Economic Theory
https://doi.org/10.1007/s00199-024-01569-6

RESEARCH ARTICLE

How strength asymmetries shape multi-sided conflicts

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Received: 12 November 2022 / Accepted: 15 March 2024

Abstract

Governments and multilateral organisations often attempt to influence multi-sided violent conflicts by supporting or undermining one of the conflicting parties. We investigate the (intended and unintended) consequences of strengthening or weakening an agent in a multi-sided conflict. Using a conflict network based on Franke and Öztürk (J Public Econ 126:104–113, 2015), we study how changing the strength of otherwise symmetric agents creates knock-on effects throughout the network. Increasing or decreasing an agent's strength has the same unintended consequences. Changes in the strength of an agent induce a relocation of conflict investments: Distant conflicts are carried out more fiercely. In line with previous results, asymmetry reduces aggregate conflict investments. In the case of bipartite networks, with two conflicting *tacit groups with aligned interests*, agents in the group of the (now) strong or weak agent face more intense conflicts. Furthermore, in conflicts where the (now strong or weak) agent is not involved, the probabilities of winning remain unchanged compared to the symmetric case.

Keywords Conflicts · Network games · Asymmetric agents

JEL Classification C72 · D74 · D85

We are grateful to the editor, Nicholas Yannelis, the co-editor, and two anonymous referees for insightful and constructive contributions which greatly improved the paper. We would also like to thank Subhasish M. Chowdhury, Matthias Dahm, Marcin Dziubiński, Christian Ewerhart, Jörg Franke, Sanjeev Goyal, Oliver Gürtler, Laura A. Harvey, David Hugh-Jones, Dan Kovenock, Maria Montero, Ivan Pastine, Amrish Patel, David Rojo Arjona, Dmitry Ryvkin and Mich Tvede for comments. Also, we are grateful for comments received at the 20th Annual SAET Conference, the 2019 Latin American Workshop in Economic Theory, and the Contest Conference and Conflict Workshop held at UEA and Bath University, respectively. All remaining errors are our own.

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Published online: 17 April 2024

1 Introduction

In wars and military struggles throughout history, parties have typically been involved in more than one conflict at a time (Huntington 2000; Maoz 2010). The arrangement of conflicts among the involved parties relative to each other-the conflict network-induces interdependencies between conflicts. This, in turn, might lead to the escalation or de-escalation of violence since altering aspects of one conflict influences seemingly unrelated conflicts. How do such *knock-on effects* influence the intensity of conflict in remote conflicts? What impact do these effects have on the intensity across all connected conflicts?

Understanding the knock-on effects of asymmetric military strength between rivals in networks is critical, not only for predicting outcomes but also for designing and implementing policies to pacify conflict-ridden regions. This becomes clear in the context of multi-sided civil wars, where third-party interventions aimed at putting an end to the overall conflict (Linke and Raleigh 2016; König et al. 2017; Silve and Verdier 2018; Aidt et al. 2019), as was the case for the Colombian conflict or the Great War of Africa.[1] For almost 15 years, the US government tried to end a long-standing multi-sided conflict between the Colombian government, several drug cartels, and various left-wing guerillas by providing funds and training to the Colombian military (Acemoglu et al. 2013),[2] Similarly, several members of the United Nations imposed targeted sanctions, such as arms embargoes,[3] in the context of the Great War of Africa with the objective of de-escalating the conflict in this region.[4] While the intention was to de-escalate the Colombian and African conflict, there exists suggestive empirical evidence of a conflict escalation in the years after these interventions (Tierney 2005; Dube and Vargas 2013; König et al. 2017). Understanding the strategic responses to the support towards one agent can, thus, help avoid such unintended future outcomes. To understand the importance of the resulting difference in military strengths in such confrontations, a theoretical analysis of the effects of asymmetries in multi-sided conflicts is necessary.

This paper aims to shed light on the effects of exogenous strength asymmetries between agents on equilibrium behaviour in conflict networks. Introducing asymmetries between agents induces knock-on effects through the connections in the network. Firstly, we investigate how asymmetry in model parameters associated with strength is related to asymmetry in conflict intensity across conflicts. Secondly, we examine

[1] There are other examples where third parties tried to influence outcomes in multi-sided conflicts. Before the more recent developments, changing the conflict to a bilateral interstate war, the Ukrainian Civil War was such an example. Russia aimed to influence the outcome of the conflict between the Ukrainian government and several rebel groups by providing them with funds. Another example is the training camps provided to several left-wing guerrilla fighters by the Soviet Union after the 1966 Tricontinental Conference in La Havana.

[2] In 1999, this initiative, called the "*Plan Colombia*" was created under Bill Clinton's administration during a broader effort to fight narco-trafficking groups across the globe.

[3] We thank the anonymous referee who suggested the example of arms embargoes to motivate the weakening of an agent.

[4] Following its attainment of independence from Belgium in 1960, the Democratic Republic of the Congo faced a large-scale conflict involving the official government, three insurgent factions, as well as a consortium of 14 foreign armed entities, including ethnically diverse groups from Rwanda and Uganda.

the resulting changes in aggregate levels of conflict investments and, thus, the overall conflict intensity in the network. Following our examples of the Colombian Civil War or the Great War of Africa, we ask under which circumstances a stronger or weaker military force induces an escalation in seemingly unrelated conflicts and a change in overall conflict intensity. Can we expect the military to increase its conflict investments against the others? Or will it fight less vigorously? More importantly, how do the other, unchanged agents react? Do their strategic incentives become more aligned, so they tacitly direct conflict investments towards the stronger or weaker agent,[5] or are their efforts redirected against each other?

We focus on two types of asymmetries in our model. The first one is related to the incentives to win a particular conflict-i.e., the prize valuations.[6] In the Colombian conflict, the value of controlling rural areas (winning conflicts) changed drastically for the Colombian government with the US interventions, as resources coming from the US were tied to dismantling drug production. The second is in terms of the power to affect the conflict outcome-i.e., the effectiveness of conflict investments. This is related to the evolution of warfare technology throughout history. In the Great War of Africa, the imposition of sanctions in the form of arms embargoes changed the warfare technologies available to different groups embedded in conflicts, decreasing the lethality of weapons used.[7]

To investigate the effects of these two types of asymmetries, we extend the model of conflict networks introduced by Franke and Öztürk (2015), (from here on FO) along two dimensions while maintaining sufficient tractability. Using a lottery *contest success function*, we show that a finite number of locally unique and interior Nash Equilibria always exists. We discuss when this extends to global uniqueness. Due to the agents' asymmetry in terms of valuations of winning a conflict, it is not possible to use previous results of n-player concave games (Rosen 1965; Goodman 1980) to guarantee the existence of a unique Nash equilibrium for all choices of valuations and effectiveness parameters.[8] Thus, we contribute to this literature by providing an alternative method to establish the existence and local uniqueness of the Nash equilibrium based on the Implicit Function Theorem. It implies that the solutions around any unique equilibrium are a smooth function from parameters into the set of actions. Thus, it conveniently also allows the study of strength asymmetries around the symmetric equilibrium as local comparative statics. The analysis of the equilibria described by this function is our main objective.

Modelling conflicts on networks is a growing stream of research (see Dziubiński et al. 2016, for a recent review). To study the local interdependencies of conflicts

5 This is different from models studying the explicit formation of alliances such as Konrad and Kovenock (2009), Bloch (2012) or Ke et al. (2015).

6 We follow Esteban and Ray (2011) approach by considering an agent to be stronger (weaker) if their prize valuation is higher (lower) than in the symmetric setting. This interpretation relies on the fact that a higher valuation is equivalent to a relatively lower marginal cost in that specific conflict.

7 Providing foreign aid to an underdog can alter the effectiveness of weapons used in a conflict and balance conflict as well. As an example, through Operation Cyclone, the United States supplied financial resources and weapons to the Afghan Mujahideen, which affected the lethality of their warfare technology against the Soviet Union.

8 An alternative approach for the case of homogeneous valuations within conflicts is to show that under fictitious play, conflict investments converge to a Nash unique equilibrium (Ewerhart and Valkanova 2020).

among *symmetric agents*, FO introduce a model of conflict networks, where identical agents are involved in multiple bilateral Tullock (1980) conflicts. Agents decide simultaneously how much to invest in each conflict to affect their probability of winning an exogenous prize. The opportunity costs of investing in a given conflict depend on the overall investments made by agents in all their other conflicts. FO provide an existence and uniqueness result of the equilibrium strategies in their setting. They also provide a closed-form characterisation of the equilibrium for two important classes of network topologies (d-regular and complete bipartite networks).

We show that compared to FO's symmetric setting as a baseline for d-regular and complete bipartite networks, the increase or decrease of one agent's strength in an otherwise strength-symmetric network could exacerbate the conflict intensity in all conflicts where strength is still symmetric. Overall conflict intensity decreases after any change in one agent's effectiveness, irrespective of whether that change made that agent stronger or weaker than this agent's rivals. Conflict intensity increases (decreases) following an increase (decrease) in one agent's valuation for winning against a specific opponent. In that case, the increase (decrease) in conflict investments is driven by that stronger (weaker) agent. In fact, we show that the knock-on effect for the remaining symmetric agents is qualitatively the same following a strengthening or weakening of an agent due to the non-monotonicity of the best-response functions in lottery contests (Tullock 1980).[9] Our analysis of the probabilities of winning reveals that an increase (decrease) in the strength of an agent increases (decreases) this agent's probability of winning. In remote conflicts, where the agent-now either strong or weak-is not directly engaged, the probabilities of winning remain unaltered compared to the symmetric setting. Notably, our findings can be interpreted in two distinct manners: Firstly, they shed light on how changes from symmetry lead to a de-escalation in the overall conflict network but a redirection of conflict investments to the more symmetric agents. Secondly, however, they also inform us about how influencing an asymmetric multi-sided conflict to become more symmetric affects individually optimal behaviour. Here, behaviour changes in exactly the opposite way, leading to increased overall but more evenly distributed conflict across the network.[10]

FO's seminal contribution triggered subsequent studies looking at similar environments. For example, König et al. (2017), Dziubiński et al. (2019), Bozbay and Vesperoni (2018), and Matros and Rietzke (2024) each study a conflict on a network with a single (univariate) choice. Jackson and Nei (2015), Hiller (2017), and Huremović (2021), like us, study conflicts on networks with multivariate choices but focus on endogenous network formation. We contribute to this growing field of conflict networks by exploring the effects of asymmetries between individual agents and across all involved agents in the network. Using a variational inequality equivalence, Xu et al. (2022) show the existence of a unique equilibrium in pure strategies on networks of multilateral conflicts (i.e., each conflict can have more than two agents fighting each other) if agents have a strictly increasing cost function.

