Arch. Rational Mech. Anal. 197 (2010) 337–356
Digital Object Identifier (DOI) 10.1007/s00205-009-0288-8

On the Brézis–Nirenberg Problem

M. Schechter & Wenming Zou

Communicated by C. Dafermos

Abstract

We study the following Brézis–Nirenberg problem (Comm Pure Appl Math 36:437–477, 1983):

$$-\Delta u = \lambda u + |u|^{2^*-2}u, \quad u \in H_0^1(\Omega),$$

where Ω is a bounded smooth domain of $\mathbf{R}^N (N \geqq 7)$ and 2^* is the critical Sobolev exponent. We show that, for each fixed $\lambda > 0$, this problem has infinitely many sign-changing solutions. In particular, if $\lambda \geqq \lambda_1$, the Brézis–Nirenberg problem has and only has infinitely many sign-changing solutions except zero. The main tool is the estimates of Morse indices of nodal solutions.

1. Introduction

We study the following Brézis–Nirenberg problem

$$-\Delta u = \lambda u + |u|^{2^*-2}u, \qquad u \in H_0^1(\Omega), \tag{1}$$

where Ω is a bounded smooth domain of $\mathbf{R}^N (N \geqq 3)$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $\lambda > 0$. The pioneering paper on equation (1) was by Brézis–Nirenberg [7] in 1983 where the authors showed that for $N \geqq 4$ and $\lambda \in (0, \lambda_1)$ problem (1) has at least one positive solution. In the sequel, λ_1 denotes the principal eigenvalue of $-\Delta$ on Ω. The same conclusion was proved in [7] for $N = 3$ when Ω is a ball and $\lambda \in (\lambda_1/4, \lambda_1)$. In this case, by using a version of the Pohozaev Identity, equation (1) has no radial solution when $\lambda \in (0, \lambda_1/4)$. It is still an open question whether there exist sign-changing non-radial solutions to (1). Note that, by using the Pohozaev Identity, (1) has no nontrivial solution when $\lambda \leqq 0$ and Ω is star-shaped. Since 1983, there has been a considerable number of papers on problem (1). Let us now briefly recap the main results obtained to date. The first multiplicity result was obtained by Cerami et al. [8], in which they proved that the number of solutions of (1) is bounded below by the number of eigenvalues of $(-\Delta, \Omega)$ lying in the open interval $(\lambda, \lambda + S|\Omega|^{-2/N})$, where S is the best constant for the Sobolev embedding $D^{1,2}(\mathbf{R}^N) \hookrightarrow L^{2^*}(\mathbf{R}^N)$ and $|\Omega|$ is the Lebesgue measure of Ω. If $N \geqq 4$ and Ω is a ball, then for any $\lambda > 0$, problem (1) has infinitely many sign-changing solutions which are built using particular symmetries of the domain Ω (see Fortunato and Jannelli [15]). If $N \geqq 7$ and Ω is a ball, then for each $\lambda > 0$, problem (1) has infinitely many sign-changing radial solutions (see Solimini [23]). In the papers [15,23], the symmetry of the ball$= \Omega$ plays an essential role, therefore their methods are invalid for general domains. In Cerami et al. [9] it was proved for $N \geqq 6$, that (1) has two pairs of solutions on any smooth bounded domain. When $4 \leqq N \leqq 6$ and Ω is a ball, there is a $\lambda^* > 0$ such that (1) has no radial solutions which change sign if $\lambda \in (0, \lambda^*)$ (see Atkinson et al. [1]). Very recently, Devillanova and Solimini [14] showed that, if $N \geqq 7$, problem (1) has infinitely many solutions for each $\lambda > 0$. In [10], Clapp–Weth got only finitely many solutions to (1) for each $\lambda > 0$ and $N \geqq 4$. Neither [14] nor [10] have information on the sign-changingness of the solutions thus obtained.

A natural question which seems to still be open is whether (1) has infinitely many sign-changing solutions on each bounded domain Ω and for any $\lambda > 0$. This is expected by many experts but no proof has yet been obtained. The results of Atkinson et al. [1] suggest that the hypothesis $N > 6$ should be imposed since it is needed in the radial case. In this paper, we give a positive answer to this open question. Precisely, we shall prove the following theorem.

Theorem 1. *If $N \geqq 7$, problem* (1) *has infinitely many sign-changing solutions.*

Remark 1. Note that if $\lambda \geqq \lambda_1$, any nontrivial solution of (1) is sign-changing. This can be seen by multiplying the first eigenfunction of the Dirichlet zero-boundary value problem and integrating both sides. Thus, by the results of [14], problem (1) has and only has infinitely many sign-changing solutions for this case.

To prove Theorem 1, it suffices to consider the case of $\lambda \in (0, \lambda_1)$. To this goal, we first establish an abstract theorem on the estimate of the Morse index for sign-changing critical points. We will introduce a new kind of linking below a level set which provides a sign-changing critical point via a minimax formula and then find the lower bound of its Morse index. Similar to the classical case, the lower bound of the Morse index is determined by the dimension of the subspace which may be chosen as large as we like. By combining this with the uniform bound theorem due to Devillanova and Solimini [14], we show that, for each fixed $\lambda > 0$, (1) indeed has infinitely many sign-changing solutions.

The study of sign-changing solutions to some elliptic equations has been an increasing interest in recent years (cf. [2–6,12,13,17,21,22,26] and the references cited therein). As when finding an existence result in a classical case, the information on Morse index of sign-changing solutions can yield new conclusions. In paper [3] by using critical group and algebraic topology arguments, two kinds of Morse indices of sign-changing solutions were obtained: one is a sign-changing solution of the mountain pass type with Morse index less than or equal to 1; the other may be degenerate and has Morse index 2. In [2], an upper-bound for the Morse index of the sign-changing solution was obtained. The results of [2] mainly rely on singular homology groups. In [6], a least energy sign-changing solution with Morse index 2 is obtained. If the nonlinear term is odd, paper [6] also obtains a sequence $\{u_k\}$ of sign-changing solutions such that each u_k has at most k nodal domains (see also [5]). But, under those assumptions, the lower bound of the Morse index for each u_k has not been determined. For many cases, the Morse indices of sign-changing critical points produced by general minimax procedure are not clear. Our goal here is to establish an estimate on the Morse index of sign-changing critical points.

2. Morse index of sign-changing critical points

Let E be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. We assume that

(A_0) there is another norm $\|\cdot\|_\star$ of E such that $\|u\|_\star \leqq C_0\|u\|$ for all $u \in E$, where $C_0 > 0$ is a constant. Moreover, we assume that $\|u_n - u^*\|_\star \to 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \|\cdot\|)$.

Let $E := \overline{\cup_{k=1}^{\infty} E_k}$ with $\dim E_k = k$, $E_k \subset E_{k+1}$. Let $G \in \mathbf{C}^2((E, \|\cdot\|), \mathbf{R})$ be an even functional which maps bounded sets to bounded sets in terms of the norm $\|\cdot\|$. In the sequel, all properties are with respect to the norm $\|\cdot\|$ if without specific indication. Assume that $G''(u_0)$ is Fredholm for any critical point u_0 of G. The gradient G' is of the form $G'(u) = u - K_G(u)$, where $K_G : E \to E$ is a continuous operator. Let $\mathcal{K} := \{u \in E : G'(u) = 0\}$ and $\tilde{E} := E\backslash\mathcal{K}$, $\mathcal{K}[a,b] := \{u \in \mathcal{K} : G(u) \in [a,b]\}$. Let $\mathcal{P}$ be a closed convex and weakly closed positive cone of E. We call the elements outside $\pm\mathcal{P}$ sign-changing. Assume that $(\pm\mathcal{P} \cap (E_k^\perp\backslash\{0\}) = \emptyset$ for all $k \geqq 2$, that is, any nonzero element of $E_k^\perp$ is sign-changing. For each $\mu > 0$, define $\mathcal{D}(\mu) := \{u \in E : \operatorname{dist}(u, \mathcal{P}) < \mu\}$. Then $\mathcal{D}(\mu)$ is an open convex set containing the positive cone $\mathcal{P}$ in its interior. Set $\mathcal{D}^* = \mathcal{D}^*(\mu) := \mathcal{D}(\mu) \cup (-\mathcal{D}(\mu))$, $\quad \mathcal{S}^* = E\backslash\mathcal{D}^*(\mu)$. We use the following assumptions.

(A_1) For any $\mu_0 > 0$ small enough, we have that $K_G(\pm\mathcal{D}(\mu_0)) \subset \pm\mathcal{D}(\mu) \subset \pm\mathcal{D}(\mu_0)$ for some $\mu \in (0, \mu_0)$. Moreover, $\pm\mathcal{D}(\mu_0) \cap \mathcal{K} \subset \pm\mathcal{P}$.

(A_2) For each k, $\lim_{\|u\|\to\infty, u\in E_k} G(u) = -\infty$.

(A_3) Assume that for any $\alpha_1, \alpha_2 > 0$ there exists an α_3 depending on α_1 and α_2 such that $\|u\| \leqq \alpha_3$ for all $u \in G^{\alpha_1} \cap \{u \in E : \|u\|_\star \leqq \alpha_2\}$, where $G^{\alpha_1} = \{u \in E : G(u) \leqq \alpha_1\}$.

Conditions similar to (A_1) have been used in [5,6,12,13,17,21,22,26]. Let

$$C_{k+1}^{**} := \sup_{E_{k+1}} G.$$

Then by condition (A_2), C_{k+1}^{**} is well-defined and $C_{k+1}^{**} < \infty$. Write $E = E_k \oplus E_k^\perp$ and let

$$\beta^*(u) = \begin{cases} \dfrac{\|u\|\,\|u\|_\star}{\|u\| + \|u\|_\star}, & u \neq 0, \\ 0, & u = 0, \end{cases}$$

then $\beta^* : E \to E$ is continuous. Set $S_0(k) := \{u \in E_k^\perp : \beta^*(u) = 1\}$, then it is easy to check that there is a constant $\alpha_4 > 0$ such that $\|u\|_\star \leqq \alpha_4$ for all $u \in S_0(k)$. Set $S(k) := S_0(k) \cap G^{C^{**}_{k+1}}$.

