Polynomial identities related to special Schubert varieties

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Abstract

Let $\mathcal{S}$ be a single condition Schubert variety with an arbitrary number of strata. Recently, an explicit description of the summands involved in the decomposition theorem applied to such a variety has been obtained in a paper of the second author. Starting from this result, we provide an explicit description of the Poincaré polynomial of the intersection cohomology of $\mathcal{S}$ by means of the Poincaré polynomials of its strata, obtaining interesting polynomial identities relating Poincaré polynomials of several Grassmannians, both by a local and by a global point of view. We also present a symbolic study of a particular case of these identities.

Keywords Leray–Hirsch theorem · Derived category · Intersection cohomology · Decomposition theorem · Schubert varieties · Resolution of singularities

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1 Introduction

In the paper [10], it was shown how one can obtain suitable polynomial identities from the study of the intersection cohomology of Schubert varieties with two strata (compare with [[10], p. 115]). The aim of our work is to extend the same approach to *Special Schubert varieties with an arbitrary number of strata*, by showing that the

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Poincaré polynomials of their intersection cohomology naturally lead to a class of tricky polynomial identities. In the final “1”, we provide some of the numerical tests for the polynomial identities that we obtained in the meantime, and a symbolic study of a particular case.

The starting point of our analysis is the main result of the paper [13], which we now summarize. Let $\mathcal{S}$ be a *single condition Schubert variety or special Schubert variety* of dimension n (see [[5], p. 328] and [[18], Example 8.4.9]). As it is well known, $\mathcal{S}$ admits two standard resolutions: a *small resolution* $\xi : \mathcal{D} \rightarrow \mathcal{S}$ [[18], Definition 8.4.6] and a (usually) non-small one $\pi : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$ [[20], Sect. 3.4 and Exercise 3.4.10]. We will describe both resolutions π and ξ in Sect. 2.4. By [[15], Sect. 6.2] and [[18], Theorem 8.4.7], we have

$$IC_{\mathcal{S}}^{\bullet} \cong R\xi_* \mathbb{Q}_{\mathcal{D}}[n] \quad \text{in} \quad D_c^b(\mathcal{S}), \quad (1)$$

where $IC_{\mathcal{S}}^{\bullet}$ denotes the *intersection cohomology complex* of $\mathcal{S}$ [[12], p. 156], and $D_c^b(\mathcal{S})$ is the constructible derived category of sheaves of $\mathbb{Q}$ -vector spaces on $\mathcal{S}$.

By the celebrated Decomposition theorem [2–4, 22], the intersection cohomology complex of $\mathcal{S}$ is also a direct summand of $R\pi_* \mathbb{Q}_{\tilde{\mathcal{S}}}[n]$ in $D_c^b(\mathcal{S})$. Specifically, the Decomposition theorem says that there is a decomposition in $D_c^b(\mathcal{S})$ [[4], Theorem 1.6.1]

$$R\pi_* \mathbb{Q}_{\tilde{\mathcal{S}}}[n] \cong \bigoplus_{i \in \mathbb{Z}} {}^p\mathcal{H}^i(R\pi_* \mathbb{Q}_{\tilde{\mathcal{S}}}[n])[-i], \quad (2)$$

where ${}^p\mathcal{H}^i(R\pi_* \mathbb{Q}_{\tilde{\mathcal{S}}}[n])$ denote the *perverse cohomology sheaves* [[4], Sect. 1.5]. Furthermore, the perverse cohomology sheaves ${}^p\mathcal{H}^i(R\pi_* \mathbb{Q}_{\tilde{\mathcal{S}}}[n])$ are semisimple, i.e. direct sums of intersection cohomology complexes of semisimple local systems, supported in the smooth strata of $\mathcal{S}$.

In the paper [13], the summands involved in (2) are explicitly described. It turns out that the semisimple local systems involved in the decomposition are constant sheaves supported in the smooth strata of π . In other words, the decomposition (2) takes the following form

$$R\pi_* \mathbb{Q}_{\tilde{\mathcal{S}}} \cong \bigoplus_{p,q} IC_{\Delta_p}^{\bullet}[q]^{\oplus m_{pq}} \quad (3)$$

for suitable multiplicities $m_{pq} \in \mathbb{N}_0$ (that are computed in [[13], Theorem 3.5]) and where the strata Δ_p are special Schubert varieties, as well.

Following the same lines as in [[10], section 4], our main aim is to deduce some classes of polynomial identities from the isomorphism (3). Specifically, we are going to prove a class of *local identities* as well as a class of *global identities*.

Our first task is accomplished in Theorem 2. In a nutshell, the argument behind our local polynomial identity rests on the remark that *each summand of (3) is a direct sum of shifted trivial local systems in $D_c^b(\Delta_p^0)$* , when restricted to the smooth part Δ_p^0 of each stratum Δ_p . This fact follows by applying the Leray-Hirsch theorem (see [[23], Theorem 7.33], [[8], Lemma 2.5]) to the summands, that are described on Δ_p^0 by means of suitable Grassmann fibrations. This implies that we are allowed to associate

a Poincaré polynomial to each summand of (3), thus providing our local identity in the stratum Δ_p^0 (for more details compare with Sect. 3).

As for the global polynomial identities, they are obtained in Theorem 3, which is the main result of our paper. The idea of the proof is very similar to that of [[10], section 4]. Here is the claim.

Theorem *For any 4-tuple of integers (i, j, k, l) such that $0 < i < j \leq k < l$ and $r = k - i < l - j = c$, we have:*

$$\begin{aligned} \frac{P_j P_{l-i}}{P_i P_{j-i} P_{k-i} P_{l-k}} &= \frac{P_{l-j} P_{k+j-i}}{P_{k-i} P_{l+i-j-k} P_k P_{j-i}} + \\ &+ \sum_{s=1}^{\min\{k-i, k-c\}} \frac{P_{k-c} P_{l-j} P_{k+j-i-s}}{P_s P_{k-c-s} P_{k-i-s} P_{l+i-j-k+s} P_k P_{j-i-s}} t^{2s(c-r+s)}, \end{aligned}$$

where we assume $P_0 = 1$ and take

$$P_\alpha = h_0 \cdot \dots \cdot h_{\alpha-1}, \quad \forall \alpha > 0 \quad \text{and} \quad h_\alpha = \sum_{i=0}^{\alpha} t^{2\alpha}, \quad \forall \alpha \geq 0.$$

It is worth spending a few words about its proof. From (3) we deduce an isomorphism among the i -th hypercohomology spaces

$$\mathbb{H}^i(R\pi_* \mathbb{Q}_{\tilde{S}}) \cong \bigoplus_{p,q} \mathbb{H}^i(IC_{\Delta_p}^\bullet[q]^{\oplus m_{pq}}), \quad (4)$$

that leads to an equality of the corresponding Poincaré polynomials

$$\sum_i t^i \dim \mathbb{H}^i(R\pi_* \mathbb{Q}_{\tilde{S}}) = \sum_{i,p,q} t^i \dim \mathbb{H}^i(IC_{\Delta_p}^\bullet[q]^{\oplus m_{pq}}). \quad (5)$$

Again, all summands of (5) are determined by means of Leray-Hirsch theorem as Poincaré polynomials of suitable Grassmann fibrations.

We also observe that an *explicit inductive algorithm for the computation of the Poincaré polynomials of the intersection cohomology of Special Schubert varieties* straightforwardly follows from our results (see Corollary 1 and Remark 2). Although these Poincaré polynomials are already known, the availability of an algorithm for their computation could be the starting point for obtaining an analogous algorithm for all Schubert varieties in a future paper, being an explicit formula in this general case not known yet.

In the “Appendix”, we give an example of elementary proof of the global polynomial identity of Theorem 3 in a particular case, by algebraic manipulation only, with a divide and conquer strategy.

2 Basic facts and notations

2.1 Preliminaries

Throughout the paper, we shall work with $\mathbb{Q}$ -coefficients *cohomology groups*; that is, for any complex variety V and any integer k , $H^k(V) = H^k(V, \mathbb{Q})$. Let $D_c^b(V)$ denote the *derived category of bounded constructible complexes of sheaves* $\mathcal{F}^\bullet$ on V ([12], Sects. 1.3 and 4.1, [4], 4, Sect. 1.5). The symbol $\mathbb{H}^k(\mathcal{F}^\bullet)$ stands for the k -th *hypercohomology group* of $\mathcal{F}^\bullet$ ([12], Definition 2.1.4), while $IC_V^\bullet$ represents the *intersection cohomology complex* of V ([12], 12, 5.4) and [4], 12, 1.5, Sect. 2.1). Lastly, the intersection cohomology groups of a pure n -dimensional complex algebraic variety V are given by ([12], Definition 5.4.3))

$$IH^k(V) = IH^k(V, \mathbb{Q}) = \mathbb{H}^k(V, IC_V^\bullet[-n]).$$

2.2 Decomposition theorem

The *Decomposition theorem*, which was proved by A. Beilinson, J. Bernstein and P. Deligne in [2], is a tool of paramount importance: most of our results descend from it directly.