[9] The best responses' non-monotonicity in the two-agent contests, as shown in Fig. 7, carries over to our setting for each bilateral conflict. Appendix B revisits this property for the two-agent case.

[10] We would like to thank one of the anonymous referees for highlighting this dual interpretation of our results, which adds to the significance of our contribution.

Their unique equilibrium is such that at least two agents invest positive amounts in each conflict. Their characterisation, by construction, collapses to FO's and ours when only bilateral interactions are feasible. In this paper, we provide general comparative static results for the class of networks studied by FO.[11]

Beyond the study of conflict networks, we also contribute to the mature literature of multi-battle contests, which provides extensive and valuable insights into the strategic aspects of conflict and war in various settings.[12] Our model contributes to this literature by allowing for larger numbers of individuals embedded in complex bilateral relations modelled with networks. Moreover, we contribute to the study of asymmetries between agents in such models. It has been documented that strengthening a single competitor in an otherwise strength-symmetric n-player contest leads to a decrease in aggregate conflict investments (i.e., "discouragement effect"). Seminal studies of this phenomenon are Schotter and Weigelt (1992), Baik (1994), and Nti (1999) for bilateral Tullock contests and Stein (2002) for n-player Tullock contests. There, the increase of asymmetries leads to a decrease in the efforts exerted in the contest. More recently, this result has been extended to all-pay-auctions (Fang et al. 2020).

We contribute to this strand of the economic literature by discussing whether insights on lower conflict intensity in asymmetric two-agent conflicts (Baik 1994; Cornes and Hartley 2005) carry over to more complex conflict settings as captured in the type of model presented here.[13] This also informs an important strand of the economics literature on how to take advantage of the "discouragement effect" to level the playing field as affirmative actions (e.g., Franke et al. 2013, 2014; Chowdhury et al. 2020). To the best of our knowledge, our paper is the first to extend these results to the setting of conflict networks under complete information. It provides the first building block to better understand the effects of strength asymmetries more generally in complex, seemingly chaotic conflict settings.

The remainder of the paper is structured as follows. Section 2 provides the model setup, and Sect. 3 explores its equilibrium properties. Using a set of selected examples, Sect. 4 provides an intuition for the general results presented in Sect. 5. Section 6 concludes. All proofs are provided in Appendix A.

2 Model setup

Let $\mathcal{I} = \{1, \ldots, n\}$ be a finite set of agents with $n \geq 2$, and let $B \subseteq \mathcal{I}^2$ (i.e., all ordered pairs of agents) be the finite set of conflicts. Agents i and j have a bilateral conflict if and only if $(ij) \in B$. The underlying conflict network $\mathcal{G}$ is represented by the graph

[11] In the most recent version of Xu et al. (2022), some additional comparative statics examples are included which also utilise the Implicit Function Theorem.

[12] These include settings with symmetric agents (e.g., Myerson 1993; Roberson 2008; Hart 2008; Olszewski and Siegel 2022); with asymmetric valuations across conflicts and/or agents (e.g., Kovenock and Roberson 2008; Washburn 2013; Roberson and Kvasov 2012; Montero et al. 2016; Thomas 2018); as well as with asymmetric agents (e.g., Friedman 1958; Robson 2005; Hortala-Vallve and Llorente-Saguer 2012; Macdonell and Mastronardi 2015; Kovenock and Rojo Arjona 2019; Kovenock and Roberson 2021). See Konrad (2009) and Dechenaux et al. (2015) for comprehensive surveys of the contest literature.

[13] For reference, we present the main insights from Baik (1994)'s two-agent asymmetric setting in Appendix B.

associated with the ordered pair of disjoint sets $(\mathcal{I}, B)$. Let P_h^l be a path of length K from Agent h to l defined as a sequence of conflicts $(i_1 i_2), (i_2 i_3), \ldots, (i_{K-1} i_K)$ such that $(i_k i_{k+1}) \in B$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = h$ and $i_K = l$. We consider networks that are *connected* ($\forall \{h, l\}$: a path P_h^l exists), *undirected* ($\forall \{i, j\} : (ij) \in B \Leftrightarrow (ji) \in B$) and *irreflexive* ($\forall i : (ii) \notin B$), implying that there are no isolated clusters of agents, each conflict is mutual, and agents cannot be enemies of themselves.[14] Let $N_i = \{j \in \mathcal{I} | (ij) \in B\}$ denote the set of i's rivals. The total number of rivals of i is given by $d_i = |N_i|$. Consequently, the total number of conflicts in $\mathcal{G}$ is $b = \frac{1}{2} \sum_{i \in \mathcal{I}} d_i$. As we are going to refer to them later, it is useful to define two prevalent, stable, and well-studied network topologies in economics: d-regular and bipartite networks.[15]

Definition

- A graph $\mathcal{G} = (\mathcal{I}, B)$ represents a **d-regular network** if and only if $d_i = d > 1 \; \forall i \in \mathcal{I}$.
- A graph $\mathcal{G} = (\mathcal{I}, B)$ with $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ represents a **bipartite network** if and only if $B \subseteq \mathcal{I}_1 \times \mathcal{I}_2$. We call $\mathcal{I}_1$ and $\mathcal{I}_2$ **groups with aligned interests** (partitions) in this type of network with sizes $|\mathcal{I}_1| = I_1$ and $|\mathcal{I}_2| = I_2$, respectively.

In each bilateral conflict, agents compete for a strictly positive and exogenous prize. Agent i's valuation of winning the prize against Agent $j \in N_i$ is denoted $v_{ij} > 0$. This framework can accommodate constant-sum bilateral conflicts, when $v_{ij} = v_{ji}$, or non-constant-sum bilateral conflicts, when $v_{ij} \neq v_{ji}$. Let $\boldsymbol{v} \in \mathbb{R}_{++}^{2b}$ be a vector that collects all valuations v_{ij} for all $i, j \in \mathcal{I}$ and $(ij) \in B$. Each Agent i makes a conflict investment $x_{ij} \in \mathbb{R}_+$ to affect Agent i's probability of winning the prize v_{ij} against Agent $j \in N_i$. We denote Agent i's action by $\mathbf{x_i} = (x_{ij})_{j \in N_i} \in \mathbb{R}_+^{d_i}$ containing all their conflict investments. Agent i's probability p_{ij} of winning against Agent j is governed by a lottery *contest success function* (CSF) $p(a_i x_{ij}, a_j x_{ji})$, where $a_i \geq 1$ captures how effectively Agent i's investments increase p_{ij}. Let $\boldsymbol{p} \in [0, 1]^{2b}$ be a vector that collects all winning probabilities and let $\boldsymbol{a} \in \mathbb{R}_{++}^n$ be a vector collecting all effectiveness parameters a_i for $i \in \mathcal{I}$.[16] The CSF is increasing and concave in its first argument and decreasing and convex in its second argument. Further, it does not depend on any x_{lk} with $(lk) \neq (ij)$.[17] Finally, if both agents in a conflict do not make any investments, the winner of that conflict is selected at random with probability $\frac{1}{2}$. Specifically, in our game, the probability of Agent i winning the prize in the conflict against Agent j is

[14] In disconnected networks, our results apply to agents of each component independently.

[15] These networks are also well-studied in more general non-zero-sum games. In conflict networks like ours, Huremović (2021) shows that the only stable networks to either unilateral or bilateral deviations in endogenous conflict network formation games are the d-regular and bipartite networks.

[16] Without loss of generality, we can order the effectiveness parameters of all the agents such that $a_1 \geq a_2 \geq .. \geq a_n$. Using this ordering, we can always normalise the effectiveness parameters of everyone by comparing it with respect to Agent n such $\frac{a_i}{a_n} \geq 1$. This normalisation simplifies our analysis and is not essential for most of our results (for our proofs, we mention when this would make a difference).

[17] Skaperdas (1996) discusses the class of CSFs satisfying these properties. See also Corchón and Dahm (2010).

$$p_{ij} = p(a_i x_{ij}, a_j x_{ji}) = \frac{f(a_i x_{ij})}{f(a_i x_{ij}) + f(a_j x_{ji})}, \tag{CSF}$$

where $f(\cdot) = g(\cdot) + \delta$ denotes the impact function for some arbitrarily small $\delta > 0$,[18] such that $g(\cdot)$ is a positive function of its argument, which has a positive but finite slope and is at least twice continuously differentiable, with $g(0) = 0$.[19] The costs of all conflict investments are captured by the function $C(X_i)$, where $X_i = \sum_{j \in N_i} x_{ij}$ denotes the total amount of investments made by Agent i across all their conflicts. The cost function is at least twice differentiable and strictly convex with $C'(0) = 0$.[20] Agent i's opportunity costs of investments across conflicts depend on the curvature of $C(X_i)$.

For each conflict $(ij) \in B$, Agent i's expected revenue is $\pi_{ij} = p_{ij} v_{ij}$. Agents are risk-neutral expected payoff maximisers, and thus, their additively separable payoff function is given by

$$\Pi_i(\mathbf{x_i}, \mathbf{x_{-i}}, \mathcal{G}) = \sum_{j \in N_i} \pi_{ij} - C(X_i).$$

Whenever the primitives in our model are such that $a_i = 1$ for all $i \in \mathcal{I}$ and $v_{ij} = v > 0$ for all $(ij) \in B$, this environment collapses to the model in FO. For ease of notation, throughout the remainder of the paper, let $\boldsymbol{\omega} = (\boldsymbol{v}, \boldsymbol{a})$ denote the combination of all the valuations and effectiveness parameters with $\boldsymbol{\omega} \in \Omega \subseteq \mathbb{R}^{2b+n}_{++}$.[21]

3 Existence and local uniqueness (everywhere)

To use the Implicit Function Theorem (IFT) for our comparative statics, we show that equilibrium existence extends to asymmetric parameters. Furthermore, as our comparative statics are discrete variations of the model's primitives, we show that these equilibria are locally unique.[22] We outline the main steps for our proof of existence and local uniqueness of strictly interior equilibrium strategies. For *balanced parametrisations* $\overline{\boldsymbol{\omega}}$ (i.e., $\forall (ij) \in B : v_{ij} = v_{ji}$), we also show that FO's global uniqueness

[18] This way of defining the impact function avoids the discontinuity of the CSF at (0, 0). This formulation was first proposed by Myerson and Wärneryd (2006) and was also used by FO. This approach has been widely used in other important contributions to the contest literature using lottery contests success functions (see Xu et al. 2022, for a comprehensive discussion of this approach).