Lemma 1. *By assumption* (A_3)*, we have a constant* $\alpha_5 = \alpha_5(\alpha_4, C^{**}_{k+1}) > 0$ *such that* $\|u\| \leqq \alpha_5$ *for all* $u \in S(k)$*. Hence, there is an* $\Lambda_0 = \Lambda_0(\alpha_5) > 0$ *such that* $\inf_{u \in S(k)} G \geqq -\Lambda_0$.

Lemma 2. *There is a constant* $\delta > 0$ *such that* $\operatorname{dist}(S(k), \pm\mathcal{P}) = \delta > 0$.

Proof. We begin with ideas initiated in [21]. By negation, we assume that

$$\operatorname{dist}(S(k), \mathcal{P}) = 0.$$

Then we find $\{u_n\} \subset S(k)$, $\{p_n\} \subset \mathcal{P}$ such that $\|u_n - p_n\| \to 0$. Then $\{u_n\}$, hence $\{p_n\}$, is bounded in both $(E, \|\cdot\|)$ and $(E, \|\cdot\|_\star)$. We assume that $u_n \rightharpoonup u^* \in E$, $p_n \rightharpoonup p^* \in \mathcal{P}$ weakly in $(E, \|\cdot\|)$; $u_n \to u^*$ strongly in $(E, \|\cdot\|_\star)$. Then we observe that $u^* \in E_k^\perp$. Since $\frac{\|u_n\|\|u_n\|_\star}{\|u_n\|+\|u_n\|_\star} = 1$ and $\|u_n - u^*\|_\star \to 0$, we see that $u^* \neq 0$. However, since $u^* = p^*$, we get a contradiction, since all nonzero elements of $E_k^\perp$ are sign-changing. □

Note that because both $\|\cdot\|_\star$ and $\|\cdot\|$ are equivalent in E_{k+1}, we have a constant ϱ_{k+1} such that $\|u\| \leqq \varrho_{k+1}\|u\|_\star$ for all $u \in E_{k+1}$. Let

$$\Gamma^*_{k+1} = \{h : \quad h \in \mathbf{C}(\Theta_{k+1}, E), h|_{\partial\Theta_{k+1}} = \mathbf{id}, h \text{ is odd}\},$$

where $\Theta_{k+1} := \{u \in E_{k+1} : \|u\| \leqq R_{k+1}\}$. Further, by Lemma 1 and condition (A_2), we may choose R_{k+1} large enough such that

$$\partial\Theta_{k+1} \cap S(k) = \emptyset, \quad \sup_{\partial\Theta_{k+1}} G << -\Lambda_0 \leqq \inf_{S(k)} G, \quad R_{k+1} \geqq \varrho_{k+1} + 2. \tag{2}$$

Without loss of generality, we may assume that $R_{k+2} > R_{k+1}$.

Lemma 3. $h(\Theta_{k+1}) \cap S_0(k) \neq \emptyset, \quad \forall\, h \in \Gamma^*_{k+1}$.

Proof. Let $U := \{u \in E_{k+1} : \beta^*(h(u)) < 1\} \cap \{u \in E_{k+1} : \|u\| < R_{k+1}\}$, then U is a symmetric neighborhood of zero in E_{k+1}. Let $P : E \to E_k$ be the orthogonal projection; then $P \circ h : \partial U \to E_k$ is odd and continuous. By Borsuk–Ulam theorem, we have that $P \circ h(u) = 0$ for some $u \in \partial U$. Hence, $h(u) \in E_k^\perp$. We claim $u \notin \partial\{u \in E_{k+1} : \|u\| < R_{k+1}\}$. Otherwise, $\|u\| = R_{k+1}$ and then $h(u) = u$, $P(u) = 0$. But $\frac{\|h(u)\|\|h(u)\|_\star}{\|h(u)\|+\|h(u)\|_\star} \leqq 1$, it follows that $R_{k+1} = \|u\| \leqq 1 + \frac{\|u\|}{\|u\|_\star} \leqq 1 + \varrho_{k+1}$. This is impossible, so our claim is true. This means that $u \in \partial\{u \in E_{k+1} : \beta^*(h(u)) < 1\}$, $\|u\| \leqq R_{k+1}$, $u \in E_{k+1}$. Hence, $h(u) \in E_k^\perp$, $\frac{\|h(u)\|\|h(u)\|_\star}{\|h(u)\|+\|h(u)\|_\star} = 1$, that is, $h(u) \in S_0(k)$. □

Combining the definition of $\mathcal{S}^*$ and Lemma 2, we may choose μ_0 in (A_1) such that $S(k) \subset \mathcal{S}^* = E \backslash \mathcal{D}^*(\mu_0)$.

Definition 1. A compact symmetric subset A of E with $\partial\Theta_{k+1} \subset A$ is said to be linked to $S_0(k)$ if, for any continuous mapping $h \in \mathbf{C}([0,1] \times A, E)$ satisfying

- $h(t,u)$ is odd in $u \in A$,
- $h(t,u) = u$ for all $u \in \partial\Theta_{k+1}$,

there holds $h(1,A) \cap S_0(k) \neq \emptyset$.

Define

$$\mathcal{L} := \{A \subset G^{C^{**}_{k+1}} : \quad A \quad \text{is linked to} \quad S_0(k)\}.$$

Then $\mathcal{L} \neq \emptyset$, since by Lemma 3 and the definition of $S_0(k)$ we see that $\Theta_{k+1} \in \mathcal{L}$. Note that $h(t,\cdot) :\equiv \mathbf{id}$ is a mapping satisfying the properties of Definition 1.

Theorem 2. *Suppose that G satisfies the (PS) condition and assumptions (A_1)–(A_3). Define*

$$C^* = \inf_{A \in \mathcal{L}} \sup_{A \cap \mathcal{S}^*} G(u).$$

Then $C^* \in [-\Lambda_0, C^{**}_{k+1}]$ and G has a sign-changing critical point $u^* \in \mathcal{S}^*$ $(u^* \neq 0)$ at level C^* and the augmented Morse index $m^*(u^*)$ of u^* is greater than or equal to k.

Proof. Since A links $S_0(k)$, by Definition 1, we observe that $A \cap S_0(k) \neq \emptyset$. Note that $A \subset G^{C^{**}_{k+1}}$, hence $A \cap S(k) \neq \emptyset$ and then C^* is well defined. Then for any $A \in \mathcal{L}$, by (2), $\sup_{A\cap\mathcal{S}^*} G \geqq \inf_{S(k)} G \geqq -\Lambda_0 >> \sup_{u \in \partial\Theta_{k+1}} G$. Therefore, $C^* \geqq -\Lambda_0$. Evidently, we have $C^* \leqq \sup G(A) \leqq C^{**}_{k+1}$. We divide the proof into steps.

Step 1. We show that there exists a sign-changing critical point $0 \neq u^* \in \mathcal{S}^*$ with $G(u^*) = C^*$. We prove that $\mathcal{K}[C^*-\bar{\varepsilon}, C^*+\bar{\varepsilon}] \cap \mathcal{S}^* \neq \emptyset$ for all $\bar{\varepsilon} > 0$ small enough. That is, there is a sign-changing critical point at level C^*. By negation, we assume that $\mathcal{K}[C^*-\bar{\varepsilon}, C^*+\bar{\varepsilon}] \cap \mathcal{S}^* = \emptyset$ for some $\bar{\varepsilon} > 0$, then by (A_1), $\mathcal{K}[C^*-\bar{\varepsilon}, C^*+\bar{\varepsilon}] \subset (-\mathcal{P} \cup \mathcal{P})$. We first assume $\mathcal{K}[C^*-\bar{\varepsilon}, C^*+\bar{\varepsilon}] \neq \emptyset$. Since $\mathcal{K}[C^*-\bar{\varepsilon}, C^*+\bar{\varepsilon}]$ is compact in E, by the definition of $\mathcal{S}^*$, we must have $\text{dist}\,(\mathcal{K}[C^*-\bar{\varepsilon}, C^*+\bar{\varepsilon}], \mathcal{S}^*) := \delta_0 > 0$. By the (PS) condition, there is an $\varepsilon_1 \in (0, \bar{\varepsilon}/3)$, $\varepsilon_1 < 1$ such that $\|G'(u)\| \geqq \varepsilon_1$ for $u \in G^{-1}[C^*-\varepsilon_1, C^*+\varepsilon_1] \backslash (\mathcal{K}[C^*-\bar{\varepsilon}, C^*+\bar{\varepsilon}])_{\delta_0/2}$, here and in the sequel, $(A)_c := \{u \in E : \text{dist}(u, A) \leqq c\}$. By decreasing ε_1, we may assume that $3\varepsilon_1 < C^* - \sup_{\partial\Theta_{k+1}} G$. Then $\langle G'(u), W(u)\rangle \geqq \varepsilon_1^2/16$ for any u with

$$u \in G^{-1}[C^*-\varepsilon_1, C^*+\varepsilon_1] \backslash \big(\mathcal{K}[C^*-\bar{\varepsilon}, C^*+\bar{\varepsilon}]\big)_{\delta_0/2},$$

where $W(u) = \frac{(1+\|u\|)^2 Y(u)}{(1+\|u\|)^2\|Y(u)\|^2+1}$ and $Y(u) = u - L_0(u)$ is the odd pseudo-gradient vector field of G : $\langle G'(u), Y(u)\rangle \geqq \frac{1}{2}\|G'(u)\|^2$ and $\|Y(u)\| \leqq 2\|G'(u)\|$ for all $u \in \tilde{E}$, where $L_0(\pm\mathcal{D}_{\mu_0}) \subset \pm\mathcal{D}_{\mu_0}$ (By (A_1), this can be done as that in [22]). Let

$$U_1 = \{u \in E : |G(u) - C^*| \geqq 3\varepsilon_1\}, \quad U_2 = \{u \in E : |G(u) - C^*| \leqq 2\varepsilon_1\}.$$