Theorem 1 (*Decomposition theorem*, [4], 4, Sect. (1.6.1)) *Let $f : X \rightarrow Y$ be a proper map of complex algebraic varieties. There is an isomorphism in the constructible bounded derived category $D_c^b(Y)$*

$$Rf_* IC_X \cong \bigoplus_{i \in \mathbb{Z}} {}^p\mathcal{H}^i(Rf_* IC_X)[-i].$$

Furthermore, the perverse sheaves ${}^p\mathcal{H}^i(Rf_ IC_X)$ are semisimple; i.e., there is a decomposition into finitely many disjoint locally closed and nonsingular subvarieties $Y = \bigsqcup S_\beta$ and a canonical decomposition into a direct sum of intersection complexes of semisimple local systems*

$${}^p\mathcal{H}^i(Rf_* IC_X) \cong \bigoplus_{\beta} IC_{S_\beta}^{\overline{}}(L_\beta).$$

Roughly speaking, Decomposition theorem states that, under mild hypotheses, the direct image of the intersection cohomology complex of a complex algebraic variety can be thought of as the direct sum of *intermediate extensions* (see [4], Sect. 2.7)) of semisimple local systems (see [12], Sect. 2.5)).

In the literature one can find different approaches to the Decomposition Theorem (see [2–4, 22, 24]), which is a very general result but also rather implicit. On the other hand, there are many special cases for which the Decomposition Theorem admits a simplified and explicit approach. One of these is the case of varieties with isolated singularities [9, 11, 21]. For instance, in the work [9], a simplified approach to the Decomposition Theorem for varieties with isolated singularities is developed, in

connection with the existence of a *natural Gysin morphism*, as defined in [[7], Definition 2.3] (see also [6] for other applications of the Decomposition Theorem to the Noether-Lefschetz Theory).

2.3 Grassmannians and Poincaré polynomials

We shall denote by $\mathbb{G}_k(\mathbb{C}^n)$ the Grassmannian of k -vector subspaces of $\mathbb{C}^n$; that is, the set of all k -dimensional subspaces of $\mathbb{C}^n$. More in general, we can extend this definition by replacing $\mathbb{C}^n$ with any complex vector space V (see [[16], 14, Sect. 6], [[14], 16, Sect. 1.5], [[20], 20, Sect. 3.1]).

Let X be a topological space. The *Poincaré polynomial* H_X of its *cohomology* and the *Poincaré polynomial* IH_X of its *intersection cohomology* (later on, they will be simply called Poincaré polynomials) are given by

$$H_X = \sum_{\alpha \in \mathbb{Z}} \dim_{\mathbb{Q}} H^{\alpha}(X) \cdot t^{\alpha} \quad \text{and} \quad IH_X = \sum_{\alpha \in \mathbb{Z}} \dim_{\mathbb{Q}} IH^{\alpha}(X) \cdot t^{\alpha},$$

respectively. When $X = \mathbb{G}_k(\mathbb{C}^l)$, we have the following explicit formula of the Poincaré polynomial (see [[5], p. 328], [[10], 10, Sect. 2 (vi), (vii), (viii)])

$$H_{\mathbb{G}_k(\mathbb{C}^l)} = \frac{P_l}{P_k P_{l-k}},$$

where we assume $P_0 = 1$ and take

$$P_{\alpha} = h_0 \cdot \dots \cdot h_{\alpha-1}, \quad \forall \alpha > 0 \quad \text{and} \quad h_{\alpha} = \sum_{i=0}^{\alpha} t^{2i}, \quad \forall \alpha \geq 0.$$

2.4 Special Schubert varieties

In this subsection we collect some facts concerning special Schubert varieties and their resolutions. For more details and explanations we refer the reader to [[13], Sect. 2.2–2.6].

Let i, j, k, l be integers such that

$$0 < i < k \leq j < l \text{ and } r = k - i < l - j = c$$

and fix a j -dimensional subspace $F \subseteq \mathbb{C}^l$. We are working with *single condition (or special) Schubert varieties*

$$\mathcal{S} = \left\{ V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F) \geq i \right\}$$

and we are considering the Whitney stratification

$$\Delta_1 \subset \dots \subset \Delta_r \subset \Delta_{r+1} = \mathcal{S}$$

where, for any p ,

$$\Delta_p = \left\{ V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F) \geq i_p = k - p + 1 \right\}$$

is a special Schubert variety, as well, and $\Delta_p = \text{Sing } \Delta_{p+1}$.

For any $0 < q < p \leq r + 1$ there is a commutative diagram

$$\begin{array}{ccc} \Delta_{pq}^0 & \hookrightarrow & \tilde{\Delta}_p \\ \rho_{pq} \downarrow & & \downarrow \pi_p \\ \Delta_q^0 & \xhookrightarrow{i} & \Delta_p \end{array}$$

where

$$\begin{aligned} \Delta_q^0 &= \Delta_q \setminus \text{Sing } \Delta_q = \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F) = i_q\}, \\ \tilde{\Delta}_p &= \left\{ (Z, V) \in \mathbb{G}_{i_p}(F) \times \mathbb{G}_k(\mathbb{C}^l) : Z \subseteq V \right\}, \\ \Delta_{pq}^0 &= \pi_p^{-1}(\Delta_q^0) \\ &= \{(Z, V) \in \mathbb{G}_{i_p}(F) \times \mathbb{G}_k(\mathbb{C}^l) : Z \subseteq V \text{ and } \dim(V \cap F) = i_q\}, \end{aligned}$$

the map

$$\pi_p : (Z, V) \in \tilde{\Delta}_p \mapsto V \in \Delta_p$$

is a resolution of singularities, and the function

$$\rho_{pq} : (Z, V) \in \Delta_{pq}^0 \mapsto V \in \Delta_q^0$$

is a fibration with fibres

$$F_{pq} = \mathbb{G}_{i_p}(\mathbb{C}^{i_q}).$$

The resolutions π_p are small when $k \leq c$ (see [[13], Remark 2.3]), whereas there are other small resolutions when $k > c$ (see [[13], Proof of Lemma 3.2]); namely

$$\xi_p : (V, U) \in \mathcal{D}_p = \{(V, U) \in \mathbb{G}_k(\mathbb{C}^l) \times \mathbb{G}_{k+j-i_p}(\mathbb{C}^l) : V + F \subseteq U\} \mapsto V \in \Delta_p.$$

3 Local polynomial identities

Before we give the proof of the first theorem, we shall fix some notations in order to make it more readable. For any pair of integers (p, q) with $0 < q < p$, we set

$$\begin{aligned} m_p &= \dim \Delta_p = (k + 1 - p)(j + p - k - 1) + (p - 1)(l - k), \\ \delta_{pq} &= \dim \mathbb{G}_{p-q}(\mathbb{C}^{k-c}) = (p - q)(k - c + q - p) \end{aligned}$$

and

$$A_{pq}^\alpha = H^\alpha(F_{pq}), \quad F_{pq} = \mathbb{G}_{i_p}(\mathbb{C}^{i_q}),$$

$$\begin{aligned} D_{pq}^\alpha &= H^\alpha(T_{pq}), & T_{pq} &= \mathbb{G}_{p-q}(\mathbb{C}^{k-c}), \\ B_{pq}^\alpha &= H^\alpha(G_{pq}), & G_{pq} &= \mathbb{G}_{p-q}(\mathbb{C}^{c-q+1}). \end{aligned}$$

Theorem 2 *For any pair of integers (p, q) with $0 < q < p$ there is a local polynomial identity*

$$\begin{aligned} \frac{P_{k-q+1}}{P_{k-p+1} P_{p-q}} &= \sum_{\tau=q+1}^{p-1} \left(\frac{P_{k-c}}{P_{p-\tau} P_{k-c-p+\tau}} \cdot \frac{P_{c-q+1}}{P_{\tau-q} P_{c-\tau+1}} \cdot t^{2d_{p\tau}} \right) \\ &+ \frac{P_{k-c}}{P_{p-q} P_{k-c-p+q}} \cdot t^{2d_{pq}} + \frac{P_{c-q+1}}{P_{p-q} P_{c-p+1}}, \end{aligned}$$

where $k \in \mathbb{Z}$ is such that $0 < i < k \leq j < l$, $c = l - j$ and $2d_{p\tau} = m_p - m_\tau - \delta_{p\tau}$.