[19] The requirement of concavity for the CSF translates into the following condition on the impact function $f(.)$: $f''(a_i x_{ij})(f(a_i x_{ij}) + f(a_j x_{ji})) - 2f'(a_i x_{ij})^2 < 0$.

[20] These assumptions make the cost function strictly increasing for every $X_i > 0$. In fact, results do not change as long as $C'(0) > 0$ is arbitrarily small.

[21] For consistency, we will stick with the word 'effectiveness' throughout the paper. Alternatively, we could refer to the latter collection of parameters as 'population weights,' as it is done in Esteban and Ray (2011).

[22] Otherwise, we could obtain one of many comparative statics that imply various effects on behaviour. Xu et al. (2022) show that for some particular class of cost function, multiplicity of equilibria occurs such that there is at least one agent active (investing a strictly positive amount) in each conflict in each of the equilibria. We focus on cost functions of the type proposed by FO to derive unique comparative statics.

extends to a neighbourhood around such parameter choices. The complete proof can be found in the appendix.

The maximisation problems of all agents depend on the network structure $\mathcal{G}$ and are structurally identical. All rivals' conflict investments are denoted $\mathbf{x}_{-i}$. Thus, every Agent i faces the following d_i-dimensional maximisation problem

$$\max_{\mathbf{x}_i \in \mathbb{R}_+^{d_i}} \Pi_i(\mathbf{x}_i, \mathbf{x}_{-i}, \mathcal{G}).$$

By defining the strategy space for each conflict investment as $[0, M]$ for a sufficiently large M,[23] our game is strategically equivalent to a continuous game with compact strategy spaces and a finite set of agents with strictly concave payoff functions. These characteristics allow us to establish equilibrium existence by using a standard fixed-point argument. For local uniqueness and interiority, we make three observations.

Firstly, we prove the following lemma.

Lemma 1 *There always exists a $\delta^* > 0$ such that $\forall \delta < \delta^*$ and $\forall$ $(ij) \in B$, an investment of $\tilde{x}_{ij} = 0$ is strictly dominated by some $x_{ij} \in [\epsilon^*, M]$ for some arbitrarily large and finite $M > 0$ and some small $\epsilon^* > 0$.*

Non-interior conflict investments lead to the existence of a profitable deviation.[24] As in FO, it is the properties of our CSF and the cost function that guarantee strict interiority.

Secondly, the system of first-order conditions of Agent i's maximisation is such that[25]

$$F_{ij} = \frac{a_i f'(a_i x_{ij}) f(a_j x_{ji})}{\big(f(a_i x_{ij}) + f(a_j x_{ji})\big)^2} v_{ij} - C'(X_i) = 0, \quad \forall \{i, j\} \in \mathcal{I} \text{ and } (ij) \in B. \quad (1)$$

We show that the Jacobian of $F = (F_{ij})_{(ij)\in B}$ has a non-zero determinant everywhere. This allows us to use the Implicit Function Theorem (IFT) to show local

[23] Define $M_i > 0$ for all i and all $(ij) \in B$ such that

$$\frac{a_i f(a_i x_{ij}) f(a_j x_{ji})}{\big(f(a_i x_{ij}) + f(a_j x_{ji})\big)^2} v_{ij} < C'(M_i).$$

Then choose $M = \max\{M_1, ..., M_n\}$. Any finite number greater than M would work as well.

[24] Whenever at least one agent invests nothing, marginal profits are large, and costs are close to zero for that agent for investing a small amount. If an agent is the sole investor in a conflict, this agent always wants to invest less to reduce costs. Investing M is not profitable either, as it is chosen such that marginal costs outweigh marginal profits on every conflict of every agent.

[25] Notice that the optimality condition is such that

$$\forall i \in \mathcal{I} \text{ and } (ij) \in B: \; v_{ij} a_i \frac{\partial p_{ij}}{\partial x_{ij}} = C'(X_i) \iff \bar{v} a_i \frac{\partial p_{ij}}{\partial x_{ij}} = \frac{\bar{v}}{v_{ij}} C'(X_i).$$

Thus, if $v_{ij} \neq \bar{v}$, and thus $\frac{\bar{v}}{v_{ij}} \neq 1$, we can consider the change in valuation v_{ij} as a change in the unit cost of the conflict investment for Agent i that is specific to conflict (ij). Therefore, Agent i is now stronger or weaker than her rivals.

uniqueness of each equilibrium and derive comparative statics. We establish finiteness of the number of equilibria since, in a compact strategy space, no sequence of equilibria can converge to a locally unique equilibrium.

Third and finally, for global uniqueness, we modify the proof in FO for a neighbourhood around their model, which is any balanced parametrisation in ours.

Proposition 1 (Existence, (local) Uniqueness and Interiority of Pure Strategies) *A finite number of locally unique, strictly interior, pure-strategy Nash equilibria exists for any* $\boldsymbol{\omega} \in \Omega$. *Around any Nash equilibrium* $x(\boldsymbol{\omega})$, *there exists a function* $x(\boldsymbol{\omega}) : \Omega \mapsto \mathbb{R}^{2b}_{++}$, *mapping any parameter* $\boldsymbol{\omega}$ *into a Nash equilibrium, which is at least twice differentiable. Its derivative is given by*

$$D_x(\boldsymbol{\omega}) = -\left[D_x F(x(\boldsymbol{\omega}); \boldsymbol{\omega})\right]^{-1} D_{\boldsymbol{\omega}} F(x(\boldsymbol{\omega}); \boldsymbol{\omega}). \tag{2}$$

For any balanced parametrisation, there exists an open neighbourhood $U(\overline{\boldsymbol{\omega}})$, *such that the equilibrium is globally unique* $\forall \boldsymbol{\omega} \in U(\overline{\boldsymbol{\omega}})$.

Our approach, using the IFT, allows us to circumvent the lack of a closed-form solution to the maximisation problem.[26] Since we have established local uniqueness of each Nash Equilibrium, we can rely on the fact that in close proximity to the respective parameter choice, conflict investment changes are unique as well. Therefore, we may seek to study $D_x(\boldsymbol{\omega})$ more closely. By applying discrete changes, we can characterise the individual and aggregate change in equilibrium investments with respect to changes in valuations and effectiveness parameters.

4 The effects of strength asymmetries: examples

To provide an intuition for our more general results, we illustrate the knock-on effects induced by the network structure by selecting some commonly used functional forms and simple networks. Our results and insights extend to more general setups and are provided in the subsequent sections.

Consider a complete network $\mathcal{G}$ of four agents (i.e., a 3-regular connected network) as depicted in Fig. 1 and let $v_{ij} = \bar{v} = 1000$ for all $(ij) \in B$ and $a_i = \bar{a} = 1$ for all $i \in \mathcal{I}$. Consider the impact function $f(x) = x$ that induces the following the CSF,[27]

$$p(x_{ij}, x_{ji}) = \frac{a_i x_{ij}}{a_i x_{ij} + a_j x_{ji}}.$$

Finally, let the cost function be given by $C(X_i) = \frac{1}{30} X_i^{30}$. In this example, each Agent i's maximisation problem is given by

[26] An earlier version of this paper (Cortes-Corrales and Gorny 2018) establishes more rigorously that such a solution does not exist for non-trivial network structures.

[27] We abstract from $f(x) = g(x) + \delta$ here for the sake of clarity. All figures presented in this section are identical up to (and beyond) the third digit presented here for $\delta = 2.2204 \times 10^{-16}$.

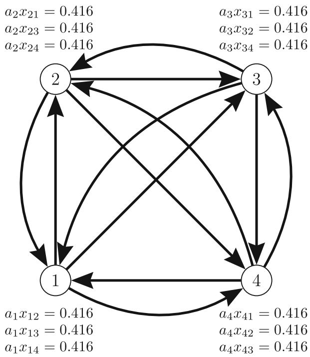


Fig. 1 Symmetric Nash equilibrium for $\bar{v} = 1000$

$$\max_{\mathbf{x}_i \in \mathbb{R}^3_+} \Pi_i(\mathbf{x}_i, \mathbf{x}_{-i}, \mathcal{G}) = \left(\sum_{j \in N_i} 1000 \frac{a_i x_{ij}}{a_i x_{ij} + a_j x_{ji}} \right) - \frac{1}{30} \left(\sum_{j \in N_i} x_{ij} \right)^{30}.$$

With these specific parameters and functional forms, the unique Nash equilibrium $(\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*)$ is characterised by the strategy $\mathbf{x}_i^* = (x_{ij}^*)_{j \in N_i}$ for every i where $x_{ij}^* = 0.4156$ for all $j \in N_i$. In equilibrium, each Agent i has an expected payoff of $\Pi_i = 1500$ with overall conflict investments of 4.992. This benchmark also allows for a closer comparison to the numerical examples presented in Fig. 2 of FO regarding the equilibrium strategies and overall conflict intensity. The order of magnitude is the same.

Changes in valuations

Based on the example depicted in Fig. 1, we consider both an increase and a decrease in the strength of Agent 1 in terms of Agent 1's conflict valuation against Agent 4. We start by inducing an increase in the valuation of Agent 1 for winning against Agent 4 from $v_{ij} = 1000$ to $\tilde{v}_{14} = 10 v_{ij}$, while all the other parameters remain unaffected. The resulting relative changes in the equilibrium conflict investment ($\Delta_{ij} = \frac{\tilde{x}_{ij}^* - x_{ij}^*}{x_{ij}^*}$) are depicted in Fig. 2a.