Let $y(u) : E \to [0, 1]$ be a locally Lipschitz continuous map such that

$$y(u) := \begin{cases} 1 & \text{for all } u \in E \backslash (\mathcal{K}[C^* - \bar{\varepsilon}, C^* + \bar{\varepsilon}])_{\delta_0/2} \\ 0 & \text{for all } u \in (\mathcal{K}[C^* - \bar{\varepsilon}, C^* + \bar{\varepsilon}])_{\delta_0/3}. \end{cases}$$

Consider $h(u) := \frac{\text{dist}(u,U_1)}{\text{dist}(u,U_2)+\text{dist}(u,U_1)}$. Let $\bar{W}(u) := y(u)h(u)W(u)$ if $u \in \tilde{E}$ and $\bar{W}(u) = 0$ otherwise, then $\bar{W}$ is a locally Lipschitz vector field on E. We consider the following Cauchy initial value problem: $\frac{\mathrm{d}\eta(t,u)}{\mathrm{d}t} = -\bar{W}(\eta(t,u))$ with $\eta(0, u) = u \in E$, which has a unique continuous odd solution $\eta(t, u)$ in E. Evidently, $G(\eta(t, u))$ is non-increasing in t. By the definition of C^*, there exists an $A \in \mathcal{L}$ such that $\sup_{A\cap\mathcal{S}^*} G \leqq C^* + \varepsilon_1$. Therefore, $A \subset G^{C^*+\varepsilon_1} \cup (E\backslash\mathcal{S}^*)$. We claim that there exists a $T_1 > 0$ such that $\eta(T_1, A) \subset G^{C^*-\varepsilon_1/4} \cup \mathcal{D}^*$. In fact, if $u \in A\cap\mathcal{D}^*$, then similar to [22] (see also [10,12]), $\eta(t, u) \in \mathcal{D}^*$ for all $t \geqq 0$. If $u \in A$, $u \notin \mathcal{D}^*$, then we see that $G(u) \leqq C^*+\varepsilon_1$. If $G(u) \leqq C^*-\varepsilon_1$, then $G(\eta(t, u)) \leqq G(u) \leqq C^*-\varepsilon_1$ for all $t \geqq 0$. If $G(u) > C^*-\varepsilon_1$, then $u \in G^{-1}[C^*-\varepsilon_1, C^*+\varepsilon_1]$. If dist $(\eta([0,\infty), u), \mathcal{K}[C^* - \bar{\varepsilon}, C^* + \bar{\varepsilon}]) \leqq \delta_0/2$, then there exists a t_m such that $\eta(t_m, u) \notin \mathcal{S}^*$. Moreover, we may choose m so that $\text{dist}(\eta(t_m, u), \mathcal{S}^*) \geqq \frac{1}{3}\delta_0 > 0$. Assume dist $(\eta([0,\infty), u), \mathcal{K}[C^* - \bar{\varepsilon}, C^* + \bar{\varepsilon}]) > \delta_0/2$. Similarly, we assume that $G(\eta(t, u)) > C^* - \varepsilon_1$ for all t (otherwise, we are done), then $\eta(t, u) \in G^{-1}[C^* - \varepsilon_1, C^* + \varepsilon_1]\backslash(\mathcal{K}[C^* - \bar{\varepsilon}, C^* + \bar{\varepsilon}])_{\delta_0/2}$, hence, $h(\eta(t, u)) = 1, y(\eta(t, u)) = 1$ and $\langle G'(\eta(t, u)), W(\eta(t, u))\rangle \geqq \varepsilon_1^2/16$ for all $t \geqq 0$. Therefore, $G(\eta(\frac{48}{\varepsilon_1}, u)) = G(u) + \int_0^{\frac{48}{\varepsilon_1}} \mathrm{d}G(\eta(s, u)) \leqq C^* - 2\varepsilon_1$. By combining the above arguments, for any $u \in A\backslash\mathcal{D}^*$, there exists a $T_u > 0$ such that either $\eta(T_u, u) \in G^{C^*-\varepsilon_1/2}$ or $\text{dist}(\eta(T_u, u), \mathcal{S}^*) \geqq \frac{1}{3}\delta_0$. By continuity, there exists a neighborhood U_u of u such that either $\eta(T_u, U_u) \subset G^{C^*-\varepsilon_1/3}$ or $\text{dist}(\eta(T_u, U_u), \mathcal{S}^*) \geqq \frac{1}{4}\delta_0$. Both cases imply that $\eta(T_u, U_u) \subset G^{C^*-\varepsilon_1/3} \cup (E\backslash\mathcal{S}^*)$. Since $A\backslash\mathcal{D}^*$ is compact in E, we get a $T_1 > 0$ such that

$$\eta(T_1, A) \subset G^{C^*-\varepsilon_1/4} \cup (E\backslash\mathcal{S}^*). \tag{3}$$

If $\mathcal{K}[C^* - \bar{\varepsilon}, C^* + \bar{\varepsilon}] = \emptyset$, then we choose $(\mathcal{K}[C^* - \bar{\varepsilon}, C^* + \bar{\varepsilon}])_{\delta_0/2} = \emptyset$ and trivially, (3) is still true. Now we show that $\eta(T_1, A) \in \mathcal{L}$. Actually, note $\partial\Theta_{k+1} \subset A \in \mathcal{L}$ and $\partial\Theta_{k+1} \subset U_1$, then $\eta(t, \cdot)|_{\partial\Theta_{k+1}} = \mathbf{id}$ for all $t \geqq 0$. Then $\partial\Theta_{k+1} \subset \eta(T_1, A)$. On the other hand, $\sup G(\eta(T_1, A)) \leqq \sup G(A) \leqq C^{**}_{k+1}$. For any $h \in \mathbf{C}([0, 1]\times\eta(T_1, A), E)$ with $h(t, u)$ odd in $u \in \eta(T_1, A)$ and $h(t, \cdot)|_{\partial\Theta_{k+1}} = \mathbf{id}$ for all $t \in [0, 1]$. We define $h^*(t, u) \in \mathbf{C}([0, 1] \times A, E)$ by $h^*(t, u) = h(t, \eta(T_1, u))$. Then $h^*(t, u)$ is odd in $u \in A$ and $h^*(t, \cdot)|_{\partial\Theta_{k+1}} = \mathbf{id}$ for all $t \in [0, 1]$. Consequently, $h^*(1, A)\cap S_0(k) \neq \emptyset$ since $A \in \mathcal{L}$. From this, it follows that $\eta(T_1, A) \in \mathcal{L}$. But by (3), $\eta(T_1, A) \cap \mathcal{S}^* \subset G^{C^*-\varepsilon_1/4}$, which contradicts the definition of C^*.

By means of the next steps, we shall obtain information about the Morse index.

Step 2. In this step and the next steps 3–5, we assume that $\mathcal{K}_{C^*}\cap\mathcal{S}^*$ contains finitely many nondegenerate critical points. We first prepare the way by stating some properties: for any $u \in \mathcal{K}_{C^*} \cap \mathcal{S}^*$ with $Morse(u) = m^*$, we will show that we can always find a closed neighborhood $N_1(u)$ of u such that $N_1(u) \cap \partial\Theta_{k+1} = \emptyset$ and find a subset $N_2(u)$ such that $N_2(u) \subset N_1(u) \subset \mathcal{S}^*$ where $N_2(u)$ is homeomorphic to a ball B_{m^*} of $\mathbf{R}^{m^*}$. To show these properties, we let

$$\mathcal{K}_{C^*} \cap \mathcal{S}^* = \{u_1, \ldots, u_m\} \cup \{-u_1, \ldots, -u_m\}. \tag{4}$$

Since $C^* \geqq -\Lambda_0 >> \sup_{u \in \partial\Theta_{k+1}} G$ and $G(\pm u_i) = C^*$ for all $i = 1, \ldots, m$, we may find a $\delta > 0$ such that $B_\delta(\pm u_i) \cap \partial\Theta_{k+1} = \emptyset$, $i = 1, 2, \ldots, m$; where $B_\delta(\pm u_i) := \{u \in E : \|u - (\pm u_i)\| \leqq \delta\}$. For each u_i or $-u_i$, there exist closed subspaces of E, E^-, $E^+ = (E^-)^\perp$ such that both subspaces are invariant under $G''(u_i)$; $G''(u_i)$ is negative (positive) definite on E^- (E^+). We write $z \in E$ as $z = z^+ + z^-$ with $z^\pm \in E^\pm$. By the Morse Lemma, there exists a Lipschitz homeomorphism H_i from a neighborhood $U_i(0)$ of 0 in E onto a neighborhood $U(u_i)$ of u_i in E with $H_i(0) = u_i$ such that

$$G(H_i(z)) = G(u_i) + \|z^+\|^2 - \|z^-\|^2, \quad \forall\, z \in U_i(0).$$

Obviously, $-H_i$ is a Lipschitz homeomorphism from the neighborhood $U_i(0)$ of 0 onto a neighborhood $-U(u_i) = U(-u_i)$ of $-u_i$ in E with $-H_i(0) = -u_i$ and

$$G(-H_i(z)) = G(-u_i) + \|z^+\|^2 - \|z^-\|^2, \quad \forall\, z \in U_i(0).$$

Let B_i^- and B_i^+ denote the closed ball of radius $r_i^- > 0$ and $r_i^+ > 0$ centered at 0 in E^- and E^+ respectively. Choose numbers r_i^- and r_i^+ so small that $2B_i^- \oplus B_i^+$ is contained in the domain $U_i(0)$ of H_i. We also assume that $4(r_i^-)^2 < (r_i^+)^2$ for $i = 1, 2, \ldots, m$. We define

$$D_i = H_i(B_i^- \oplus B_i^+), \quad D_i' = H_i(2B_i^- \oplus B_i^+), \qquad i = 1, \ldots, m. \tag{5}$$