Proof By the Decomposition theorem [[4], Theorem 1.6.1], we know that

$$R\pi_{p*} \mathbb{Q}_{\tilde{\Delta}_p} [m_p] \cong \bigoplus_{\alpha \in \mathbb{Z}} {}^p\mathcal{H}^\alpha(R\pi_{p*} \mathbb{Q}_{\tilde{\Delta}_p} [m_p]) [-\alpha].$$

In [[13], Remark 3.1] it is shown how the Leray-Hirsch theorem implies that

$${}^p\mathcal{H}^\alpha \left(i^* R\pi_{p*} \mathbb{Q}_{\tilde{\Delta}_p} \right) |_{\Delta_q^0} \cong A_{pq}^{\alpha-m_q} \otimes \mathbb{Q}_{\Delta_q^0} [m_q],$$

that is,

$${}^p\mathcal{H}^\alpha \left(i^* R\pi_{p*} \mathbb{Q}_{\tilde{\Delta}_p} [m_p] \right) |_{\Delta_q^0} \cong A_{pq}^{\alpha+m_p-m_q} \otimes \mathbb{Q}_{\Delta_q^0} [m_q]$$

(for a generalization of the Leray-Hirsch theorem in a categorical framework we refer to [[23], Theorem 7.33] and [[8], Lemma 2.5]). In addition, in [[13], Theorem 3.5] it is proved that

$${}^p\mathcal{H}^\alpha \left(R\pi_{p*} \mathbb{Q}_{\tilde{\Delta}_p} [m_p] \right) |_{\Delta_q^0} \cong \bigoplus_{\tau=0}^p D_{p\tau}^{\delta_{p\tau}+\alpha} \otimes Ri_{p\tau*} IC_{\Delta_\tau}^\bullet |_{\Delta_q^0},$$

where $i_{p\tau} : \Delta_\tau \hookrightarrow \Delta_p$ is the inclusion. By [[13], Remark 3.3], we also have

$$IC_{\Delta_\tau}^\bullet |_{\Delta_q^0} \cong \bigoplus_{\beta \in \mathbb{Z}} B_{\tau q}^\beta \otimes \mathbb{Q}_{\Delta_q^0} [m_\tau - \beta] \cong \bigoplus_{\beta \in \mathbb{Z}} B_{\tau q}^{\beta+m_\tau} \otimes \mathbb{Q}_{\Delta_q^0} [-\beta].$$

Combining these results, we obtain

$$\begin{aligned} &\bigoplus_{\alpha \in \mathbb{Z}} A_{pq}^{\alpha+m_p-m_q} \otimes \mathbb{Q}_{\Delta_q^0} [m_q - \alpha] \cong \\ &\cong \bigoplus_{\alpha \in \mathbb{Z}} \left(\bigoplus_{\tau=q}^p D_{p\tau}^{\delta_{p\tau}+\alpha} \otimes \bigoplus_{\beta \in \mathbb{Z}} B_{\tau q}^{\beta+m_\tau} \otimes \mathbb{Q}_{\Delta_q^0} [-\beta] \right) [-\alpha], \end{aligned}$$

where $\tau \in \{q, \dots, p\}$ because $\Delta_\tau \setminus \Delta_q^0 = \emptyset$ whenever $\tau < q$. Since the γ -th cohomology group of a topological space is trivial when $\gamma < 0$, we obtain

$$\begin{aligned} & \bigoplus_{\alpha \geq -m_p} A_{pq}^{\alpha+m_p} \otimes \mathbb{Q}_{\Delta_q^0}[-\alpha] \cong \\ & \cong \bigoplus_{\alpha \geq -m_p} \left(\bigoplus_{\tau=q}^p D_{p\tau}^{\delta_{p\tau}+\alpha} \otimes \bigoplus_{\beta \geq -m_p} B_{\tau q}^{\beta+m_\tau} \otimes \mathbb{Q}_{\Delta_q^0}[-\beta] \right) [-\alpha]. \end{aligned} \quad (6)$$

The right-hand complex can be rewritten as follows

$$\begin{aligned} & \bigoplus_{\alpha \geq -m_p} \left(\bigoplus_{\tau=q}^p D_{p\tau}^{\delta_{p\tau}+\alpha} \otimes \bigoplus_{\beta \geq -m_p} B_{\tau q}^{\beta+m_\tau} \otimes \mathbb{Q}_{\Delta_q^0}[-\beta] \right) [-\alpha] \\ & \cong \bigoplus_{\alpha, \beta \geq -m_p} \left(\bigoplus_{\tau=q}^p D_{p\tau}^{\delta_{p\tau}+\alpha} \otimes B_{\tau q}^{\beta+m_\tau} \otimes \mathbb{Q}_{\Delta_q^0}[-\alpha-\beta] \right) \end{aligned}$$

and, for any γ , its γ -th term is

$$\bigoplus_{\alpha+\beta=\gamma} \left(\bigoplus_{\tau=q}^p D_{p\tau}^{\delta_{p\tau}+\alpha} \otimes B_{\tau q}^{\beta+m_\tau} \otimes \mathbb{Q}_{\Delta_q^0} \right).$$

The isomorphism (6) implies that the γ -th terms of those complexes are isomorphic for any $\gamma \geq -m_p$; i.e.

$$A_{pq}^{\gamma+m_p} \otimes \mathbb{Q}_{\Delta_q^0} \cong \bigoplus_{\alpha+\beta=\gamma} \left(\bigoplus_{\tau=q}^p D_{p\tau}^{\delta_{p\tau}+\alpha} \otimes B_{\tau q}^{\beta+m_\tau} \otimes \mathbb{Q}_{\Delta_q^0} \right).$$

We shall observe one last thing before we compute Poincaré polynomials. For any $n \in \mathbb{N}$, $\mathbb{G}_0(\mathbb{C}^n) = \{0\}$, as it is the space of 0-dimensional subspaces through the origin. As a consequence,

$$H^k(\mathbb{G}_0(\mathbb{C}^n)) \cong \begin{cases} \mathbb{Q} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, when $\tau = p$,

$$\delta_{pp} = 0 \quad \text{and} \quad D_{pp}^\alpha \cong \begin{cases} \mathbb{Q} & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$$

and, consequently,

$$\bigoplus_{\alpha+\beta=\gamma} D_{pp}^{\alpha} \otimes B_{pq}^{\beta+m_p} \otimes \mathbb{Q}_{\Delta_q^0} \cong B_{pq}^{\gamma+m_p} \otimes \mathbb{Q}_{\Delta_q^0}.$$

Similarly, when $\tau = q$, we have

$$B_{pq}^{\beta+m_q} \cong \begin{cases} \mathbb{Q} & \text{if } \beta = -m_q \\ 0 & \text{otherwise} \end{cases}$$

and

$$\bigoplus_{\alpha+\beta=\gamma} D_{pq}^{\delta_{pq}+\alpha} \otimes B_{qq}^{\beta+m_q} \otimes \mathbb{Q}_{\Delta_q^0} \cong D_{pq}^{\delta_{pq}+m_q+\gamma} \otimes \mathbb{Q}_{\Delta_q^0}.$$

In conclusion, for any $\gamma \geq -m_p$, we have

$$\begin{aligned} A_{pq}^{\gamma+m_p} \otimes \mathbb{Q}_{\Delta_q^0} &\cong \bigoplus_{\alpha+\beta=\gamma} \left(\bigoplus_{\tau=q+1}^{p-1} D_{p\tau}^{\delta_{p\tau}+\alpha} \otimes B_{\tau q}^{\beta+m_{\tau}} \otimes \mathbb{Q}_{\Delta_q^0} \right) \oplus \\ &\oplus \left(D_{pq}^{\delta_{pq}+m_q+\gamma} \otimes \mathbb{Q}_{\Delta_q^0} \right) \oplus \left(B_{pq}^{\gamma+m_p} \otimes \mathbb{Q}_{\Delta_q^0} \right). \end{aligned}$$

Let $s = \gamma + m_p$ and recall that $m_p - m_{\tau} - \delta_{p\tau} = 2d_{p\tau}$ (see [[13], 13, Sect. 2.6]).