In Fig. 2a, we observe, not surprisingly, that Agent 1 (the strong agent) increases the conflict investment against Agent 4, as winning conflict (14) has become more profitable. Due to the convexity of the cost function, the strong agent reduces the conflict investments on all other conflicts. Now turn to Agent 4. This agent is discouraged and reduces the conflict investment against Agent 1, who invests more than twice as much in the conflict against Agent 4 than before. At the same time, Agent 4 increases conflict investments against Agents 2 and 3. Since Agents 2 and 3 receive lower conflict investments from Agent 1, their investment profitability in the conflicts with Agent 1 is lower and relatively higher in all conflicts with the remaining agents, including Agent 4. Conflict intensity thus relocates away from Agent 1's conflicts toward the weak agents' conflicts. The new expected payoffs, after the tenfold increase of v_{14}, now are naturally unequal such that $\Pi_1 = 8145$, $\Pi_2 = \Pi_3 = 1750$ and $\Pi_4 = 1239$

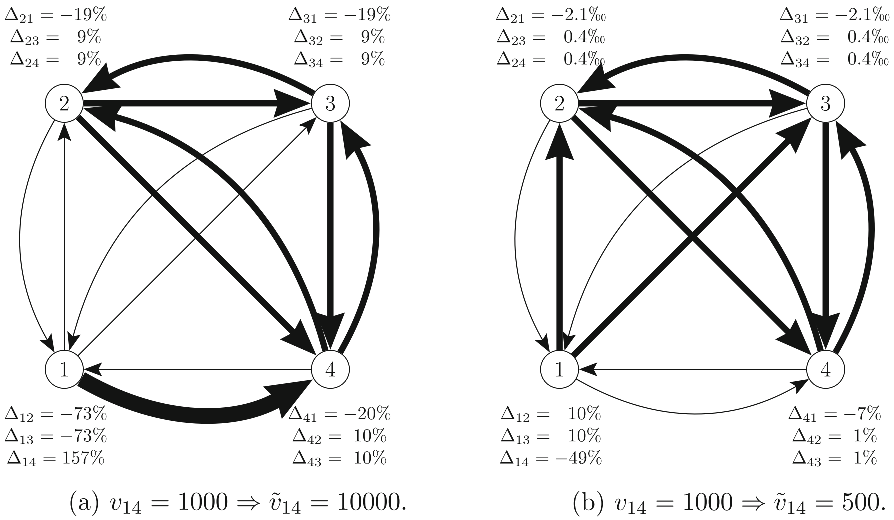


Fig. 2 Changes in the equilibrium strategies in a complete d-regular network due to changes in valuations

with overall conflict investments of 5.021. Notice that a change in the conflict between Agents 1 and 4 ended up benefiting other agents by increasing their expected payoffs by 250 units (17%), and the overall conflict intensity increased by 0.029 (0.6%). The increase is entirely driven by Agent 1 since all other agents do not increase their sum of conflict investments.

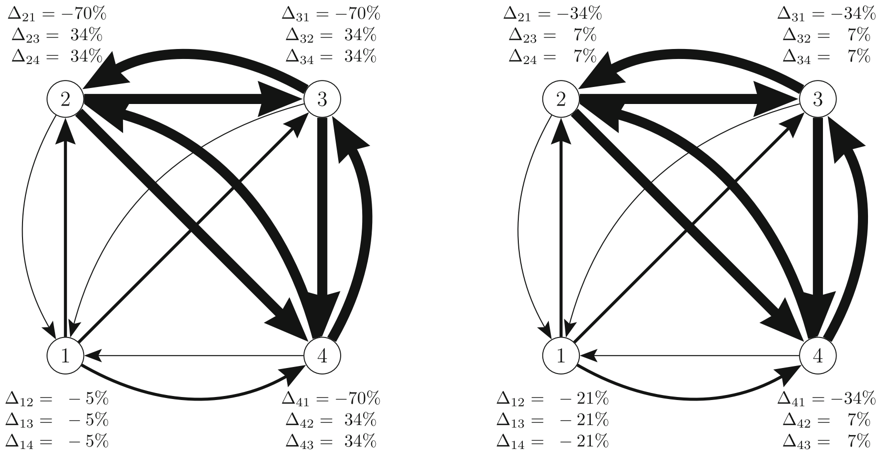
Alternatively, we could also consider a decrease in v_{14} as depicted in Fig. 2b. In this case, Agent 1 decreases the conflict investment against Agent 4 as expected, while the other agents qualitatively exhibit the same changes as when we considered the increase of v_{14}. The only relevant difference is related to the magnitudes.[28] These examples already illustrate the basic interplay of forces driving our more general comparative statics presented in Sect. 5.

Changes in investment effectiveness

Instead of changing the valuations from our symmetric benchmark, we now consider a change (i.e., an increase or a decrease) in Agent 1's effectiveness to influence the probabilities of winning (i.e., a_1). Notice that changes in valuations are battlefield-specific, while changes in abilities are agent-specific. We start by considering an increase from $a_1 = 1$ to $\tilde{a}_1 = 5a_1$ while all other parameters remain unchanged. The resulting relative changes in the equilibrium conflict investments (i.e., Δ_{ij}) due to the increase in a_1 are shown in Fig. 3a.

As all (now) weak agents are discouraged from fighting against the strong Agent 1, they reduce their conflict investments against Agent 1 and increase them against

[28] As per Proposition 1, we can consider the parameter changes in $U(\bar{\omega})$. Note that this neighbourhood is not necessarily symmetric around $\bar{\omega}$. In fact, for our example here, we can consider a ten-fold increase in v_{ij}, whereas we cannot consider a ten-fold decrease.



(a) $a_1 = 1 \Rightarrow \tilde{a}_1 = 5$ (b) $a_1 = 1 \Rightarrow \tilde{a}_1 = 1/5$

Fig. 3 Changes in the equilibrium strategies in a complete d-regular network due to investment effectiveness

each other. Again, conflict investments move away from the strong agent towards the remaining agents. The new expected payoffs after the increase of a_1 are $\Pi_1 = 2826$ and $\Pi_2 = \Pi_3 = \Pi_4 = 1058$. In this case, Agent 1 increases the expected payoff by 442 per conflict, while the weak agents lose the same amount in their conflicts against Agent 1. In this case, the overall conflict intensity (4.888) is lower than the benchmark (4.992). Nonetheless, the effective conflict intensity (i.e., $\sum_{i \in \mathcal{I}} a_i \sum_{j \in N_i} x_{ij}$) went up substantially to 9.63, given that the reduction in the conflict investment of Agent 1 (-5%) is overcompensated by an increase in Agent 1's effectiveness ($\tilde{a}_1 = 5a_1$). This leads to $\tilde{a}_1 \tilde{x}_{1j} = 5 \cdot 0.3951 = 1.9755$ for $j = 2, 3, 4$ (an increase of 375% in the *effective conflict investment*). The difference from the previous case is that only one attacked agent received increased effective conflict investments in each conflict. In this case, this is true for every weak agent.

Alternatively, we could also consider a decrease in a_1 as shown in Fig. 3b. In this case, we observe the same qualitative patterns as those described in Fig. 3a when a_1 was increased. The main difference is in the changes' magnitude; a decrease of a_1 triggers a stronger reaction for Agent 1's investments than for the other agents.

This result also applies in the model's general framework except for the investments of Agent 1. The knock-on effects are also relatively large since we changed the effectiveness parameter by a large factor for a clearer exposition of our example. This, in turn, also has a higher-order strategic effect on Agent 1's investment choices. At the symmetric equilibrium, our general results predict that Agent 1 does not change the investments in any conflict after such a change in effectiveness, which is consistent with the literature (we revisit the relevant result of the seminal paper: Baik 1994, p.375, in Appendix B). Therefore, a positive or negative change in the effectiveness of an agent triggers the same type of knock-on effects.

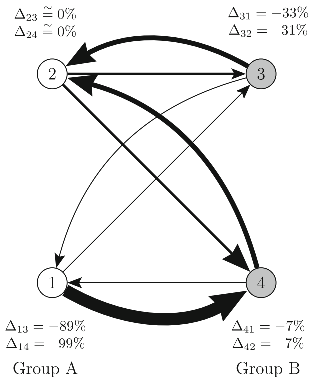


Fig. 4 Changes in the equilibrium strategies in a bipartite network due to $v_{14} = 1000 \Rightarrow \tilde{v}_{14} = 10000$

Effects in a bipartite network

In bipartite networks, agents are divided into two *groups with aligned incentives* (or partitions), so conflicts only occur between agents from these different groups. These networks are of specific interest as they lend themselves to the study of inter-group conflict and alliance formation as documented in previous work done by Jackson and Nei (2015) or König et al. (2017). In the context of civil wars, bipartite networks can be observed when temporary agreements (informal alliances) between rebel groups in terms of bilateral cease-of-fire periods are established. This was the case during the Syrian War after the Arab Spring in 2011 when Islamic rebel factions agreed to a cease of fire to focus attention against the secular regime (Gade et al. 2019). As our benchmark for this example, we start with a bipartite network of four symmetric agents with two groups of two agents with aligned incentives each. In this case, the expected payoff of every agent is $\Pi_i = 983$, and their chances of winning a conflict are $\frac{1}{2}$ in each conflict. Similarly to our previous examples, considering both increases or decreases in either conflict valuations or conflict effectiveness leads qualitatively to the same first- and second-order effects as presented formally in Propositions 4 and 5. Therefore, we exemplify the main second-order effects in bipartite networks with the strengthening of an agent. Consider an increase of v_{14} as depicted in Fig. 4. Whereas behavioural responses to a change in Agent 1's v_{14} are similar to the ones mentioned above, outcomes differ qualitatively for the other agents.

As in the first example, Agent 4 is discouraged by the almost doubled investment of Agent 1 (it increased by 99%). Similarly, Agent 3 faces an 89% lower investment from Agent 1. Thus, the marginal payoff on the conflicts against Agent 1 is reduced for Agents 3 and 4. Since marginal costs are the same for Agents 3 and 4 before the change in v_{14}, they increase their investments against Agent 2 and decrease their investments against Agent 1. This time, from Agent 2's perspective, compared to the example in the complete network in Fig. 2a, there is no reduced investment from Agent 1 since they are not engaged in a conflict. Payoffs are given by $\Pi_1 = 6891$ and $\Pi_2 = 899$, $\Pi_3 = 1411$ and $\Pi_4 = 817$.[29]

[29] The changes in the behaviour of Agent 2 are lower than 1% so for ease of exposition they are approximated to zero omitting the signs.

