} D(\rho\|\sigma) &= \int_0^1 \int_0^s \|\rho - \sigma\|_{\rho_t^{-1}}^2 dt ds \\ &\leq \int_0^1 \int_0^s \frac{1}{1 - t} \|\rho - \sigma\|_{\sigma^{-1}}^2 dt ds \\ &= \int_0^1 \int_0^s \frac{1}{1 - t} dt ds \|\rho - \sigma\|_{\sigma^{-1}}^2 = \|\rho - \sigma\|_{\sigma^{-1}}^2. \end{aligned}$$

□

Remark 2.3. Note that the upper bound does not require the assumption $\rho \leq c\sigma$.

Now given a conditional expectation $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$, it follows immediately from the above that for any state ρ and $\rho_{\mathcal{N}} = E_{\mathcal{N}*}(\rho)$,

$$k(C(E_{\mathcal{N}})) \|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 \leq D(\rho\|\rho_{\mathcal{N}}) \leq \|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2, \tag{23}$$

where $C(E_{\mathcal{N}})$ is the index defined in (16). We also have an variant of the lower bound with another weighting state.

Lemma 2.4. *Let ρ, σ and ω be three full-rank density operators and suppose $\rho, \sigma \leq c\omega$ for some $c > 0$. Then*

$$\|\rho - \sigma\|_{\omega^{-1}}^2 \leq 2c D(\rho\|\sigma). \tag{24}$$

Proof. Take $\rho_t = (1 - t)\sigma + t\rho, t \in [0, 1]$. By the assumption and Lemma 2.1, we have $\rho_t \leq c\omega$ and hence

$$c \|\rho - \sigma\|_{\rho_t^{-1}}^2 \geq \|\rho - \sigma\|_{\omega^{-1}}^2$$

for each t . Therefore,

$$\begin{aligned} D(\rho\|\sigma) &= \int_0^1 \int_0^s \|\rho - \sigma\|_{\rho_t^{-1}}^2 dt ds \\ &\geq \int_0^1 \int_0^s \frac{1}{c} \|\rho - \sigma\|_{\omega^{-1}}^2 dt ds \\ &\geq \frac{1}{2c} \|\rho - \sigma\|_{\omega^{-1}}^2 . \end{aligned}$$

□

2.4. Detailed Balance

We shall now discuss the detailed balance condition and its connection to the spectral gap. Given a full-rank state σ and $0 \leq s \leq 1$, we define the multiplication operator

$$\Gamma_{\sigma,s}(X) = \sigma^{1-s} X \sigma^s .$$

$\Gamma_{\sigma,s}$ is a positive operator on the Hilbert–Schmidt space and induces the following weighted inner product

$$\langle X, Y \rangle_{\sigma,s} := \text{tr}(X^\dagger \sigma^{1-s} Y \sigma^s), \quad \|X\|_{\sigma,s}^2 = \langle X, X \rangle_{\sigma,s} .$$

We denote by $L_2(\sigma, s)$ the corresponding L_2 space. A map $\Phi^* : \mathcal{M} \rightarrow \mathcal{M}$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\sigma,s}$ if

$$\Phi \circ \Gamma_{\sigma,s} = \Gamma_{\sigma,s} \circ \Phi^* ,$$

where Φ is the adjoint of Φ^* for the trace inner product. Denote

$$H = -\ln \sigma, \quad \Delta_\sigma(X) = \sigma X \sigma^{-1}, \quad \alpha_t(X) = e^{itH} X e^{-itH}, \quad t \in \mathbb{C}$$

as the modular generator, modular operator, and modular automorphism group of σ respectively. It was proved in [21, Theorem 2.9] that under the assumption $\Phi^*(a^\dagger) = (\Phi^*(a))^\dagger$, Φ^* is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\sigma,s}$ for some $s \neq 1/2$ if and only if Φ^* commutes with Δ_σ and is self-adjoint for $s = 1/2$, and hence Φ^* is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\sigma,s}$ for all $s \in [0, 1]$. We say that a map Φ^* satisfies σ -DBC (detailed balance condition) if Φ^* is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\sigma,1}$. Note that

$$\Gamma_\sigma = \int_0^1 \Gamma_{\sigma,s} ds .$$

Thus, we also have $\Gamma_\sigma \circ \Phi^* = \Phi \circ \Gamma_\sigma$ and hence $\Gamma_\sigma^{-1} \circ \Phi = \Phi^* \circ \Gamma_\sigma^{-1}$ if Φ^* satisfies the σ -DBC.

Let $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ be a conditional expectation. It can be readily seen that $E_{\mathcal{N}}$ satisfies the σ -DBC condition for all $\sigma \in \mathcal{D}(E_{\mathcal{N}})$ (invariant state satisfying $\sigma = E_{\mathcal{N}*}(\sigma)$). Hence

$$\forall s \in [0, 1], \Gamma_{\sigma,s} \circ E_{\mathcal{N}} = E_{\mathcal{N}*} \circ \Gamma_{\sigma,s} \text{ and } \Gamma_\sigma \circ E_{\mathcal{N}} = E_{\mathcal{N}*} \circ \Gamma_\sigma .$$

In particular, $E_{\mathcal{N}}$ is the projection onto \mathcal{N} for the L_2 -norms $\|\cdot\|_{\sigma,s}$ for any $s \in [0, 1]$ and $\|\cdot\|_{\sigma}$, for all $\sigma \in \mathcal{D}(E_{\mathcal{N}})$. Indeed, for any $X \in \mathcal{M}$,

$$\begin{aligned} \langle E_{\mathcal{N}}(X), X - E_{\mathcal{N}}(X) \rangle_{\sigma,s} &= \langle \Gamma_{\sigma,s} \circ E_{\mathcal{N}}(X), X - E_{\mathcal{N}}(X) \rangle_{\text{HS}} \\ &= \langle E_{\mathcal{N}^*} \circ \Gamma_{\sigma,s}(X), X - E_{\mathcal{N}}(X) \rangle_{\text{HS}} \\ &= \langle \Gamma_{\sigma,s}(X), E_{\mathcal{N}}(X - E_{\mathcal{N}}(X)) \rangle_{\text{HS}} = 0. \end{aligned}$$

Now, let $\Phi : \mathcal{M}_* \rightarrow \mathcal{M}_*$ be a quantum channel and \mathcal{N} be the multiplicative domain of Φ^* . Then,

$$\mathcal{N} := \{a \in \mathcal{M} \mid \Phi^*(aa^\dagger) = \Phi^*(a)\Phi^*(a^\dagger), \Phi^*(a^\dagger a) = \Phi^*(a^\dagger)\Phi^*(a)\}.$$

There always exists an invariant state σ such that $\Phi(\sigma) = \sigma$. The next lemma shows that if Φ^* satisfy σ -DBC, then Φ^* restricted to \mathcal{N} is a $*$ -involution.

Lemma 2.5. *Let $\Phi : \mathcal{M}_* \rightarrow \mathcal{M}_*$ be a quantum channel and let \mathcal{N} be the multiplicative domain of Φ^* . Then,*

(i) *There exists an invariant state σ such that $\Phi(\sigma) = \sigma$*

If, in addition, σ is full-rank and Φ^ satisfies σ -DBC,*

(ii) *Φ^* is a contraction on $L_2(\sigma, s)$ for any $s \in [0, 1]$ and $L_2(\sigma)$. Φ^* restricted to \mathcal{N} is a $*$ -isomorphism and an L_2 -isometry on $L_2(\sigma, s)$ for all $s \in [0, 1]$, as well as on $L_2(\sigma)$.*

(iii) *Let $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ be the conditional expectation such that $E_{\mathcal{N}^*}(\sigma) = \sigma$. Then*

$$\Phi^* \circ E_{\mathcal{N}} = E_{\mathcal{N}} \circ \Phi^*, (\Phi^*)^2 \circ E_{\mathcal{N}} = E_{\mathcal{N}} \circ (\Phi^*)^2 = E_{\mathcal{N}}.$$

Proof. (i) Viewing Φ as a linear map, Φ has eigenvalue 1 because $\Phi^*(\mathbb{1}) = \mathbb{1}$. Since Φ preserves self-adjointness, we have an operator $a = a^\dagger$ such that $\Phi(a) = a$. Let a_+ (resp. a_-) be the positive (resp. negative) part of a . We have $\Phi(a) = \Phi(a_+) - \Phi(a_-) = a$. Because Φ is positive and trace preserving, $\Phi(a_+)$ and $\Phi(a_-)$ are positive and

$$\text{tr}(\Phi(a_+)) + \text{tr}(\Phi(a_-)) = \text{tr}(a_+) + \text{tr}(a_-) = \|a\|_1.$$

We show that this implies $\Phi(a_+) = a_+$ and $\Phi(a_-) = a_-$, which proves (i). Indeed, let b_1 and b_2 be any two positive operators such that

$$b_1 - b_2 = b \quad \text{and} \quad \text{tr}(b_1) + \text{tr}(b_2) = \|b\|_1.$$

Then by duality, there exists a self-adjoint operator $-1 \leq X \leq 1$ such that

$$\|b\|_1 = \text{tr}(bX) = \text{tr}(b_1X) - \text{tr}(b_2X) \leq \|b_1\|_1 + \|b_2\|_1 = \text{tr}(b_1) + \text{tr}(b_2) = \|b\|_1.$$

This implies that $\text{supp}(b_1)$ is contained in the spectrum projection of X for eigenvalue $+1$ and similarly $\text{supp}(b_2)$ is contained in the spectrum projection of X for eigenvalue -1 . Hence $\text{supp}(b_1)$ and $\text{supp}(b_2)$ are mutually orthogonal, which implies $b_+ = b_1$ and $b_- = b_2$. The result follows after choosing $b = \Phi(a)$ with

$b_1 = \Phi(a_+)$, $b_2 = \Phi(a_-)$, $b_+ = \Phi(a)_+$ and $b_- = \Phi(a)_-$. For (ii), consider, for $X \in \mathcal{M}$,

$$\begin{aligned} \|\Phi^*(X)\|_{\sigma,s}^2 &= \text{tr} \left(\Phi^*(X^\dagger) \sigma^{1-s} \Phi^*(X) \sigma^s \right) \\ &= \text{tr} \left(\Phi^*(\alpha_{i \frac{1-s}{2}}(X)^\dagger) \Phi^*(\alpha_{i \frac{1-s}{2}}(X)) \sigma \right) \\ &\leq \text{tr} \left(\Phi^*(\alpha_{i \frac{1-s}{2}}(X)^\dagger \alpha_{i \frac{1-s}{2}}(X)) \sigma \right) \\ &= \text{tr} \left(\alpha_{i \frac{1-s}{2}}(X)^\dagger \alpha_{i \frac{1-s}{2}}(X) \sigma \right) \\ &= \|X\|_{\sigma,s}^2 . \end{aligned}$$

In the above inequality, we used the Kadison–Schwarz inequality and the second to last equality follows from $\Phi(\sigma) = \sigma$. Note that $\alpha_s(\mathcal{N}) = \mathcal{N}$ for any $s \in \mathbb{C}$. Then for any $X \in \mathcal{N}$, $\Phi^*(\alpha_{i \frac{1-s}{2}}(X)^\dagger) \Phi^*(\alpha_{i \frac{1-s}{2}}(X)) = \Phi^*(\alpha_{i \frac{1-s}{2}}(X)^\dagger \alpha_{i \frac{1-s}{2}}(X))$ and the above inequality becomes an equality. This proves (ii) for $L_2(\sigma, s)$ for all $s \in [0, 1]$. The assertion for $L_2(\sigma)$ follows by integration. For (iii), we first note that, for any $X \in \mathcal{N}$, $(\Phi^*)^2(X) = X$. Indeed,

$$\langle (\Phi^*)^2(X), X \rangle_{\sigma,s} = \langle \Phi^*(X), \Phi^*(X) \rangle_{\sigma,s} = \|X\|_{\sigma,s}^2 .$$

This further implies that $\Phi^*(X) \in \mathcal{N}$ is in the multiplicative domain because

$$\Phi^*(\Phi^*(X^\dagger) \Phi^*(X)) = \Phi^*(\Phi^*(X^\dagger X)) = X^\dagger X = (\Phi^*)^2(X^\dagger) (\Phi^*)^2(X) .$$

Also, Φ^* is invariant on the orthogonal complement of \mathcal{N} because, for any $Y \in \mathcal{M}$,

$$\langle X, \Phi^* \circ (\text{id} - E_{\mathcal{N}})(Y) \rangle_{\sigma,s} = \langle \Phi^*(X), (\text{id} - E_{\mathcal{N}})(Y) \rangle_{\sigma,s} = 0 .$$

This completes the proof. \square

We see from the above lemma that under σ -DBC, Φ^* is a self-adjoint contraction on $L_2(\sigma, s)$ (also $L_2(\sigma)$), and \mathcal{N} is the union of the eigenspace of Φ^* for eigenvalue 1 and -1 . The eigenspace for eigenvalue 1 is the fixed point space of Φ^* , which is a subalgebra $\mathcal{F} \subset \mathcal{N}$. For each invariant state $\sigma = \Phi(\sigma)$, we have $\sigma = E_{\mathcal{F}^*}(\sigma)$. In finite dimensions, there always exists $0 < \varepsilon < 1$ such that

$$\|\Phi^*(\text{id} - E_{\mathcal{N}}) : L_2(\sigma, s) \rightarrow L_2(\sigma, s)\| \leq (1 - \varepsilon) ,$$

which is a spectral gap condition. The next lemma shows that this spectral gap condition is independent of $s \in [0, 1]$ and of the choice of invariant state σ .

Lemma 2.6. *Let $\Phi : \mathcal{M}_* \rightarrow \mathcal{M}_*$ be a quantum channel and Φ^* be its adjoint. Suppose Φ^* satisfy σ -DBC for some full-rank invariant state σ such that $\Phi(\sigma) = \sigma$. Then,*

- (i) $(\Phi^*)^2$ satisfies ρ -DBC for all states $\rho \in \mathcal{D}(E_{\mathcal{N}})$ and Φ^* satisfies ρ -DBC for all invariant states ρ .

(ii) For each full-rank state $\rho \in \mathcal{D}(E_{\mathcal{N}})$, denote $\lambda(\rho, s) = \| \Phi^*(\text{id} - E_{\mathcal{N}}) : L_2(\rho, s) \rightarrow L_2(\Phi(\rho), s) \|^2$. Then for all $s \in [0, 1]$

$$\lambda(\rho, s) = \lambda(\sigma, 1) .$$

(iii) For each full-rank state $\rho \in \mathcal{D}(E_{\mathcal{N}})$, denote $\lambda(\rho) := \| \Phi^*(\text{id} - E_{\mathcal{N}}) : L_2(\rho) \rightarrow L_2(\Phi(\rho)) \|^2$. Then

$$\begin{aligned} \lambda(\rho) &:= \| \Phi(\text{id} - E_{\mathcal{N}^*}) : L_2(\rho^{-1}) \rightarrow L_2(\Phi(\rho)^{-1}) \| \\ &= \| \Phi^*(\text{id} - E_{\mathcal{N}}) : L_2(\rho) \rightarrow L_2(\Phi(\rho)) \| = \lambda(\sigma, 1) = \lambda(\sigma) . \end{aligned}$$

Proof. By Lemma 2.5, $(\Phi^*)^2|_{\mathcal{N}}$ is the identity map and we have the module property

$$(\Phi^*)^2(aXb) = a(\Phi^*)^2(X)b , \quad \forall a, b \in \mathcal{N} .$$

Note that for any two states $\rho, \sigma \in \mathcal{D}(E_{\mathcal{N}})$, $\rho^{-s}\sigma^s \in \mathcal{N}$ for any $s \in \mathbb{C}$. Therefore, we have for all $s \in [0, 1]$,

$$\Gamma_{\rho, s} \circ (\Phi^*)^2 \circ \Gamma_{\rho, s}^{-1} = \Gamma_{\sigma, s} \circ (\Phi^*)^2 \circ \Gamma_{\sigma, s}^{-1} = \Phi^2 .$$