$$\begin{aligned} &\bigoplus_{\alpha+\beta=\gamma} \left(\bigoplus_{\tau=q+1}^{p-1} D_{p\tau}^{\delta_{p\tau}+\alpha} \otimes B_{\tau q}^{\beta+m_{\tau}} \otimes \mathbb{Q}_{\Delta_q^0} \right) \\ &\cong \bigoplus_{\alpha'+\beta=s} \left(\bigoplus_{\tau=q+1}^{p-1} D_{p\tau}^{\delta_{p\tau}+\alpha'-m_p} \otimes B_{\tau q}^{\beta+m_{\tau}} \otimes \mathbb{Q}_{\Delta_q^0} \right) \\ &\cong \bigoplus_{\alpha''+\beta'=s} \left(\bigoplus_{\tau=q+1}^{p-1} D_{p\tau}^{\delta_{p\tau}+\alpha''-m_p+m_{\tau}} \otimes B_{\tau q}^{\beta'} \otimes \mathbb{Q}_{\Delta_q^0} \right) \\ &\cong \bigoplus_{\alpha''+\beta'=s} \left(\bigoplus_{\tau=q+1}^{p-1} D_{p\tau}^{\alpha''-2d_{p\tau}} \otimes B_{\tau q}^{\beta'} \otimes \mathbb{Q}_{\Delta_q^0} \right), \end{aligned}$$

where we set $\alpha' = \alpha + m_p$ and $\alpha'' = \alpha' - m_{\tau}$, $\beta' = \beta + m_{\tau}$. Hence, we have

$$\begin{aligned} A_{pq}^s \otimes \mathbb{Q}_{\Delta_q^0} &\cong \bigoplus_{\alpha+\beta=s} \left(\bigoplus_{\tau=q+1}^{p-1} D_{p\tau}^{\alpha-2d_{p\tau}} \otimes B_{\tau q}^{\beta} \otimes \mathbb{Q}_{\Delta_q^0} \right) \\ &\oplus \left(D_{pq}^{s-2d_{pq}} \otimes \mathbb{Q}_{\Delta_q^0} \right) \oplus \left(B_{pq}^s \otimes \mathbb{Q}_{\Delta_q^0} \right) \end{aligned}$$

for any $s \geq 0$. At long last, if we denote by

$$a_{pq}^s = \dim A_{pq}^s, \quad d_{pq}^s = \dim D_{pq}^s, \quad b_{pq}^s = \dim B_{pq}^s,$$

we obtain identities

$$a_{pq}^s = \sum_{\tau=q+1}^{p-1} \left(\sum_{\alpha+\beta=s} d_{p\tau}^{\alpha-2d_{p\tau}} \cdot b_{\tau q}^{\beta} \right) + d_{pq}^{s-2d_{pq}} + b_{pq}^s.$$

If we formally multiply both sides by t^s ,

$$\begin{aligned} a_{pq}^s \cdot t^s &= \sum_{\tau=q+1}^{p-1} \left(\sum_{\alpha+\beta=s} d_{p\tau}^{\alpha-2d_{p\tau}} \cdot b_{\tau q}^{\beta} \right) \cdot t^s + d_{pq}^{s-2d_{pq}} \cdot t^s + b_{pq}^s \cdot t^s \\ &= \sum_{\tau=q+1}^{p-1} \left(\sum_{\alpha+\beta=s} (d_{p\tau}^{\alpha-2d_{p\tau}} \cdot t^{\alpha-2d_{p\tau}}) \cdot (b_{\tau q}^{\beta} \cdot t^{\beta}) \right) \cdot t^{2d_{p\tau}} \\ &\quad + (d_{pq}^{s-2d_{pq}} \cdot t^{s-2d_{pq}}) \cdot t^{2d_{pq}} + b_{pq}^s \cdot t^s \end{aligned}$$

and if we take the sum over $s \in \mathbb{Z}$

$$\begin{aligned} \sum_s a_{pq}^s \cdot t^s &= \sum_{\tau=q+1}^{p-1} \left(\sum_{\alpha} (d_{p\tau}^{\alpha-2d_{p\tau}} \cdot t^{\alpha-2d_{p\tau}}) \cdot \sum_{\beta} (b_{\tau q}^{\beta} \cdot t^{\beta}) \right) \cdot t^{2d_{p\tau}} \\ &\quad + \sum_s (d_{pq}^{s-2d_{pq}} \cdot t^{s-2d_{pq}}) \cdot t^{2d_{pq}} + \sum_s b_{pq}^s \cdot t^s. \end{aligned}$$

We are done, because, by definition of Poincaré polynomials, the above equality becomes

$$H_{F_{pq}} = \sum_{\tau=q+1}^{p-1} (H_{T_{p\tau}} \cdot H_{G_{\tau q}}) \cdot t^{2d_{p\tau}} + H_{T_{pq}} \cdot t^{2d_{pq}} + H_{G_{pq}};$$

that is,

$$\begin{aligned} \frac{P_{k-q+1}}{P_{k-p+1} P_{p-q}} &= \sum_{\tau=q+1}^{p-1} \left(\frac{P_{k-c}}{P_{p-\tau} P_{k-c-p+\tau}} \cdot \frac{P_{c-q+1}}{P_{\tau-q} P_{c-\tau+1}} \cdot t^{2d_{p\tau}} \right) \\ &\quad + \frac{P_{k-c}}{P_{p-q} P_{k-c-p+q}} \cdot t^{2d_{pq}} + \frac{P_{c-q+1}}{P_{p-q} P_{c-p+1}}. \end{aligned}$$

4 Global polynomial identities

We shall begin by introducing some further notations:

$$\begin{aligned} H_p &= \sum_{\alpha \in \mathbb{Z}} \dim_{\mathbb{Q}} H^{\alpha}(\tilde{\Delta}_p) \cdot t^{\alpha}; \\ I_p &= \sum_{\alpha \in \mathbb{Z}} \dim_{\mathbb{Q}} I H^{\alpha}(\Delta_p) \cdot t^{\alpha} = \sum_{\alpha \in \mathbb{Z}} \dim_{\mathbb{Q}} \mathbb{H}^{\alpha} \left(IC_{\Delta_p}^{\bullet}[-m_p] \right) \cdot t^{\alpha}; \\ f_{pq} &= \sum_{\alpha \in \mathbb{Z}} d_{pq}^{\alpha} \cdot t^{\alpha} = \sum_{\alpha \in \mathbb{Z}} \dim_{\mathbb{Q}} H^{\alpha}(T_{pq}) \cdot t^{\alpha}. \end{aligned}$$

Recall that we defined a small resolution $\xi_p : \mathcal{D}_p \rightarrow \Delta_p$ as

$$\xi_p : (V, U) \in \mathcal{D}_p = \{(V, U) \in \mathbb{G}_k(\mathbb{C}^l) \times \mathbb{G}_{k+j-i_p}(\mathbb{C}^l) : V + F \subseteq U\} \mapsto V \in \Delta_p.$$

Remark 1 The map

$$\varphi : U \in G := \{U \in \mathbb{G}_{k+j-i_p}(\mathbb{C}^l) \mid F \subseteq U\} \mapsto U/F \in \mathbb{G}_{k-i_p}(\mathbb{C}^{l-j}),$$

provides an isomorphism between G and $\mathbb{G}_{k-i_p}(\mathbb{C}^{l-j})$. Therefore, we recognize $\mathcal{D}_p$ as the Grassmannian of subspaces of dimension k of the restriction of the tautological bundle $\mathcal{T}$ over $\mathbb{G}_{k+j-i_p}(\mathbb{C}^l)$ to the subspace G :

$$\mathcal{D}_p \cong \mathbb{G}_k(\mathcal{T}|_G).$$

By the Leray-Hirsch theorem [[23], Theorem 7.33] and the Künneth formula, we have

$$\begin{aligned} H^{\alpha}(\mathcal{D}_p) &\cong \bigoplus_{\beta \in \mathbb{Z}} H^{\beta}(\mathbb{G}_{k-i_p}(\mathbb{C}^{l-j})) \otimes H^{\alpha-\beta}(\mathbb{G}_k(\mathbb{C}^{k+j-i_p})) \cong \\ &\cong H^{\alpha}(\mathbb{G}_{k-i_p}(\mathbb{C}^{l-j}) \times \mathbb{G}_k(\mathbb{C}^{k+j-i_p})), \end{aligned}$$

(compare also with [[8], 8, Sect. 2] for a discussion of the Leray-Hirsch theorem in a context which is closely related with that considered here).