It is a common observation in the literature on alliances (for examples, see Sánchez-Pagés 2007; Konrad and Kovenock 2009; Kovenock and Roberson 2012; Bloch 2012) that due to free-riding incentives, alliances can end up with lower chances of winning than individual agents. In alliances, agents typically maximise the sum of their payoffs, leading to a tension between individual and collective incentives. In turn, this results in free-riding behaviour and lower individual payoffs in the alliance compared to agents fending for themselves. The literature suggests that asymmetries within alliances can exacerbate this problem (Esteban and Ray 1999). Even though the agents in our *groups of aligned interests* do not maximise the sum of their payoffs, we observe a qualitatively similar outcome for Agent 2 in Group A. Whereas Agent 1's payoff has increased substantially as a result of increasing v_{14}, we see that in bipartite networks, on average, asymmetry can reduce the chances of winning for each group member. At symmetry, Agents 1 and 2 can expect to win 50% of their conflicts (or win each conflict with a probability of 50%). After introducing asymmetry in valuations, this number drops to 44%. Accordingly, the share of conflicts Agents 3 and 4 can expect to win increases from 50% to 56%. Having an agent in the group who has a particular agenda for investing most of their resources in a conflict with a specific opponent can thus be detrimental to the other group members' chances of winning. This is at odds with findings in sequential "team" contests, where outcomes of past battles do not affect the outcomes of subsequent battles (Fu et al. 2015). This shows that even when there are *groups with aligned interests* in opposition to explicit alliances or teams that share the winning payoff among themselves, similar outcomes are observed even when the reasons for them are different.

Asymmetry in the effectiveness of one agent affects rivals in a non-selective way. Thus, when increasing a_1 in a bipartite network in the same way as in the previous example, we see that Group B suffers. This is true both in terms of payoffs and winning probabilities.

The expected payoffs after the change in a_1 are given by $\Pi_1 = 1796$ and $\Pi_2 = 803$, $\Pi_3 = 679$ and $\Pi_4 = 679$. The strength of Agent 1 still hurts Agent 2, but it hurts Agents 3 and 4 more. The expected share of conflicts won is 66% for Agents 1 and 2, and thus only 34% for Agents 3 and 4.

5 The effects of strength asymmetries: results

For the general results, we denote $S \subseteq \mathcal{I}$ a clique of $\mathcal{G}$ with s agents if every pair of Agents i and j in S is involved in a bilateral conflict-i.e. $(ij) \in B$ for all $\{i, j\} \in S$ with $s = |S|$. We use the term *effective conflict investment* for $a_i x_{ij}$. Note that we can derive the effects of asymmetries on conflict investments ignoring the reactions of agents outside the clique using an intermediate result to our earlier propositions. We show that effect sizes diminish with the length of a path originating at Agent i. Thus, for a neighbourhood around any baseline parametrisation, any effects in the clique dominate the effects in the remaining network in terms of their magnitude. Consequently, the resulting changes in conflict investments from agents outside the clique are negligible. As in the examples, it is important to note that even though effects can be of second- or higher-order mathematically, all effects occur simultaneously.

Denoting a $k \times 1$ vector of ones with $\mathbf{1}_k$, let $\bar{\bar{\boldsymbol{\omega}}} = \{(\mathbf{1}_{2b}\bar{v}, \mathbf{1}_n\bar{a}) \mid (\bar{v}, \bar{a}) \in \mathbb{R}^2_+\}$ be our *baseline parametrisation* (FO's framework)-the benchmark for our comparative statics. In such a baseline parametrisation, all valuations and effectiveness parameters are the same across agents and conflicts. For a given conflict network structure, we denote $\bar{\bar{\mathbf{x}}} = (\bar{\bar{\mathbf{x}}}_1, \ldots, \bar{\bar{\mathbf{x}}}_n)$ and $\mathbf{x}' = (\mathbf{x}'_1, \ldots, \mathbf{x}'_n)$ the equilibrium profiles associated with parametrisation $\bar{\bar{\boldsymbol{\omega}}}$ and $\boldsymbol{\omega}'$, respectively.

5.1 Individual conflict investments

Starting from a baseline parametrisation $\bar{\bar{\boldsymbol{\omega}}}$, we consider a change (either positive or negative) in some Agent i's valuation for winning the conflict against some specific rival j.

Proposition 2 *Let $\boldsymbol{\omega}' = (\boldsymbol{v}', \mathbf{1}_n\bar{a})$ be such that $v'_{ij} = \bar{v}+\epsilon$ and $v_{hl} = \bar{v}$ $\forall (hl) \in B$ such that $(hl) \neq (ij)$. For every S in any d-regular conflict network $\mathcal{G}$, $\exists \underline{\epsilon} \in (-\bar{v}, 0)$ and $\exists \bar{\epsilon} \in (0, \infty)$ where $\forall \epsilon \in (\underline{\epsilon}, \bar{\epsilon})$ such that $\epsilon \neq 0$, the equilibrium profile $\mathbf{x}'$ associated with $\boldsymbol{\omega}'$ compares to the equilibrium profile $\bar{\bar{\mathbf{x}}}$ associated with $\bar{\bar{\boldsymbol{\omega}}}$ in the following way:*

- *First-order effects: $\forall k \in S$ such that $k \notin \{i, j\}$,*

$$\text{if } \epsilon > 0 \text{ then } x'_{ij} > \bar{\bar{x}}_{ij} \text{ and } x'_{ik} < \bar{\bar{x}}_{ik}.$$
$$\text{if } \epsilon < 0 \text{ then } x'_{ij} < \bar{\bar{x}}_{ij} \text{ and } x'_{ik} > \bar{\bar{x}}_{ik}.$$

- *Second-order effects: $\forall k, j \in S$ such that $i \notin \{k, j\}$,*

$$\text{if } \epsilon \neq 0 \text{ then } x'_{kj} > \bar{\bar{x}}_{jk} \text{ and } x'_{ki} < \bar{\bar{x}}_{ki}.$$

The explicit derivation of $D_{\bar{\bar{\mathbf{x}}}}(\bar{\bar{\boldsymbol{\omega}}})$ (i.e., Eq. (2) evaluated at $\bar{\bar{\boldsymbol{\omega}}}$) indicates that Agent i increases (decreases) the conflict investment against Agent j and reduces (increases) the investments in all other conflicts, following an increase (decrease) in v_{ij}. The consideration of second-order effects reveals that all other agents reduce investments against Agent i and increase their investments against each other. Notice that the second-order effects follow the same patterns independently of whether we are considering an increase or decrease of the particular conflict value. This is the behaviour depicted in Fig. 2a and b. One might wonder about the width of the parameter range in which Proposition 2 holds. Using numerical investigations of the examples presented in Fig. 1, we find that for all values of v_{14} between 75 and 50,000 (that is, roughly between a 13-fold decrease and a 50-fold increase), each equilibrium is in line with Proposition 2.[30]

Qualitatively, this result is similar when we consider a change (positive or negative) in the effectiveness a_i of some Agent i. Effectiveness enters the CSF through the impact functions, resulting in non-linear effects. Thus, we limit our analysis to impact functions of type $f(a_i x_{ij}) = (ca_i x_{ij})^r$ for some $c > 0$ and $r \in (0, 2)$ to obtain the following result.

[30] Note that the numerical thresholds are bounded by the machine's precision when applying a Newton–Raphson algorithm rather than the proposition's qualitative results. This is the case for all numerical thresholds in the remainder of this section unless mentioned otherwise.

Proposition 3 *Let $\boldsymbol{\omega}' = (\mathbf{1}_{2b}\bar{v}, \boldsymbol{a}')$ be such that $a'_i = \bar{a} + \epsilon$ and $a_j = \bar{a}$ $\forall j \in \mathcal{I}$ such that $j \neq i$. For every S in any d-regular conflict network $\mathcal{G}$ and any $\mu > 0$, $\exists \underline{\epsilon} \in (-\bar{a}, 0)$ and $\exists \bar{\epsilon} \in (0, \infty)$ where $\forall \epsilon \in (\underline{\epsilon}, \bar{\epsilon})$ such that $\epsilon \neq 0$, the equilibrium profile $\mathbf{x}'$ associated with $\boldsymbol{\omega}'$ compares to the equilibrium profile $\bar{\bar{\mathbf{x}}}$ associated with $\bar{\bar{\boldsymbol{\omega}}}$ in the following way:*

- *First-order effects: $\forall k \in S$ such that $k \neq i$,*

$$\text{if } \epsilon \neq 0 \text{ then } \left|x'_{ik} - \bar{\bar{x}}_{ik}\right| < \mu.$$
$$\text{if } \epsilon \neq 0 \text{ then } x'_{ki} < \bar{\bar{x}}_{ki}.$$

- *Second-order effects: $\forall k, j \in S$ such that $i \notin \{k, j\}$,*

$$\text{if } \epsilon \neq 0 \text{ then } x'_{kj} > \bar{\bar{x}}_{jk}.$$

With impact functions of the type considered here, $D_{\bar{\bar{\mathbf{x}}}}(\bar{\bar{\boldsymbol{\omega}}})$ implies that $\frac{\partial x_{ij}}{\partial a_i}|_{\omega=\bar{\bar{\omega}}} = 0$. Thus, at $\bar{\bar{\boldsymbol{\omega}}}$, the effect of an increase in a_i on the equilibrium investments of Agent i is negligible. Yet, the effective investments of Agent i, $a_i x_{ik}$, increase for all $k \in S$.[31] Other agents again reduce their investments against Agent i and increase investments against each other. This is the behaviour depicted in Fig. 3a. Using the examples in Sect. 4 as a benchmark again, we numerically investigate the size of the neighbourhood in which Proposition 3 holds. The behavioural prediction implied by Proposition 3 applies to efficiency values for a_1 between 0.1 and 12 (i.e., between a ten-fold decrease and a twelve-fold increase).

A particularly relevant type of network to study is the complete bipartite network with two *groups with aligned interests* (two partitions).[32] This type of structure has been used repeatedly for the study of inter-group conflict and alliances (see, e.g., Fu et al. 2015; Jackson and Nei 2015; König et al. 2017). As opposed to this literature, we do not model explicit alliances or teams that maximise the sum of their payoffs but use the notion of *groups with aligned interests*. This stems from the network structure directing their conflict investments towards the same set of rivals ($N_i = N_j$ for $i, j \in \mathcal{I}_1$) which resonates with the saying that "the enemy of my enemy is my friend." Following our previous results, we start by analysing the behavioural changes due to an increase of v_{ij} in such network structures.