We may choose r_i^-, r_i^+, δ small enough such that

$$\pm D_i \subset \pm D_i' \subset B_\delta(\pm u_i) \subset \pm U(u_i) \subset \mathcal{S}^*, \quad i = 1, \ldots, m. \tag{6}$$

We may assume that $\{\pm D_i\}_{i=1}^m$ are disjoint from each other. The same is true of $\{\pm D_i'\}_{i=1}^m$, $\{B_\delta(\pm u_i)\}_{i=1}^m$ and $\{\pm U(u_i)\}_{i=1}^m$. Let $\phi : \mathbf{R} \to [0, 1]$ be a Lipschitz function such that $\phi(t) = 0$ for $t \leqq 0$ and $\phi(t) = 1$ for $t \geqq 1$. Inspired by Lazer and Solimini [16], we define $h : \mathbf{R} \times E \to E$ by

$$h(t, z) = \begin{cases} z \\ \text{for } z \notin \left(\cup_{i=1}^m D_i'\right) \cup \left(\cup_{i=1}^m (-D_i')\right), \\ H_1\left[t\phi\left(\frac{\|H_1^{-1}(z)^-\|}{r_1^-} - 1\right) H_1^{-1}(z)^+ + (1-t)H_1^{-1}(z)^+ + H_1^{-1}(z)^-\right], \\ \text{for } z \in D_1', \\ -H_1\left[t\phi\left(\frac{\|H_1^{-1}(-z)^-\|}{r_1^-} - 1\right) H_1^{-1}(-z)^+ + (1-t)H_1^{-1}(-z)^+ + H_1^{-1}(-z)^-\right], \\ \text{for } z \in -D_1', \\ \ldots\ldots \\ H_m\left[t\phi\left(\frac{\|H_m^{-1}(z)^-\|}{r_m^-} - 1\right) H_m^{-1}(z)^+ + (1-t)H_m^{-1}(z)^+ + H_m^{-1}(z)^-\right], \\ \text{for } z \in D_m', \\ -H_m\left[t\phi\left(\frac{\|H_m^{-1}(-z)^-\|}{r_m^-} - 1\right) H_m^{-1}(-z)^+ + (1-t)H_m^{-1}(-z)^+ + H_m^{-1}(-z)^-\right], \\ \text{for } z \in -D_m'. \end{cases}$$

Then $h(t, z)$ is odd in $z \in E$ and continuous in

$$E \backslash \left(\cup_{i=1}^{m} \left(-H_i(2B_i^- \oplus \partial B_i^+) \cup H_i(2B_i^- \oplus \partial B_i^+) \right) \right). \tag{7}$$

Choose $0 < \varepsilon < \min_{i=1,\ldots,m}\{(r_i^+)^2 - 4(r_i^-)^2\} := r^*$. Note that for any

$$u \in \cup_{i=1}^{m} \left(\left(-H_i(2B_i^- \oplus \partial B_i^+) \cup H_i(2B_i^- \oplus \partial B_i^+) \right) \right),$$

say, $u \in H_i(2B_i^- \oplus \partial B_i^+)$, then $u = H_i(w)$, $w = w^+ + w^- \in 2B_i^- \oplus \partial B_i^+$ with $\|w^+\| = r_i^+$, $\|w^-\| \leqq 2r_i^-$, it follows that

$$G(u) = G(H_i(w)) = C^* + \|w^+\|^2 - \|w^-\|^2 \geqq C^* + (r_i^+)^2 - 4(r_i^-)^2 > C^* + \varepsilon.$$

We observe that

$$G^{C^*+\varepsilon} \subset E \backslash \cup_{i=1}^{m} \left(\left(-H_i(2B_i^- \oplus \partial B_i^+) \cup H_i(2B_i^- \oplus \partial B_i^+) \right) \right) \tag{8}$$

for all $0 < \varepsilon < r^*$. Furthermore,

- $h(0, z) = z$ for all $z \in E$;
- $h(t, z) = z$ for all $z \notin \left(\cup_{i=1}^{m} B_\delta(u_i) \right) \cup \left(\cup_{i=1}^{m} B_\delta(-u_i) \right)$;
- $\frac{\mathrm{d}G(h(t,z))}{\mathrm{d}t} \leqq 0$, this is, $G(h(t, z))$ is non-increasing in t for any fixed $z \in E$.

Take $M_i = H_i(\frac{1}{3}B_i^- \oplus \frac{1}{3}B_i^+)$, $i = 1, 2, \ldots, m$. Then by (5) $M_i \subset D_i \subset D_i' \subset \mathcal{S}^*$. Define

$$N_2(u_i) = h(1, G^{C^*+\varepsilon}) \cap M_i. \tag{9}$$

Then $N_2(u_i) \subset N_1(u_i) := D_i \subset \mathcal{S}^*$. Note that h is continuous on $G^{C^*+\varepsilon}$ by (7) and (8). For $w \in N_2(u_i)$, then $w \in M_i$, $w = H_i(v)$, $v = H_i^{-1}(w) = v^+ \oplus v^-$, $v^\pm \in \frac{1}{3}B_i^\pm$; w is in the neighborhood $U(u_i)$ of u_i. On the other hand, $w = h(1, x)$, $x \in G^{C^*+\varepsilon}$. By (8), we see that

$$x \in E \backslash \cup_{i=1}^{m} \left(\left(-H_i(2B_i^- \oplus \partial B_i^+) \cup H_i(2B_i^- \oplus \partial B_i^+) \right) \right).$$

If $x \notin \left(\cup_{i=1}^{m} D_i' \right) \cup \left(\cup_{i=1}^{m} (-D_i') \right)$, then $w = h(1, x) = x \in M_i \subset D_i'$, a contradiction. That is, there must exist some D_{k_0}' (or $-D_{k_0}'$) such that $x \in D_{k_0}'$ (or $x \in -D_{k_0}'$) and hence

$$\begin{aligned} w &= h(1, x) \\ &= H_{k_0}\left(\phi\left(\frac{\|H_{k_0}^{-1}(x)^-\|}{r_{k_0}^-} - 1 \right) H_{k_0}^{-1}(x)^+ + H_{k_0}^{-1}(x)^- \right) \end{aligned}$$

or respectively,

$$\begin{aligned} w &= h(1, x) \\ &= -H_{k_0}\left(\phi\left(\frac{\|H_{k_0}^{-1}(-x)^-\|}{r_{k_0}^-} - 1 \right) H_{k_0}^{-1}(-x)^+ + H_{k_0}^{-1}(-x)^- \right). \end{aligned}$$

It implies that w is in the neighborhood of u_{k_0} (or $-u_{k_0}$). Then we must have $w \in U(u_{k_0})$ and $i = k_0$ and

$$w = h(1, x) = H_i\left(\phi\left(\frac{\|H_i^{-1}(x)^-\|}{r_i^-} - 1\right)H_i^{-1}(x)^+ + H_i^{-1}(x)^-\right).$$

Then $v^+ + v^- = H_i^{-1}(w)$ satisfying

$$v^- = H_i^{-1}(w)^- = (H_i^{-1}(x))^-, \quad \|v^-\| = \|H_i^{-1}(x)^-\| \leqq \frac{r_i^-}{3};$$

it follows by the definition of ϕ that $v^+ = \phi\left(\frac{\|H_i^{-1}(x)^-\|}{r_i^-} - 1\right)H_i^{-1}(x)^+ = 0$. So $w = H_i(v^-)$ with $v^- \in \frac{1}{3}B_i^-$. Moreover,

$$G(w) = G(u_i) - \|v^-\|^2 \leqq C^*. \tag{10}$$

These arguments imply that

$$N_2(u_i) = h(1, G^{C^*+\varepsilon}) \cap M_i \subset G^{C^*}. \tag{11}$$

In particular, if $G(w) = C^*$ for $w \in h(1, G^{C^*+\varepsilon}) \cap M_i \cap G^{-1}(C^*)$, then by (10) $v^- = 0$ and then $w = H_i(v^-) = H_i(0) = u_i$. Therefore,

$$h(1, G^{C^*+\varepsilon}) \cap M_i \cap G^{-1}(C^*) \cap \mathcal{S}^* = \{u_i\}. \tag{12}$$

So, for all $w \in N_2(u_i)$ we see that $w = H_i(v^-)$, $v^- \in \frac{1}{3}B_i^-$, $H_i^{-1}(w) = v^-$. Conversely, for any $v^- \in \frac{1}{3}B_i^-$ we have $H_i(v^-) \in M_i = H_i(\frac{1}{3}B_i^- \oplus \frac{1}{3}B_i^+) \subset D_i \subset D_i'$. Let $z = H_i(v^-)$, then $v^- = H_i^{-1}(z)$ and hence

$$\begin{aligned}
&h(1, z)\\
&= H_i\left(t\phi\left(\frac{\|H_i^{-1}(z)^-\|}{r_i^-} - 1\right)H_i^{-1}(z)^+ + (1-t)H_i^{-1}(z)^+ + H_i^{-1}(z)^-\right)\Big|_{t=1}\\
&= H_i(H_i^{-1}(z)^-)\\
&= H_i(v^-)\\
&= z.
\end{aligned}$$

Moreover, $G(z) = G(H_i(v^-)) = C^* - \|v^-\|^2 \leqq C^*$, that is, $z = h(1, z) \in h(1, G^{C^*+\varepsilon})$ and $H_i(v^-) \in M_i \cap h(1, G^{C^*+\varepsilon}) = N_2(u_i)$ (cf. (9)). Summing up, we obtain that

$$N_2(u_i) \text{ is homeomorphic to the ball } \frac{1}{3}B_i^-. \tag{13}$$

If we choose

$$M_i^* = H_i\left(\frac{1}{4}B_i^- \oplus \frac{1}{4}B_i^+\right), \quad i = 1, 2, \ldots, m,$$

we may prove analogously that

$$N_2^*(u_i) := h(1, G^{C^*+\varepsilon}) \cap M_i^* \text{ is homeomorphic to the ball } \frac{1}{4}B_i^-. \tag{14}$$

Now we choose

$$N_3(u_i) := h(1, G^{C^*+\varepsilon}) \cap \partial M_i, \tag{15}$$

$$M_i' = H_i\left(\frac{2}{3}B_i^- \oplus \frac{2}{3}B_i^+\right), \quad i = 1, 2, \ldots, m, \tag{16}$$

then similar to (10)–(12), we may show that

$$h(1, G^{C^*+\varepsilon}) \cap M_i' \subset G^{C^*}; \tag{17}$$

$$h(1, G^{C^*+\varepsilon}) \cap M_i' \cap G^{-1}(C^*) \cap \mathcal{S}^* = \{u_i\}; \tag{18}$$

and, by (5)–(6), $\pm M_i^* \subset \pm M_i \subset \pm M_i' \subset \pm D_i \subset \pm D_i' \subset B_\delta(\pm u_i) \subset \mathcal{S}^*$.
Step 3. Finding another $A^* \in \mathcal{L}$ which has properties stated in Step 4.