This shows $(\Phi^*)^2$ satisfies ρ -DBC. Now consider a state ρ such that $\Phi(\rho) = \rho$. Because both $\rho, \sigma \in \mathcal{D}(E_{\mathcal{F}})$, we have $\rho^{-s}\sigma^s \in \mathcal{F}$ for any $s \in \mathbb{C}$. Then it follows from the same argument above that Φ^* satisfies ρ -DBC. For (ii), we denote $\iota = \Phi^*|_{\mathcal{N}}$ to be the involution Φ^* restricted to \mathcal{N} . Note that for any $s \in \mathbb{C}$, it can be verified by the finite dimensional direct sum structure in (13) that

$$\iota(\rho^{-s}\sigma^s) = \Phi(\rho)^{-s}\sigma^s , \quad (25)$$

where $\rho \circ \iota = \Phi(\rho)$. For a mean zero element $Y = X - E_{\mathcal{N}}(X)$,

$$\begin{aligned} \| Y \|_{\rho, s}^2 &= \| \Gamma_{\rho, s}^{1/2}(Y) \|_2^2 \\ &= \| \Gamma_{\sigma, s}^{1/2} \Gamma_{\sigma, s}^{-1/2} \Gamma_{\rho, s}^{1/2}(Y) \|_2^2 \\ &= \| \Gamma_{\sigma, s}^{1/2}(Y_0) \|_2^2 \\ &= \| Y_0 \|_{\sigma, s}^2 , \end{aligned}$$

where $Y_0 = \Gamma_{\sigma, s}^{-1/2} \Gamma_{\rho, s}^{1/2}(Y)$ is also a mean zero element in \mathcal{N}^\perp . Moreover,

$$\begin{aligned} \| \Phi^*(Y_0) \|_{\sigma, s}^2 &= \| \Gamma_{\sigma, s}^{1/2} \Gamma_{\sigma, s}^{-1/2} \Gamma_{\Phi(\rho), s}^{1/2} \Phi^*(Y_0) \|_2^2 \\ &= \| \Gamma_{\Phi(\rho), s}^{1/2} \Phi^*(Y_0) \|_2^2 \\ &= \| \Phi^*(Y_0) \|_{\Phi(\rho), s}^2 , \end{aligned}$$

where we used (25) in the first line. This proves $\lambda(\rho, s) = \lambda(\sigma, s)$ for each s . For the independence of s , we have, for $r \in [0, 1]$,

$$\begin{aligned} \| \Phi^*(Y) \|_{\sigma, s}^2 &= \text{tr} [\Phi^*(Y)^\dagger \sigma^{1-s} \Phi^*(Y) \sigma^s] \\ &= \text{tr} [\Phi^*(\alpha_{i, \frac{r-s}{2}}(Y))^\dagger \sigma^{1-r} \Phi^*(\alpha_{i, \frac{r-s}{2}}(Y)) \sigma^r] \\ &= \| \Phi^*(\alpha_{i, \frac{r-s}{2}}(Y)) \|_{\sigma, r}^2 , \end{aligned}$$

where $\alpha_{i \frac{r-s}{2}}(Y) = \alpha_{i \frac{r-s}{2}}(X - E_{\mathcal{N}}(X)) = \alpha_{i \frac{r-s}{2}}(X) - E_{\mathcal{N}}(\alpha_{i \frac{r-s}{2}}(X))$ is also in \mathcal{N}^{\perp} . Moreover,

$$\|Y\|_{\sigma,s}^2 = \|\alpha_{i \frac{r-s}{2}}(Y)\|_{\sigma,r}^2 .$$

For (iii), the inequality $\lambda(\rho) \leq \lambda(\sigma, 1)$ follows from integrating the $\langle \cdot, \cdot \rangle_{\rho,s}$ inner product to obtain $\langle \cdot, \cdot \rangle_{\rho}$. The equality $\lambda(\sigma, 1) = \lambda(\sigma)$ follows from the fact that the map $\Phi^*(\text{id} - E_{\mathcal{N}})$ is self-adjoint with respect to both $\langle \cdot, \cdot \rangle_{\sigma}$ and $\langle \cdot, \cdot \rangle_{\sigma,s}$ for any $s \in [0, 1]$. Then the quantity $\|\Phi^*(\text{id} - E_{\mathcal{N}})\|$, which is equal to the maximal eigenvalue of $\Phi^*(\text{id} - E_{\mathcal{N}})$, is independent of the choice of Hilbert space norm $\|\cdot\|$. We note that, by (25),

$$\begin{aligned} \Gamma_{\Phi(\rho),s} \circ \Phi^* \circ \Gamma_{\rho,s}^{-1} &= \Gamma_{\Phi(\rho),s} \circ \Phi^* \circ \Gamma_{\rho,s}^{-1} \Gamma_{\sigma,s} \Gamma_{\sigma,s}^{-1} \\ &= \Gamma_{\Phi(\rho),s} \Gamma_{\Phi(\rho),s}^{-1} \Gamma_{\sigma,s} \circ \Phi^* \circ \Gamma_{\sigma,s}^{-1} = \Phi . \end{aligned}$$

and

$$\Gamma_{\Phi(\rho)} \circ \Phi^* \circ \Gamma_{\rho}^{-1} = \Phi ,$$

This implies $\Gamma_{\Phi(\rho)} \circ \Phi^*(\text{id} - E_{\mathcal{N}}) \circ \Gamma_{\rho}^{-1} = \Phi(\text{id} - E_{\mathcal{N}^*})$, and hence

$$\begin{aligned} &\|\Phi(\text{id} - E_{\mathcal{N}^*}) : L_2(\rho^{-1}) \rightarrow L_2(\Phi(\rho)^{-1})\| \\ &= \|\Gamma_{\Phi(\rho)} \circ \Phi^*(\text{id} - E_{\mathcal{N}}) \circ \Gamma_{\rho}^{-1} : L_2(\rho^{-1}) \rightarrow L_2(\Phi(\rho)^{-1})\| \\ &= \|\Phi^*(\text{id} - E_{\mathcal{N}}) : L_2(\rho) \rightarrow L_2(\Phi(\rho))\| . \end{aligned}$$

Moreover, since both σ and ρ are invariant to Φ^2 , we have by (ii),

$$\begin{aligned} \lambda(\sigma) &= \|(\Phi^2)^*(\text{id} - E_{\mathcal{N}}) : L_2(\sigma) \rightarrow L_2(\sigma)\| = \|(\Phi^2)^*(\text{id} - E_{\mathcal{N}}) : L_2(\rho) \rightarrow L_2(\rho)\| \\ &\leq \|\Phi^*(\text{id} - E_{\mathcal{N}}) : L_2(\rho) \rightarrow L_2(\Phi(\rho))\| \|\Phi^*(\text{id} - E_{\mathcal{N}}) : L_2(\Phi(\rho)) \rightarrow L_2(\rho)\| \\ &\leq \|\Phi^*(\text{id} - E_{\mathcal{N}}) : L_2(\sigma) \rightarrow L_2(\sigma)\|^2 = \lambda(\sigma) . \end{aligned}$$

This verifies (iii). □

3. Modified Logarithmic Sobolev Inequalities

In this section, we prove the complete modified logarithmic Sobolev inequality (CMLSI) for quantum Markov semigroups on finite dimensional matrix algebras. The argument is a simple application of the key estimates in Section 2.3. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a finite dimensional von Neumann algebra. A quantum Markov semigroup (QMS) $(\mathcal{P}_t)_{t \geq 0} : \mathcal{M} \rightarrow \mathcal{M}$ is a continuous parameter semigroup of completely positive, unital maps such that $\mathcal{P}_0 = \text{id}_{\mathcal{M}}$ and $\mathcal{P}_s \circ \mathcal{P}_t = \mathcal{P}_{s+t}$ for all $s, t \geq 0$. Such a semigroup is characterised by its generator, called the Lindbladian \mathcal{L} , which is defined as

$$\mathcal{L}(X) = \lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{P}_t(X) - X) , \quad \forall X \in \mathcal{M} ,$$

so that $\mathcal{P}_t = e^{t\mathcal{L}}$ for all $t \geq 0$. A QMS is said to be *primitive* if it admits a unique full-rank invariant state σ . In this section, we consider semigroups $\mathcal{P}_t : \mathcal{M} \rightarrow \mathcal{M}$ on a finite dimensional von Neumann algebra \mathcal{M} and exclusively study QMS that satisfy the following *detailed balance condition* with respect to some (possibly non-unique) full-rank invariant state σ : if, for any $X, Y \in \mathcal{M}$ and any $t \geq 0$,

$$\text{tr}(\sigma X^\dagger \mathcal{P}_t(Y)) = \text{tr}(\sigma \mathcal{P}_t(X)^\dagger Y), \tag{\sigma\text{-DBC}}$$

then we say that a semigroup \mathcal{P}_t is GNS-symmetric if \mathcal{P}_t satisfies σ -DBC for a full-rank invariant state σ . It was proved (see [21, 61]) that for a GNS-symmetric QMS $(\mathcal{P}_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}))_{t \geq 0}$ on $\mathcal{B}(\mathcal{H})$, its generator \mathcal{L} can be written as

$$\mathcal{L}(X) = \sum_j \left(e^{-\omega_j/2} A_j^\dagger [X, A_j] + e^{\omega_j/2} [A_j, X] A_j^\dagger \right). \tag{26}$$

Here $A_j \in \mathcal{B}(\mathcal{H})$ and ω_j are some real parameters such that for any invariant state σ , $\Delta_\sigma(A_j) := \sigma A_j \sigma^{-1} = e^{-\omega_j} A_j$. Very recently, this result was extended by Wirth [80] to all uniform continuous semigroups $(\mathcal{P}_t : \mathcal{M} \rightarrow \mathcal{M})_{t \geq 0}$ on a von Neumann algebra \mathcal{M} . Since we focus on finite dimensions, all the (continuous) semigroups considered in this paper admit the above Lindbladian form (26). Moreover, there exists a conditional expectation $E_{\mathcal{F}} : \mathcal{M} \rightarrow \mathcal{F}$ onto the fixed point algebra $\mathcal{F} = \{X \in \mathcal{M} \mid [A_j, X] = 0 \forall j\}$ such that [34]

$$e^{t\mathcal{L}} \xrightarrow{t \rightarrow \infty} E_{\mathcal{F}}.$$

We are interested in the exponential convergence to this limit in terms of relative entropy. Recall that the *entropy production* (sometimes also referred as *Fisher information*) for a state $\rho \in \mathcal{D}(\mathcal{M})$ is defined as

$$\text{EP}_{\mathcal{L}}(\rho) := - \left. \frac{d}{dt} \right|_{t=0} D(\mathcal{P}_{t*}(\rho) \| E_{\mathcal{F}*}(\rho)) = - \text{tr}(\mathcal{L}_*(\rho)(\ln \rho - \ln E_{\mathcal{F}*}(\rho))), \tag{27}$$

which is the opposite of the derivative of the relative entropy with respect to the equilibrium state. Here and in what follows, \mathcal{L}_* (resp. \mathcal{P}_{t*} and $E_{\mathcal{F}*}$) denotes the adjoint maps of the generator \mathcal{L} (resp. semigroup map \mathcal{P}_t and conditional expectation $E_{\mathcal{F}}$). We say that a QMS $\mathcal{P}_t : \mathcal{M} \rightarrow \mathcal{M}$ satisfies the *modified logarithmic Sobolev inequality* (MLSI) with $\alpha > 0$ if, for any $\rho \in \mathcal{D}(\mathcal{M})$,

$$\alpha D(\rho \| E_{\mathcal{F}*}(\rho)) \leq \text{EP}_{\mathcal{L}}(\rho). \tag{MLSI}$$

The best constant α achieving this bound is called the modified logarithmic Sobolev constant of the semigroup, and is denoted by $\alpha_{\text{MLSI}}(\mathcal{L})$. It turns out that this inequality is equivalent to the following exponential decay of relative entropy;

$$D(\mathcal{P}_{t*}(\rho) \| E_{\mathcal{F}*}(\rho)) \leq e^{-\alpha t} D(\rho \| E_{\mathcal{F}*}(\rho)).$$

We also consider the *complete modified logarithmic Sobolev inequality* (CMLSI), which requires

$$\alpha D(\rho \| (E_{\mathcal{F}*} \otimes \text{id})(\rho)) \leq \text{EP}_{(\mathcal{L} \otimes \text{id})}(\rho) \tag{CMLSI}$$

to hold for all states ρ on $\mathcal{M} \otimes \mathcal{B}(\mathcal{H})$ and any finite dimensional Hilbert space \mathcal{H} as a reference system (or even $\mathcal{B}(\mathcal{H})$ replaced by a finite von Neumann algebra). We denote the best constant α achieving (MLSI) as $\alpha_{\text{CMLS}}(\mathcal{L})$. In [46], it was shown that the proof of the positivity of α_{CMLS} for all GNS-symmetric quantum Markov semigroups can be reduced to that for (trace) symmetric quantum Markov semigroups, that is to those for which $\mathcal{L} = \mathcal{L}_*$. However, the problem of the positivity of the CMLS constant for symmetric QMS has been left open despite considerable work delved on that topic in the recent years (see e.g. [17, 18, 37, 81]). Here, we provide a positive answer to the question via a simple application of our key estimates from Section 2.3.

First, we recall that the Dirichlet form associated to \mathcal{L} takes the following simple form [21, Section 5]: for any invariant state $\sigma = E_{\mathcal{F}_*}(\sigma)$,

$$\mathcal{E}_\sigma(X) := -\langle X, \mathcal{L}(X) \rangle_\sigma = \sum_j \int_0^1 e^{(\frac{1}{2}-s)\omega_j} \langle \partial_j(X), \partial_j(X) \rangle_{\sigma,s} ds, \tag{28}$$

where $\partial_j(X) := [A_j, X]$. We denote

$$\|X\|_{\sigma,\omega_j} := \int_0^1 e^{(\frac{1}{2}-s)\omega_j} \langle \partial_j(X), \partial_j(X) \rangle_{\sigma,s} ds \Rightarrow \mathcal{E}_\sigma(X) = \sum_j \|\partial_j(X)\|_{\sigma,\omega_j}^2. \tag{29}$$

Then the entropy production associated to \mathcal{L} can be written as (see [46, Lemma 2.3])

$$\text{EP}_{\mathcal{L}}(\rho) = \sum_j \|\Gamma_{\sigma,\frac{1}{2}} \circ \partial_j \circ \Gamma_{\sigma,\frac{1}{2}}^{-1}(\rho)\|_{\rho^{-1},\omega_j}^2, \tag{30}$$

where, for any $X \in \mathcal{M}$,

$$\|X\|_{\rho^{-1},\omega_j}^2 = \int_0^\infty \text{tr} \left[X^\dagger (e^{-\frac{\omega_j}{2}} \rho + u)^{-1} X (e^{\frac{\omega_j}{2}} \rho + u)^{-1} \right] du.$$

We denote the kernels corresponding to the inner products $\|\cdot\|_{\sigma,\omega_j}$ and $\|\cdot\|_{\sigma^{-1},\omega_j}$ by Γ_{σ,ω_j} and $\Gamma_{\sigma^{-1},\omega_j}$, respectively.

Lemma 3.1. *The following relation holds for any full-rank state σ :*

$$\Gamma_{\sigma,\omega_j}^{-1} = \Gamma_{\sigma^{-1},\omega_j}. \tag{31}$$

Moreover, whenever $\sigma = E_{\mathcal{F}_*}(\sigma)$,

$$\Gamma_{\sigma,\frac{1}{2}} \circ \partial_j \circ \Gamma_{\sigma,\frac{1}{2}}^{-1} = \Gamma_{\sigma,\omega_j} \circ \partial_j \circ \Gamma_{\sigma}^{-1}. \tag{32}$$

Proof. The first identity follows from Lemma 5.8 in [21]. The proof of the second identity follows by direct computation using the commutation relation $\sigma A_j = e^{-\omega_j} A_j \sigma$. \square

We recall that the spectral gap $\lambda(\mathcal{L})$ of the Lindbladian \mathcal{L} is characterized as

$$\lambda(\mathcal{L}) := \inf_X \frac{\mathcal{E}_\sigma(X)}{\|X - E_{\mathcal{F}}(X)\|_\sigma^2} \quad (33)$$

for a given full-rank invariant state σ .

Lemma 3.2. *Suppose \mathcal{P}_t is GNS-symmetric to a full-rank invariant state $\sigma = E_{\mathcal{F}_*}(\sigma)$. Then the infimum in (33) is independent of the choice of the full-rank invariant state σ .*

Proof. By assumption the generator \mathcal{L} is symmetric with respect to the GNS inner product (σ -DBC), which also implies self-adjointness with respect to the inner products $\langle \cdot, \cdot \rangle_\sigma$ (cf. [21, Theorem 2.9]). Moreover, self-adjointness with respect to the GNS inner product is independent of the invariant state chosen. Therefore, \mathcal{L} is self-adjoint with respect to $\langle \cdot, \cdot \rangle_\sigma$ for any full-rank invariant state σ . Now, the spectral gap (33) is the difference between the smallest eigenvalue (here, 0) and the second smallest eigenvalue of $-\mathcal{L}$, hence a quantity independent of the inner product with respect to which \mathcal{L} is self-adjoint, which allows us to conclude. \square

We are now ready to prove Theorem 1.1, which is the main theorem of this section.

Theorem 3.3. *Any GNS-symmetric quantum Markov semigroup on a finite dimensional von Neumann algebra \mathcal{M} satisfies the complete modified logarithmic Sobolev inequality. More precisely, given such a QMS $(\mathcal{P}_t = e^{t\mathcal{L}} : \mathcal{M} \rightarrow \mathcal{M})_{t \geq 0}$ with fixed point algebra \mathcal{F} , the following bound holds true:*

$$\frac{\lambda(\mathcal{L})}{C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{F})} \leq \alpha_{\text{CMLS}}(\mathcal{L}) \leq 2\lambda(\mathcal{L}). \quad (34)$$

Similarly, the modified logarithmic Sobolev inequality constant is controlled by

$$\frac{\lambda(\mathcal{L})}{C_\tau(\mathcal{M} : \mathcal{F})} \leq \alpha_{\text{MLS}}(\mathcal{L}) \leq 2\lambda(\mathcal{L}). \quad (35)$$

Proof. The proof of the upper bounds is standard and can be found in [6, 50], so we focus on the lower bounds. We first provide a bound on the MLSI constant. For this we use the upper bound in Lemma 2.2 that, for $X := \Gamma_{E_{\mathcal{F}_*}(\rho)}^{-1}(\rho)$,

$$D(\rho \| E_{\mathcal{F}_*}(\rho)) \leq \|\rho - E_{\mathcal{F}_*}(\rho)\|_{E_{\mathcal{F}_*}(\rho)^{-1}}^2 = \|X - \mathbf{1}\|_{E_{\mathcal{F}_*}(\rho)}^2 \leq \lambda(\mathcal{L})^{-1} \mathcal{E}_{E_{\mathcal{F}_*}(\rho)}(X),$$

where $\lambda(\mathcal{L})$ is the spectral gap of \mathcal{L} . Next, we have, by (29), that

$$\begin{aligned} \mathcal{E}_{E_{\mathcal{F}^*}(\rho)}(X) &= \sum_j \|\partial_j(X)\|_{E_{\mathcal{F}^*}(\rho), \omega_j}^2 \\ &\stackrel{(1)}{=} \sum_j \|\Gamma_{E_{\mathcal{F}^*}(\rho), \omega_j} \circ \partial_j \circ \Gamma_{E_{\mathcal{F}^*}(\rho)^{-1}}(\rho)\|_{E_{\mathcal{F}^*}(\rho)^{-1}, \omega_j}^2 \\ &\stackrel{(2)}{=} \sum_j \|\Gamma_{E_{\mathcal{F}^*}(\rho), \frac{1}{2}} \circ \partial_j \circ \Gamma_{E_{\mathcal{F}^*}(\rho)^{-1}, \frac{1}{2}}(\rho)\|_{E_{\mathcal{F}^*}(\rho)^{-1}, \omega_j}^2 \\ &\stackrel{(3)}{\leq} C_\tau(\mathcal{M} : \mathcal{F}) \sum_j \|\Gamma_{E_{\mathcal{F}^*}(\rho), \frac{1}{2}} \circ \partial_j \circ \Gamma_{E_{\mathcal{F}^*}(\rho)^{-1}, \frac{1}{2}}(\rho)\|_{\rho^{-1}, \omega_j} \\ &\stackrel{(4)}{=} C_\tau(\mathcal{M} : \mathcal{F}) \text{EP}_{\mathcal{L}}(\rho). \end{aligned}$$

For the above equality (1), we used the inverse relation (31); in (2) we used the relation (32); (3) is an application of Lemma 2.1 with the weights $\mu_1 := \exp(-\frac{\omega_j}{2})$ and $\mu_2 := \exp(\frac{\omega_j}{2})$; finally (4) follows from (30). The proof of CMLSI (34) follows the exact same steps, up to replacing the constant $C_\tau(\mathcal{M} : \mathcal{F})$ by its completely bounded version $C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{F})$. \square

Remark 3.4. The above theorem applies for the derivation triples introduced in Carlen-Maas’s work [22] as well as the symmetric quantum Markov semigroup on finite von Neumann algebra considered in [17, 25, 30] whenever the index $C_{\text{cb}}(\mathcal{M} : \mathcal{F})$ is finite. Nevertheless, $C_{\text{cb}}(\mathcal{M} : \mathcal{F}) = C(\mathcal{M} : \mathcal{F}) = +\infty$ whenever \mathcal{M} is infinite dimensional and \mathcal{F} is finite dimensional, which limits its applicability in infinite dimensional settings. In infinite dimensions, other tools like curvature have been introduced to obtain CMLSI (see [17, 18, 56, 81]).