The following formula, which represents the main result of this paper, provides a strong generalization of [[10], Sect. 2, Remark 4.2].

Theorem 3 *With the same notations and conditions as Theorem 2, we have*

$$\begin{aligned} \frac{P_j P_{l-i}}{P_i P_{j-i} P_{k-i} P_{l-k}} &= \frac{P_{l-j} P_{k+j-i}}{P_{k-i} P_{l+i-j-k} P_k P_{j-i}} + \\ &+ \sum_{s=1}^{\min\{k-i, k-c\}} \frac{P_{k-c} P_{l-j} P_{k+j-i-s}}{P_s P_{k-c-s} P_{k-i-s} P_{l+i-j-k+s} P_k P_{j-i-s}} t^{2s(c-r+s)}. \end{aligned}$$

Proof The first part of the proof is similar to that of Theorem 2. We combine the Decomposition theorem [[4], Theorem 1.6.1] with [[13], Theorem 3.5] so as to obtain

$$R\pi_{p*}\mathbb{Q}_{\tilde{\Delta}_p}[m_p] \cong \bigoplus_{\alpha \in \mathbb{Z}} \left(\bigoplus_{q=0}^p D_{pq}^{\delta_{pq}+\alpha} \otimes Ri_{pq*}IC_{\Delta_q}^\bullet \right)[- \alpha];$$

that is,

$$\begin{aligned} R\pi_{p*}\mathbb{Q}_{\tilde{\Delta}_p} &\cong \bigoplus_{\alpha \in \mathbb{Z}} \left(\bigoplus_{q=1}^{p-1} D_{pq}^{\delta_{pq}+\alpha} \otimes Ri_{pq*}IC_{\Delta_q}^\bullet[-m_p - \alpha] \right) \oplus \\ &\oplus \bigoplus_{\alpha \in \mathbb{Z}} \left(D_{p0}^{\delta_{p0}+\alpha} \otimes Ri_{p0*}IC_{\Delta_0}^\bullet[-m_p - \alpha] \right) \oplus \\ &\oplus \bigoplus_{\alpha \in \mathbb{Z}} \left(D_{pp}^{\delta_{pp}+\alpha} \otimes Ri_{pp*}IC_{\Delta_p}^\bullet[-m_p - \alpha] \right). \end{aligned}$$

We have already met the term $D_{pp}^{\delta_{pp}+\alpha}$ and the second summand of the right-hand side is the zero complex since $\Delta_0 = \emptyset$. Hence, we have

$$\begin{aligned} R\pi_{p*}\mathbb{Q}_{\tilde{\Delta}_p} &\cong \bigoplus_{\alpha \in \mathbb{Z}} \left(\bigoplus_{q=1}^{p-1} D_{pq}^{\delta_{pq}+\alpha} \otimes Ri_{pq*}IC_{\Delta_q}^\bullet[-m_p - \alpha] \right) \oplus IC_{\Delta_p}^\bullet[-m_p] \\ &= \bigoplus_{\alpha \in \mathbb{Z}} \left(\bigoplus_{q=1}^{p-1} D_{pq}^{\alpha-2d_{pq}} \otimes Ri_{pq*}IC_{\Delta_q}^\bullet[-m_q - \alpha] \right) \oplus IC_{\Delta_p}^\bullet[-m_p], \end{aligned}$$

where we have also taken account of $m_p - m_r - \delta_{pr} = 2d_{pr}$ (see [[13], 13, Sect. 2.6]).

If we apply hypercohomology, we obtain (for any $\beta \in \mathbb{Z}$)

$$\begin{aligned} \mathbb{H}^\beta(R\pi_{p*}\mathbb{Q}_{\tilde{\Delta}_p}) &\cong \bigoplus_{\alpha \in \mathbb{Z}} \left(\bigoplus_{q=1}^{p-1} D_{pq}^{\alpha-2d_{pq}} \otimes \mathbb{H}^{\beta-\alpha}(Ri_{pq*}IC_{\Delta_q}^\bullet[-m_q]) \right) \\ &\oplus \mathbb{H}^\beta(IC_{\Delta_p}^\bullet[-m_p]) \end{aligned}$$

By [[12], Definition 5.4.3],

$$IH^\beta(\Delta_p) = \mathbb{H}^\beta(IC_{\Delta_p}^\bullet[-m_p])$$

and, by [[12], Definition 2.1.4] and [[17], Chapter II, (4.5)],

$$\begin{aligned}\mathbb{H}^\beta(Ri_{pq*}IC_{\Delta_q}^\bullet) &= H^\beta(\Gamma(\Delta_p, i_{pq*}I^\bullet)) = H^\beta(\Gamma(i_{pq*}^{-1}(\Delta_p), I^\bullet)) = \\ &= H^\beta(\Gamma(\Delta_q, I^\bullet)) = \mathbb{H}^\beta(IC_{\Delta_q}^\bullet),\end{aligned}$$

where $IC_{\Delta_q}^\bullet \rightarrow I^\bullet$ is an injective resolution of $IC_{\Delta_q}^\bullet$ and Γ is the global section functor. Similarly,

$$\begin{aligned}\mathbb{H}^\beta(R\pi_{p*}\mathbb{Q}_{\tilde{\Delta}_p}) &= H^\beta(\Gamma(\Delta_p, R\pi_{p*}\mathbb{Q}_{\tilde{\Delta}_p})) = H^\beta(\Gamma(\Delta_p, \pi_{p*}I^\bullet)) \\ &= H^\beta(\Gamma(\pi_{p*}^{-1}(\Delta_p), I^\bullet)) = H^\beta(\Gamma(\tilde{\Delta}_p, I^\bullet)) \\ &= H^\beta(\tilde{\Delta}_p, \mathbb{Q}_{\tilde{\Delta}_p}) = H^\beta(\tilde{\Delta}_p),\end{aligned}$$

where $\mathbb{Q}_{\tilde{\Delta}_p} \rightarrow I^\bullet$ is an injective resolution and the last equality is [[17], Theorem 7.12, p. 242].

Substituting in the above isomorphism, we obtain

$$H^\beta(\tilde{\Delta}_p) \cong \bigoplus_{\alpha \in \mathbb{Z}} \left(\bigoplus_{q=1}^{p-1} D_{pq}^{\alpha-2d_{pq}} \otimes IH^{\beta-\alpha}(\Delta_q) \right) \oplus \mathbb{H}^\beta(IC_{\Delta_p}^\bullet[-m_p]).$$

As we did in the proof of Theorem 2, we conclude

$$\begin{aligned}\sum_{\beta \in \mathbb{Z}} \dim H^\beta(\tilde{\Delta}_p) \cdot t^\beta &= \sum_{q=1}^{p-1} \sum_{\alpha, \beta \in \mathbb{Z}} d_{pq}^{\alpha-2d_{pq}} \cdot t^{\alpha-2d_{pq}} \\ &\quad \cdot \dim IH^{\beta-\alpha}(\Delta_q) \cdot t^{\beta-\alpha} \cdot t^{2d_{pq}} + \sum_{\beta \in \mathbb{Z}} \dim IH^\beta(\Delta_p) \cdot t^\beta,\end{aligned}$$

which can be compactly rewritten as

$$H_p = I_p + \sum_{q=1}^{p-1} f_{pq} \cdot I_q \cdot t^{2d_{pq}}. \quad (7)$$

Again, Leray-Hirsch theorem implies that $\tilde{\Delta}_p$ has the same Poincaré polynomial as $\mathbb{G}_{i_p}(F) \times \mathbb{G}_{k-i_p}(\mathbb{C}^{l-i_p})$. Thus, the left-hand side is

$$H_p = H_{\mathbb{G}_{i_p}(F) \times \mathbb{G}_{k-i_p}(\mathbb{C}^{l-i_p})} = H_{\mathbb{G}_{i_p}(F)} \cdot H_{\mathbb{G}_{k-i_p}(\mathbb{C}^{l-i_p})}.$$