Given two neighborhoods $\mathcal{O}_1 \subset \mathcal{O}$ of $\mathcal{K}_{C^*}$, we may then find an odd and continuous descending flow $\eta_0 : [0, 1] \times E \to E$ such that $\eta_0(t, u) = u$ for all $u \in \partial\Theta_{k+1}$ and all u with $|G(u) - C^*| \geqq 2\varepsilon$; $\eta_0(1, u) = u$ for all $u \in \mathcal{O}_1$. In particular,

$$\eta_0(1, G^{C^*+\varepsilon}\backslash\mathcal{O}) \subset G^{C^*-\varepsilon}.$$

This essentially comes from the proof of Theorem A.4 in [19]. Moreover, we may require that $\eta_0(t, \mathcal{D}^*) \subset \mathcal{D}^*$ for all $t \in [0, 1]$, since the odd pseudo-gradient vector field $Y(u) = u - L_0(u)$ satisfying $L_0(\pm\mathcal{D}_{\mu_0}) \subset \pm\mathcal{D}_{\mu_0}$ (this is done in [22,26]). We may assume that $\mathcal{O}$ has two parts: $\mathcal{O}_{pn} \cup \mathcal{O}_s$, where $\mathcal{O}_{pn}$ is the neighborhood of all negative or positive critical points at level C^* such that $\mathcal{O}_{pn} \subset \mathcal{D}^*$, and $\mathcal{O}_s$ is the neighborhood of all sign-changing critical points at level C^*. Hence,

$$G^{C^*+\varepsilon}\backslash\mathcal{O}_s \subset \left(G^{C^*+\varepsilon}\backslash\mathcal{O}\right) \cup \mathcal{D}^*, \quad \eta_0(1, G^{C^*+\varepsilon}\backslash\mathcal{O}_s) \subset G^{C^*-\varepsilon} \cup \mathcal{D}^*.$$

Let

$$F_0 = \mathcal{S}^*\backslash\left(\cup_{i=1}^m M_i'\right) \cup \left(\cup_{i=1}^m(-M_i')\right),$$
$$G_0 = \left(\cup_{i=1}^m M_i\right) \cup \left(\cup_{i=1}^m(-M_i)\right) \subset \mathcal{S}^*.$$

In particular, we choose

$$\mathcal{O}_s := \left(\cup_{i=1}^m \mathrm{int} M_i'\right) \cup \left(\cup_{i=1}^m \mathrm{int}(-M_i')\right)$$

and

$$\left(\cup_{i=1}^m M_i^*\right) \cup \left(\cup_{i=1}^m(-M_i^*)\right) \subset \mathcal{O}_1. \tag{19}$$

Then

$$\eta_0(1, G^{C^*+\varepsilon} \cap F_0) \subset G^{C^*-\varepsilon} \cup \mathcal{D}^*. \tag{20}$$

Now we choose $A \in \mathcal{L}$ such that

$$\sup_{A\cap\mathcal{S}^*} G \leqq C^* + \varepsilon. \tag{21}$$

Now we let

$$A^* := \eta_0(1, h(1, A)). \tag{22}$$

We show that $A^* \in \mathcal{L}$. Obviously, $\partial\Theta_{k+1} \subset A^*$, $\sup G(A^*) \leqq C^{**}_{k+1}$. Take any h^* satisfying the assumptions of Definition 1, that is, $h^* \in \mathbf{C}([0,1]\times A^*, E)$ satisfying

- $h^*(t,u)$ is odd in u;
- $h^*(t,u) = u$ for all $u \in \partial\Theta_{k+1}$.

Let

$$f^* = h^*(t, \eta_0(1, h(1,u))), \quad u \in A.$$

Then $f^*(t,u)$ is odd in u; $f^*(t,u) = u$, for all $u \in \partial\Theta_{k+1}$. Hence, $f^*(1, A) \cap S_0(k) \neq \emptyset$. That is, $h^*(1, A^*) \cap S_0(k) \neq \emptyset$. Therefore, $A^* \in \mathcal{L}$.
Step 4. We now show that

$$\sup_{A^*\cap\mathcal{S}^*} G = C^* \tag{23}$$

and

$$G(u) < C^*, \quad \forall\, u \in \left(A^*\backslash\mathcal{K}_{C^*}\right)\cap\mathcal{S}^*. \tag{24}$$

In particular, there exists $u \in \mathcal{K}_{C^*}\cap\mathcal{S}^*\cap A^*$.

Indeed, for any $u \in A^*\cap\mathcal{S}^*$, write $u = \eta_0(1,v) \in \mathcal{S}^*$, then $v \in \mathcal{S}^*$ and $v = h(1,a)$ with $a \in \mathcal{S}^*\cap A$.

If $v \in F_0$ then $G(v) = G(h(1,a)) \leqq G(a) \leqq C^* + \varepsilon$. It follows that $v \in F_0 \cap G^{C^*+\varepsilon}$. Hence, $u = \eta_0(1,v) \in G^{C^*-\varepsilon}\cup\mathcal{D}^*$ by (20). Then, $u = \eta_0(1,v) \in G^{C^*-\varepsilon}$, $G(u) < C^*$, $u \notin \mathcal{K}_{C^*}$.

If $v \notin F_0$, then $v \in \left(\cup_{i=1}^m M_i'\right)\cup\left(\cup_{i=1}^m(-M_i')\right)$. Say $v \in M_i'$ for some i. Since $v = h(1,a)$ with $a \in \mathcal{S}^*$ and $A\cap\mathcal{S}^* \subset G^{C^*+\varepsilon}$ by (21), we have that $a \in G^{C^*+\varepsilon}$, $v = h(1,a) \in h(1, G^{C^*+\varepsilon})$. Then $v \in M_i' \cap h(1, G^{C^*+\varepsilon})$, which implies by (17) that $G(v) \leqq C^*$, $G(u) = G(\eta_0(1,v)) \leqq G(v) \leqq C^*$. If $G(v) = C^*$, by (18) we have that $v = u_i$, hence $u = v = u_i$. If $G(v) < C^*$, then we see that $G(u) < c$. On the other hand, if $G(v) < C^*$ for all such v, then $G(u) < C^*$ for all $u \in A^*\cap\mathcal{S}^*$, which contradicts the definition of C^* since $A^* \in \mathcal{L}$. These arguments imply that we must have a $v \in M_i'\cap h(1, G^{C^*+\varepsilon})$ such that $G(v) = C^*$ and therefore $v = u_i$, hence, $u = \eta_0(1,v) = \eta_0(1,u_i) = u_i$. That is, we have found a sign-changing critical point u_i in $A^*\cap\mathcal{S}^*$ such that $G(u) = C^*$. This completes the step.
Step 5. We show that there is an $u \in \mathcal{K}_{C^*}\cap\mathcal{S}^*\cap A^*$ such that $Morse(u) \geqq k$. Note that $\mathcal{K}_{C^*}\cap\mathcal{S}^* = \{u_1, u_2, \ldots, u_m\}\cup\{-u_1, -u_2, \ldots, -u_m\}$(see (4)). By negation we assume that

$$Morse(\pm u_i) < k, \quad i = 1, 2, \ldots, m. \tag{25}$$

Next, we show that

$$A^{**} := A^* \backslash \left(\left(\cup_{i=1}^m \mathrm{int} M_i^* \right) \cup \left(\cup_{i=1}^m \mathrm{int}(-M_i^*) \right) \right) \in \mathcal{L}. \tag{26}$$

If (26) were true, then we would have a contradiction to (23)–(24). Thus, (25) must be not true and the proof for the non-degenerate case of the Theorem is finished.