Remark 3.5. When $\mathcal{M} := \mathcal{B}(\mathcal{H})$ and the semigroup is primitive, comparison to the logarithmic Sobolev constant α_{LSI} combined with standard interpolation inequalities provide the following bounds for $\alpha_{\text{MLS I}}$ [20, 50, 66]:

$$\frac{\lambda(\mathcal{L})}{\ln(\mu_{\min}(\sigma)^{-1}) + 2} \leq \alpha_{\text{LSI}}(\mathcal{L}) \leq \frac{\alpha_{\text{MLS I}}(\mathcal{L})}{2} \leq \lambda(\mathcal{L}). \tag{36}$$

The lower bound can be compared with the one provided in (35) together with (19) and (15) to give that

$$\frac{\mu_{\min}(\sigma)\lambda(\mathcal{L})}{d_{\mathcal{H}}} \leq \alpha_{\text{MLS I}}(\mathcal{L}), \quad \frac{\mu_{\min}(\sigma)\lambda(\mathcal{L})}{d_{\mathcal{H}}^2} \leq \alpha_{\text{CMLS I}}(\mathcal{L}). \tag{37}$$

Clearly, the lower bounds in (36) are asymptotically tighter. However, we emphasise that our bounds (37) are the first generic non-trivial lower bounds for non-primitive QMS, and the CMLSI bound are independent of the size of the environment and hence stable under tensorization, which is even new for primitive semigroup. For classical Markov semigroups (equivalently, graph Laplacians of a weighted graph), (37) gives an alternative CMLSI bounds to the one proved in [56]. In Sects. 6 and 6.2, we will use the approximate tensorization bounds, which is the subject of Section 5,

to derive bounds on the CMLSI constant that are sharper than (37) above. As we will see, in some cases, the CMLSI lower bounds can scale similarly to the LSI bounds in the primitive setting. It remain opens whether the CMLSI constant admits asymptotic bounds better than $\mathcal{O}(d_{\mathcal{H}}^{-2})\lambda(\mathcal{L})$ in general.

4. Strong Data Processing Inequalities

In this section, we study the complete strong data processing inequality for a quantum channel, which is a discrete time analog of CMLSI. We recall the definition of the weighted L_2 -norm corresponding to a full-rank state ω :

$$\|X\|_{\omega^{-1}}^2 = \int_0^\infty \text{tr} \left(X^\dagger \frac{1}{\omega+s} X \frac{1}{\omega+s} \right) ds, \quad X \in \mathcal{M}_* .$$

If $X = \rho - \omega$ for some other state ρ ,

$$\chi_2(\rho, \omega) := \|\rho - \omega\|_{\omega^{-1}}^2$$

is a special case of the quantum χ_2 -divergence studied in [82]. It is known that χ_2 also satisfies the data processing inequality: for a quantum channel Φ ,

$$\chi_2(\Phi(\rho), \Phi(\omega)) \leq \chi_2(\rho, \omega) . \tag{38}$$

Indeed, the data processing inequality of relative entropy follows from (38) and the argument used in Lemma 2.2. We shall now discuss how to control relative entropy contraction coefficients by their χ_2 analogues.

Let $\Phi : \mathcal{M}_* \rightarrow \mathcal{M}_*$ be a quantum channel and Φ^* be the adjoint map of Φ . We denote by \mathcal{N} the multiplicative domain of Φ^* . Suppose Φ admits a full-rank invariant state σ and Φ satisfies σ -DBC. Then by Lemma 2.5, Φ^* restricted to \mathcal{N} is a $*$ -isomorphism. Denote by $E : \mathcal{M} \rightarrow \mathcal{N}$ the σ -preserving condition expectation and by E_* its pre-adjoint on \mathcal{M}_* . For a full-rank state ω , we have discussed the following L_2 -contraction constant in Lemma 2.6:

$$\lambda(\omega) := \|\Phi(\text{id} - E_*) : L_2(\omega^{-1}) \rightarrow L_2(\Phi(\omega)^{-1})\|^2 .$$

Equivalently, $\lambda(\omega)$ gives the contraction coefficient of χ_2 :

$$\lambda(\omega) = \sup_{E_*(\rho) = E_*(\omega), \rho \neq \omega} \frac{\chi_2(\Phi(\rho), \Phi(\omega))}{\chi_2(\rho, \omega)} .$$

Here the supremum is over all state $\rho \neq \omega$ with $E_*(\rho) = E_*(\omega)$, and we restrict our optimization to states ρ and ω with the same ‘‘mean’’ (also called decoherence free part) given by the map E_* . This is because if $\mathcal{N} \neq \mathbb{C}\mathbb{1}$ is not trivial, then for any two invariant states $\sigma, \sigma' \in \mathcal{D}(\mathcal{N})$,

$$\chi_2(\Phi(\sigma'), \Phi(\sigma)) = \chi_2(\sigma', \sigma), \quad D(\Phi(\sigma') \| \Phi(\sigma)) = D(\sigma' \| \sigma),$$

and hence $\lambda(\sigma) = 1$ for any invariant state $\Phi(\sigma) = \sigma$.

The next theorem is a quantum analog of [73, Theorem 3.4] which shows that the χ_2 contraction coefficient implies local strong data processing inequality.

Theorem 4.1. *Let $\Phi : \mathcal{M}_* \rightarrow \mathcal{M}_*$ be a quantum channel that admits some full-rank invariant state $\sigma = \Phi(\sigma)$. Let ω be a full-rank state and denote $\lambda(\omega) := \|\Phi(\text{id} - E_*) : L_2(\omega^{-1}) \rightarrow L_2(\Phi(\omega)^{-1})\|^2$. Then, for any state ρ with $E_*(\omega) = E_*(\rho)$,*

$$D(\Phi(\rho) \|\Phi(\omega)) \leq c D(\rho \|\omega), \quad (39)$$

where c is a constant such that

$$\lambda(\omega) \leq c \leq c(C(\rho : \omega), \lambda(\omega)). \quad (40)$$

Here $C(\rho : \omega) := \inf\{C|\rho \leq C\omega\}$ and $c(C, \lambda)$ is an explicit function such that $c(C, \lambda) < 1$ whenever $\lambda < 1$. In particular, for any state ρ , $c(C(\rho : \omega), \lambda(\omega)) \leq c(\mu_{\min}(\omega)^{-1}, \lambda(\omega))$ where $\mu_{\min}(\omega)$ is the minimum eigenvalue of ω .

Proof. We first show the lower bound. Write $\lambda \equiv \lambda(\omega)$. Let ρ be a state with $E_*(\rho) = E_*(\omega)$. Take the linear interpolation of states $\omega_t := (1-t)\omega + t\rho$, $t \in [0, 1]$. Now assume Φ satisfies (39) for $c > 0$. We have

$$D(\Phi(\omega_t) \|\Phi(\omega)) \leq c D(\omega_t \|\omega),$$

since $E_*(\omega_t) = E_*(\omega)$. Consider the function $f(t) = c D(\omega_t \|\omega) - D(\Phi(\omega_t) \|\Phi(\omega))$. Taking derivatives, we have $f(0) = f'(0) = 0$ and [55]

$$f''(0) = c \|\rho - \omega\|_{\omega^{-1}}^2 - \|\Phi(\rho) - \Phi(\omega)\|_{\Phi(\omega)^{-1}}^2.$$

Note that $f''(0) \geq 0$, because $f(t) \geq 0$ for $t \in [0, \varepsilon]$. Therefore,

$$\|\Phi(\rho - \omega)\|_{\Phi(\omega)^{-1}}^2 \leq c \|\rho - \omega\|_{\omega^{-1}}^2.$$

This proves the lower bound

$$\lambda(\omega) \leq c.$$

For the upper bound, denote $\rho_t = t\rho + (1-t)\omega$ and $g(t) = D(\rho_t \|\omega) - D(\Phi(\rho_t) \|\Phi(\omega))$. We have $g(0) = g'(0) = 0$, and

$$g''(t) = \|\rho - \omega\|_{\rho_t^{-1}}^2 - \|\Phi(\rho - \omega)\|_{\Phi(\rho_t)^{-1}}^2.$$

It follows from (38) (see also [55, Example 2]) that $g''(t) \geq 0$. Using Lemma 2.1 and the definition of $\lambda(\omega)$, we also have that

$$\begin{aligned} g''(t) &= \|\rho - \omega\|_{\rho_t^{-1}}^2 - \|\Phi(\rho - \omega)\|_{\Phi(\rho_t)^{-1}}^2 \\ &\geq \frac{1}{1 + (C-1)t} \|\rho - \omega\|_{\omega^{-1}}^2 - \frac{1}{1-t} \|\Phi(\rho - \omega)\|_{\Phi(\omega)^{-1}}^2 \\ &\geq \left(\frac{1}{1 + (C-1)t} - \frac{\lambda^2}{1-t} \right) \|\rho - \omega\|_{\omega^{-1}}^2, \end{aligned}$$

where $C = \inf\{C \mid \rho \leq C\omega\}$. Thus, we have, for $t_0 := \frac{1-\lambda^2}{1+\lambda^2(C-1)}$,

$$g''(t) \geq \begin{cases} \left(\frac{1}{1+(C-1)t} - \frac{\lambda^2}{1-t} \right) \|\rho - \omega\|_{\omega^{-1}}^2, & t \leq t_0 \\ 0, & t > t_0 \end{cases}.$$

Denote $a(s) := \int_0^s \frac{1}{1+(C-1)t} - \frac{\lambda^2}{1-t} dt = \frac{\ln(1+(C-1)s)}{C-1} + \lambda^2 \ln(1-s)$. Since $g'(0) = 0$, we have $g'(s) \geq a(s) \|\rho - \omega\|_{\omega^{-1}}^2$ if $s \leq t_0$ and $g'(s) \geq a(t_0) \|\rho - \omega\|_{\omega^{-1}}^2$ if $s \geq t_0$. Denote

$$b(t) := \int_0^t a'(s) ds = \frac{(1+(C-1)t) \ln(1+(C-1)t) - (C-1)t}{(C-1)^2} - \lambda^2((1-t) \ln(1-t) + t).$$

We have

$$\begin{aligned} D(\rho \|\omega) - D(\Phi(\rho) \|\Phi(\omega)) &= g(1) - g(0) = \int_0^1 g'(s) ds \\ &\geq ((1-t_0)a(t_0) + b(t_0)) \|\rho - \omega\|_{\omega^{-1}}^2 \\ &\geq ((1-t_0)a(t_0) + b(t_0)) D(\rho \|\omega), \end{aligned}$$

where the last inequality follows from Lemma 2.2. The SDPI constant is then upper bounded by

$$c = 1 - (1-t_0)a(t_0) - b(t_0) < 1.$$

It is clear from the derivation that c as a function depending on C and λ satisfies

$$c(C, \lambda) \geq c(C', \lambda), \quad C' \geq C \geq 1.$$

Then the last assertion follows from $\rho \leq \mathbb{1} \leq \mu_{\min}(\omega)^{-1}\omega$. \square

Next, we consider strong data processing inequality for a quantum channel $\Phi : \mathcal{M}_* \rightarrow \mathcal{M}_*$ with respect to its decoherence free states $\mathcal{D}(E_{\mathcal{N}})$. We say Φ satisfies a α -strong data processing inequality (α -SDPI) for some $0 < \alpha < 1$ if for any state $\rho \in \mathcal{D}(\mathcal{H})$,

$$D(\Phi(\rho) \|\Phi \circ E_{\mathcal{N}*}(\rho)) \leq \alpha D(\rho \|\mathcal{E}_{\mathcal{N}*}(\rho)). \quad (41)$$

We say that Φ satisfies the α -complete strong data processing inequality (α -CSDPI) for some $0 < \alpha < 1$ if, for any $n \in \mathbb{N}$ and all bipartite states $\rho \in \mathcal{D}(\mathbb{M}_n(\mathcal{M}))$,

$$D((\Phi \otimes \text{id}_n)(\rho) \|\Phi \circ E_{\mathcal{N}*} \otimes \text{id}_n(\rho)) \leq \alpha D(\rho \|(E_{\mathcal{N}*} \otimes \text{id}_n)(\rho)), \quad (42)$$

where id_n denotes the identity channel on the matrix algebra \mathbb{M}_n . We denote the best (smallest) constant achieving SDPI (41) (resp. CSDPI (42)) as $\alpha_{\text{SDPI}}(\Phi)$ (resp. as $\alpha_{\text{CSDPI}}(\Phi)$). The advantage of the CSDPI constant is that it is stable under tensorization.

Proposition 4.2. *Let $\Phi_1 : \mathcal{M}_{1*} \rightarrow \mathcal{M}_{1*}$ and $\Phi_2 : \mathcal{M}_{2*} \rightarrow \mathcal{M}_{2*}$ be two quantum channels. Denote $E_j : \mathcal{M}_j \rightarrow \mathcal{N}_j$, $j = 1, 2$ as the condition expectation onto the multiplicative domain of Φ_j^* respectively. Then*

$$\alpha_{\text{CSDPI}}(\Phi_1 \otimes \Phi_2) \leq \max\{\alpha_{\text{CSDPI}}(\Phi_1), \alpha_{\text{CSDPI}}(\Phi_2)\}.$$

Namely, for any $n \geq 1$ and states $\rho \in \mathcal{D}(\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathbb{M}_n)$

$$\begin{aligned} & D(\Phi_1 \otimes \Phi_2 \otimes \text{id}_{\mathbb{M}_n}(\rho) \| (\Phi_1 \circ E_{1*}) \otimes (\Phi_2 \circ E_{2*}) \otimes \text{id}_{\mathbb{M}_n}(\rho)) \\ & \leq \max\{\alpha_{\text{CSDPI}}(\Phi_1), \alpha_{\text{CSDPI}}(\Phi_2)\} D(\rho \| E_{1*} \otimes E_{2*} \otimes \text{id}_{\mathbb{M}_n}(\rho)). \end{aligned} \quad (43)$$

Proof. The proof is a natural application of the data processing inequality. For ease of notations, we argue for $n = 1$ as the case for general $n \geq 1$ follows the same argument. Note that for $j = 1, 2$, $\Phi_j \circ E_{j*} = E_{j*} \circ \Phi_j$. Write $\alpha_1 := \alpha_{\text{CSDPI}}(\Phi_1)$ and $\alpha_2 := \alpha_{\text{CSDPI}}(\Phi_2)$. We have

$$\begin{aligned} & D(\Phi_1 \otimes \Phi_2(\rho) \| (\Phi_1 \circ E_{1*}) \otimes (\Phi_2 \circ E_{2*})(\rho)) \\ & = D(\Phi_1 \otimes \Phi_2(\rho) \| E_{1*} \otimes E_{2*}(\Phi_1 \otimes \Phi_2(\rho))) \\ & = D(\Phi_1 \otimes \Phi_2(\rho) \| E_{1*} \Phi_1 \otimes \Phi_2(\rho)) \\ & \quad + D(E_{1*} \Phi_1 \otimes \Phi_2(\rho) \| E_{1*} \otimes E_{2*}(\Phi_1 \otimes \Phi_2(\rho))) \\ & \leq \alpha_1 D(\text{id} \otimes \Phi_2(\rho) \| E_{1*} \otimes \Phi_2(\rho)) \\ & \quad + \alpha_2 D((E_{1*} \circ \Phi_1) \otimes \text{id}(\rho) \| (E_{1*} \circ \Phi_1) \otimes E_{2*}(\rho)) \\ & \leq \alpha_1 D(\rho \| E_{1*} \otimes \text{id}(\rho)) \\ & \quad + \alpha_2 D(E_{1*} \otimes \text{id}(\rho) \| E_{1*} \otimes E_{2*}(\rho)) \\ & \leq \max\{\alpha_1, \alpha_2\} D(\rho \| E_{1*} \otimes E_{2*}(\rho)), \end{aligned}$$

where in the second equality and the last inequality, we used the chain rule (12) and the second last inequality uses data processing inequality for the map $\text{id} \otimes \Phi_2$ and $\Phi_1 \otimes \text{id}$, respectively. \square

As an application of Theorem (4.1), we have $\alpha_{\text{SDPI}}(\Phi)$ and $\alpha_{\text{CSDPI}}(\Phi)$ are two-sided bounded by the spectral gap in finite dimensions.

Corollary 4.3. *Let $\Phi : \mathcal{M}_* \rightarrow \mathcal{M}_*$ be a quantum channel and \mathcal{N} be the multiplicative domain of Φ^* . Assume that Φ^* satisfies the σ -DBC for some full-rank invariant state $\sigma = \Phi(\sigma)$. Denote the spectral gap $\lambda(\Phi) := \|\Phi^*(\text{id} - E_{\mathcal{N}}) : L_2(\sigma) \rightarrow L_2(\sigma)\|^2 < 1$. There exists an explicit constant $c(C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N}), \lambda) < 1$ such that*

$$\lambda(\Phi) \leq \alpha_{\text{CSDPI}}(\Phi) \leq c(C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N}), \lambda(\Phi)). \quad (44)$$

The same estimate holds for $\alpha_{\text{SDPI}}(\Phi)$ simply replacing $C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N})$ by $C_{\tau}(\mathcal{M} : \mathcal{N})$.

Proof. We have shown in Lemma 2.6 that $\lambda(\Phi) = \lambda(\sigma)$ and $\lambda(\sigma) \geq \lambda(\rho)$ for all decoherence free state $\omega \in \mathcal{D}(E_{\mathcal{N}^*})$. Then (44) follows from Theorem 4.1 and the fact that $\rho \leq C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N})E_{\mathcal{N}^*}(\rho)$ for any $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathbb{C}^n)$. \square

Remark 4.4. For a primitive unital quantum channel $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, it was proved in [63] that

$$\alpha_{\text{SPDI}}(\Phi) \leq 1 - \alpha_{\text{LSI}}(\Phi^* \Phi - \text{id}) \leq 1 - \frac{\lambda(\Phi)}{\ln d + 2},$$

where $\alpha_{\text{LSI}}(\Phi^* \Phi - \text{id})$ is the log-Sobolev constant of the map $\Phi^* \Phi - \text{id}$ seen as the generator of a quantum Markov semigroup. This is generically better than the bounds found in Corollary 4.3. Nevertheless, our results give explicit SDPI constants for general non-ergodic GNS-symmetric quantum channels, independently of the size of the environment. Moreover, the CSDPI constant satisfies the tensorization property.