By virtue of [[13], Formula (19)] and Remark 1, we have

$$IH^\alpha(\Delta_p) = \mathbb{H}^\alpha(\Delta_p, IC_{\Delta_p}^\bullet[-m_p]) = \mathbb{H}^\alpha(\Delta_p, R\xi_{p*}\mathbb{Q}_{D_p})$$

$$= H^\alpha(\mathcal{D}_p) \cong \bigoplus_{\beta \in \mathbb{Z}} H^\beta(\mathbb{G}_{k-i_p}(\mathbb{C}^{l-j})) \otimes H^{\alpha-\beta}(\mathbb{G}_k(\mathbb{C}^{k+j-i_p}));$$

in other words,

$$I_p = H_{\mathbb{G}_{k-i_p}(\mathbb{C}^{l-j}) \times \mathbb{G}_k(\mathbb{C}^{k+j-i_p})} = H_{\mathbb{G}_{k-i_p}(\mathbb{C}^{l-j})} \cdot H_{\mathbb{G}_k(\mathbb{C}^{k+j-i_p})}.$$

Adopting the same notations as §2.3, we have

$$H_p = \frac{P_j}{P_{i_p} P_{j-i_p}} \cdot \frac{P_{l-i_p}}{P_{k-i_p} P_{l-k}},$$

$$I_p = \frac{P_{l-j}}{P_{k-i_p} P_{l-j-k+i_p}} \cdot \frac{P_{k+j-i_p}}{P_k P_{j-i_p}}$$

and

$$f_{pq} = \frac{P_{k-c}}{P_{p-q} P_{k-c-(p-q)}}.$$

Formula (7) becomes

$$\frac{P_j P_{l-i_p}}{P_{i_p} P_{j-i_p} P_{k-i_p} P_{l-k}} = \frac{P_{l-j} P_{k+j-i_p}}{P_{k-i_p} P_{l-j-k+i_p} P_k P_{j-i_p}} +$$

$$+ \sum_{q=1}^{\min\{p-1, k-c-p\}} \frac{P_{k-c} P_{l-j} P_{k+j-i_q}}{P_{p-q} P_{k-c-(p-q)} P_{k-i_q} P_{l-j-k+i_q} P_k P_{j-i_q}} t^{2d_{pq}}.$$

Since we are interested in studying the Poincaré polynomials of the Schubert variety $\mathcal{S}$, we are going to take $p = r + 1$. Bearing in mind that $i_q = k - q + 1$ (in particular, $i_p = i_{r+1} = k - r = i$) and $c = l - j$, if we set $s = p - q = r + 1 - q$, we have (from left to right, numerators first)

$$l - i_p = l - i,$$

$$k + j - i_p = k + j - i,$$

$$k + j - i_q = j + q - 1 = j + r - s = j + k - i - s,$$

$$j - i_p = j - i,$$

$$k - i_p = k - i,$$

$$l - j - k + i_p = l + i - j - k,$$

$$k - i_q = q - 1 = r - s = k - i - s,$$

$$l - j - k + i_q = l - j - q + 1 = l - j + s - r = l - j + s - k + i$$

$$j - i_q = j - k + q - 1 = j - k + r - s = j - i - s$$

and the previous equality becomes $(2d_{pq} = 2(p - q)(c + 1 - q) = 2s(c + s - r))$

$$\frac{P_j P_{l-i}}{P_i P_{j-i} P_{k-i} P_{l-k}} = \frac{P_{l-j} P_{k+j-i}}{P_{k-i} P_{l+i-j-k} P_k P_{j-i}} +$$

$$+ \sum_{s=1}^{\min\{k-i, k-c\}} \frac{P_{k-c} P_{l-j} P_{k+j-i-s}}{P_s P_{k-c-s} P_{k-i-s} P_{l+i-j-k+s} P_k P_{j-i-s}} t^{2s(c-r+s)}.$$

Corollary 1 For all $p = 2, \dots, r + 1$ one has:

$$I_p = H_p - \sum_{q=1}^{p-1} P_{pq}(t) = H_p - \sum_{q=1}^{p-1} t^{2d_{pq}} f_{pq} I_q.$$

Remark 2 From the previous corollary we get an *explicit inductive algorithm for the computation of Poincaré polynomials of the intersection cohomology of Special Schubert varieties*, which is described by the following equality, where we put $g_{pq} = t^{2d_{pq}} f_{pq}$ in order to simplify the notation:

$$\begin{bmatrix} I_{r+1} \\ I_r \\ \cdot \\ \cdot \\ \cdot \\ I_2 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & g_{r+1,r} & g_{r+1,r-1} & g_{r+1,r-2} & \cdots & g_{r+1,1} \\ 0 & 1 & g_{r,r-1} & g_{r,r-2} & \cdots & g_{r,1} \\ 0 & 0 & 1 & g_{r-1,r-2} & \cdots & g_{r-1,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & g_{21} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} H_{r+1} \\ H_r \\ \cdot \\ \cdot \\ \cdot \\ H_2 \\ H_1 \end{bmatrix}$$

$$= \sum_{k=0}^r (-1)^k \begin{bmatrix} 0 & g_{r+1,r} & g_{r+1,r-1} & g_{r+1,r-2} & \cdots & g_{r+1,1} \\ 0 & 0 & g_{r,r-1} & g_{r,r-2} & \cdots & g_{r,1} \\ 0 & 0 & 0 & g_{r-1,r-2} & \cdots & g_{r-1,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & g_{21} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^k \begin{bmatrix} H_{r+1} \\ H_r \\ \cdot \\ \cdot \\ \cdot \\ H_2 \\ H_1 \end{bmatrix}.$$

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Appendix: a symbolic point of view

In this “Appendix”, we consider the global polynomial identity of Theorem 3 from a symbolic point of view. Recall that the requests over the integers i, j, k, j, l and the values $r := k - i, c := l - j$ are

$$0 < i < k \leq j < l \text{ and } 0 < r < c < k,$$

or, equivalently,

$$0 < i < j \text{ and } 0 < r < c < r + i \leq j. \quad (8)$$

Nevertheless, if we also assume $P_\alpha = 0$ for every $\alpha < 0$, the polynomial identity of Theorem 3 symbolically makes sense, i.e. the denominators do not vanish, under the following weaker assumptions:

$$0 \leq i \leq k \leq j \text{ and } 0 \leq r \leq c \leq k$$

or, equivalently,

$$0 \leq i \leq j \text{ and } 0 \leq r \leq c \leq r + i \leq j. \quad (9)$$

However, in the few further cases $r = 0$ or $c = r + i$ or $i = 0$ or $i = j$, which we have under the assumptions (9), the polynomial identity trivially holds. For the remaining case $c = r$, we obtain some experimental evidences verifying the polynomial identity for the 4-uples (i, j, c, r) , where $c = r$ varies in $[2, 20]$, i in $[1, 20]$ and j in $[r + i, 40]$, by direct computations performed in CoCoA5 (see [1]).

In the following, we consider the assumptions (8) (except for some special cases that will be highlighted), and give a proof of that identity by a mere algebraic manipulation when $2 = \min\{k - i, k - c\}$, which is the first case with a significant geometric meaning in the context of this paper.

By direct computations, we also verified that the polynomial identity of Theorem 3 holds for all the 4-uples (i, j, c, r) where i varies in $[1, 20]$, r in $[2, 20]$, j in $[r + i, 40]$ and c in $[r + 1, r + i - 1]$. Some CoCoA5 functions that perform such computations are available at <http://wpage.unina.it/cioffifr/PolynomialIdentity>.

Case 2 = $k - i \leq k - c$

With the same notation of Theorem 3 and recalling that $k = i + 2$ and $\ell = j + c$, the global polynomial identity becomes:

$$\begin{aligned} \frac{P_j P_{j+c-i}}{P_i P_{j-i} P_2 P_{j+c-i-2}} &= \frac{P_c P_{j+2}}{P_2 P_{c-2} P_{i+2} P_{j-i}} + \\ &+ t^{2(c-1)} \frac{P_{i-c+2} P_c P_{j+1}}{P_{i-c+1} P_{c-1} P_{i+2} P_{j-i-1}} + t^{4c} \frac{P_{i-c+2} P_j}{P_2 P_{i-c} P_{i+2} P_{j-i-2}} \end{aligned} \quad (10)$$

where only the parameters i, j, c appear. Note that formula (10) does make sense for every $j \geq i + 2$ and $c \geq 2$. Let

$$H := \frac{P_j P_{j+c-i}}{P_i P_{j-i} P_2 P_{j+c-i-2}}, \quad F := \frac{P_c P_{j+2}}{P_2 P_{c-2} P_{i+2} P_{j-i}},$$

$$F_1 := t^{2(c-1)} \frac{P_{i-c+2} P_c P_{j+1}}{P_{i-c+1} P_{c-1} P_{i+2} P_{j-i-1}}, \quad F_2 := t^{4c} \frac{P_{i-c+2} P_j}{P_2 P_{i-c} P_{i+2} P_{j-i-2}}.$$