Proposition 4 *For every complete bipartite conflict network $\mathcal{G}$ with groups of equal size $|\mathcal{I}_1| = |\mathcal{I}_2|$, let $i \in \mathcal{I}_1$ and $\boldsymbol{\omega}' = (\boldsymbol{v}', \mathbf{1}_n\bar{a})$ be such that $v'_{ij} = \bar{v} + \epsilon$ and $v_{ql} = \bar{v}$ $\forall (ql) \in B$ such that $(ql) \neq (ij)$. Then, $\exists \underline{\epsilon} \in (-\bar{v}, 0)$ and $\exists \bar{\epsilon} \in (0, \infty)$ where $\forall \epsilon \in (\underline{\epsilon}, \bar{\epsilon})$ such that $\epsilon \neq 0$, the equilibrium profile $\mathbf{x}'$ associated with $\boldsymbol{\omega}'$ compares to the equilibrium profile $\bar{\bar{\mathbf{x}}}$ associated with $\bar{\bar{\boldsymbol{\omega}}}$ in the following way:*

(i) If $\epsilon > 0$ ($\epsilon < 0$), Agent $i \in \mathcal{I}_1$ increases (decreases) the conflict investments against Agent j and decreases (increases) the conflict investments against all other Agents $k \neq j \in \mathcal{I}_2$.

[31] The fact that a discrete change of a_1 to $(1+\Delta_a)a_1$ decreases x_{1k} to $(1-\Delta_x)x$ in the example is in line with this result as long as $(1+\Delta_a)(1+\Delta_x) > 1$. Due to the continuity of the solution function derived in Proposition 1, this is guaranteed to hold in some, potentially small, neighbourhood around $\bar{\bar{\boldsymbol{\omega}}}$. The example illustrates that the results can even apply to an increase of $a_1 = 1$ to $a'_1 = 5$ or to a decrease of a_1 to $a_1 = \frac{1}{5}$.

[32] By construction, this network structure induces an even number of agents in the conflict network.

(ii) *All Agents $h \in \mathcal{I}_1$ for $h \neq i$ decrease the conflict investments against Agent j and increase the conflict investments against all other Agents $k \in \mathcal{I}_2$ such that $k \neq j \in \mathcal{I}_2$ (i.e., $x'_{hj} < \bar{\bar{x}}_{hj}$ and $x'_{hk} > \bar{\bar{x}}_{hk}$).*

(iii) *All Agents $k \in \mathcal{I}_2$ (including Agent j) decrease the conflict investments against Agent i and increase the conflict investments against all Agents $h \in \mathcal{I}_1$ such that $h \neq i$ (i.e., $x'_{ki} < \bar{\bar{x}}_{ki}$ and $x'_{kh} > \bar{\bar{x}}_{kh}$).*

As in our early example presented in Fig. 4, Proposition 4 generally shows that a higher v_{ij} increases conflict investments against the other group members. Using the parameters our our examples in Sect. 4, we numerically find that this proposition holds for changes of v_{ij} to values between 70 and almost up to 1150 (i.e., between about a 14-fold decrease and a 1.15-fold increase). The same qualitative result holds in general whenever we observe changes in the effectiveness of a specific agent.

Proposition 5 *For every complete bipartite conflict network $\mathcal{G}$ with groups of equal size $|\mathcal{I}_1| = |\mathcal{I}_2|$, let $i \in \mathcal{I}_1$ and $\boldsymbol{\omega}' = (\mathbf{1}_{2b}\bar{v}, \boldsymbol{a}')$ be such that $a'_i = \bar{a} + \epsilon$ and $a_l = \bar{a}$ $\forall l \in \mathcal{I}$ such that $l \neq i$. Then, for any $\mu > 0$, $\exists \underline{\epsilon} \in (-\bar{a}, 0)$ and $\exists \bar{\epsilon} \in (0, \infty)$ where $\forall \epsilon \in (\underline{\epsilon}, \bar{\epsilon})$ such that $\epsilon \neq 0$, the equilibrium profile $\mathbf{x}'$ associated with $\boldsymbol{\omega}'$ compares to the equilibrium profile $\bar{\bar{\mathbf{x}}}$ associated with $\bar{\bar{\boldsymbol{\omega}}}$ in the following way:*

(i) *Agent $i \in \mathcal{I}_1$ does not change the conflict investments by more than μ.*

(ii) *All Agents $h \neq i \in \mathcal{I}_1$ decrease their conflict investments (i.e., $x'_{hk} < \bar{\bar{x}}_{hk}$, $\forall k \in \mathcal{I}_2$).*

(iii) *All Agents $k \in \mathcal{I}_2$ decrease the conflict investments against Agent i and increase the conflict investments against all Agents $h \in \mathcal{I}_1$ such that $h \neq i$ (i.e., $x'_{ki} < \bar{\bar{x}}_{ki}$ and $x'_{kh} > \bar{\bar{x}}_{kh}$).*

As mentioned earlier, the behaviour of the (now) either strong or weak agent changes marginally following a change in this agent's effectiveness. This is not captured in our examples, as we apply discrete changes to the parameters there. We did so to gain a better intuition. Nonetheless, in both the examples and our propositions, the introduction of asymmetries in a multi-sided conflict with two groups results in reduced winning probabilities for the agents that are part of the (now) strong agent's group. Furthermore, based on our example depicted in Fig. 5, our proposition describes the changes in numerical investigations for values of a_i to values between 0.005 and 13 (i.e., between a 200-fold decrease and a 13-fold increase).

5.2 Aggregate conflict investments

In the earlier examples, we provided cases where aggregate conflict investments changed after changing one agent's strength. Furthermore, we established more generally that such a change decreases the conflict investments directed at this agent, whereas investments among the other agents increase. Now, we compare those changes to determine the effects of strength asymmetries on the sum of investments made across the network, that is, across conflicts and agents $\mathcal{X} = \sum_{i \in \mathcal{I}} X_i$ and on the investments made by all agents other than i, $\mathcal{X}_{-i} = \mathcal{X} - X_i$. Let $\bar{\bar{\mathcal{X}}}$ and $\mathcal{X}'$ be the aggregate equilibrium conflict investments associated with $\bar{\bar{\boldsymbol{\omega}}}$ and $\boldsymbol{\omega}'$, respectively.

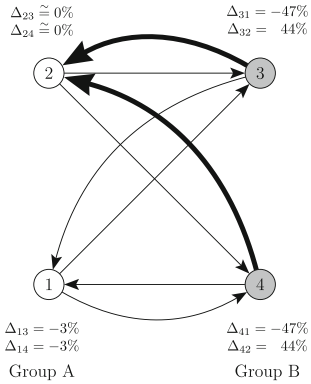


Fig. 5 Changes in the equilibrium strategies in a bipartite network due to $a_1 = 1 \Rightarrow \tilde{a}_1 = 5$

In canonical models of n-player contests with a single indivisible prize, Stein (2002) shows that strengthening a single competitor in an otherwise strength-symmetric n-player contest, in terms of effectiveness, decreases the overall conflict investments (i.e., $\mathcal{X}$). In fact, a similar observation can be made for conflict networks of the type considered here.[33] In fact, any change in effectiveness induces a decrease in aggregate investments.

Proposition 6 *Let* $\boldsymbol{\omega}' = (\mathbf{1}_{2b}\bar{v}, \boldsymbol{a}')$ *be such that* $a_i' = \bar{a} + \epsilon$ *and* $a_j = \bar{a}\ \forall j \neq i \in \mathcal{I}$. *In every* d*-regular conflict network* $\exists \underline{\epsilon} \in (-\bar{a}, 0)$ *and* $\exists \bar{\epsilon} \in (0, \infty)$ *where* $\forall \epsilon \in (\underline{\epsilon}, \bar{\epsilon})$ *such that* $\epsilon \neq 0$, *we have* $\Delta\mathcal{X}_{-i} < 0$ *where* $\Delta\mathcal{X}_{-i} = (\mathcal{X}'_{-i} - \bar{\bar{\bar{\mathcal{X}}}}_{-i})$.

In the symmetric equilibrium, opponents' investments coincide with the maximiser of each agent's best response. Thus, any change, whether an increase or decrease in an opponent's investment, leads to a reduction in own investment. There is one exception: many opponents reduce their investments against some agent in a heterogeneous way. In our example illustrated in Fig. 3a, the increase in Agent 1's effective investments was about 500%, while it was only an increase of 34% by the other agents. Thus, from the perspective of, say, Agent 4, there is a massive drop in the marginal probability of winning against Agent 1 but only a slight drop against Agent 2 or 3, respectively. The subsequent reduction in investments against Agent 1 results in low marginal costs, leading to an increase in investments against Agents 2 and 3, where Agent 4 is now on the increasing side of the marginal expected payoff graphs. Thus, in any d-regular network around a symmetric equilibrium, strength asymmetries decrease the aggregate amount of conflict investments. Our earlier example in Fig. 3a, where a_i increased by a factor of 5, does indeed show that a reduction is even possible for fairly substantial asymmetries. Making an agent either stronger or weaker to reduce overall conflict is thus a sound strategy close to symmetry and possible even for a wider neighbourhood of parameters. Furthermore, our example considers a complete network that is not part of a larger network, whereas our propositions on changes in individual conflict

[33] If we impose further restrictions on the degree of homogeneity (degree zero) of the CSFs, we can reach qualitatively similar results to Stein (2002).

investments apply to cliques within a larger network. As we have argued earlier, effects in the clique dominate higher-order effects down any path from Agent i.

It is important to note that our proposition can also be read to be informative about the change from asymmetry-e.g., increasing a weak agent's a_i to level it with the other symmetric agents-to complete symmetry. In this case, the support of Agent i increases the aggregate conflict investments, in line with Baik (1994). We can make a similar statement in the case of asymmetries in valuations.

Proposition 7 *Let* $\boldsymbol{\omega}' = (\boldsymbol{v}', \mathbf{1}_n\bar{a})$ *be such that* $v'_{ij} = \bar{a} + \epsilon$ *and* $v_{kl} = \bar{v}\ \forall (kl) \neq (ij) \in B$. *In every d-regular conflict network* $\exists \underline{\epsilon} \in (-\bar{a}, 0)$ *and* $\exists \bar{\epsilon} \in (0, \infty)$ *where* $\forall \epsilon \in (\underline{\epsilon}, \bar{\epsilon})$ *such that* $\epsilon \neq 0$, *we have* $\Delta \mathcal{X}_{-i} < 0$, *where* $\Delta \mathcal{X}_{-i} = (\mathcal{X}'_{-i} - \bar{\bar{\mathcal{X}}}_{-i})$.

An increase in v_{ij} can lead to higher aggregate investments, but only due to the increase in Agent i's investment against Agent j. The sum of all other agents' investments drops. We can thus summarise that the discouragement effect is predominant in the type of conflict network considered here. Again, this proposition also implies that a change from asymmetry ($v_{ij} \neq \bar{v} = v_{qk}\ \forall (qk) \in B$ such that $q \neq i$ or $k \neq i$) to symmetry, actually increases aggregate conflict investments.