5. Approximate Tensorization

In this section, we consider the approximate tensorization of the relative entropy in a general setting. Let \mathcal{M} be a finite dimensional von Neumann algebra equipped with a faithful trace tr . Let $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{M}$ be two subalgebras of \mathcal{M} and $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$. Let $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ and $E_i : \mathcal{M} \rightarrow \mathcal{N}_i, i = 1, 2$, be conditional expectations such that $E_{\mathcal{N}} \circ E_i = E_{\mathcal{N}}$. If ρ is a state that satisfies $E_{\mathcal{N}^*}(\rho) = \rho$, then

$$\rho = E_{\mathcal{N}^*}(\rho) = E_{i^*} \circ E_{\mathcal{N}^*}(\rho) = E_{i^*}(\rho), i = 1, 2.$$

Namely, every $E_{\mathcal{N}}$ invariant state is both E_1 and E_2 invariant. Denote that $\rho_{\mathcal{N}} = E_{\mathcal{N}^*}(\rho)$ and $\rho_i = E_{i^*}(\rho), i = 1, 2$. We are interested in the following *approximate tensorization* property:

$$D(\rho \| \rho_{\mathcal{N}}) \leq c(D(\rho \| \rho_1) + D(\rho \| \rho_2)), \forall \rho \in \mathcal{D}(E_{\mathcal{M}}). \quad (45)$$

It was proved in [36, Corollary 2.3] that the constant c equals to 1 if and only if E_1 and E_2 form a commuting square, i.e. $E_1 \circ E_2 = E_2 \circ E_1 = E_{\mathcal{N}}$. Using the chain rule $D(\rho \| \rho_{\mathcal{N}}) = D(\rho \| \rho_i) + D(\rho_i \| \rho_{\mathcal{N}})$, the inequality (45) is equivalent to the tensoring uncertainty relation

$$D(\rho \| \rho_{\mathcal{N}}) \geq \alpha(D(\rho_1 \| \rho_{\mathcal{N}}) + D(\rho_2 \| \rho_{\mathcal{N}})), \forall \rho \in \mathcal{D}(E_{\mathcal{M}}), \quad (46)$$

where $\alpha = \frac{c}{2c-1} > 1/2$. Take $\rho(t) = t\rho + (1-t)\rho_{\mathcal{N}}$ and the function

$$f(t) = D(\rho(t) \| \rho_{\mathcal{N}}) - \alpha(D(\rho_1(t) \| \rho_{\mathcal{N}}) + D(\rho_2(t) \| \rho_{\mathcal{N}})).$$

Then we have $f(0) = f'(0) = 0$ and

$$f''(0) = \|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 - \alpha(\|\rho_1 - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 + \|\rho_2 - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2),$$

so a necessary condition for (46) and, equivalently, (45), is that for any state ρ ,

$$\|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 \geq \alpha \left(\|\rho_1 - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 + \|\rho_2 - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 \right).$$

In particular, if we choose $\rho = \rho_1 = E_{1*}(\rho)$, we have

$$\frac{(1-\alpha)}{\alpha} \|\rho_1 - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 \geq \|E_{2*}(\rho_1) - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2.$$

Because $1/2 < \alpha \leq 1$, for $\lambda = \frac{1-\alpha}{\alpha}$ this can be reformulated as the L_2 -clustering condition

$$\begin{aligned} & \|E_{2*} \circ E_{1*} - E_{\mathcal{N}*} : L_2(\rho_{\mathcal{N}}^{-1}) \rightarrow L_2(\rho_{\mathcal{N}}^{-1})\| \\ & = \|E_1 \circ E_2 - E_{\mathcal{N}} : L_2(\rho_{\mathcal{N}}) \rightarrow L_2(\rho_{\mathcal{N}})\| := \lambda < 1. \end{aligned}$$

Since $E_2 \circ E_1$ is the identity on \mathcal{N} and satisfies the $\rho_{\mathcal{N}}$ -DBC condition, the above definition is independent of the choice of invariant state $\rho_{\mathcal{N}}$ (see Lemma 2.6, also [8, Theorem 2]). Note that in finite dimensions, the constant λ is always strictly less than 1, otherwise there would exist a nonzero $X \notin \mathcal{N}$ such that $E_1(X) = X$, $E_2(X) = X$ and hence $X \in \mathcal{N}$, which leads to a contradiction. We now show that the L_2 -clustering condition is also a sufficient condition for (45).

Theorem 5.1. *Let $\sigma \in \mathcal{D}(E_{\mathcal{N}})$. Denote $\|E_1 \circ E_2 - E_{\mathcal{N}} : L_2(\sigma) \rightarrow L_2(\sigma)\| = \lambda < 1$ as the L_2 -clustering constant. Then for any state ρ ,*

$$D(\rho \| \rho_{\mathcal{N}}) \leq c(D(\rho \| \rho_1) + D(\rho \| \rho_2)), \quad (47)$$

where

(i) *the constant c satisfies*

$$\frac{1}{1-\lambda^2} \leq c \leq \frac{2C_{\tau}(\mathcal{M} : \mathcal{N})}{(1-\lambda)^2}; \quad (48)$$

(ii) *if, in addition, $\lambda < \frac{1}{\sqrt{2}}$,*

$$c \leq 1 + \left(\frac{\lambda}{1-\lambda} + \frac{\lambda^2}{1-2\lambda^2} \right) C_{\tau}(\mathcal{M} : \mathcal{N}). \quad (49)$$

Similarly, for any $n \in \mathbb{N}$ and all states $\rho \in \mathcal{D}(\mathcal{M} \otimes \mathbb{M}_n)$, we have

$$D(\rho \| (E_{\mathcal{N}*} \otimes \text{id})(\rho)) \leq c_{\text{cb}} \left(D(\rho \| (E_{1*} \otimes \text{id})(\rho)) + D(\rho \| (E_{2*} \otimes \text{id})(\rho)) \right), \quad (50)$$

where c_{cb} satisfies either (48) or (49) after replacing $C_{\tau}(\mathcal{M} : \mathcal{N})$ by $C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N})$.

Proof. The lower bound was proven at the beginning of the section, so we focus on the upper bound. Note that E_1 , E_2 and $E_{\mathcal{N}}$ are all projections on $L_2(\rho_{\mathcal{N}})$. For a state ρ , we write $\rho_{12} = E_{1*}E_{2*}(\rho)$ and $\rho_{21} = E_{2*}E_{1*}(\rho)$. By the L_2 -clustering condition

$$\begin{aligned} \|\rho - \rho_{21}\|_{\rho_{\mathcal{N}}^{-1}} &\geq \|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}} - \|\rho_{\mathcal{N}} - \rho_{21}\|_{\rho_{\mathcal{N}}^{-1}} \\ &\geq (1 - \lambda) \|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}. \end{aligned} \quad (51)$$

Moreover, since E_{1*} , E_{2*} and $E_{\mathcal{N}*}$ are projections on $L_2(\rho_{\mathcal{N}}^{-1})$,

$$\begin{aligned} &\|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 - \|\rho - \rho_1\|_{\rho_{\mathcal{N}}^{-1}}^2 - \|\rho - \rho_2\|_{\rho_{\mathcal{N}}^{-1}}^2 \\ &\leq \|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 - \|\rho - \rho_2\|_{\rho_{\mathcal{N}}^{-1}}^2 - \|\rho_2 - E_{2*}(\rho_1)\|_{\rho_{\mathcal{N}}^{-1}}^2 \\ &\leq \|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 - \|\rho - E_{2*}(\rho_1)\|_{\rho_{\mathcal{N}}^{-1}}^2 \\ &= \|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 - \|\rho - \rho_{21}\|_{\rho_{\mathcal{N}}^{-1}}^2 \\ &\leq (1 - (1 - \lambda)^2) \|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2, \end{aligned}$$

where the last line follows from (51). Namely, we have

$$\|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 \leq \frac{1}{(1 - \lambda)^2} (\|\rho - \rho_1\|_{\rho_{\mathcal{N}}^{-1}}^2 + \|\rho - \rho_2\|_{\rho_{\mathcal{N}}^{-1}}^2).$$

Now, using Lemma 2.2,

$$\begin{aligned} D(\rho\|\rho_{\mathcal{N}}) &\leq \|\rho - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 \\ &\leq \frac{1}{(1 - \lambda)^2} (\|\rho - \rho_1\|_{\rho_{\mathcal{N}}^{-1}}^2 + \|\rho - \rho_2\|_{\rho_{\mathcal{N}}^{-1}}^2) \\ &\leq \frac{C_{\tau}(\mathcal{M} : \mathcal{N})}{(1 - \lambda)^2} (\|\rho - \rho_1\|_{\rho_1(t)^{-1}}^2 + \|\rho - \rho_2\|_{\rho_2(t)^{-1}}^2), \end{aligned}$$

where $\rho_1(t) = t\rho + (1 - t)\rho_1$ and $\rho_2(t) = t\rho + (1 - t)\rho_2$. As in Lemma 2.2, for $i = 1, 2$ $D(\rho\|\rho_i) = \int_0^1 \int_0^s \|\rho - \rho_i\|_{\rho_i(t)}^2 dt ds$. Then, integrating the above inequality, we have

$$D(\rho\|\rho_{\mathcal{N}}) \leq \frac{2C_{\tau}(\mathcal{M} : \mathcal{N})}{(1 - \lambda)^2} (D(\rho\|\rho_1) + D(\rho\|\rho_2)),$$

which proves (i). For (ii), by the chain rule ([46, Lemma 3.4]), we have

$$\begin{aligned} D(\rho\|\rho_{\mathcal{N}}) &= D(\rho\|\rho_1) + D(\rho_1\|\rho_{\mathcal{N}}) \\ &= D(\rho\|\rho_1) + D(\rho_1\|\rho_{12}) + \text{tr}(\rho_1(\ln \rho_{12} - \ln \rho_{\mathcal{N}})), \end{aligned}$$

and similarly,

$$\begin{aligned} D(\rho\|\rho_{\mathcal{N}}) &= D(\rho\|\rho_2) + D(\rho_2\|\rho_{\mathcal{N}}) \\ &= D(\rho\|\rho_2) + D(\rho_2\|\rho_{21}) + \text{tr}(\rho_2(\ln \rho_{21} - \ln \rho_{\mathcal{N}})). \end{aligned}$$

It suffices to estimate the error term $\text{tr}(\rho_1(\ln \rho_{12} - \ln \rho_{\mathcal{N}}))$ and $\text{tr}(\rho_2(\ln \rho_{21} - \ln \rho_{\mathcal{N}}))$. Recall the integral identity that for positive $A, B > 0$

$$\ln A - \ln B = \int_0^\infty \frac{1}{A+s} (A-B) \frac{1}{B+s} ds.$$

Thus, by Cauchy–Schwarz inequality and Lemma 2.2,

$$\begin{aligned} & \text{tr}(\rho_1(\ln \rho_{12} - \ln \rho_{\mathcal{N}})) \\ &= \text{tr}((\rho_1 - \rho_{12})(\ln \rho_{12} - \ln \rho_{\mathcal{N}})) + \text{tr}(\rho_{12}(\ln \rho_{12} - \ln \rho_{\mathcal{N}})) \\ &= \int_0^\infty \text{tr} \left((\rho_1 - \rho_{12}) \frac{1}{\rho_{12}+s} (\rho_{12} - \rho_{\mathcal{N}}) \frac{1}{\rho_{\mathcal{N}}+s} \right) ds + D(\rho_{12} \parallel \rho_{\mathcal{N}}) \\ &\leq \|\rho_1 - \rho_{12}\|_{\rho_{12}^{-1}} \|\rho_{12} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}} + \|\rho_{12} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2. \end{aligned}$$

Similarly,

$$\text{tr}(\rho_2(\ln \rho_{21} - \ln \rho_{\mathcal{N}})) \leq \|\rho_2 - \rho_{21}\|_{\rho_{21}^{-1}} \|\rho_{21} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}} + \|\rho_{21} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2.$$

Note that by the L_2 -clustering condition,

$$\begin{aligned} \|\rho_{12} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}} &\leq \lambda \|\rho_2 - \rho_{21}\|_{\rho_{\mathcal{N}}^{-1}} \leq \lambda (\|\rho_2 - \rho_{21}\|_{\rho_{\mathcal{N}}^{-1}} + \|\rho_{21} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}), \\ \|\rho_{21} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}} &\leq \lambda \|\rho_1 - \rho_{12}\|_{\rho_{\mathcal{N}}^{-1}} \leq \lambda (\|\rho_1 - \rho_{12}\|_{\rho_{\mathcal{N}}^{-1}} + \|\rho_{12} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}). \end{aligned}$$

Thus

$$\begin{aligned} \|\rho_{12} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 &\leq 2\lambda^2 (\|\rho_2 - \rho_{21}\|_{\rho_{\mathcal{N}}^{-1}}^2 + \|\rho_{21} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2), \\ \|\rho_{21} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 &\leq 2\lambda^2 (\|\rho_1 - \rho_{12}\|_{\rho_{\mathcal{N}}^{-1}}^2 + \|\rho_{12} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2). \end{aligned}$$

Therefore, for $\lambda < \frac{1}{\sqrt{2}}$, by Lemma 2.4,

$$\begin{aligned} \|\rho_{12} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 + \|\rho_{21} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 &\leq \frac{2\lambda^2}{1-2\lambda^2} (\|\rho_1 - \rho_{12}\|_{\rho_{\mathcal{N}}^{-1}}^2 + \|\rho_2 - \rho_{21}\|_{\rho_{\mathcal{N}}^{-1}}^2) \\ &\leq \frac{2\lambda^2}{1-2\lambda^2} (2C_1 D(\rho_1 \parallel \rho_{12}) + 2C_2 D(\rho_2 \parallel \rho_{21})), \end{aligned}$$

$C_1 = C_\tau(\mathcal{N}_1 : \mathcal{N})$, and $C_2 = C_\tau(\mathcal{N}_2 : \mathcal{N})$. On the other hand, denoting

$$M := \max\{\sqrt{2C_1 D(\rho_1 \parallel \rho_{12})}, \sqrt{2C_2 D(\rho_2 \parallel \rho_{21})}\},$$

we have

$$\begin{aligned} & \|\rho_1 - \rho_{12}\|_{\rho_{12}^{-1}} \|\rho_{12} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}} + \|\rho_2 - \rho_{21}\|_{\rho_{21}^{-1}} \|\rho_{21} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}} \\ &\leq M (\|\rho_{12} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}} + \|\rho_{21} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}) \\ &\leq M \frac{\lambda}{1-\lambda} (\|\rho_2 - \rho_{21}\|_{\rho_{\mathcal{N}}^{-1}} + \|\rho_1 - \rho_{12}\|_{\rho_{\mathcal{N}}^{-1}}) \\ &\leq M \frac{\lambda}{1-\lambda} (\sqrt{2C_2 D(\rho_2 \parallel \rho_{21})} + \sqrt{2C_1 D(\rho_1 \parallel \rho_{12})}) \\ &\leq \frac{2\lambda}{1-\lambda} C_{\max} (D(\rho \parallel \rho_1) + D(\rho \parallel \rho_2)), \end{aligned}$$

where $C_{\max} := \max\{C_{\tau}(\mathcal{N}_1 : \mathcal{N}), C_{\tau}(\mathcal{N}_2 : \mathcal{N}), C_{\tau}(\mathcal{M} : \mathcal{N}_1), C_{\tau}(\mathcal{M} : \mathcal{N}_2)\} \leq C_{\tau}(\mathcal{M} : \mathcal{N})$. Therefore, we have

$$\begin{aligned} & 2D(\rho \| \rho_{\mathcal{N}}) - D(\rho \| \rho_1) - D(\rho \| \rho_2) - D(\rho_1 \| \rho_{12}) - D(\rho_2 \| \rho_{21}) \\ &= \text{tr}(\rho_1(\ln \rho_{12} - \ln \rho_{\mathcal{N}})) + \text{tr}(\rho_2(\ln \rho_{21} - \ln \rho_{\mathcal{N}})) \\ &\leq \|\rho_1 - \rho_{12}\|_{\rho_{12}^{-1}} \|\rho_{12} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}} + \|\rho_2 - \rho_{21}\|_{\rho_{21}^{-1}} \|\rho_{21} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}} \\ &\quad + \|\rho_{12} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 + \|\rho_{21} - \rho_{\mathcal{N}}\|_{\rho_{\mathcal{N}}^{-1}}^2 \\ &\leq \frac{2\lambda}{1-\lambda} C_{\max}(D(\rho \| \rho_1) + D(\rho \| \rho_2)) + \frac{2\lambda^2}{1-2\lambda^2} (C_1 D(\rho_1 \| \rho_{12}) + C_2 D(\rho_2 \| \rho_{21})) \\ &\leq \left(\frac{2\lambda^2}{1-2\lambda^2} + \frac{2\lambda}{1-\lambda} \right) C_{\max}(D(\rho \| \rho_1) + D(\rho \| \rho_2)). \end{aligned}$$

follows after rearranging the terms in the outer bounds and a last use of the data processing inequality. The proof of (50) follows the exact same lines after replacing $C_{\tau}(\mathcal{M} : \mathcal{N})$ by $C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N})$. \square

Remark 5.2. By using $\rho_i(t) = t\rho + (1-t)\rho_i \leq (tC_{\tau}(\mathcal{M} : \mathcal{N}) + (1-t)C_{\tau}(\mathcal{N}_i : \mathcal{N}))\rho_{\mathcal{N}}$, the constant c in (i) can be improved to

$$c = \frac{K(C_{\tau}(\mathcal{M} : \mathcal{N}), \max\{C_{\tau}(\mathcal{N}_1 : \mathcal{N}), C_{\tau}(\mathcal{N}_2 : \mathcal{N})\})}{(1-\lambda)^2},$$

where $K(c_1, c_2) := \frac{c_1 \ln c_1 - c_1 + c_2}{(c_1 - c_2)^2}$. c_{cb} . For (ii), as shown in the proof, the constant $C_{\tau}(\mathcal{M} : \mathcal{N})$ in (ii) can be improved to

$$\begin{aligned} C_{\max} &:= \max\{C_{\tau}(\mathcal{N}_1 : \mathcal{N}), C_{\tau}(\mathcal{N}_2 : \mathcal{N}), C_{\tau}(\mathcal{M} : \mathcal{N}_1), C_{\tau}(\mathcal{M} : \mathcal{N}_2)\} \\ &\leq C_{\tau}(\mathcal{M} : \mathcal{N}), \end{aligned}$$

The same remark holds for c_{cb} in both cases.