First of all, we keep in mind that $\partial\Theta_{k+1} \subset A^{**}$ and $A^{**} \subset A^* \subset G^{C_{k+1}^{**}}$. Now by negation, if (26) does not hold, we find a g such that $g \in \mathbf{C}([0,1] \times A^{**}, E)$ satisfying: $g(t,u)$ is odd in u; $g(t,u) = u$ for all $u \in \partial\Theta_{k+1}$, but

$$g[1, A^{**}] \cap S_0(k) = \emptyset. \tag{27}$$

Note that A^{**} is compact. Since $A^* \cap M_i^* \subset A^* \cap \mathcal{S}^*$, we have

$$\begin{aligned} A^* \cap M_i^* &= \eta_0(1, h(1, A)) \cap M_i^* \\ &\subset \eta_0 (1, h(1, A)) \cap \mathcal{S}^* \\ &\subset \eta_0 \left(1, h(1, A) \cap \mathcal{S}^*\right) \cap \mathcal{S}^* \\ &\subset \eta_0 \left(1, h(1, A \cap \mathcal{S}^*) \cap \mathcal{S}^*\right) \cap \mathcal{S}^* \\ &\subset \eta_0 \left(1, h(1, G^{C^*+\varepsilon}) \cap \mathcal{S}^*\right) \cap \mathcal{S}^* \quad (by \ \ (21)) \\ &\subset \eta_0 \left(1, h(1, G^{C^*+\varepsilon}) \cap \mathcal{S}^*\right). \end{aligned}$$

Since (19) implies that $\eta_0(1,\cdot)|_{M_i^*} = \mathbf{id}$ for $i = 1, \ldots, m$ and $\eta_0(1,\cdot)$ is a homeomorphism, we have

$$A^* \cap M_i^* = \eta_0^{-1}(1,\cdot)|_{A^* \cap M_i^*} \subset h(1, G^{C^*+\varepsilon}) \cap \mathcal{S}^* \subset h(1, G^{C^*+\varepsilon}),$$

therefore, $A^* \cap M_i^* \subset h(1, G^{C^*+\varepsilon}) \cap M_i^*$. Recall (14), we see that $h(1, G^{C^*+\varepsilon}) \cap M_i^*$ is homeomorphic to the ball $\frac{1}{4}B_i^-$, hence we obtain that $A^* \cap M_i^*$ is homeomorphic to a compact subset of $\mathbf{R}^{Morse(u_i)}$. We now observe that $g_i = g : [0,1] \times \left(A^* \cap \partial M_i^*\right) \to E$ can be extended to

$$g_i : \left([0,1] \times (A^* \cap \partial M_i^*)\right) \cup \left(\{0\} \times (A^* \cap M_i^*)\right) \to E$$

Let $\tilde{g}_i(1,\cdot)$ be the extension of $g_i(1,\cdot)$ from the domain $A^* \cap \partial M_i^*$ to the domain $A^* \cap M_i^*$. For each $u \in A^* \cap M_i^*$, we write $\tilde{g}_i(1,u) = \tilde{g}_i(1,u)^- + \tilde{g}_i(1,u)^+$ with $\tilde{g}_i(1,u)^+ \in E_k^\perp$, $\tilde{g}_i(1,u)^- \in E_k$. Note that $\dim E_k = k$. Define

$$\Omega_i = \left\{ u \in A^* \cap M_i^* : \begin{array}{l} \dfrac{\|\tilde{g}_i(1,u)^+\| \|\tilde{g}_i(1,u)^+\|_\star}{\|\tilde{g}_i(1,u)^+\| + \|\tilde{g}_i(1,u)^+\|_\star} = 1, \\ \tilde{g}_i(1,u)^- = 0 \end{array} \right\}. \tag{28}$$

We see that $\tilde{g}_i(1,\Omega_i) \subset S_0(k)$. Since by (27),

$$\tilde{g}_i(1, A^* \cap \partial M_i^*) \cap S_0(k) = g_i(1, A^* \cap \partial M_i^*) \cap S_0(k) = \emptyset,$$

we see that $\Omega_i \cap (A^* \cap \partial M_i^*) = \emptyset$, Ω_i is a compact subset of $A^* \cap M_i^*$. We can find an $\varepsilon' >$ small enough and define

$$\Omega_i^{\varepsilon'} = \left\{ u \in A^* \cap M_i^* : \begin{array}{l} \|\tilde{g}_i(1,u)^-\| < \varepsilon'; \\ 1 - \varepsilon' < \dfrac{\|\tilde{g}_i(1,u)^+\| \|\tilde{g}_i(1,u)^+\|_\star}{\|\tilde{g}_i(1,u)^+\| + \|\tilde{g}_i(1,u)^+\|_\star} < 1 + \varepsilon', \end{array} \right\}$$

such that $\Omega_i^{\varepsilon'} \cap (A^* \cap \partial M_i^*) = \emptyset$. Inspired by [20], we consider the mapping $\beta_i : \partial\Omega_i^{\varepsilon'} \to (E_k \times \mathbf{R}) \backslash \{(0,1)\}$ by

$$\beta_i(u) = \left(\tilde{g}_i(1,u)^-, \quad \frac{\|\tilde{g}_i(1,u)^+\| \|\tilde{g}_i(1,u)^+\|_\star}{\|\tilde{g}_i(1,u)^+\| + \|\tilde{g}_i(1,u)^+\|_\star} \right).$$

Since $\Omega_i^{\varepsilon'}$ is a compact subset of $\mathbf{R}^{Morse(u_i)}$ and $Morse(u_i) < k < k+1 = \dim(E_k \times \mathbf{R})$, it has an the extension $\tilde{\beta}_i : \Omega_i^{\varepsilon'} \to (E_k \times \mathbf{R}) \backslash \{(0,1)\}$, so we write

$$\tilde{\beta}_i(u) = \left(g_i^*(1,u), \rho(u) \right).$$

Then for any $u \in \partial\Omega_i^{\varepsilon'}$ we have that $\tilde{\beta}_i(u) = \beta_i(u)$, $g_i^*(1,u) = \tilde{g}_i(1,u)^-$;

$$\rho(u) = \frac{\|\tilde{g}_i(1,u)^+\| \|\tilde{g}_i(1,u)^+\|_\star}{\|\tilde{g}_i(1,u)^+\| + \|\tilde{g}_i(1,u)^+\|_\star}.$$

Define $T_i(u) : A^* \cap M_i^* \to E$ as follows:

$$T_i(u) = \begin{cases} g_i^*(1,u) + \rho(u) \dfrac{\tilde{g}_i(1,u)^+}{\frac{\|\tilde{g}_i(1,u)^+\| \|\tilde{g}_i(1,u)^+\|_\star}{\|\tilde{g}_i(1,u)^+\| + \|\tilde{g}_i(1,u)^+\|_\star}}, & \text{for } u \in \Omega_i^{\varepsilon'}, \\ \tilde{g}_i(1,u), & \text{for } u \notin \Omega_i^{\varepsilon'}. \end{cases}$$

On the other hand, if $u \notin \Omega_i^{\varepsilon'}$, then $u \notin \Omega_i$, then either

$$\frac{\|\tilde{g}_i(1,u)^+\| \|\tilde{g}_i(1,u)^+\|_\star}{\|\tilde{g}_i(1,u)^+\| + \|\tilde{g}_i(1,u)^+\|_\star} \neq 1$$

or $\tilde{g}_i(1,u)^- \neq 0$. In this case, $T_i(u) = \tilde{g}_i(1,u) = \tilde{g}_i(1,u)^- + \tilde{g}_i(1,u)^+ \notin S_0(k)$. Assume $u \in \Omega_i^{\varepsilon'}$. If

$$T_i(u) = g_i^*(1,u) + \rho(u) \frac{\tilde{g}_i(1,u)^+}{\frac{\|\tilde{g}_i(1,u)^+\| \|\tilde{g}_i(1,u)^+\|_\star}{\|\tilde{g}_i(1,u)^+\| + \|\tilde{g}_i(1,u)^+\|_\star}} \in S_0(k),$$

we must have $g_i^*(1,u) = 0$ and $\|\rho(u)\| = 1$, which contradicts their definitions. Therefore,

$$T_i(A^* \cap M_i^*) \cap S_0(k) = \emptyset. \tag{29}$$

Let

$$\bar{g}_i(t,u) = \begin{cases} g_i(t,u), & \text{for others,} \\ \\ T_i(u), & \text{for } t = 1, u \in A^* \cap M_i^*. \end{cases}$$

Then $\bar{g}_i$ is the extension of g_i from the domain $\big([0,1] \times A^* \cap \partial M_i^*\big) \cup (\{0\} \times A^* \cap M_i^*)$ to the domain $\big([0,1] \times A^* \cap \partial M_i^*\big) \cup \big(\{0,1\} \times A^* \cap M_i^*\big)$. By (29), we have

$$\bar{g}_i(1, A^* \cap M_i^*) \cap S_0(k) = T_i(A^* \cap M_i^*) \cap S_0(k) = \emptyset. \tag{30}$$

Next we show that

$$\bar{g}_i(t,u) : \big([0,1] \times (A^* \cap \partial M_i^*)\big) \cup \big(\{0,1\} \times (A^* \cap M_i^*)\big) \to E$$

has an extension $\Psi_i^* : [0,1] \times (A^* \cap M_i^*) \to E$ such that

$$\Psi_i^*(1, A^* \cap M_i^*) \cap S_0(k) = \emptyset. \tag{31}$$

First we let $G_i(t,u) : [0,1] \times (A^* \cap M_i^*) \to E$ be an extension of $\bar{g}_i(t,u)$. We write $G_i(t,u) = G_i(t,u)^- + G_i(t,u)^+$ with $G_i(t,u)^+ \in E_k^\perp$ and $G_i(t,u)^- \in E_k$. Define

$$\Lambda_i := \left\{ (t,u) \in [0,1] \times (A^* \cap M_i^*) : \begin{array}{c} G_i(t,u)^- = 0, \\ \dfrac{\|G_i(t,u)^+\| \|G_i(t,u)^+\|_\star}{\|G_i(t,u)^+\| + \|G_i(t,u)^+\|_\star} = 1 \end{array} \right\}.$$

Then Λ_i is a compact subset of $\mathbf{R}^{Morse(u_i)+1}$ and by (30), we observe that

$$\Lambda_i \cap \big(\{1\} \times (A^* \cap \partial M_i^*)\big) = \emptyset.$$

As before, for $\varepsilon' > 0$ small enough, let

$$\Lambda_i^{\varepsilon'} := \left\{ (t,u) \in [0,1] \times (A^* \cap M_i^*) : \begin{array}{l} \|G_i(t,u)^-\| < \varepsilon', \\ 1 - \varepsilon' < \dfrac{\|G_i(t,u)^+\| \|G_i(t,u)^+\|_\star}{\|G_i(t,u)^+\| + \|G_i(t,u)^+\|_\star} \\ < 1 + \varepsilon' \end{array} \right\}.$$

Consider the map $\Psi_i : \partial \Lambda_i^{\varepsilon'} \to (E_k \times \mathbf{R}) \backslash \{(0,1)\}$ given by

$$\Psi_i(t,u) = \left(G_i(t,u)^-, \frac{\|G_i(t,u)^+\| \|G_i(t,u)^+\|_\star}{\|G_i(t,u)^+\| + \|G_i(t,u)^+\|_\star} \right).$$