Since $\frac{P_\alpha}{P_{\alpha-h}} = h_{\alpha-1} \dots h_{\alpha-h}$, for every $\alpha > h \geq 0$, we compute $F := \frac{h_{c-2} h_{c-1} P_{j+2}}{h_1 P_{i+2} P_{j-i}}$ and observe that

$$H = F \left[\frac{h_{j+c-i-2} h_{j+c-i-1} h_i h_{i+1}}{h_j h_{j+1} h_{c-2} h_{c-1}} \right], \quad F_1 = F \left[t^{2(c-1)} \frac{h_1 h_{i-c+1} h_{j-i-1}}{h_{j+1} h_{c-2}} \right],$$

$$F_2 = F \left[t^{4c} \frac{h_{i-c} h_{i-c+1} h_{j-i-2} h_{j-i-1}}{h_{c-2} h_{c-1} h_j h_{j+1}} \right].$$

Hence, letting

$$F(i, j, c) := \frac{h_{j+c-i-2} h_{j+c-i-1} h_i h_{i+1}}{h_j h_{j+1} h_{c-2} h_{c-1}} +$$

$$- t^{2(c-1)} \frac{h_1 h_{i-c+1} h_{j-i-1}}{h_{j+1} h_{c-2}} - t^{4c} \frac{h_{i-c} h_{i-c+1} h_{j-i-2} h_{j-i-1}}{h_{c-2} h_{c-1} h_j h_{j+1}},$$

formula (10) holds if and only if $F(i, j, c) = 1$. Observe that $F(i, j, c)$ does make sense for every $c \geq 2$ and for every positive integers i, j . In the following, we repeatedly apply the equality $t^{2\alpha} h_\beta = h_{\alpha+\beta} - h_{\alpha-1}$, which holds for every $\alpha, \beta \geq 0$.

We now show that $F(i, j, 3) = 1$ for $j \geq i$, where

$$F(i, j, 3) = \frac{h_i h_{i+1}}{h_j h_{j+1}} \frac{h_{j-i+1} h_{j-i+2}}{P_3} - t^4 \frac{h_{i-2} h_{j-i-1}}{h_{j+1}} - t^{12} \frac{h_{j-i-2} h_{j-i-1} h_{i-2} h_{i-3}}{P_3 h_j h_{j+1}}.$$

Although we should consider $i \geq c \geq 3$ due to (8), symbolically we easily obtain

$$F(2, j, 3) = \frac{h_2 h_3}{h_j h_{j+1}} \frac{h_j h_{j-1}}{h_1 h_2} - t^4 \frac{h_{j-3} h_1}{h_{j+1} h_1} = \frac{h_3}{h_{j+1}} \frac{h_{j-1}}{h_1} - \frac{(h_{j-1} - h_1) h_1}{h_1 h_{j+1}} =$$

$$\frac{(h_3 - h_1) h_{j-1} + h_1^2}{h_1 h_{j+1}} = \frac{t^4 (1 + t^2) h_{j-1} + h_1^2}{h_1 h_{j+1}} = \frac{h_{j+1} - h_1 + h_1}{h_{j+1}} = 1.$$

Hence, by induction, we can assume $F(i-1, j, 3) = 1$ and prove that $F(i, j, 3) - F(i-1, j, 3)$ is null, for every $i > 2$. We compute

$$F(i, j, 3) - F(i-1, j, 3) = \frac{h_i h_{j-i+2} (h_{i+1} h_{j-i+1} - h_{i-1} h_{j-i+3})}{h_j h_{j+1} P_3} +$$

$$- \frac{t^4 h_j P_3 (h_{i-2} h_{j-i-1} - h_{i-3} h_{j-i})}{h_j h_{j+1} P_3} - \frac{t^{12} h_{j-i-1} h_{i-3} (h_{j-i-2} h_{i-2} - h_{j-i} h_{i-4})}{h_j h_{j+1} P_3}.$$

So, letting $G_0 := h_i h_{j-i+2} (h_{i+1} h_{j-i+1} - h_{i-1} h_{j-i+3})$

$$G_1 := -t^4 h_j P_3 (h_{i-2} h_{j-i-1} - h_{i-3} h_{j-i})$$

$$G_2 := -t^{12} h_{j-i-1} h_{i-3} (h_{j-i-2} h_{i-2} - h_{j-i} h_{i-4}),$$

we obtain $F(i, j, 3) - F(i - 1, j, 3) = 0$ if and only if $G_0 + G_1 + G_2 = 0$.

Being $h_{i+1} = t^4 h_{i-1} + h_1$ e $h_{j-i+3} = t^4 h_{j-i+1} + h_1$, G_0 becomes:

$$- G_0 = h_i h_{j-i+2} h_1 (h_{j-i+1} - h_{i-1}).$$

Being $h_{i-2} = t^2 h_{i-3} + h_0$, $h_{j-i} = t^2 h_{j-i-1} + h_0$ and then $t^4 h_{i-3} = h_{i-1} - h_1$, $t^4 h_{j-i-1} = h_{j-i+1} - h_1$, G_1 becomes:

$$- G_1 = h_j P_3(h_{i-1} - h_{j-i+1}).$$

Being $h_{j-i} = t^4 h_{j-i-2} + h_1$ e $h_{i-2} = t^4 h_{i-4} + h_1$, G_2 becomes

$$- G_2 = t^6 h_{j-i-1} h_{i-3} h_1 (h_{i-1} - h_{j-i+1}).$$

So, we obtain $G_0 + G_1 + G_2 = h_1(h_{i-1} - h_{j-i+1})(h_i h_{j-i+2} - h_j h_2 - t^6 h_{j-i+1} h_{i-3}) = 0$.

Recall that we are assuming $k - i \leq k - c$, hence $i \geq c$, and we know that $F(i, j, 3) = 1$. So, our thesis now is $F(i, j, c) - F(i, j, c - 1) = 0$. Assuming $c > 3$, we compute

$$\begin{aligned} F(i, j, c) - F(i, j, c - 1) &= \frac{h_{j+c-i-2} h_{j+c-i-1} h_i h_{i+1}}{h_j h_{j+1} h_{c-2} h_{c-1}} - \frac{h_{j+c-i-3} h_{j+c-i-2} h_i h_{i+1}}{h_j h_{j+1} h_{c-3} h_{c-2}} + \\ &\quad - t^{2(c-1)} \frac{h_1 h_{i-c+1} h_{j-i-1}}{h_{j+1} h_{c-2}} + t^{2(c-2)} \frac{h_1 h_{i-c+2} h_{j-i-1}}{h_{j+1} h_{c-3}} + \\ &\quad - t^{4c} \frac{h_{i-c} h_{i-c+1} h_{j-i-2} h_{j-i-1}}{h_{c-2} h_{c-1} h_j h_{j+1}} + t^{4(c-1)} \frac{h_{i-c+1} h_{i-c+2} h_{j-i-2} h_{j-i-1}}{h_{c-3} h_{c-2} h_j h_{j+1}}. \end{aligned}$$

So, letting

$$Q_0 := h_{j+c-i-2} h_{j+c-i-1} h_i h_{i+1} h_{c-3} - h_{j+c-i-3} h_{j+c-i-2} h_i h_{i+1} h_{c-1}$$

$$Q_1 := -t^{2(c-1)} h_1 h_{i-c+1} h_{j-i-1} h_j h_{c-1} h_{c-3} + t^{2(c-2)} h_1 h_{i-c+2} h_{j-i-1} h_j h_{c-1} h_{c-2}$$

$$Q_2 := -t^{4c} h_{i-c} h_{i-c+1} h_{j-i-2} h_{j-i-1} h_{c-3} + t^{4(c-1)} h_{i-c+1} h_{i-c+2} h_{j-i-2} h_{j-i-1} h_{c-1},$$

we have $F(i, j, c) - F(i, j, c - 1) = 0$ if and only if $Q_0 + Q_1 + Q_2 = 0$.