The above theorem gives the equivalence of L_2 -clustering condition and complete approximate tensorization c_{cb} given finite index. Moreover, the (ii) above recovers the optimal constant $c = 1$ in the case of commuting squares ($\lambda = 0$).

In the classical literature [24, 27], approximate tensorization constants were found under the strong condition of smallness of the norm $\|E_1 \circ E_2 - E_{\mathcal{N}} : L_1 \rightarrow L_{\infty}\|$ instead of the L_2 -condition $\|E_1 \circ E_2 - E_{\mathcal{N}} : L_2 \rightarrow L_2\|$ that we use. In that setting, the approximate tensorization constants obtained in Theorem 5.1 are not tight because the Pimsner–Popa indices coincide with the dimension bounds for the $L_1 \rightarrow L_{\infty}$ norm. Quantum extensions using $L_1 \rightarrow L_{\infty}$ cluster condition were recently found in [8], however they yield additive error terms in generic noncommutative situations, e.g. when the algebra \mathcal{N} is not trivial. This generalization however was found fruitful in deriving the positivity of the MLSI constant for some classes of Gibbs samplers in [19], where the multiplicative constant could be related to the notion of clustering of correlations in the equilibrium Gibbs state. There, the analysis could be reduced to the case of states ρ for which the additive error vanishes. However, the problem of the vanishing of the additive constant for general states remained open.

After the preprint submission of a preliminary version of the present paper, LaRacuente in [53] introduced a method based on our Lemma 2.2 to find asymptotically tight approximate tensorization constants. The main idea from [53] resides in an application of the map $(E_1 \circ E_2)^k$ in order to control the $L_1 \rightarrow L_\infty$ norm via $\|(E_1 \circ E_2)^k - E_{\mathcal{N}} : L_2(\sigma) \rightarrow L_2(\sigma)\| = \lambda^k$, at the cost of having to multiply the approximate tensorization constant by k . Our next two results can be interpreted as a merging of these different contributions to improve the tightness of the approximate tensorization constant. We first start with a theorem which generalizes the original results of [24, 27] without the additive error terms found in [8], at the cost of having to replace the $L_1 \rightarrow L_\infty$ condition by the completely bounded version of it. Although the condition is closer to that of [53], our proof is arguably more straightforward and resembles the ones of [8, 19, 24, 27].

Theorem 5.3. *Let $\mathcal{N} \subset \mathcal{M}$ be a finite dimensional von Neumann subalgebra and $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ be a conditional expectation. Let $\Phi : \mathcal{M}_* \rightarrow \mathcal{M}_*$ be a quantum channel such that Φ^* is GNS-symmetric to a full-rank $E_{\mathcal{N}}$ -invariant states and satisfies $\Phi^* \circ E_{\mathcal{N}} = E_{\mathcal{N}} \circ \Phi^* = E_{\mathcal{N}}$. Suppose, for some $0 < \varepsilon < \sqrt{2 \ln 2} - 1$, that*

$$(1 - \varepsilon)E_{\mathcal{N}} \leq_{\text{cp}} \Phi^* \leq_{\text{cp}} (1 + \varepsilon)E_{\mathcal{N}},$$

where the inequalities hold in completely positive order. Then, for all $n \in \mathbb{N}$ and states $\rho \in \mathcal{D}(\mathcal{M} \otimes \mathbb{M}_n)$,

$$D(\rho \| E_{\mathcal{N}*}(\rho)) \leq \frac{1}{1 - \varepsilon^2(2 \ln 2 - 1)^{-1}} D(\rho \| \Phi^2(\rho)). \quad (52)$$

Proof. Let $\rho \in \mathcal{D}(\mathcal{M} \otimes \mathbb{M}_n)$ and $\rho_{\mathcal{N}} := E_{\mathcal{N}*}(\rho)$. Then,

$$\begin{aligned} D(\rho \| \rho_{\mathcal{N}}) - D(\rho \| \Phi^2(\rho)) &= \text{tr} [\rho (-\ln(\rho) + \ln A)] \\ &= -D\left(\rho \left\| \frac{A}{\text{tr}(A)}\right.\right) + \ln \text{tr}(A) \leq \ln \text{tr}(A) \end{aligned}$$

for $A := \exp(\ln \Phi^2(\rho) - \ln \rho_{\mathcal{N}} + \ln \rho)$. Here the last inequality follows from the positivity of the relative entropy. Using Lieb's triple matrix inequality (see [57, Theorem 7]),

$$\ln \text{tr}(A) \leq \ln \int_0^\infty \text{tr} \left(\Phi^2(\rho) \frac{1}{\rho_{\mathcal{N}} + s} \rho \frac{1}{\rho_{\mathcal{N}} + s} \right) ds.$$

Then, by the GNS-symmetry of Φ^* ,

$$\begin{aligned} D(\rho \| \rho_{\mathcal{N}}) &\leq D(\rho \| \Phi^2(\rho)) + \ln \int_0^\infty \text{tr} \left(\Phi(\rho) \frac{1}{\rho_{\mathcal{N}} + s} \Phi(\rho) \frac{1}{\rho_{\mathcal{N}} + s} \right) ds \\ &\stackrel{(1)}{\leq} D(\rho \| \Phi^2(\rho)) + \int_0^\infty \text{tr} \left((\Phi(\rho) - E_{\mathcal{N}*}(\rho)) \right. \\ &\quad \left. \times \frac{1}{\rho_{\mathcal{N}} + s} (\Phi(\rho) - E_{\mathcal{N}*}(\rho)) \frac{1}{\rho_{\mathcal{N}} + s} \right) ds, \end{aligned}$$

where (1) arises from the basic inequality $\ln(x) \leq x - 1$ and the trace preserving property of Φ and $E_{\mathcal{N}_*}$. Now, since $\Phi^* \geq_{\text{cp}} (1 - \varepsilon)E_{\mathcal{N}}$, there exists a quantum channel $\Psi : \mathcal{M}_* \rightarrow \mathcal{M}_*$ such that $\Phi = (1 - \varepsilon)E_{\mathcal{N}_*} + \varepsilon\Psi$. Therefore,

$$\begin{aligned} D(\rho \| \rho_{\mathcal{N}}) &\leq D(\rho \| \Phi^2(\rho)) + \varepsilon^2 \int_0^\infty \text{tr} \left((\Psi(\rho) - E_{\mathcal{N}_*}(\rho)) \right. \\ &\quad \left. \times \frac{1}{\rho_{\mathcal{N}} + s} (\Psi(\rho) - E_{\mathcal{N}_*}(\rho)) \frac{1}{\rho_{\mathcal{N}} + s} \right) ds \\ &= D(\rho \| \Phi^2(\rho)) + \varepsilon^2 \|(\Psi - E_{\mathcal{N}_*})(\rho)\|_{\rho_{\mathcal{N}}}^2 \\ &\stackrel{(2)}{\leq} D(\rho \| \Phi^2(\rho)) + \varepsilon^2 k(2)^{-1} D(\Psi(\rho) \| \rho_{\mathcal{N}}) \\ &\leq D(\rho \| \Phi^2(\rho)) + \varepsilon^2 k(2)^{-1} D(\rho \| \rho_{\mathcal{N}}), \end{aligned}$$

where (2) comes from Lemma 2.2 and the fact that $\Phi^* \leq (1 + \varepsilon)E_{\mathcal{N}}$, so that

$$\Psi(\rho) \leq \varepsilon^{-1}(1 + \varepsilon - (1 - \varepsilon))\rho_{\mathcal{N}} = 2\rho_{\mathcal{N}}.$$

The result follows after rearranging the terms in the last inequality. □

The above theorem can be used to derive approximate tensorization bounds. We are grateful to Chi-Fang Chen for pointing out that the approximate tensorization constant in the older version can be improved with a factor $\frac{1}{m}$.

Corollary 5.4. *Let $\mathcal{N}_1, \dots, \mathcal{N}_m \subset \mathcal{M}$ be finite dimensional von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$, and let $\mathcal{N} = \bigcap_{i=1}^m \mathcal{N}_i$. Let $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ and $E_i : \mathcal{M} \rightarrow \mathcal{N}_i$ be some corresponding conditional expectations. Suppose for some full-rank $E_{\mathcal{N}}$ -invariant state $\sigma = E_{\mathcal{N}_*}(\sigma)$, $\sigma = E_{i_*}(\sigma)$ for each i . Then for $\Phi^* = \frac{1}{m} \sum_{i=1}^m E_i$, we have that, for all $n \in \mathbb{N}$ and all states $\rho \in \mathcal{D}(\mathcal{M} \otimes \mathbb{M}_n)$,*

$$D(\rho \| E_{\mathcal{N}_*}(\rho)) \leq \frac{2k}{1 - \varepsilon^2(2 \ln(2) - 1)^{-1}} \sum_{i=1}^m D(\rho \| E_{i_*}(\rho)), \tag{53}$$

whenever $0 < \varepsilon < \sqrt{2 \ln(2) - 1}$ and k satisfies

$$(1 - \varepsilon)E_{\mathcal{N}} \leq_{\text{cp}} (\Phi^*)^k \leq_{\text{cp}} (1 + \varepsilon)E_{\mathcal{N}}. \tag{54}$$

If, additionally, E_i and $E_{\mathcal{N}}$ are trace preserving conditional expectations, we have that for all $n \in \mathbb{N}$ and all state $\rho \in \mathcal{D}(\mathcal{M} \otimes \mathbb{M}_n)$,

$$D(\rho \| E_{\mathcal{N}_*}(\rho)) \leq \frac{4}{m} \left\lceil \frac{\ln C_{\text{cb}}(\mathcal{M} : \mathcal{N}) + 1}{\ln(\lambda^{-1})} \right\rceil \sum_{i=1}^m D(\rho \| E_{i_*}(\rho)), \tag{55}$$

where $\lambda := \|\Phi - E_{\mathcal{N}} : L_2(\text{tr}) \rightarrow L_2(\text{tr})\|$ and $\lceil s \rceil$ denotes the smallest integer greater than or equal to s .

Proof. Equation (53) is a direct consequence of Theorem 5.3 and successive applications of convexity of the relative entropy and the chain rule [53, Lemma 3.2],

$$\begin{aligned} D(\rho \|\Phi^{2k}(\rho)) &\leq \frac{1}{m} \sum_{i=1}^m D(\rho \|\mathcal{E}_{i*} \Phi^{2k-1}(\rho)) \\ &\leq \frac{1}{m} \sum_{i=1}^m D(\rho \|\mathcal{E}_{i*}(\rho)) + D(\rho \|\Phi^{2k-1}(\rho)) \\ &\leq \frac{2k}{m} \sum_{i=1}^m D(\rho \|\mathcal{E}_{i*}(\rho)). \end{aligned}$$

In the case when all \mathcal{E}_i and E are trace preserving, we prove in ‘‘Appendix A’’ that ε can be chosen as (see Lemma A.1)

$$\varepsilon = \lambda^k C_{\text{cb}}(\mathcal{M} : \mathcal{N})$$

for k large enough so that the condition (54) is satisfied. Therefore, we can choose $\varepsilon \leq \sqrt{\ln 2 - \frac{1}{2}}$ and $1 - \varepsilon^2(2 \ln(2) - 1)^{-1} \geq 1/2$ by taking

$$k = \left\lceil (-\ln \lambda)^{-1} \left(\ln C - \ln \sqrt{\ln 2 - \frac{1}{2}} \right) \right\rceil \leq \left\lceil \frac{\ln C + 1}{-\ln \lambda} \right\rceil, \quad C := C_{\text{cb}}(\mathcal{M} : \mathcal{N}).$$

Then (55) follows from (53). □

Remark 5.5. Although the bound (53) does not recover the exact tensorization for commuting conditional expectations, it has the merit over our other bounds to be independent of the index $C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N})$. In Section 6, we use the bound (55) to derive sharper CMLSI constants than Theorem 3.3 for several examples.

Remark 5.6. Another natural choice is $\Phi^* = \prod_{i=1}^m \mathcal{E}_i$, where the product can be interpreted with any order. Indeed, by the chain rule,

$$D\left(\rho \left\| \prod_{i=1}^m \mathcal{E}_i(\rho) \right.\right) \leq \sum_{i=1}^m D(\rho \|\mathcal{E}_i(\rho)).$$

By a similar argument as to that above, we obtain the approximate tensorization

$$D(\rho \|\mathcal{E}_{\mathcal{N}*}(\rho)) \leq 4k \sum_{i=1}^m D(\rho \|\mathcal{E}_{i*}(\rho)) \quad \text{where } k \leq \left\lceil \frac{\ln C_{\text{cb}}(\mathcal{E}_{\mathcal{N}}) + 1}{-\ln \lambda_p} \right\rceil$$

with $\lambda_p := \|\prod_{i=1}^m \mathcal{E}_i - \mathcal{E}_{\mathcal{N}}\|$. Let us also denote $\lambda_s := \frac{1}{m} \sum_{i=1}^m \mathcal{E}_i - \mathcal{E}_{\mathcal{N}}$. By the detectability lemma [4], we have

$$\lambda_p = \left\| \prod_{i=1}^m \mathcal{E}_i - \mathcal{E}_{\mathcal{N}} \right\| \leq \frac{1}{\frac{\lambda_E}{g^2} + 1},$$

where $\lambda_E = m - m\lambda_s$ is the spectral gap of $\sum_i \mathbb{1} - \mathcal{E}_i$ and g is the integer such each \mathcal{E}_i commute with all but at most g \mathcal{E}_j 's. Then when $\lambda_E \rightarrow 0$ is small, the approximate tensorization constant from $\Phi^* = \prod_{i=1}^m \mathcal{E}_i$ is always weaker than the one from $\Phi^* = \frac{1}{m} \sum_{i=1}^m \mathcal{E}_i$.

6. Applications and Examples

6.1. Symmetric Lindbladians

In this section, we use the approximate tensorization results of Section 5 to give tighter bounds on the CMLSI constant for generators \mathcal{L} which can be expressed as a summation of a family of generators

$$\mathcal{L} = \sum_{k=1}^m \mathcal{L}_k.$$

For simplicity, here we only consider symmetric generators. We say that a generator $\mathcal{L} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is symmetric if

$$\mathrm{tr}(Y^\dagger \mathcal{L}(X)) = \mathrm{tr}(\mathcal{L}(Y^\dagger)X), \quad \forall X, Y \in \mathcal{B}(\mathcal{H}).$$

Hence, $\mathcal{L} = \mathcal{L}_*$, $\mathcal{P}_t = \mathcal{P}_{t*}$ and the conditional expectations $E = E_*$ are all trace symmetric (or equivalently, \mathcal{P}_t is GNS-symmetric with respect to the completely mixed state $\frac{\mathbb{1}}{d_{\mathcal{H}}}$). In this case, we do not distinguish \mathcal{P}_t (resp. E and \mathcal{L}) from its predual \mathcal{P}_{t*} (resp. E_* and \mathcal{L}_*) because one can identify $\mathcal{B}(\mathcal{H}) \cong \mathcal{B}(\mathcal{H})_*$ via the Hilbert–Schmidt inner product in finite dimensions. The main idea here consists in using the approximate tensorization of the CMLSI constant.

Proposition 6.1. *Let $(\mathcal{L}_k : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}))_{k=1}^m$ be a family of symmetric generators and $\mathcal{L} = \sum_{k=1}^m \mathcal{L}_k$. Denote E , resp. E_k , as the conditional expectation onto the kernel of \mathcal{L} , resp. \mathcal{L}_k . Then the CMLSI constants of \mathcal{L} and \mathcal{L}_k satisfy*

$$\alpha_{\mathrm{CMLSI}}(\mathcal{L}) \geq C^{-1} \left(\min_k \alpha_{\mathrm{CMLSI}}(\mathcal{L}_k) \right),$$

where C satisfies the complete approximate tensorization

$$D(\rho \| E(\rho)) \leq C \sum_{i=1}^m D(\rho \| E_i(\rho)). \quad (56)$$

Moreover, denote $\lambda(\mathcal{L}_k)$ as the spectral gap of \mathcal{L}_k and λ_E as the spectral gap of $\mathcal{L}_E = \sum_k (E_k - \mathrm{id})$. Then,

$$C \leq \frac{4}{m} \left\lceil \frac{\ln C_{\mathrm{cb}}(E) + 1}{\ln(\lambda^{-1})} \right\rceil, \quad (57)$$

where $C_{\mathrm{cb}}(E)$ is the index of E and $\lambda := 1 - \frac{\lambda_E}{m}$.

Proof. For any state, we have

$$\begin{aligned}
 D(\rho \| E(\rho)) &\leq C \sum_{i=1}^m D(\rho \| E_i(\rho)) \\
 &\leq C \sum_{i=1}^m \alpha_{\text{CMLS I}}(\mathcal{L}_k)^{-1} \text{EP}_{\mathcal{L}_k}(\rho) \\
 &\leq C (\min_k \alpha_{\text{CMLS I}}(\mathcal{L}_k))^{-1} \sum_k \text{EP}_{\mathcal{L}_k}(\rho) \\
 &= C (\min_k \alpha_{\text{CMLS I}}(\mathcal{L}_k))^{-1} \text{EP}_{\mathcal{L}}(\rho).
 \end{aligned}$$

Here the first inequality above uses the assumption (56), and the last line uses the linearity of the entropy production (27). For the second assertion, we denote \mathcal{N} as the range of E , which coincides with the kernel of \mathcal{L} . Then $\frac{1}{m} \sum_k E_k - E$ is a positive contraction supported on $L_2(\mathcal{N})^\perp \subset L_2(\mathcal{B}(\mathcal{H}))$. Since

$$\frac{1}{m} \sum_k E_k - E_{\mathcal{N}} + \frac{1}{m} \mathcal{L}_E = \frac{1}{m} \sum_k E_k - E + \frac{1}{m} \sum_k \text{id} - E_k = \text{id} - E,$$

we have

$$\left\| \frac{1}{m} \sum_{k=1}^m E_k - E : L_2(\mathcal{B}(\mathcal{H})) \rightarrow L_2(\mathcal{B}(\mathcal{H})) \right\| = 1 - \frac{\lambda_E}{m} =: \lambda.$$

Then the approximate tensorization constant (57) follows from using $\Psi = \frac{1}{m} \sum_k E_k$ in Corollary 5.4. \square

Remark 6.2. The above proposition can be extended to a family of generator \mathcal{L}_k that are GNS-symmetric to a common full rank state σ . Indeed, the content in “Appendix B” can be extended to state preserving conditional expectations (see [7, Section 5]), hence Corollary (55) and the above Proposition 6.1 also extend to that setting. Nevertheless, the symmetric setting is sufficient for the examples discussed in this section.