All our results hold for connected networks like those described in the model section. Nonetheless, our results apply to each component independently for disconnected networks with more than one component. For example, in a network with three separate triads-three components with three agents connected to each other-our results describe the effect of strength asymmetry on each triad on its own, as the behaviour of each agent is independent of agents not connected by a path to them.

5.3 Payoffs and winning probabilities

We finally turn our attention to how the probabilities of winning are affected by the type of asymmetries we consider in this paper. To allow this analysis, we focus on impact functions of the type $f(ax) = (kax)^r$ for all $k > 0$ and $r \in (0, 2)$ in the remainder of this paper.[34] We start by considering how changes in effectiveness affect the probability of winning.

Proposition 8 *Let* $\boldsymbol{\omega}' = (\mathbf{1}_{2b}\bar{v}, \boldsymbol{a}')$ *be such that* $a'_i = \bar{a} + \epsilon$ *and* $a_j = \bar{a}\ \forall j \in \mathcal{I}$ *such that* $j \neq i$. *For every S in any d-regular conflict network and any* $\mu > 0$, $\exists \underline{\epsilon} \in (-\bar{a}, 0)$ *and* $\exists \bar{\epsilon} \in (0, \infty)$ *where* $\forall \epsilon \in (\underline{\epsilon}, \bar{\epsilon})$ *such that* $\epsilon \neq 0$, *the winning probabilities* p' *in the equilibrium associated with* $\boldsymbol{\omega}'$ *compare to the winning probabilities* $\bar{\bar{p}}$ *in the equilibrium associated with* $\bar{\bar{\boldsymbol{\omega}}}$ *in the following way:*

- *If* $\epsilon > 0$: $p'_{ik} > \bar{\bar{p}}_{ik}$, $\forall k \in N_i$.
- *If* $\epsilon < 0$: $p'_{ik} < \bar{\bar{p}}_{ik}$, $\forall k \in N_i$.
- $\forall \epsilon \in (\underline{\epsilon}, \bar{\epsilon})$ *and* $\forall k, q \neq i$ *such that* $(kq) \in B$: $p'_{kq} = \bar{\bar{p}}_{qk} = \frac{1}{2}$.

Under the impact functions assumed here and as an intermediate step to the above result, we show that Agent i does not change any conflict investments following a

[34] Using $k = 1$, this is the impact function axiomatised by Skaperdas (1996).

change in a_i; thus, Agent i's effective conflict investments increase (decrease) with no changes in costs.

Corollary 1 *If a_i increases (decreases), the payoff of Agent i increases (decreases).*

We carry out the same analysis with regard to changes in the valuation as we have done previously.

Proposition 9 *Let $\boldsymbol{\omega}' = (\boldsymbol{v}', \mathbf{1}_n \bar{a})$ be such that $v'_{ij} = \bar{v} + \epsilon$ and $v_{hl} = \bar{v}\ \forall (hl) \in B$ such that $(hl) \neq (ij)$. For every S in any d-regular conflict network $\exists \underline{\epsilon} \in (-\bar{v}, 0)$ and $\exists \bar{\epsilon} \in (0, \infty)$ where $\forall \epsilon \in (\underline{\epsilon}, \bar{\epsilon})$ such that $\epsilon \neq 0$, the winning probabilities p' in the equilibrium associated with $\boldsymbol{\omega}'$ compare to the winning probabilities $\bar{\bar{p}}$ in the equilibrium associated with $\bar{\bar{\boldsymbol{\omega}}}$ in the following way:*

- *If $\epsilon > 0$: $p'_{ij} > \bar{\bar{p}}_{ij}$, $p'_{ik} < \bar{\bar{p}}_{ik}$, and $p'_{jk} > \bar{\bar{p}}_{jk}$, $\forall k \in N_i$ such that $k \neq j$.*
- *If $\epsilon < 0$: $p'_{ij} < \bar{\bar{p}}_{ij}$, $p_{ik} > \bar{\bar{p}}_{ik}$, and $p'_{jk} < \bar{\bar{p}}_{jk}$, $\forall k \in N_i$ such that $k \neq j$.*
- *$\forall \epsilon \in (\underline{\epsilon}, \bar{\epsilon})$ and $\forall k, q \neq i, j$ such that $(kq) \in B$: $p'_{kq} = \bar{\bar{p}}_{kq} = \frac{1}{2}$.*

Here, the change in investments leads to both a change in winning probabilities and costs. Only for an increase in strength can we derive how these changes affect Agent i's overall payoff. To see this, consider the following argument. Agent i could stick to $\bar{\bar{\mathbf{x}}}_i$ without a response from the other agents in the network (remember that their responses were second-order effects originating from Agent i's response to the change in v_{ij}). This way, the winning probabilities would not change, but the rewards on conflict (ij) would increase. If Agent i chooses another set of investments in equilibrium, it must be a profitable deviation from this strategy.

Corollary 2 *If v_{ij} increases, the payoff of Agent i increases.*

In the case of a decrease in v_{ij}, such a profitable deviation from the symmetric equilibrium strategy profile creates something we could call a *substitution effect* on Agent i's payoff. This substitution effect is certainly positive because otherwise, Agent i could simply stick to the symmetric equilibrium strategy profile. However, the reduction in v_{ij} constitutes a negative *income effect*, as, leaving conflict investments unchanged, a lower valuation in a conflict leads to a lower payoff. How these two effects compare generally depends on the specific cost function and the parameter values at symmetry.

6 Conclusion

We modelled multi-sided conflicts with a network of agents competing in lottery contests. The agents can invest in their specific conflicts to affect their chance of winning them. We find that an agent does not change their conflict investments if their effectiveness changes, independently of the direction. Yet, since this affects effective conflict investments, such a change affects the optimal investments of all their rivals. The change in the strength of an existing agent induces other agents to increase their investments against each other while reducing investments against the targeted agent. Conflict intensity is thus relocated away from the changed agent and towards the unchanged agents independently of whether the changed agent is now stronger or weaker. For the winning probabilities, we found that an increase (decrease) in an agent's strength increases (decreases) this agent's probabilities of winning. Furthermore, in conflicts where the (now strong or weak) agent is not involved, the probabilities of winning remain unchanged compared to the symmetric case. Our findings offer a dual interpretation: they inform the analysis of the impact of asymmetries but also how transforming an asymmetric multi-sided conflict into a more symmetric one influences optimal behaviour.

Our results have implications for foreign policies that aim to support or hinder an agent in otherwise balanced conflicts to influence conflict outcomes. Since distant conflicts are affected, and conflict can escalate among the unchanged agents, our analysis suggests that such policies can be ill-advised from a strategic vantage point. In fact, according to König et al. (2017), utilising data from the Great War of Africa, arms embargoes demonstrated the capability to decrease fighting activity within the targeted group by 40–60% (first-order effect) while simultaneously leading to an increase in conflict between non-targeted groups (second-order effects). This outcome aligns with the cautious findings of studies regarding the effects of arms embargoes, similar to Tierney (2005) and our results. Our findings bear relevance to the impact of foreign combat support in civil wars. As highlighted by Sullivan and Karreth (2015), in a bilateral conflict involving an incumbent and rebel groups, the influence of foreign aid on the conflict's outcome hinges on the initial strength levels of each party. Our results echo these insights by considering a novel aspect of the conflict-the network structure.

While grand contests or models with only two agents are technically appealing tools to model behaviour in conflict, their non-linear functional forms impede the analysis of asymmetries in more complex settings. Our analysis is one of the first building blocks in establishing results on asymmetries more broadly and in models that allow for more complex interactions. Two such extensions come to mind. In models of endogenous network formation, agents decide on whether they want to participate in a conflict with another agent (see, e.g., Song and van der Schaar 2015). Our model can provide a theoretical foundation for the order in which conflicts are started by describing how the expected benefits of initiating a conflict vary with asymmetries. That can lead to a richer environment in which a discussion of Balancing or Bandwagoning in International Relations is possible in a stylised model. In fact, our framework can be used to extend the insights of Kimbrough et al. (2014) by studying how the network structure, aside from the strength asymmetry, can promote the peaceful resolution of conflicts. This would allow an analysis of peaceful outcomes beyond the mere absence of conflict investment.

It is also worth keeping the broader optimal design question from a social planner's perspective in mind. Answering this question is the next natural and important step to continue the analysis of multi-sided conflicts, especially under asymmetries. Our model contributes to this broader research agenda by laying the foundation for analysing how strength relocation leads to a strategic relocation of conflict investments. Our model can be the building block of a two-stage game in which a social planner chooses the optimal intervention first, and conflicting agents choose conflict investments in the second stage. Our paper characterises the behaviour in this second stage, which is necessary for being able to address this broader question. Our results show the richness of consequences that such interventions can have for distinct parts of the network. Making one agent stronger or weaker reduces the conflict investments that this specific agent is facing, as well as the aggregate conflict intensity across the network. Still, some remote parts of the network can see a local-potentially unanticipated and unintended-increase in conflict intensity. Since the objectives of social planners differ (such as de-escalation, stabilisation, pacification, etc.), this remains an open question for future studies to explore. It further remains an empirical exercise to gauge the importance of this phenomenon in actual conflicts and across many relevant domains, such as cybersecurity, criminal gangs, and information warfare, amongst others. Historical examples of such interventions are numerous, and for the most recent ones, data sets are available.

A Proofs of propositions in the main text

As a notational convention, we use $p^1_{ij} = \frac{\partial p_{ij}}{\partial(a_i x_{ij})}$, $p^2_{ij} = \frac{\partial p_{ij}}{\partial(a_j x_{ji})}$, $p^{12}_{ij} = \frac{\partial^2 p_{ij}}{\partial(a_i x_{ij})\partial(a_j x_{ji})}$, $p^{11}_{ij} = \frac{\partial^2 p_{ij}}{\partial(a_i x_{ij})^2}$, and so on as short-hands for the derivatives of the CSF. Note that this implies, for example, $\frac{\partial p_{ij}}{\partial x_{ij}} = \frac{\partial p_{ij}}{\partial(a_i x_{ij})}\frac{\partial(a_i x_{ij})}{\partial x_{ij}} = a_i p^1_{ij}$, due to our definition of conflict investments and effective conflict investments.