Being $h_{j+c-i-1} = t^4 h_{j+c-i-3} + h_1$ and $t^4 h_{c-3} = h_{c-1} - h_1$, then Q_0 becomes

$$- Q_0 = h_{j+c-i-2} h_1 h_i h_{i+1} (h_{c-3} - h_{j+c-i-3}) = -t^{2(c-2)} h_{j+c-i-2} h_1 h_i h_{i+1} h_{j-i-1}.$$

Being $t^{2(c-1)} h_{i-c+1} = h_i - h_{c-2}$ and $t^{2(c-2)} h_{i-c+2} = h_i - h_{c-3}$, then Q_1 becomes

$$- Q_1 = t^{2(c-2)} h_1 h_i h_j h_{c-1} h_{j-i-1}.$$

Being $t^4 h_{i-c} = h_{i-c+2} - h_1$, then Q_2 becomes

$$- Q_2 = t^{4(c-1)} h_{i-c+1} h_{j-i-2} h_{j-i-1} ((h_{i-c+2} - h_1) h_{c-3} + h_{i-c+2} h_{c-1}).$$

Note that, if $j = i$ or $j = i + 1$, then $Q_0 + Q_1 = 0 = Q_2$. So, we now consider the less obvious case $j > i + 1$. Observe that

$$\begin{aligned} Q_1 + Q_0 &= t^{2(c-2)} h_1 h_i h_{j-i-1} (h_j h_{c-1} - h_{i+1} h_{j+c-i-2}) = \\ &\quad t^{2(c-2)} h_1 h_i h_{j-i-1} h_{j-i-2} (h_{c-1} - h_{i+1}) = -t^{2(c-2)} h_1 h_i h_{j-i-1} h_{j-i-2} t^{2c} h_{i-c+1}. \end{aligned}$$

We are done, because:

$$\begin{aligned} Q_1 + Q_0 + Q_2 &= t^{4(c-1)} h_{i-c+1} h_{j-i-2} h_{j-i-1} (-h_1 h_i + (h_{i-c+2} - h_1) h_{c-3} + h_{i-c+2} h_{c-1}) = \\ &= t^{4(c-1)} h_{i-c+1} h_{j-i-2} h_{j-i-1} h_{i-c+2} (t^{2(c-1)} + t^{2(c-2)} - h_1 t^{2(c-2)}) = 0. \end{aligned}$$

Case 2 = $k - c < k - i$

In this case we have $r + i = k = c + 2$, $\ell = j + c$, $c = r + i - 2$, and the global polynomial identity becomes:

$$\begin{aligned} \frac{P_j P_{j+r-2}}{P_i P_{j-i} P_r P_{j-2}} &= \frac{P_{r+i-2}}{P_r P_{i-2}} \frac{P_{r+j}}{P_{r+i} P_{j-i}} + \\ &+ t^{2(i-1)} \frac{P_2 P_{r+i-2} P_{r+j-1}}{P_{r-1} P_{i-1} P_{r+i} P_{j-i-1}} + t^{4i} \frac{P_2 P_{r+i-2} P_{r+j-2}}{P_2 P_{r-2} P_i P_{r+i} P_{j-i-2}} \end{aligned} \quad (11)$$

where only the parameters i, j, r appear. Note that formula (11) does make sense for every $2 \leq r$ and $2 \leq i \leq j - 2$. Let

$$\begin{aligned} K &:= \frac{P_j P_{j+r-2}}{P_i P_{j-i} P_r P_{j-2}}, \quad E := \frac{P_{r+i-2}}{P_r P_{i-2}} \frac{P_{r+j}}{P_{r+i} P_{j-i}}, \\ E_1 &:= t^{2(i-1)} \frac{P_2 P_{r+i-2} P_{r+j-1}}{P_{r-1} P_{i-1} P_{r+i} P_{j-i-1}}, \quad E_2 := t^{4i} \frac{P_2 P_{r+i-2} P_{r+j-2}}{P_2 P_{r-2} P_i P_{r+i} P_{j-i-2}}. \end{aligned}$$

Analogously to the previous case, we compute $K = E \left[\frac{h_{j-1} h_{j-2} h_{r+i-1} h_{r+i-2}}{h_{i-1} h_{i-2} h_{r+j-1} h_{r+j-2}} \right]$,
 $E_1 = E \left[t^{2(i-1)} \frac{h_{r-1} h_1 h_{j-i-1} h_{i-1} h_{r+j-2}}{h_{r+j-1} h_{i-2} h_{i-1} h_{r+j-2}} \right]$, $E_2 = E \left[t^{4i} \frac{h_{r-2} h_{r-1} h_{j-i-2} h_{j-i-1}}{h_{r+j-2} h_{r+j-1} h_{i-2} h_{i-1}} \right]$.
Hence, letting

$$\begin{aligned} FF(i, j, r) &:= \frac{h_{j-1} h_{j-2} h_{r+i-1} h_{r+i-2}}{h_{i-1} h_{i-2} h_{r+j-1} h_{r+j-2}} + \\ &- t^{2(i-1)} \frac{h_{r-1} h_1 h_{j-i-1} h_{i-1} h_{r+j-2}}{h_{r+j-1} h_{i-2} h_{i-1} h_{r+j-2}} - t^{4i} \frac{h_{r-2} h_{r-1} h_{j-i-2} h_{j-i-1}}{h_{r+j-2} h_{r+j-1} h_{i-2} h_{i-1}}, \end{aligned}$$

formula (11) holds if and only if $FF(i, j, r) = 1$. Observe that $FF(i, j, r)$ does make sense for all integers $j \geq i \geq 2$ and $r \geq 0$. We easily check that $FF(i, j, 0) = 1$. So, we now assume $r > 0$ and prove that $FF(i, j, r) - FF(i, j, r - 1) = 0$. Arguing as in the case $k - i = 2$, we let

$$\begin{aligned} H_0 &:= h_{r+j-3} h_{j-1} h_{j-2} h_{r+i-1} h_{r+i-2} - h_{r+j-1} h_{j-1} h_{j-2} h_{r+i-2} h_{r+i-3} \\ H_1 &:= t^{2(i-1)} h_{r+j-1} h_{r-2} h_1 h_{j-i-1} h_{i-1} h_{r+j-3} + \\ &- t^{2(i-1)} h_{r+j-3} h_{r-1} h_1 h_{j-i-1} h_{i-1} h_{r+j-2} \\ H_2 &:= t^{4i} h_{r+j-1} h_{r-3} h_{r-2} h_{j-i-2} h_{j-i-1} - t^{4i} h_{r+j-3} h_{r-2} h_{r-1} h_{j-i-2} h_{j-i-1} \end{aligned}$$

so that the thesis now is $H_0 + H_1 + H_2 = 0$.

We apply in H_0 the following replacements, in the given order: first $h_{r+i-1} = t^4 h_{r+i-3} + h_1$ and $h_{r+j-1} = t^4 h_{r+j-3} + h_1$, then $h_{r+j-2} - h_{r+i-2} = t^{2(r+i-2)} h_{j-i-1}$. Hence, H_0 becomes

$$- H_0 = h_{j-1} h_{j-2} h_{r+i-2} h_1 t^{2(r+i-2)} h_{j-i-1}.$$

We apply in H_1 the following replacements, in the given order: first $h_{r+j-1} = t^2 h_{r+j-2} + h_0$ and $h_{r-1} = t^2 h_{r-2} + h_0$, then $h_{r-2} - h_{r+j-2} = -t^{2(r-1)} h_{j-1}$. Hence, H_1 becomes

$$- H_1 = t^{2(i-1)} h_{r+j-3} h_{j-i-1} h_{i-1} (-t^{2(r-1)} h_{j-1}) h_1.$$

We apply in H_2 the following replacements, in the given order: first $h_{r+j-1} = t^4 h_{r+j-3} + h_1$ and $h_{r-1} = t^4 h_{r-3} + h_1$, then $h_{r-3} - h_{r+j-3} = -t^{2(r-2)} h_{j-1}$. Hence, H_2 becomes

$$- H_2 = -t^{2i} t^{2(r+i-2)} h_{r-2} h_{j-i-2} h_{j-i-1} h_1 h_{j-1}.$$

Thus, we now have

$$H_0 + H_1 + H_2 = t^{2(r+i-2)} h_1 h_{j-1} h_{j-i-1} (h_{j-2} h_{r+i-2} - h_{r+j-3} h_{i-1} - t^{2i} h_{r-2} h_{j-i-2}).$$

Being $h_{j-2} = t^{2(j-i-1)} h_{i-1} + h_{j-i-2}$ and $h_{r+j-3} = t^{2(j-i-1)} h_{r+i-2} + h_{j-i-1}$, we finally obtain

$$H_0 + H_1 + H_2 = t^{2(r+i-2)} h_1 h_{j-1} h_{j-i-1} h_{j-i-2} (h_{r+i-2} - h_{i-1} - t^{2i} h_{r-2}) = 0.$$

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