We now illustrate the above bound on several examples. First, we note that, due to symmetry, the Lindbladian (26) takes a simple form

$$\mathcal{L}(\rho) = - \sum_{k=1}^m [a_k, [a_k, \rho]], \tag{58}$$

where $a_1, \dots, a_m \in \mathcal{B}(\mathcal{H})$ form a family of self-adjoint operators. Let us first consider a single term in the generator

$$\mathcal{L}_a(\rho) = -[a, [a, \rho]]$$

for a self-adjoint $a \in \mathcal{B}(\mathcal{H})$. Let $a = \sum_{i=1}^n \kappa_i P_i$ be the spectral decomposition of a , where P_i is the spectral projection with respect to the eigenvalue κ_i . One calculates that

$$[a, \rho] = \sum_{i,j} (\kappa_i - \kappa_j) P_i \rho P_j, \quad \mathcal{L}_a(\rho) = -[a, [a, \rho]] = - \sum_{i,j} (\kappa_i - \kappa_j)^2 P_i \rho P_j.$$

Then \mathcal{L}_a generates the semigroup

$$e^{\mathcal{L}_a t}(\rho) = \sum_{i,j} e^{-(\kappa_i - \kappa_j)^2 t} P_i \rho P_j,$$

which is a Schur multiplier semigroup (also called generalized dephasing semigroup). The invariant subalgebra $\mathcal{N}_a = \oplus_{i=1}^n \mathcal{B}(P_i \mathcal{H})$ and the indices are

$$C(\mathcal{B}(\mathcal{H}), \mathcal{N}_a) = C_{\text{cb}}(\mathcal{B}(\mathcal{H}), \mathcal{N}_a) = n \leq d_{\mathcal{H}}.$$

Viewing \mathcal{L}_a as a self-adjoint operator on $L_2(\mathcal{B}(\mathcal{H}), \text{tr})$, $P_i \mathcal{B}(\mathcal{H}) P_j$ corresponds to the eigenspace associated to the eigenvalue $-(\kappa_i - \kappa_j)^2$. Thus the norm and spectral gap of \mathcal{L}_a are

$$\|\mathcal{L}_a : L_2(\mathcal{B}(\mathcal{H}), \text{tr}) \rightarrow L_2(\mathcal{B}(\mathcal{H}), \text{tr})\| = \max_{i,j} |\kappa_i - \kappa_j|^2, \quad \lambda(\mathcal{L}_a) = \min_{i \neq j} |\kappa_i - \kappa_j|^2.$$

It was proved in [17, Theorem 4.23] that Schur multiplier semigroups admit the following estimates on their CMLS constant (note that our normalization of the CMLS constants differs with [17] by a factor of 2):

$$\frac{\lambda(\mathcal{L}_a)}{2 \ln(2n)} \leq \alpha_{\text{CMLS}}(\mathcal{L}_a) \leq 2\lambda(\mathcal{L}_a). \tag{59}$$

Moreover, for a commuting family $\{a_1, \dots, a_m\}$, the Lindbladian $\mathcal{L}(\rho) = -\sum_{k=1}^m [a_k, [a_k, \rho]]$ also generates a Schur multiplier semigroup and the above estimate (59) remains valid.

Applying Proposition 6.1, we extend the above estimate to general Lindbladians of the form $\mathcal{L} = \sum_k \mathcal{L}_{a_k}$ for not necessarily commuting operators a_k .

Corollary 6.3. *Let $(\mathcal{P}_t = e^{\mathcal{L}t} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}))_{t \geq 0}$ be a symmetric quantum Markov semigroup, and let its generator be given by*

$$\mathcal{L}(X) = \sum_{k=1}^m \mathcal{L}_{a_k}(X) = - \sum_{k=1}^m [a_k, [a_k, X]].$$

Denote $E_{\mathcal{N}}$, resp. E_k , as the conditional expectation onto the kernel of \mathcal{L} , resp. that of \mathcal{L}_{a_k} . Denote $\lambda(\mathcal{L}_{a_k})$ as the spectral gap of \mathcal{L}_{a_k} and λ_E as the spectral gap of $\mathcal{L}_E = \sum_k (E_k - \text{id})$. Then,

$$\alpha_{\text{CMLS}}(\mathcal{L}) \geq m \left\lceil \frac{2 \ln d + 1}{\ln(\lambda^{-1})} \right\rceil^{-1} \frac{\min_k \lambda(\mathcal{L}_{a_k})}{8 \ln(2d)},$$

where $d = \dim(\mathcal{H})$ and $\lambda := 1 - \frac{\lambda_E}{m}$.

Remark 6.4. For the spectral gap, we have that, for each k ,

$$\lambda(\mathcal{L}_{a_k}) \leq \lambda(\mathcal{L}) .$$

Also, the spectral gap of $\mathcal{L}_E = \sum_i E_i - \text{id}$ satisfies

$$\lambda(\mathcal{L}_E) \leq \| \mathcal{L} : L_2(\text{tr}) \rightarrow L_2(\text{tr}) \|^{-1} \lambda(\mathcal{L}) ,$$

because $\text{id} - E_k \geq - \| \mathcal{L}_{a_k} : L_2 \rightarrow L_2 \|^{-1} \mathcal{L}_{a_k}$ and $\| \mathcal{L}_{a_k} : L_2 \rightarrow L_2 \| \leq \| \mathcal{L} : L_2 \rightarrow L_2 \|$, since each generator \mathcal{L}_{a_k} is a non-positive operator in the L_2 sense. Therefore the constant obtained in Theorem 6.3 can be further estimated by

$$\alpha_{\text{CMLS}}(\mathcal{L}) \geq m \left[\frac{2 \ln d + 1}{-\ln(1 - \frac{\lambda(\mathcal{L})}{m})} \right]^{-1} \frac{\min_k \lambda(\mathcal{L}_{a_k})}{8 \ln(2d)} = \Omega\left(\frac{\lambda(\mathcal{L}) \min_k \lambda(\mathcal{L}_{a_k})}{(\ln d)^2}\right).$$

Thus, in general, the constant obtained in Corollary 6.3 has better asymptotics in terms of the dimension d than that of Theorem 3.3. We note that this approach does not apply to GNS-symmetric Lindbladians because in general the decomposition (58) in terms of Schur multipliers is not available.

Example 6.5. We now illustrate Theorem 6.3 for transference semigroups from the sub-Laplacian on $\text{SU}(2)$. We refer to [37, Section 4.2] for the transference technique for quantum Markov semigroups. Let $\text{SU}(2)$ be the special unitary group on \mathbb{C}^2 . Denote

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

as the anti-selfadjoint Pauli matrices. Then

$$\text{SU}(2) = \{ aX + bY + cZ + d\mathbb{1} \mid |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \} ,$$

which is isomorphic to the 3-sphere \mathbb{S}^3 . Its Lie algebra is the anti-selfadjoint matrix space

$$i(\mathbb{M}_2)_{\text{sa}} := \text{span}\{X, Y, Z\},$$

equipped with the Lie bracket relations

$$[X, Y] = 2Z, \quad [Y, Z] = 2X, \quad [Z, X] = 2Y. \tag{60}$$

The canonical bi-invariant Riemannian metric on $\text{SU}(2)$ admits $\{X, Y, Z\}$ as an orthonormal basis. The representation theory of $\text{SU}(2)$ gives the well-known spin structure of quantum mechanics, where any irreducible representation of $\text{SU}(2)$ is indexed by an integer $m \in \mathbb{N}^+$. Let $\eta_m : \mathfrak{su}(2) \rightarrow i(\mathbb{M}_m)_{\text{sa}}$ be the Lie algebra homomorphism induced by the m -th irreducible representation, and let $\{|j\rangle \mid j = 1, \dots, m\}$ be the eigenbasis of $\eta_m(Z)$. Denote that $X_m := \eta_m(X)$, and similarly for Y_m and Z_m as short notations. Under the normalization of (60),

$$\begin{aligned} X_m |j\rangle &= \sqrt{(j-1)(m-j+1)} |j-1\rangle - \sqrt{(j+1)(m-j-1)} |j+1\rangle, \\ Y_m |j\rangle &= i\sqrt{(j-1)(m-j+1)} |j-1\rangle + i\sqrt{(j+1)(m-j-1)} |j+1\rangle, \\ Z_m |j\rangle &= (m-2j+1) |j\rangle. \end{aligned}$$

For each irreducible representation, one can consider quantum Markov semigroups transferred from the heat semigroup $(e^{\Delta t})_{t \geq 0}$ on $SU(2)$, which is given by the Casimir operator

$$\Delta = X^2 + Y^2 + Z^2.$$

It follows from the complete Bakry-Emery Theorem [56, Theorem 4.3] that the heat semigroup has $\alpha_{\text{CMLS}}(\Delta) \geq 2$ because its Ricci curvature is 1. Therefore, the transferred Lindbladian on \mathbb{M}_m

$$\mathcal{L}_m^\Delta(\rho) = -[X_m, [X_m, \rho]] - [Y_m, [Y_m, \rho]] - [Z_m, [Z_m, \rho]]$$

admits a uniform CMLS constant $\alpha_{\text{CMLS}}(\mathcal{L}_m^\Delta) \geq 2$ by transference principle [37, Proposition 4.7]. Here we consider the canonical sub-Laplacian on $SU(2)$ given by

$$\Delta_H = X^2 + Y^2.$$

It is known (see e.g. [10]) that Δ_H is hypoelliptic and generates a classical Markov semigroup $(e^{\Delta_H t})_{t \geq 0}$; its transferred Lindbladian on \mathbb{M}_m is given by

$$\mathcal{L}_m^H(\rho) = -[X_m, [X_m, \rho]] - [Y_m, [Y_m, \rho]].$$

Although the CMLS constant for the sub-Laplacian Δ_H is currently still unknown (MLS is known from hypercontractivity), we can use approximate tensorization to obtain a uniform lower bound on $\alpha_{\text{CMLS}}(\mathcal{L}_m^H)$ for each m .

It suffices to consider $m \geq 3$. Denote \mathcal{X} and \mathcal{Y} as the kernel of $\mathcal{L}_{X_m}(\cdot) = [X_m, [X_m, \cdot]]$ (resp. \mathcal{L}_{Y_m}). Since both X_m and Y_m have distinct spectrum $\{m - 1, m - 3, m - 5, \dots, -(m - 1)\}$, \mathcal{X} and \mathcal{Y} are subalgebra generated by X_m and Y_m respectively. Then the conditional expectation $E_{\mathcal{X}}$ (resp. $E_{\mathcal{Y}}$) are the pinching map

$$E_{\mathcal{X}}(\rho) = \sum_i \langle x_i | \rho | x_i \rangle |x_i\rangle\langle x_i|, \quad E_{\mathcal{Y}}(\rho) = \sum_y \langle y_j | \rho | y_j \rangle |y_j\rangle\langle y_j|,$$

where $\{|x_i\rangle\}$ and $\{|y_j\rangle\}$ are the eigenbasis of X_m and respectively Y_m . Also denote $E_{\mathbb{C}} := \text{tr}(\cdot) m^{-1} \mathbb{1}$ the conditional expectation onto $\mathbb{C}1$. Note that it is sufficient to consider the map $E_{\mathcal{X}} E_{\mathcal{Y}} - E_{\mathbb{C}} : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_A)$ from its support \mathcal{Y} to the range \mathcal{X} , which is given by the matrix

$$O_{\mathcal{X}, \mathcal{Y}} = \left(| \langle x_i | y_j \rangle |^2 - \frac{1}{m} \right)_{i,j}.$$

Therefore, we have that

$$\begin{aligned} \|E_{\mathcal{X}} E_{\mathcal{Y}} - E_{\mathbb{C}} : L_1(\mathcal{B}(\mathcal{H}_A), d_A^{-1} \mathbb{1}) \rightarrow \mathcal{B}(\mathcal{H}_A)\|_{\text{cb}} &= \|O_{\mathcal{X}, \mathcal{Y}} : l_1(\mathcal{Y}) \rightarrow l_\infty(\mathcal{X})\| \\ &=: \lambda. \end{aligned}$$

Since all the eigenvectors $\{|x_i\rangle\}$ and $\{|y_j\rangle\}$ are explicit, one can numerically see that the constant λ in Corollary 5.4 satisfies $\lambda < 1/5$. Combining Corollary 5.4 and Lemma A.1, we get the approximate tensorization

$$D(\rho \| m^{-1} \mathbb{1}) \leq c (D(\rho \| E_{\mathcal{X}}(\rho)) + D(\rho \| E_{\mathcal{Y}}(\rho)))$$

for $c = 2(1 - \frac{1}{25}(2 \ln 2 - 1)^{-1})^{-1} \approx 2.231 < \frac{9}{4}$. Furthermore, since both X_m and Y_m have integer spectrum, they can be transferred from the Laplace operator $\Delta_{\mathbb{T}}$ on the unit torus \mathbb{T} . Then by transference and [17, Theorem 4.12] (note that our normalization differs by a factor of 2),

$$\alpha_{\text{CMLS}}(\mathcal{L}_{X_m}) = \alpha_{\text{CMLS}}(\mathcal{L}_{Y_m}) \geq \frac{1}{2 \ln(3)}$$

for each m . Therefore we obtain the dimension-free numerical bound

$$\alpha_{\text{CMLS}}(\mathcal{L}_m^H) \geq \left(\frac{9}{2} \ln(3)\right)^{-1} \approx 0.2,$$

which is obviously tighter than the dimension dependent estimate from Theorem 3.3 (\mathcal{L}_m^H has a uniform spectral gap 1 for all m by transference, see [10, Proposition 3.1]). It remains open whether the sub-Laplacian Δ_H itself satisfies CMLS.

6.2. Subsystem Lindbladians

In this subsection, we consider symmetric Markov semigroups whose Lindblad operators act on edges of a given graph. Such generators have been extensively studied for functional inequalities in the classical setting. (see e.g. [76] and the references therein). Here, we use (i) the CMLS constants found in Theorem 3.3 to control the local interaction in combination with (ii) the sharpening of the approximate tensorization constant found in Section 6 to derive asymptotically tight lower bounds on the CMLS constant for various models of relevance.

Let $G := (V, E)$ be a finite, connected and undirected graph with vertex set V , of cardinality $|V| = n$, and edge set $E := \{(v, w) \in V \times V : v \sim w\}$. We recall that the degree $\text{deg}(v)$ of a vertex $v \in V$ is the number of edges that are incident to v . Moreover, G is said to be γ -regular if all vertices $v \in V$ have same degree γ . Important examples include finite groups through the scope of their Cayley graphs. Given a graph $G := (V, E)$, the graph Laplacian Δ_G acting on the function spaces $\{f | f : V \rightarrow \mathbb{C}\}$ is defined as

$$\Delta_G f(v) = \sum_{w:(w,v) \in E} (f(w) - f(v)), \tag{61}$$

where Δ_G is a negative semi-definite matrix on $l_2(V)$, which generates the heat semigroup $T_t = e^{\Delta_G t}$ on $l_\infty(V)$. Note that here we choose Δ_G to be negative to match our convention for quantum Lindbladians. The spectral gap is defined as the gap between the largest and second largest eigenvalues of Δ_G .

Here we consider a quantum Markov semigroup with the locality structure of $G = (V, E)$ consists in introducing a local evolution on the n -fold tensor product $\mathcal{H}_V := \bigotimes_{v \in V} \mathcal{H}_v$ of a given finite dimensional local Hilbert space \mathcal{H} , namely, an n -qudit system for $d = \text{dim}(\mathcal{H})$. The Lindblad operators are supported on the edges $e \in E$ as

$$\mathcal{L}_G := \sum_{e \in E} \mathcal{L}_e, \text{ where } \mathcal{L}_e(\rho) := \sum_{j \in J^{(e)}} L_j^{(e)} \rho L_j^{(e)} - \frac{1}{2} \{L_j^{(e)} L_j^{(e)}, \rho\}, \tag{62}$$

where for any edge $e = (v, w)$ and $\{L_j^{(e)}\}_{j \in J(e)}$ are the family of local Lindblad operators that act trivially on subsystems other than $\mathcal{H}_v \otimes \mathcal{H}_w$. We call (62) a subsystem Lindbladian. In what follows, we denote by $\mathcal{M}_V := \mathcal{B}(\mathcal{H}_V)$ (resp. $\mathcal{M}_e := \mathcal{B}(\mathcal{H}_v \otimes \mathcal{H}_w)$) the algebra of operators on which \mathcal{L}_G (resp. \mathcal{L}_e for $e = (v, w) \in E$) acts. We also denote by E_e (resp. E_G) the conditional expectation projecting onto the kernel of \mathcal{L}_e for $e \in E$ (resp. of \mathcal{L}_G). Finally, we introduce the indices

$$C_G := C_{\text{cb}}(\mathcal{M}_V : E_G(\mathcal{M}_V)), \quad C_e := C_{\text{cb}}(\mathcal{M}_e : E_e(\mathcal{M}_e)),$$

and c_G as the minimum non-zero eigenvalue of the Choi-Jamiolkowski state J_{E_G} of E_G . It is also useful to introduce the generator

$$\tilde{\mathcal{L}}_G := \sum_{e \in E} E_e - \text{id}.$$

As previously discussed, the lower bounds on the complete modified logarithmic Sobolev constant derived in Theorem 3.3 are asymptotically not tight when the total dimension is large. For instance, in the case of a primitive semigroup on n -qudit systems, the completely bounded Pimsner–Popa index is equal to d^n , where d is the dimension of the local Hilbert space \mathcal{H} . This gives lower bounds on the CMLSI constant of a subsystem Lindbladian that are exponentially small in the number of vertices. In the next theorem, we essentially leverage the locality structure of \mathcal{L}_G to provide exponentially tighter bounds by combining Theorem 3.3, Corollary 5.4, Lemma A.1 and the detectability lemma [1, 2, 4, 49].