Since $Morse(u_i) + 1 < k + 1$, Ψ_i has an extension $\bar{\Psi}_i : \Lambda_i^{\varepsilon'} \to (E_k \times \mathbf{R}) \backslash \{(0,1)\}$. We denote $\bar{\Psi}_i$ by $\bar{\Psi}_i(t,u) = \big(\bar{G}_i(t,u)^-, \xi(t,u)\big)$ with $\bar{G}_i(t,u)^- \in E_k$ and $\xi(t,u) \in \mathbf{R}$. Define $\Psi_i^*(t,u) : [0,1] \times (A^* \cap M_i^*) \to E$ by

$$\Psi_i^*(t,u) = \begin{cases} \bar{G}_i(t,u)^- + \xi(t,u) \dfrac{G_i(t,u)^+}{\frac{\|G_i(t,u)^+\| \|G_i(t,u)^+\|_\star}{\|G_i(t,u)^+\| + \|G_i(t,u)^+\|_\star}}, & \text{for } (t,u) \in \Lambda_i^\varepsilon, \\ \\ G_i(t,u), & \text{for } (t,u) \notin \Lambda_i^\varepsilon. \end{cases}$$

Then Ψ_i^* from the domain $[0,1]\times(A^*\cap M_i^*)$ is an extension of $\bar{g}_i(t,u)$ such that (31) holds. Now, we define

$$H(t,u)=\begin{cases}\Psi_1^*(t,u), & \text{for } u\in A^*\cap M_1^*,\\ -\Psi_1^*(t,-u), & \text{for } u\in A^*\cap(-M_1^*),\\ \cdots\cdots & \\ \Psi_m^*(t,u), & \text{for } u\in A^*\cap M_m^*,\\ -\Psi_m^*(t,-u), & \text{for } u\in A^*\cap(-M_m^*),\\ g(t,u), & \text{for } u\in A^*\backslash\big((\cup_{i=1}^m \mathrm{int} M_i^*)\cup(\cup_{i=1}^m \mathrm{int}(-M_i^*))\big)\,.\end{cases}$$

Then $H(t,u)$ is odd in u and $H(t,u)|_{\partial\Theta_{k+1}}=g(t,u)|_{\partial\Theta_{k+1}}=\mathbf{id}$, $t\in[0,1]$. That is, H satisfies the properties of Definition 1 with respect to A^*. But by (27) and (31), $H(1,A^*)\cap S_0(k)=\emptyset$, which contradicts the fact that $A^*\in\mathcal{L}$.

Step 6. To finish the proof of this theorem, we have only to consider the degenerate critical points of $\mathcal{K}_{C^*}\cap\mathcal{S}^*$. The idea is classical. But since we want to obtain sign-changing critical points, we must say more. We will apply the ideas of the Marino–Prodi type perturbation methods (see Marino and Prodi [18] and also Lazer and Solimini [16] and Viterbo [25]) and a simple modification of the arguments in Section 3 of Solimini [24].

Assume by negation that each critical point of G in $\mathcal{K}_{C^*}\cap\mathcal{S}^*$ has an augmented Morse index less than k. Then for any $w\in\mathcal{K}_{C^*}\cap\mathcal{S}^*$, there is a decomposition of $E: E=E^-(w)\oplus E^0(w)\oplus E^+(w)$ and there exists a constant $\rho_w>0$ such that

$$\begin{aligned}&\langle G''(w)u,u\rangle\geqq\rho_w\|u\|^2,\quad\forall u\in E^+(w),\\ &\langle G''(w)u,u\rangle\leqq-\rho_w\|u\|^2,\quad\forall u\in E^-(w),\\ &\langle G''(w)u,u\rangle=0,\quad\forall u\in E^0(w),\end{aligned}\tag{32}$$

where

$$\dim E^-(w)+\dim E^0(w)<k,\quad\forall\, w\in\mathcal{K}_{C^*}\cap\mathcal{S}^*.\tag{33}$$

By the Sard–Smale Theorem, we find regular values C_1 and C_2 of G such that $C_1<C^*<C_2$. Let $M_0:=\mathcal{K}[C_1,C_2]\cap\mathcal{S}^*$. Then M_0 is compact and isolated from the rest of the critical points of G. Since G is of $\mathbf{C}^2$, for each $w\in M_0$ we may find a small open ball B_w centered at w satisfying

$$\|G''(w)-G''(u)\|<\rho_w/2,\quad\forall\, u\in B_w.\tag{34}$$

We may assume that $M_0\subset\cup_{i=1}^m B_{w_i}$. Define $\rho^*:=\frac{1}{2}\min\{\rho_{w_1},\rho_{w_2},\ldots,\rho_{w_m}\}$. By (32) and (34) we have

$$\langle G''(v)u,u\rangle\geqq\rho^*\|u\|^2,\quad\forall\, v\in B_{w_i},\quad\forall\, u\in E^+(w_i),\quad i=1,\ldots,m.\tag{35}$$

Now by the Marino–Prodi–Solimini's perturbation methods mentioned above, for any $\nu>0$ small enough and a functional $G_\nu\in\mathbf{C}^2(E,\mathbf{R})$ such that

(1) $U_\nu(M_0) \subset \cup_{i=1}^m B_{w_i}$, where $U_\nu(M_0) := \{u \in E : \text{dist}(u, M_0) < \nu\}$,
(2) G_ν satisfies the (PS) condition,
(3) $G(u) = G_\nu(u)$ when $u \notin U_\nu(M_0)$,
(4) any critical point of G_ν in $U_\nu(M_0)$ is non-degenerate,
(5) $|G_\nu(u) - G(u)| \leqq \nu$, $\|G'_\nu(u) - G'(u)\| \leqq \nu$, $\|G''_\nu(u) - G''(u)\| \leqq \nu$ for all $u \in E$. In particular, we may let G_ν be an even functional since G is also.

On the other hand, for the perturbed functional G_ν, we still have conditions (A_2)–(A_3) if ν small enough. To keep the flow-invariance, we write

$$G'_\nu(u) = G'(u) - \theta(u) = u - (K_G(u) + \theta(u)) := u - K^*_G(u).$$

Since $\|\theta(u)\| \leqq \nu$ is small enough, we may suppose that $2\theta(u) \in (-\mathcal{D}^* \cap \mathcal{D}^*)$. Then by assumption (A_1) for G, we still have that $K^*_G(\pm\mathcal{D}(\mu_0)) \subset \pm\mathcal{D}(\mu) \subset \pm\mathcal{D}(\mu_0)$ for another $\mu \in (0, \mu_0)$. That is, (A_1) holds for G_ν. Hence, we may apply the results of Step 1 to G_ν to conclude that G_ν has a sign-changing critical point at level C^*_ν defined as C^*. By item (5) above, we see that $C^*_\nu \in (C_1, C_2)$ since ν is very small. By item (3), we have that

$$\mathcal{K}_{C^*_\nu}(G_\nu) \cap \mathcal{S}^* := \{u \in E : G'_\nu(u) = 0, G_\nu(u) = C^*_\nu\} \cap \mathcal{S}^* \subset U_\nu(M_0)$$

By item (4), $\mathcal{K}_{C^*_\nu}(G_\nu) \cap \mathcal{S}^*$ consists of non-degenerate critical points of G_ν and hence, is a finite set. Hence, by the results of steps 2–5, G_ν has a sign-changing critical point $w_0 \in \mathcal{K}_{C^*_\nu}(G_\nu) \cap \mathcal{S}^*$ whose Morse index is not less than k. Assume $w_0 \in B_{w_i}$ for some i with $1 \leqq i \leqq m$, then by item (5) and (35), we have

$$\langle G''_\nu(w_0)u, u\rangle \geqq \langle G''(w_0)u, u\rangle - \frac{\rho^*}{2}\|u\|^2 \geqq \frac{\rho^*}{2}\|u\|^2, \quad \forall u \in E^+(w_i).$$

By (33), we have codim $E^+(w_i) = \dim E^-(w_i) + \dim E^0(w_i) < k$. Therefore, the Morse index of w_0 is less than k. This is a contradiction which completes the proof of the Theorem 2. □

3. Proof of Theorem 1

We fix a $p_0 \in (2, 2^*)$ and choose a sequence $\{p_n\}_{n\in\mathbf{N}}$ in $(p_0, 2^*)$ such that $p_n \to 2^*$. Consider the functional

$$G_{n,\lambda}(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 \mathrm{d}x - \frac{1}{p_n}\int_\Omega |u|^{p_n}\mathrm{d}x, \quad u \in H^1_0(\Omega),$$

where $H^1_0(\Omega)$ is the usual Sobolev space with the inner product $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x$ and the corresponding norm $\|u\| = \langle u, u\rangle^{1/2}$. Let $0 < \lambda_1 < \lambda_2 \leqq \lambda_3 \leqq \cdots \leqq \lambda_k \leqq \cdots$ be the eigenvalues of $(-\Delta, \Omega)$ and $\phi_k(x)$ be the eigenfunction corresponding to λ_k. Denote $E_k := \text{span}\{\phi_1, \phi_2, \ldots, \phi_k\}$. For each $p_n \in (2, 2^*)$, we let $\|\cdot\|_\star = \|\cdot\|_{p_n}$ (the usual L^{p_n} norm). Then (A_0) of Section 2 is satisfied. The gradient $G'_{n,\lambda}$ is of the form $G'_{n,\lambda}(u) = u - K_{n,\lambda}(u)$, where $K_{n,\lambda} : E \to E$ is a continuous operator. Let $\mathcal{P} := \{u \in H^1_0(\Omega) : u \geqq 0\}$. For each $\mu > 0$, define $\mathcal{D}(\mu) := \{u \in E : \text{dist}(u, \mathcal{P}) < \mu\}$. Set $\mathcal{D}^* = \mathcal{D}^*(\mu) := \mathcal{D}(\mu) \cup (-\mathcal{D}(\mu))$, $\quad \mathcal{S}^* = E\backslash\mathcal{D}^*(\mu)$.