Theorem 6.6. *Let $G = (V, E)$ be a finite, connected, undirected graph with maximum degree γ , and let \mathcal{L}_G be a symmetric subsystem Lindbladian of the form (62). Then, for all $m \in \mathbb{N}$ and any state $\rho \in \mathcal{D}(\mathcal{H}_V \otimes \mathbb{C}^{\otimes m})$,*

$$D(\rho \| E_G(\rho)) \leq 4 \left\lceil \frac{\ln(C) + 1}{\ln\left(\frac{\tilde{\mathcal{L}}_G}{4(\gamma-1)^2} + 1\right)} \right\rceil \sum_{e \in E} D(\rho \| E_e(\rho)), \tag{63}$$

where $C := \min\{C_G, c_G^{-1}\}$ and $\lambda(\tilde{\mathcal{L}}_G)$ is the spectral gap of $\tilde{\mathcal{L}}_G$. Moreover, the CMLSI constant for the generator \mathcal{L}_G satisfies

$$\begin{aligned} \frac{1}{4} \left\lceil \frac{\ln(C) + 1}{\ln\left(\frac{\lambda(\tilde{\mathcal{L}}_G)}{4(\gamma-1)^2} + 1\right)} \right\rceil^{-1} \min_{e \in E} \frac{\lambda(\mathcal{L}_e)}{C_e} &\leq \frac{1}{4} \left\lceil \frac{\ln(C) + 1}{\ln\left(\frac{\lambda(\tilde{\mathcal{L}}_G)}{4(\gamma-1)^2} + 1\right)} \right\rceil^{-1} \min_{e \in E} \alpha_{\text{CMLSI}}(\mathcal{L}_e) \\ &\leq \alpha_{\text{CMLSI}}(\mathcal{L}_G). \end{aligned} \tag{64}$$

As a consequence, whenever $\lambda(\tilde{\mathcal{L}}_G)$ is uniformly lower bounded by a constant independent of $|V|$, $\alpha_{\text{CMLSI}}(\tilde{\mathcal{L}}_G) = \Omega\left(\frac{1}{\ln(C)}\right)$, hence recovering the asymptotics of classical expanders.

Proof. We first establish (63): by Remark 5.6, the approximate tensorization constant of the family $\{E_e\}_{e \in E}$ of conditional expectations can be upper bounded by the constant $4k$, where k is the integer such that

$$(1 - \varepsilon) E_G \leq \left(\prod_{e \in E} E_e \right)^k \leq (1 + \varepsilon) E_G$$

for $\varepsilon := \sqrt{\ln(2) - \frac{1}{2}}$. Here the ordering in the product $\prod_{e \in E} E_e$ is arbitrary. Next, we have from Lemmas A.1 and A.2 in ‘‘Appendix A’’ that k can be chosen as

$$k = \left\lceil \frac{1 + \ln C}{\ln \frac{1}{\lambda}} \right\rceil, \quad \text{where } \lambda := \left\| \prod_{e \in E} E_e - E_G : L_2 \rightarrow L_2 \right\|,$$

$$C := \min\{C_G, c_G^{-1}\}.$$

Finally, the L_2 -constant λ can be controlled by the gap of the generator $\tilde{\mathcal{L}}_G$ using the detectability Lemma from [4, Corollary 3]:

$$\lambda = \left\| \prod_{e \in E} E_e - E_G : L_2 \rightarrow L_2 \right\| \leq \frac{1}{\lambda(\tilde{\mathcal{L}}_G)/4(\gamma - 1)^2 + 1}.$$

Here γ is the maximum degree of the graph. We then use Theorem 3.3 to further lower bound $\alpha_{\text{CMLS}}(\mathcal{L}_e)$ in terms of the spectral gap $\lambda(\mathcal{L}_e)$ and corresponding index C_e . □

Remark 6.7. A sequence of γ -regular graphs $\{G_i = (V_i, E_i)\}_{i \in \mathbb{N}}$ of increasing size with $\lim_i |V_i| = +\infty$ is called a *family of expander graphs* if there exists $\lambda_0 > 0$ such that the spectral gaps $\lambda(\Delta_{G_i}) \geq \lambda_0$ uniformly for all i [45]. Modified logarithmic Sobolev inequalities for such generators have been widely considered in the classical literature, (see e.g. the survey [14]). In the limit $i \rightarrow \infty$, using Corollary 5.4 for the average map $\Phi = \frac{1}{|E|} \sum_e E_e$ instead of resorting to the detectability Lemma for the product map $\Phi = \prod_{e \in E} E_e$ would lead to a similar bound:

$$\frac{|E|}{4} \left\lceil \frac{\ln(C) + 1}{\ln(1 - \frac{\lambda(\tilde{\mathcal{L}}_G)}{|E|})} \right\rceil^{-1} \min_{e \in E} \frac{\lambda(\mathcal{L}_e)}{C_e} \leq \alpha_{\text{CMLS}}(\mathcal{L}_G).$$

In [15, Remark 5.6], it was shown that the MLSI constant of a random walk on an expander graph G of degree γ satisfies that

$$\frac{2\lambda_0}{\ln |G|} \leq \alpha_{\text{MLSI}}(\Delta_G) \leq \frac{2\gamma \ln(\gamma)^2}{\ln |G|}.$$

Since, for these graphs, $C = \ln |G|$, the above estimates show that our lower bounds recover the right dependence on $|G|$ up to multiplicative constants.

6.3. Random Permutations

We now illustrate the bounds derived in Theorem 6.6 on random permutations models. We consider quantum Markov semigroups introduced in [9, Section IV.D] which represent the action of a random transposition gate applied to two registers i, j on the n qudit system $\mathcal{H}_V = \mathcal{H}^{\otimes n}$ with $\dim(\mathcal{H}) = d$. Let $G = (V, E)$ be a finite graph with $|V| = n$. Denote the swap gate $S_{i,j}$ acting on registers of vertex i and j as

$$S_{i,j}(|\psi\rangle \otimes |\varphi\rangle) = |\varphi\rangle \otimes |\psi\rangle \tag{65}$$

for any two $|\psi\rangle, |\varphi\rangle \in \mathcal{H}$. The generator of the *quantum nearest neighbour random transposition* model [9, Section IV.D] is defined as

$$\mathcal{L}_G^{\text{NNRT}} := \sum_{e \in E} \mathcal{L}_e, \text{ where } \mathcal{L}_e(\rho) = \frac{1}{2} S_{i,j} \rho S_{i,j} - \rho$$

for $e = (i, j)$. The above generator can simply be understood as that of the natural action of the permutation group \mathcal{S}_n on $\mathcal{H}_V = \mathcal{H}^{\otimes n}$, which allows transitions between random adjacent registers connected by edges. In other words, $\mathcal{L}_G^{\text{NNRT}}$ is the subsystem Lindbladian of the graph $G = (V, E)$ with the local Lindbladian \mathcal{L}_e at edge $(i, j) \in E$.

Corollary 6.8. *Let $G = (V, E)$ be a connected finite graph and let $\mathcal{L}_G^{\text{NNRT}}$ be the generator of the quantum nearest neighbour random transposition model defined as above. Then*

$$\alpha_{\text{CMLS}}(\mathcal{L}_G^{\text{NNRT}}) \geq \frac{1}{2} \left[\frac{(\ln(n!) + 1)}{\ln\left(\frac{\lambda(\mathcal{L}_G^{\text{NNRT}})}{4(\gamma-1)^2} + 1\right)} \right]^{-1},$$

where γ is the maximal degree of G and $\lambda(\mathcal{L}_G^{\text{NNRT}})$ is the spectral gap.

Proof. We first note that, for each edge,

$$\mathcal{L}_e(\rho) = \frac{1}{2}(S_{i,j} \rho S_{i,j} - \rho) = E_{i,j}(\rho) - \rho,$$

where $E_{i,j}(\rho) = \frac{1}{2}(S_{i,j} \rho S_{i,j} + \rho)$ is a conditional expectation onto the symmetric space on $\mathcal{H}_i \otimes \mathcal{H}_j$. The maps $E_{i,j}$ are implemented by the self-adjoint unitaries $S_{i,j}$. Then by [81, Theorem 5.1], the local CMLS constant $\alpha_{\text{CMLS}}(\mathcal{L}_e) = 2$. Given the bound derived in Theorem 6.6, it suffices to calculate the index $C_G := C_{\text{cb}}(\mathcal{B}(\mathcal{H}_V) : \mathcal{N})$, where \mathcal{N} is the fixed point subalgebra of $\mathcal{L}_G^{\text{NNRT}}$. Since G is connected, then $\{\sigma_{i,j} \mid (i, j) \in E\}$ is a generating set for \mathcal{S}_n . Thus \mathcal{N} is the commutant of the representation

$$\pi : \mathcal{S}_n \rightarrow \mathcal{B}(\mathcal{H}_V), \quad \pi(\sigma_{i,j}) = S_{i,j}.$$

As discussed in Example A.4 of ‘‘Appendix A’’, the index is

$$C_{\text{cb}}(\mathcal{B}(\mathcal{H}_V) : \mathcal{N}) = \sum_{\pi_i \in \text{Int}(\mathcal{S}_n), \pi_i \subset \pi} m_i^2,$$

where m_i is the dimension of irreducible representation π_i and the summation is over all irreducible representations in the decomposition of $\pi = \oplus_i \pi_i \otimes \text{id}_{n_i}$. By the expression provided in (14), we know all irreducible representations (up to unitary equivalent) are contained in π . Then, by the Schur–Weyl Theorem,

$$C_G = C_{\text{cb}}(\mathcal{B}(\mathcal{H}_V) : \mathcal{N}) = \sum_{\pi_i \in \text{Irr}(\mathcal{S}_n)} m_i^2 = |\mathcal{S}_n| = n! .$$

□

The quantum random transposition model above corresponds to a classical random transposition on the permutation group \mathcal{S}_n on $[n] := \{1, \dots, n\}$. Indeed, consider the classical generator $\Delta_{\mathcal{S}_n}^G$ on \mathcal{S}_n ,

$$\Delta_{\mathcal{S}_n}^G f(\sigma) = \frac{1}{2} \sum_{(i,j) \in E} (f(\sigma_{ij}\sigma) - f(\sigma)) , \quad \sigma \in \mathcal{S}_n ,$$

where $\sigma_{ij} \in \mathcal{S}_n$ is the 2-permutation switching i and j . One can verify that $\mathcal{L}_G^{\text{NNRT}}$ is the transferred generator of $\Delta_{\mathcal{S}_n}^G$, which is the the graph Laplacian $\Delta_{\mathcal{S}_n}^G$ of the Cayley graph of \mathcal{S}_n with generating set $\{\sigma_{ij} \mid (i, j) \in E\}$.

For a general graph $G = (V, E)$, the transference goes through the graph Laplacian $\Delta_{\mathcal{S}_n}^G$ of the Cayley graph of \mathcal{S}_n with generating set $\{\sigma_{ij} \mid (i, j) \in E\}$. When a sharp estimate for $\alpha_{\text{CMLS I}}(\Delta_{\mathcal{S}_n})$ is missing, our Theorem 6.6 can provide tighter bounds than transference. It was proved in [56] that for a graph Laplacian Δ_G ,

$$\alpha_{\text{CMLS I}}(\Delta_G) \geq \frac{2}{45 \gamma (|G| + 1)^2} ,$$

where γ is the maximum degree of G . Here, we have $V = \mathcal{S}_n$ with $|\mathcal{S}_n| = n!$ growing exponentially. This exponential growth also appears if we use Theorem 3.3,

$$\frac{\lambda(\Delta_{\mathcal{S}_n}^G)}{n!} \leq \alpha_{\text{CMLS I}}(\Delta_{\mathcal{S}_n}^G) \leq 2\lambda(\Delta_{\mathcal{S}_n}^G) ,$$

since $C_{\text{cb}}(l_\infty(\mathcal{S}_n); \mathbb{C}) = |\mathcal{S}_n| = n!$. Compared with these, Theorem 6.6 gives a lower bound on the CMLS I constant for $\mathcal{L}_G^{\text{NNRT}}$ that has exponentially better dependence of $|G| = n$ (and is also independent of $d = \dim(\mathcal{H})$).

7. Discussion and Open Problems

We end our paper by discussing some open problems. Theorem 3.3 proves that any GNS-symmetric quantum Markov semigroup $(\mathcal{P}_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}))_{t \geq 0}$ on a finite dimensional Hilbert space \mathcal{H} satisfies the complete modified log-Sobolev inequality (CMLS I) with constant

$$\alpha_{\text{CMLS I}}(\mathcal{L}) \geq \frac{\lambda(\mathcal{L})}{C_{\text{cb}}(E)} , \tag{66}$$

where \mathcal{L} is the generator and $E = \lim_t \mathcal{P}_t$. In the primitive case (unique invariant state), $C_{\text{cb},\tau}(E) \sim d^2$ with $d = \dim(\mathcal{H})$. On the other hand, it was proven that, for a primitive semigroup with invariant state σ ,

$$\frac{\lambda(\mathcal{L})}{\ln(\mu_{\min}(\sigma)^{-1}) + 2} \leq \frac{\alpha_{\text{LSI}}(\mathcal{L})}{2} \leq \alpha_{\text{MLSI}}(\mathcal{L}), \tag{67}$$

where α_{MLSI} is the optimal constant for the modified log-Sobolev inequality and α_{LSI} is the optimal constant for the L_2 -log-Sobolev inequality (LSI), which is known to be equivalent to hypercontractivity [66]. Here $\mu_{\min}(\sigma)$ is the minimal eigenvalue of σ and $\ln(\mu_{\min}(\sigma)^{-1}) \sim \ln d$. Our lower bound on α_{CMLSI} controls any amplification $\mathcal{P}_t \otimes \text{id}_{\mathbb{M}_n}$, in contrast with L_2 -log-Sobolev inequality/hypercontractivity bound which fails for $\mathcal{P}_t \otimes \text{id}_{\mathbb{M}_n}$ for any $n > 1$. It remains open whether the bound (66) can be improved asymptotically.

Problem 7.1. Does there exist a general lower bound on the CMLSI constant of the form $\alpha_{\text{CMLSI}}(\mathcal{L}) \geq \lambda(\mathcal{L}) \mathcal{O}(\ln d)^{-1}$?

Our second question concerns the strong data processing inequality (SDPI). It was proven in [63] that for a primitive unital quantum channel Φ ,

$$\alpha_{\text{SDPI}}(\Phi) \geq 1 - \frac{1}{2} \alpha_{\text{LSI}}(\Phi^* \Phi - \text{id}), \tag{68}$$

where $\alpha_{\text{LSI}}(T^*T - \text{id})$ is the LSI constant of the map $T^*T - \text{id}$ seen as the generator of a QMS. This combined with (66) gives upper bounds on SDPI constant for primitive unital channel. Nevertheless, since LSI generally fails for non-primitive semigroups, this approach does not apply to CSDPI. In order to find better (C)SDPI constant, we propose the following question:

Problem 7.2. Can we find a lower bound on $\alpha_{\text{SDPI}}(\Phi)$ in terms of the modified log-Sobolev constant $\alpha_{\text{MLSI}}(\Phi^* \Phi - \text{id})$ for any non-primitive quantum channel Φ ?

For classical Markov chains, such a result was obtained in [31]. Note that in general $2\alpha_{\text{MLSI}}(\mathcal{L}) \geq \alpha_{\text{LSI}}(\mathcal{L})$, so a positive answer to Problem 7.2 would be stronger than (68). Moreover, combined with our Theorem 3.3, such a positive solution would also give a lower estimate on the CSDPI constant in terms of the spectral gap and the index.

Recall that our SDPI constant α_{SDPI} is defined as the optimal constant $0 \leq \alpha \leq 1$ such that

$$D(\Phi(\rho) \parallel \Phi \circ E_*(\rho)) \leq \alpha D(\rho \parallel E_*(\rho))$$

for any state ρ . Here $E_*(\rho)$ is the decoherence free part of the state ρ in the sense that for a GNS symmetric channel Φ , $\Phi^2 \circ E_*(\rho) = E_*(\rho)$ and $\lim_{n \rightarrow \infty} \|\Phi^n(\rho) - \Phi^n \circ E_*(\rho)\| = 0$. This is a natural choice analogous to MLSI. Nevertheless, the data processing inequality asserts that $D(\Phi(\rho) \parallel \Phi(\sigma)) \leq D(\rho \parallel \sigma)$ for any two states ρ and σ . Indeed, in Theorem 4.1, we prove that for a state ω , the best local constant $\alpha'(\sigma)$, which satisfies

$$D(\Phi(\rho) \parallel \Phi(\sigma)) \leq \alpha'(\sigma) D(\rho \parallel \sigma)$$

for all states ρ with $E_*(\rho) = E_*(\omega)$, can be two-sided controlled by the corresponding χ_2 contraction coefficient. (Here the restriction $E_*(\rho) = E_*(\omega)$ is needed because without it the constant $\alpha(\omega)$ would often be trivially equal to 1). It is then natural to ask whether for a quantum channel Φ , there is a non-trivial upper bound on $\alpha(\omega)$ uniformly in ω . Namely,

Problem 7.3. For a finite dimensional quantum channel Φ , does there exist a constant $\alpha'(\Phi) < 1$ such that

$$D(\Phi(\rho) \parallel \Phi(\sigma)) \leq \alpha'(\Phi) D(\rho \parallel \sigma)$$

for all states ρ, σ with $E_*(\rho) = E_*(\omega)$?

Such a constant $\alpha'(\Phi)$ leads to a stronger notion of contraction than our definition of α_{SDPI} ; it is closer to the classical strong data processing inequality studied in [71, 73], which was proven to be equivalent to the contraction coefficient of (classical) χ_2 -divergence. Note that, by our Theorem 4.1, it also suffices to show that there is a global contraction coefficient on the quantum χ_2 divergence

$$\chi_2(\rho, \sigma) = \|\rho - \sigma\|_{\sigma^{-1}}^2 = \int_0^\infty \text{tr} \left((\rho - \sigma) \frac{1}{\sigma + s} (\rho - \sigma) \frac{1}{\sigma + s} \right) ds$$

for all states σ .