Then it is easy to check that (A_1) is satisfied by $K_{n,\lambda}$, see for example [10, Lemma 2] (and also [5,12,13,22]). The assumptions (A_2)–(A_3) are evidently true for $G_{n,\lambda}$. Let

$$C^{**}_{k+1}(n,\lambda) := \sup_{E_{k+1}} G_{n,\lambda}.$$

Lemma 4. *There exists a constant $T_1 > 0$ independent of k and n such that*

$$C^{**}_{k+1}(n,\lambda) \leqq T_1 \lambda_{k+1}^{\frac{p_0}{2(p_0-2)}}.$$

Proof. This is standard. We include the proof here for completeness. The definition of E_{k+1} implies that $\|u\|^2 \leqq \lambda_{k+1}\|u\|_2^2$. Note that with $p_n > p_0$, we have $\|u\|_{p_0} \leqq D_1\|u\|_{p_n}$, where $D_1 > 0$ is a constant independent of n and k. Therefore, $G_{n,\lambda}(u) \leqq \frac{1}{2}\|u\|^2 - D_2\int_\Omega |u|^{p_0}\mathrm{d}x + D_3$, where $D_2 > 0$, $D_3 > 0$ are constant, independent of n and k. Since there is a constant $D_4 > 0$ such that $\|u\|_2 \leqq D_4\|u\|_{p_0}$, therefore we may have $D_5 > 0$ such that $\|u\|^{p_0} \leqq D_5\lambda_{k+1}^{p_0/2}\|u\|_{p_0}^{p_0}$ for all $u \in E_{k+1}$. Then

$$\begin{aligned} G_{n,\lambda}(u) &\leqq \frac{1}{2}\|u\|^2 - D_6\lambda_{k+1}^{-p_0/2}\|u\|^{p_0} + D_3 \\ &\leqq D_7\lambda_{k+1}^{\frac{p_0}{2(p_0-2)}} + D_3 \\ &\leqq T_1\lambda_{k+1}^{\frac{p_0}{2(p_0-2)}}, \end{aligned}$$

where D_i $(i = 1, \ldots, 7)$ and T_1 are positive constants independent of k and n. □

By Theorem 2, there is a sign-changing critical point $u^*(n,\lambda,k) \neq 0$ of $G_{n,\lambda}$ such that

$$G_{n,\lambda}\left(u^*(n,\lambda,k)\right) = C^*(n,\lambda,k) \leqq C^{**}_{k+1}(n,\lambda) \leqq T_1\lambda_{k+1}^{\frac{p_0}{2(p_0-2)}} \tag{36}$$

and the augmented Morse index $m^*(u^*(n,\lambda,k))$ of $u^*(n,\lambda,k)$ is $\geqq k$. We now must estimate the lower bound of $C^*(n,\lambda,k)$. By the proof of Theorem 2, for $G_{n,\lambda}$ we choose $\|\cdot\|_\star = \|\cdot\|_{p_n}$ and then

$$\beta^*(u) := \beta^*_{n,\lambda}(u) = \begin{cases} \dfrac{\|u\|\|u\|_{p_n}}{\|u\| + \|u\|_{p_n}}, & u \neq 0, \\ 0, & u = 0, \end{cases}$$

and $S_0(n,k) := \{u \in E_k^\perp : \beta^*_{n,\lambda}(u) = 1\}$. Since $\|u\|_{p_n} \leqq \kappa_1\|u\|$ for all $u \in E$, where κ_1 is independent of n and k. Then $\|u\|_{p_n} \leqq \kappa_1 + 1 := \alpha_4$ (as in Section 2) for all $u \in S_0(n,k)$. Recall $S(k) := S(n,k) := S_0(n,k) \cap G_{n,\lambda}^{C^{**}_{k+1}(n,\lambda)}$. Then

$$\|u\|^2 \leqq \frac{\lambda_1}{\lambda_1 - \lambda}\left((\kappa_1 + 1)^{p_n} + 2C^{**}_{k+1}(n,\lambda)\right), \quad \forall\, u \in S(n,k). \tag{37}$$

Hence, as in Lemma 1, we find $\alpha_5 > 0$ such that such that $\|u\| \leqq \alpha_5$ for all $u \in S(n,k)$. In particular, by (36)–(37), $\alpha_5 > 0$ is independent of n. Using Lemma 1, there is a $\Lambda_0 = \Lambda_0(\alpha_5) > 0$ such that $\inf_{u\in S(n,k)} G_{n,\lambda} \geqq -\Lambda_0$, where Λ_0 is independent of n. By Theorem 2 and (36),

$$C^*(n,\lambda,k) \in [-\Lambda_0, T_1\lambda_{k+1}^{\frac{p_0}{2(p_0-2)}}]. \tag{38}$$

For the fixed λ, k, $\{u^*(n,\lambda,k)\}_{n\in\mathbf{N}}$ is a sequence of solutions to the following equation:

$$-\Delta u = \lambda u + |u|^{p_n-2}u, \qquad u \in H_0^1(\Omega), \tag{39}$$

for p_n varying in $[p_0, 2^*]$. By Lemma 4, it is easy to check that $\{u^*(n,\lambda,k)\}_{n\in\mathbf{N}}$ is bounded in $H_0^1(\Omega)$. Here we insert the following proposition attributable to Devillanova and Solimini [14].

Proposition 1. (Uniform bound through concentration estimates) *Let $N \geqq 7$ and U be a bounded set in $H_0^1(\Omega)$ whose elements are solutions, for a fixed $\lambda > 0$, to the problem*

$$-\Delta u = \lambda u + |u|^{p-2}u, \qquad u \in H_0^1(\Omega), \tag{40}$$

for p varying in $[2, 2^*]$. *Then U is uniformly bounded, that is, there exists a constant $C > 0$ such that*

$$\sup_{u\in U}\sup_{x\in\Omega} |u(x)| \leqq C.$$

By this proposition, we know that $\{u^*(n,\lambda,k)\}_{n\in\mathbf{N}}$ is uniformly bounded. Then, by standard compactness arguments we have a convergent subsequence having limit $u^*(\lambda,k)$ which is a solution of (1) at level $C^*(\lambda,k)$. Since (1) has no solution with negative energy, we know that $C^*(\lambda,k) \geqq 0$. By (38),

$$0 \leqq \lim_{n\to\infty} C^*(n,\lambda,k) = C^*(\lambda,k) \leqq T_1\lambda_{k+1}^{\frac{p_0}{2(p_0-2)}}.$$

Moreover, we claim that $u^*(\lambda,k)$ is still sign-changing. Indeed, since $\{u^*(n,\lambda,k)\}_{n\in\mathbf{N}}$ is a sequence of sign-changing solutions to (39), let

$$u^*(n,\lambda,k)^{\pm} := \max\{\pm u^*(n,\lambda,k), 0\}.$$

Then $\|u^*(n,\lambda,k)^{\pm}\|^2 = \lambda\|u^*(n,\lambda,k)^{\pm}\|_2^2 + \int_\Omega |u^*(n,\lambda,k)^{\pm}|^{p_n}\mathrm{d}x$. It follows that $\|u^*(n,\lambda,k)^{\pm}\|^2 \leqq \varepsilon_0\|u^*(n,\lambda,k)^{\pm}\|_{p_n}^{p_n}$. Hence, $\|u^*(n,\lambda,k)^{\pm}\| \geqq s_0 > 0$, where ε_0, s_0 are constants independent of n. This implies that the limit $u^*(\lambda,k)$ of the subsequence of $\{u^*(n,\lambda,k)\}_{n\in\mathbf{N}}$ is still sign-changing. Next, we have only to show that $\lim_{k\to\infty} C^*(\lambda,k) = \infty$. Otherwise, we assume that $\{C^*(\lambda,k)\}_{k\in\mathbf{N}}$ is bounded and then $\lim_{k\to\infty} C^*(\lambda,k) = c' < \infty$. For any $k \in \mathbf{N}$ we may find an n_k (we may assume $n_k > k$) such that $|C^*(n_k,\lambda,k) - C^*(\lambda,k)| < 1/k$. It follows that $\lim_{k\to\infty} C^*(n_k,\lambda,k) = \lim_{k\to\infty} C^*(\lambda,k) = c' < \infty$. Since $u^*(n_k,\lambda,k)$ is a sign-changing critical point of $G_{n_k,\lambda}$ such that $G_{n_k,\lambda}(u^*(n_k,\lambda,k)) = C^*(n_k,\lambda,k)$ and the augmented Morse index $m^*(u^*(n_k,\lambda,k))$ of $u^*(n_k,\lambda,k)$ is greater than or equal to k. Once again, we may show that $\{u^*(n_k,\lambda,k)\}$ is bounded in $H_0^1(\Omega)$

since $c' < \infty$ is independent of k. Hence, by Proposition 1, $\{u^*(n_k, \lambda, k)\}_{k\in\mathbf{N}}$ is uniformly bounded, hence the Morse index of $\{u^*(n_k, \lambda, k)\}_{k\in\mathbf{N}}$ is bounded. This contradiction is caused by the assumption that $\{C^*(\lambda, k)\}_{k\in\mathbf{N}}$ is bounded. This finishes the proof of Theorem 1. □

Acknowledgments. W. Zou thanks Professor M. Ramos so much for many enlightening discussions when he was visiting CMAF (Lisboa, Portugal). Ramos' suggestions are very highly appreciated. Schechter is supported by NSF. Zou is supported by NSFC (10871109, 10571096) and the program of the Ministry of Education in China for NCET in Universities of China.

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Department of Mathematics,
University of California,
Irvine, CA 92697-3875,
USA.
e-mail: mschecht@math.uci.edu

and

Department of Mathematical Sciences,
Tsinghua University,
100084 Beijing, China.
e-mail: wzou@math.tsinghua.edu.cn

(*Received November 18, 2008 / Accepted October 11, 2009*)
Published online January 16, 2010 –