Appendix A: Completely Positive Order Relations from Norm Estimates

In this appendix, we provide two generic strategies to derive the order relations needed in Theorem 5.3 in the case of symmetric generators. Our first method is a variant of [37, Lemma 3.15] (see also [53, Corollary 1.8]). Let \mathcal{M} be a finite dimensional von Neumann algebra equipped with trace τ and let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra. Note that we do not fix the normalization of τ at this point. For $p \geq 1$, the space of p -integrable operators in \mathcal{M} is denoted by $L_p(\mathcal{M}, \tau) \equiv L_p(\tau)$, with associated norm

$$\|x\|_{L_p(\mathcal{M})} \equiv \|x\|_p := \tau(|x|^p)^{\frac{1}{p}}.$$

We also need the notion of an amalgamated L_p norm [47]: for $1 \leq p \leq \infty$ the $L_\infty^p(\mathcal{N} \subset \mathcal{M})$ norm of $X \in \mathcal{M}$ is given by

$$\|X\|_{L_\infty^p(\mathcal{N} \subset \mathcal{M})} := \sup \|aXb\|_{L_p(\mathcal{M})},$$

where the supremum is taken over $a, b \in \mathcal{N}$ with $\|a\|_{2p} = \|b\|_{2p} = 1$. Its operator space structure is given by (see [38, Appendix])

$$\mathbb{M}_n(L_\infty^p(\mathcal{N} \subset \mathcal{M})) = L_\infty^p(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M})).$$

It was proved in [38, Theorem 3.9] that

$$\|\text{id} : L_\infty^2(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M}\|_{\text{cb}}^2 = \|\text{id} : L_\infty^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M}\|_{\text{cb}} = C_{\text{cb}}(\mathcal{M} : \mathcal{N}). \tag{69}$$

Let $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ be the trace preserving conditional expectation onto \mathcal{N} . Recall that there exists a module basis $\{\xi_i\}_{i=1}^n \in \mathcal{M}$ satisfying ([67, Theorem 3.15], see also [5, Conséquence 1.8]):

$$E_{\mathcal{N}}(\xi_i^\dagger \xi_j) = \delta_{ij} p_i,$$

where $p_i \in \mathcal{N}$ are some projections. Also recall that $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ is a \mathcal{N} -bimodule map if $\Phi(aXb) = a\Phi(X)b$ for all $a, b \in \mathcal{N}$ and $X \in \mathcal{M}$. We have $\Phi \circ E_{\mathcal{N}} = E_{\mathcal{N}} \circ \Phi$ if Φ is unital and $E_{\mathcal{N}} \circ \Phi = E_{\mathcal{N}}$ if Φ is trace preserving. For a \mathcal{N} -bimodule map Φ , we have by [37, Lemma 3.12]

$$\begin{aligned} \lambda &= \|\Phi : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{M})\| = \|\Phi : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{M})\|_{\text{cb}} \\ &= \|\Phi : L_\infty^2(\mathcal{M}) \rightarrow L_\infty^2(\mathcal{M})\|_{\text{cb}}. \end{aligned} \tag{70}$$

Next, we define the *module Choi operator* of a bi-modular map Φ as

$$\chi_\Phi = \sum_{i,j=1}^n |i\rangle\langle j| \otimes \Phi(\xi_i^\dagger \xi_j) \in \mathcal{B}(l_2^n) \otimes \mathcal{M},$$

where l_2^n denotes the space of n -dimensional vectors with associated norm

$$\|u\|_{l_2^n} = \left(\sum_i |u_i|^2\right)^{\frac{1}{2}}$$

and $\{|i\rangle\}_{i=1}^n$ is a fixed orthonormal basis in l_2^n . Thus Φ and χ_Φ determine each other because for each $x \in \mathcal{M}$, we have a unique decomposition $x = \sum_i \xi_i x_i$ with $x_i \in \mathcal{N}$ satisfying $p_i x_i = x_i$. Indeed, we have $x_i = E_{\mathcal{N}}(\xi_i^\dagger x)$. Moreover, Φ is completely positive if and only if χ_Φ is a positive operator in $\mathcal{B}(l_2^n) \otimes \mathcal{M}$. Indeed, for any finite family $y_1, \dots, y_m \in \mathcal{M}$, we assume the decomposition $y_j = \sum_l \xi_l x_{jl}$ with $x_{jl} \in \mathcal{N}$. Then

$$\begin{aligned} (\text{id} \otimes \Phi) \left(\sum_{i,j} |i\rangle\langle j| \otimes y_i^\dagger y_j \right) &= \sum_{i,j} |i\rangle\langle j| \otimes \Phi(y_i^\dagger y_j) \\ &= \sum_{i,j,k,l} |i\rangle\langle j| \otimes x_{ik}^\dagger \Phi(\xi_k^\dagger \xi_l) x_{jl} \\ &= \left(\sum_{i,k} |i\rangle\langle k| \otimes x_{ik}^\dagger \right) \chi_\Phi \left(\sum_{j,l} |l\rangle\langle j| \otimes x_{jl} \right), \end{aligned}$$

from which the equivalence claimed directly follows. We also recall [37, Lemma 3.14] that for a \mathcal{N} -bimodule map Φ ,

$$\|\chi_\Phi\|_{\mathcal{B}(l_2^n) \otimes \mathcal{M}} = \|\Phi : L_\infty^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M}\|_{\text{cb}}. \tag{71}$$

We are now ready to state and prove the first main Lemma of this section:

Lemma A.1. *Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra and let $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ be the corresponding trace preserving conditional expectation. Let $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ be a \mathcal{N} -bimodule map.*

(i) *if $\|\chi_{\Phi} - \chi_{E_{\mathcal{N}}}\|_{\mathcal{B}((l_2^n)) \otimes \mathcal{M}} \leq \varepsilon < 1$, then*

$$(1 - \varepsilon)E_{\mathcal{N}} \leq_{\text{cp}} \Phi \leq_{\text{cp}} (1 + \varepsilon)E_{\mathcal{N}}.$$

(ii) *if Φ is a τ -symmetric quantum channel and $\lambda := \|\Phi - E_{\mathcal{N}} : L_2(\mathcal{M}, \tau) \rightarrow L_2(\mathcal{M}, \tau)\| < 1$, then for $k > \frac{\ln C_{\text{cb}}(\mathcal{M} : \mathcal{N})}{-\ln \lambda}$,*

$$(1 - \varepsilon)E_{\mathcal{N}} \leq_{\text{cp}} \Phi^k \leq_{\text{cp}} (1 + \varepsilon)E_{\mathcal{N}}$$

for $\varepsilon = \lambda^k C_{\text{cb}}(\mathcal{M} : \mathcal{N}) < 1$.

Proof. For (i) we first observe that the Choi operator of $E_{\mathcal{N}}$

$$\chi_{E_{\mathcal{N}}} = \sum_{i,j=1}^n |i\rangle\langle j| \otimes E_{\mathcal{N}}(\xi_i^\dagger \xi_j) = \sum_i |i\rangle\langle i| \otimes p_i$$

is a projection. By the assumption $\|\chi_{\Phi} - \chi_{E_{\mathcal{N}}}\|_{\mathcal{B}((l_2^n)) \otimes \mathcal{M}} \leq \varepsilon < 1$, we may write $\chi_{E_{\mathcal{N}}} - \chi_{\Phi} = \alpha - \beta$ with $0 \leq \alpha, \beta \leq \varepsilon$. Let $\{y_1, \dots, y_m\}$ be a finite family of elements in \mathcal{M} with decomposition $y_j = \sum_l \xi_l x_{jl}$ with $x_{jl} = p_j x_{jl} \in \mathcal{N}$, and denote $Y = \sum_j |1\rangle\langle j| \otimes y_j$. Then,

$$\begin{aligned} & (\text{id} \otimes \Phi)(Y^\dagger Y) - (\text{id} \otimes E_{\mathcal{N}})(Y^\dagger Y) \\ &= \sum_{i,j,k,l} |i\rangle\langle j| \otimes x_{ik}^\dagger p_k (\Phi - E_{\mathcal{N}})(\xi_k^\dagger \xi_l) p_j x_{jl} \\ &= \left(\sum_{i,k} |i\rangle\langle k| \otimes x_{ik}^\dagger \right) (\chi_{\Phi} - \chi_{E_{\mathcal{N}}}) \left(\sum_{j,l} |l\rangle\langle j| \otimes x_{jl} \right) \\ &= \left(\sum_{i,k} |i\rangle\langle k| \otimes x_{ik}^\dagger p_k \right) (\alpha - \beta) \left(\sum_{j,l} |l\rangle\langle j| \otimes p_j x_{jl} \right). \end{aligned}$$

Since $0 \leq \alpha \leq \varepsilon$,

$$\begin{aligned} & \left(\sum_{i,k} |i\rangle\langle k| \otimes x_{ik}^\dagger p_k \right) \alpha \left(\sum_{j,l} |l\rangle\langle j| \otimes p_j x_{jl} \right) \\ & \leq \varepsilon \left(\sum_{i,k} |i\rangle\langle k| \otimes x_{ik}^\dagger p_k \right) \left(\sum_{j,l} |l\rangle\langle j| \otimes p_j x_{jl} \right) \\ & = \varepsilon \sum_{i,j,k} |i\rangle\langle j| \otimes x_{ik}^\dagger p_k x_{jk} \\ & = \varepsilon \sum_{i,j,l,k} |i\rangle\langle j| \otimes x_{ik}^\dagger E_{\mathcal{N}}(\xi_k \xi_l) x_{jl} = \varepsilon (\text{id} \otimes E_{\mathcal{N}})(Y^\dagger Y) \end{aligned}$$

and similarly for β . Thus we showed that

$$-\varepsilon (\text{id} \otimes E_{\mathcal{N}})(Y^\dagger Y) \leq (\text{id} \otimes \Phi)(Y^\dagger Y) - (\text{id} \otimes E_{\mathcal{N}})(Y^\dagger Y) \leq \varepsilon (\text{id} \otimes E_{\mathcal{N}})(Y^\dagger Y)$$

for any $Y = \sum_j |1\rangle\langle j| \otimes y_j$, which proves (i) because any $X \geq 0$ in $\mathbb{M}_m \otimes \mathcal{M}$ can be written as a sum $X = \sum_j Y_j^\dagger Y_j$ of such Y 's.

For (ii), we first observe that since Φ is trace perserving and unital, we have by bimodule property that $\Phi \circ E_{\mathcal{N}} = E_{\mathcal{N}} \circ \Phi = E_{\mathcal{N}}$ and hence $(\Phi - E_{\mathcal{N}})^k = \Phi^k - E_{\mathcal{N}}$. Then, we can use identity (71) between the L_∞ norm of the module Choi operator of Φ^k and its L_∞^1 norm, so that:

$$\begin{aligned} & \| \chi_{\Phi^k} - \chi_{E_{\mathcal{N}}} \|_{\mathcal{B}(\mathcal{M}_2^m) \otimes \mathcal{M}} \\ &= \| \Phi^k - E_{\mathcal{N}} : L_\infty^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M} \|_{\text{cb}} \\ &= \| (\Phi - E_{\mathcal{N}})^k : L_\infty^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M} \|_{\text{cb}} \\ &= \| \text{id} : L_\infty^1(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty^2(\mathcal{N} \subset \mathcal{M}) \|_{\text{cb}} \\ &\quad \times \| (\Phi - E_{\mathcal{N}})^k : L_\infty^2(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty^2(\mathcal{N} \subset \mathcal{M}) \|_{\text{cb}} \\ &\quad \times \| \text{id} : L_\infty^2(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M} \|_{\text{cb}} \\ &= C_{\text{cb}}(\mathcal{M} : \mathcal{N})^{1/2} \cdot \lambda^k \cdot C_{\text{cb}}(\mathcal{M} : \mathcal{N})^{1/2} = C_{\text{cb}}(\mathcal{M} : \mathcal{N}) \lambda^k, \end{aligned}$$

where we also used (69) and (70). Then the assertion (ii) follows from (i). □

The next main result of the present appendix is an approach to obtain cp orders which follows the idea of [16, Appendix A]. Here, we restrict ourselves to the algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and take $\tau := \text{tr}$ to be the standard matrix trace. Given a map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, the standard (normalized) Choi-Jamiołkowski matrix J_Φ is

$$J_\Phi := (\text{id} \otimes \Phi)(|\psi\rangle\langle\psi|) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}),$$

where $|\psi\rangle = \frac{1}{\sqrt{d_{\mathcal{H}}}} \sum_i |i\rangle|i\rangle$ is the maximally entangled state on $\mathcal{H} \otimes \mathcal{H}$. It is well-known that $J_\Phi \geq 0$ if and only if Φ is completely positive, and J_Φ is a state if and only if Φ is a quantum channel.

Lemma A.2. *Let $E_{\mathcal{N}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}$ be the trace preserving conditional expectation onto a subalgebra \mathcal{N} . Let $\Phi : \mathcal{T}_1(\mathcal{H}) \rightarrow \mathcal{T}_1(\mathcal{H})$ be a quantum channel such that $\Phi \circ E_{\mathcal{N}} = E_{\mathcal{N}} \circ \Phi = E_{\mathcal{N}}$. Suppose $J_{E_{\mathcal{N}}} \geq C_E^{-1} P$ for some $C_E > 0$ where P is the support projection of $J_{E_{\mathcal{N}}}$.*

(i) *If J_Φ has support $P_\Phi \leq P$ and for some $0 \leq \varepsilon < 1$*

$$\|J_\Phi - J_{E_{\mathcal{N}}}\|_\infty \leq \varepsilon C_E^{-1},$$

then

$$(1 - \varepsilon)E_{\mathcal{N}} \leq_{\text{cp}} \Phi \leq_{\text{cp}} (1 + \varepsilon)E_{\mathcal{N}}.$$

(ii) *If for each k , J_{Φ^k} has support projection P and $\lambda := \| \Phi - E_{\mathcal{N}} : L_2(\mathcal{B}(\mathcal{H}), \text{tr}) \rightarrow L_2(\mathcal{B}(\mathcal{H}), \text{tr}) \| < 1$. Then for $k > \frac{\ln(C_E)}{-\ln(\lambda)}$,*

$$(1 - \varepsilon)E_{\mathcal{N}} \leq_{\text{cp}} \Phi^k \leq_{\text{cp}} (1 + \varepsilon)E_{\mathcal{N}},$$

where $\varepsilon = \lambda^k C_E < 1$.

Proof. If J_Φ has support $P_\Phi \leq P$, then

$$J_\Phi - J_{E_{\mathcal{N}}} \leq \|J_\Phi - J_{E_{\mathcal{N}}}\|_\infty P \leq \|J_\Phi - J_{E_{\mathcal{N}}}\|_\infty C_E J_{E_{\mathcal{N}}} \leq \varepsilon J_{E_{\mathcal{N}}},$$

Similarly,

$$J_{E_{\mathcal{N}}} - J_\Phi \leq \varepsilon J_{E_{\mathcal{N}}},$$

Thus we have

$$(1 - \varepsilon)J_{E_{\mathcal{N}}} \leq J_\Phi \leq (1 + \varepsilon)J_{E_{\mathcal{N}}}$$

which implies (i). For (ii), we note that the maximally entangled state $|\psi\rangle\langle\psi|$ is a pure state, so that

$$\begin{aligned} \|J_{\Phi^k} - J_{E_{\mathcal{N}}}\|_\infty &\leq \|J_{\Phi^k} - J_{E_{\mathcal{N}}}\|_2 = \|(\text{id} \otimes (\Phi^k - E_{\mathcal{N}}))(|\psi\rangle\langle\psi|)\|_2 \\ &= \|(\text{id} \otimes (\Phi - E_{\mathcal{N}})^k)(|\psi\rangle\langle\psi|)\|_2 \\ &\leq \|(\text{id} \otimes (\Phi - E_{\mathcal{N}})^k)(|\psi\rangle\langle\psi|)\|_2 \\ &\leq \|\Phi - E_{\mathcal{N}} : L_2(\mathcal{B}(\mathcal{H}), \text{tr}) \rightarrow L_2(\mathcal{B}(\mathcal{H}), \text{tr})\|^k = \lambda^k \end{aligned}$$

Then assertion (ii) follows from (i). \square

We end this appendix by comparing the approaches proposed in Lemmas A.1 and A.2 in two cases.

Example A.3. (Trivial subalgebra) Consider the trivial subalgebra $\mathbb{C}1 \subset \mathcal{B}(\mathcal{H})$. Then for $\dim(\mathcal{H}) = d$,

$$C_{\text{cb}}(\mathcal{B}(\mathcal{H}), \mathbb{C}) = C_E = d^2.$$

Example A.4. (Unitary Representations of compact groups) Let G be a compact group and let $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ be a unitary representation. The fixed point space for the conjugation action $\alpha : G \curvearrowright \mathcal{B}(\mathcal{H})$ defined as $\alpha_g(x) = \pi(g)x\pi(g)^\dagger$ is $\mathcal{N} = \pi(G)' = \{x \in \mathcal{B}(\mathcal{H}) \mid x\pi(g) = \pi(g)x\}$, that is the commutant of the representation. Suppose π admits the decomposition $\pi = \bigoplus_{i \in \text{Irr}(G)} (\pi_i \otimes \text{id}_{n_i})$ as a direct sum of irreducible representations $\pi_i : G \rightarrow \mathbb{M}_{m_i}$ with multiplicity n_i . Then $\mathcal{N} = \bigoplus_{i \in \text{Irr}(G)} (\mathbb{C}\mathbb{1}_{m_i} \otimes \mathbb{M}_{n_i})$ is a direct sum of matrix algebras \mathbb{M}_{n_i} with multiplicity m_i . The trace preserving conditional expectation is

$$E_{\mathcal{N}} = \bigoplus_i (\text{tr}_{m_i} \otimes \text{id}_{n_i}),$$

where $\text{tr}_{m_i}(x) = \text{tr}(x) \frac{\mathbb{1}}{m_i}$ is the partial trace map and id_{n_i} is the identity map on \mathbb{M}_{n_i} . The Choi-Jamiolkowski state of $E_{\mathcal{N}}$ is

$$J_{E_{\mathcal{N}}} = \frac{1}{d_{\mathcal{H}}} \bigoplus_i \left(\frac{n_i}{m_i} (\mathbb{1}_{m_i} \otimes \mathbb{1}_{m_i}) \otimes |\psi_{n_i}\rangle\langle\psi_{n_i}| \right)$$

where $\mathbb{1}_{m_i} \in \mathbb{M}_{m_i}$ is the identity operator and $|\psi_{n_i}\rangle = \frac{1}{\sqrt{n_i}} \sum_{j=1}^{n_i} |j\rangle|j\rangle$ is the maximally entangled state in $\mathbb{M}_{n_i} \otimes \mathbb{M}_{n_i}$. The support projection is

$$P = \bigoplus_i \left((\mathbb{1}_{m_i} \otimes \mathbb{1}_{m_i}) \otimes |\psi_{n_i}\rangle\langle\psi_{n_i}| \right).$$

Then $J_{E_{\mathcal{N}}} \geq C_E^{-1} P$ for

$$C_E = d_{\mathcal{H}} \max_i \frac{m_i}{n_i}.$$

On the other hand, the cb-index is

$$C_{\text{cb}}(\mathcal{B}(\mathcal{H}) : \mathcal{N}) = \sum_i m_i^2.$$

In particular, when G is a finite group and $\pi_L : G \rightarrow \mathcal{B}(l_2(G))$ is its the left regular representation, we have by Schur–Weyl Theorem that $n_i = m_i$ and

$$C_E = d_{\mathcal{H}} = \sum_i m_i^2 = C_{\text{cb}}(\mathcal{B}(l_2(G)) : \mathcal{N}),$$

where \mathcal{N} is isomorphic to the group algebra $\mathbb{C}G$.

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