A.1 Proof of proposition 1

The proof proceeds in five lemmas. First, we show that a pure strategy Nash Equilibrium exists for all $\boldsymbol{\omega} \in \Omega$. Second, by means of contradiction, we show that every such equilibrium must be bounded and strictly interior. Third, we show that the determinant of the Jacobian of $F = (F_{ij})_{(ij)\in B}$ is strictly positive for any equilibrium to apply the IFT to conclude local uniqueness. Fourth, we use a result due to Rosen (1965), according to which the equilibrium is globally unique if

$$\sigma(\mathbf{x}, \mathbf{r}) = \sum_{i\in\mathcal{I}} r_i \pi_i, \quad r_i \geq 0, \tag{3}$$

is *strictly diagonally concave* for some $r_i > 0$ for all $i \in \mathcal{I}$. Goodman (1980) shows that a sufficient condition for this to hold within our setting is that π_i is concave in $\mathbf{x}_i$, convex in $\mathbf{x}_{-i}$ for all $i \in \mathcal{I}$, and that $\sigma(x, r)$ is concave in $\mathbf{x}$. We show that this is true at any balanced parametrisation $\overline{\boldsymbol{\omega}}$, and thus, the uniqueness result of Franke and Öztürk (2015) carries over to our setting for such parameters. Due to the earlier application of the IFT at all $\omega \in \Omega$ including all possible $\overline{\boldsymbol{\omega}}$, it follows that there exists a neighbourhood around any equilibrium for which it is locally unique. The resulting matrix of comparative statics in expression (2) is a direct result of the application of the IFT and holds for any equilibrium.

Fifth and finally, to show finiteness of equilibrium points, we demonstrate that the set of equilibria is compact. In this compact set, there cannot be any (infinite and non-trivial) sequence of equilibria, as this would lead to a contradiction with local uniqueness.

Lemma 2 *A pure strategy Nash Equilibrium exists for all* $\boldsymbol{\omega} \in \Omega$.

Proof The game, with $f(a_i x_{ij}) = g(a_i x_{ij}) + \delta$ for some arbitrarily small $\delta > 0$, is a continuous game with a finite set of players. The set of strategies for each agent and conflict investment is $[0, M]$ and thus non-empty, convex and compact, and the payoff functions are continuously differentiable. It suffices to show that these are quasi-concave to apply Kakutani (1941)'s theorem to conclude existence.
Denote the Hessian on the payoff function of agent i with H_i. Using the Leibniz formula for determinants, the general formula for $\det(H_i)$ obtains as

$$\det(H_i) = \left(\prod_{j \in N_i} a_i^2 p_{ij}^{11} v_{ij}\right) - C''(X_i)\left(\sum_{j \in N_i} \prod_{l \neq j} a_i^2 p_{il}^{11} v_{il}\right). \tag{4}$$

Since this determinant of H_i is positive whenever d_i is even and negative whenever d_i is odd, and the signs of its principal minors are alternating,[35] the claim follows as this shows that H_i is negative definite. The payoff functions are concave and, thus, quasi-concave. □

Lemma 3 *In any pure strategy Nash Equilibrium we have* $x_{ij} \in [\epsilon_i^*, M_i]$ *for some finite* $M_i > 0$ *and some small* $\epsilon_i^* > 0$ *for all* $i, j \in \mathcal{I}$.

Proof We need to verify three claims.

Claim A *In every equilibrium, there exists a bound* M_i *for all* $i \in \mathcal{I}$ *such that for every* $j \in N_i$ *we have* $x_{ij} < M_i$.

Consider the highest possible revenue agent i can get from winning all of i's conflicts,

$$V_i := \sum_{j \in N_i} v_{ij}.$$

Agent i's investment levels are thus bounded by $M_i = C^{-1}(V_i)$, for otherwise, her payoff would be negative, and not investing would result in a higher payoff.

[35] These are obtained by removing the respective co-factors from the formula, deleting one negative factor from both multiplications. This changes the sign of the entire expression, as the resulting matrix changes from odd to even dimension or vice versa.

Claim B *Any strategy profile with* $x_{ij} = x_{ji} = 0$ *for any* $(ij) \in B$ *can never be an equilibrium.*

Suppose by contradiction that it can. Agent i's marginal payoff for these investments is given by

$$\frac{\delta}{\delta^2} = \frac{1}{\delta}.$$

Since her investment levels are bounded above by M_i, her highest marginal costs are $C'(M_i)$. We can thus always find a $\delta^* > 0$ such that for any $\delta \in (0, \delta^*)$ we have

$$\frac{1}{\delta} > C'(M_i).$$

Thus, there is a profitable deviation. A contradiction.

Claim C *Any strategy profile with* $x_{ij} > 0$ *and* $x_{ji} = 0$ *for any* $(ij) \in B$ *can never be a Nash Equilibrium.*

Suppose not and assume $x_{ji} = 0$. Let agent i's strategy profile be given by $\mathbf{x}_i = (x_{i1}, ..., x_{ij}, ..., x_{id_i})$ with $x_{ij} > 0$. For small $\delta > 0$, the probability of winning for i is close to 1. Now consider the alternative profile $\mathbf{x}_i'$ which is such that $x_{ij}' = x_{ij} - \epsilon > 0$. Costs have reduced and $p_{ij}' v_{ij} < p_{ij} v_{ij}$. Still, there exists a δ^{**} such that for any $\delta \in (0, \delta^{**})$ we have $p_{ij}' v_{ij} - C(X_i') > p_{ij} v_{ij} - C(X_i)$, thus constituting a profitable deviation. A contradiction. □

We can thus find some ϵ_i for each agent, such that $x_{ij} > \epsilon_i$ for all $j \in N_i$.

Lemma 4 *For every* $\boldsymbol{\omega} \in \Omega$ *we have* $\det(H) > 0$.

Proof Aside from the diagonal blocks, H is a sparse matrix with only one (potentially) non-zero element in each of the off-diagonal blocks (the cross-derivatives $a_i a_j p_{ji}^{21} v_{ij}$ and $a_i a_j p_{ij}^{12} v_{ji}$ in the symmetric pairs of blocks). The determinant can thus be expressed as the sum of the determinant of the diagonal matrix and the additional possible permutations with the respective rows containing these non-zero elements.[36] Each of these possible permutations that leaves a non-zero diagonal product is associated with one or more conflicts.

Let the set of all permutations and their combinations be denoted S_n with typical element ϕ. It contains all sets of additional row permutations that correspond to some set of conflicts $(ij) \in B$.[37] The signum function $\text{sgn}(\phi)$ is negative when $|\phi|$ is odd and positive when $|\phi|$ is even, where $|\phi|$ is the number of permutations in ϕ. Using the Leibniz formula for determinants, we get

$$\det(H) = \prod_{i \in \mathcal{I}} \det(H_i) + \sum_{\phi \in S_n} \text{sgn}(\phi) \prod_{(ij) \in \phi} - \left(a_i a_j p_{ij}^{12}\right)^2 v_{ij} v_{ji} \prod_{i \in \mathcal{I}} \det(H_i(\phi))$$

[36] For all other permutations, the summand to the determinant would vanish.

[37] When rows i and j are swapped, this is counted as one permutation.

$$= \prod_{i\in\mathcal{I}} \det(H_i) + \sum_{\phi\in S_n} \operatorname{sgn}(\phi)(-1)^{|\phi|} \prod_{(ij)\in\phi} \left(a_i a_j p_{ij}^{12}\right)^2 v_{ij} v_{ji} \prod_{i\in\mathcal{I}} \det(H_i(\phi))$$

$$= \prod_{i\in\mathcal{I}} \det(H_i) + \sum_{\phi\in S_n} \prod_{(ij)\in\phi} \left(a_i a_j p_{ij}^{12}\right)^2 v_{ij} v_{ji} \prod_{i\in\mathcal{I}} \det(H_i(\phi)). \tag{5}$$

We know that $\prod_{i\in\mathcal{I}} \det(H_i) > 0$ as all H_i associated with an even number of conflicts have $\det(H_i) > 0$ and all H_i associated with an odd number of conflicts have $\det(H_i) < 0$, due to negative definiteness. But since the total number of investment choices is $2b$, there must be an even number of the latter type of H_i. Thus, $\prod_{i\in\mathcal{I}} \det(H_i) > 0$ is true.[38]

Whenever we delete a leading cofactor (some row j and column j) of any H_i, the sign changes since two negative factors are deleted, but the number of elements in the product changes from odd to even or vice versa. Since the considered permutations always affect exactly two such Hessians, the sign of the last product in (5) cannot change for any possible non-zero permutation. Since $\left(\prod_{(ij)\in\phi} a_i a_j p_{ij}^{12}\right)^2 v_{ij} v_{ji} \geq 0$, this shows that $\det(H) > 0$ for all $\omega \in \Omega$. □

Lemma 5 *Every pure strategy Nash Equilibrium is locally unique, and the comparative statics at any such equilibrium are given by expression (2). Furthermore, for an open neighbourhood around any balanced parametrisation, there exists a globally unique, interior, pure-strategy Nash Equilibrium.*

Proof The first sentence follows immediately from applying the Implicit Function Theorem (IFT). Since $\det(H) > 0$ and thus $\det(H) \neq 0$, and F being continuously differentiable on $\mathbb{R}^{4b+n}$, it implies that the solution at any equilibrium is locally continuous in its parameters and that the derivatives are given by expression (2).

For the second part, we apply the results by Rosen (1965) and Goodman (1980) at an arbitrary equilibrium for a balanced parametrisation. The individual payoff functions are strictly concave in own strategies (H_i is negative definite for all $\mathbf{x} \in \mathbb{R}^{2b}_{+}$) and strictly convex in the strategies of others, as each payoff function for agent i is the sum of convex CSFs in $\mathbf{x}_{-i}$. Function (3) for $r_i = r$ for all $i \in \mathcal{I}$ can be rewritten as

$$\begin{aligned}
\sigma(\mathbf{x}, \mathbf{r}) &= \sum_{i\in\mathcal{I}} r p_{ij} v_{ij} - \sum_{i\in\mathcal{I}} r C(X_i) = r \sum_{i\in\mathcal{I}} p_{ij} v_{ij} - r \sum_{i\in\mathcal{I}} C(X_i) \\
&= r \sum_{(ij)\in B} \left(p_{ij} v_{ij} + (1 - p_{ij}) v_{ji}\right) - r \sum_{i\in\mathcal{I}} C(X_i) \\
&= r \sum_{(ij)\in B} v_{ji} + r \sum_{(ij)\in B} p_{ij}(v_{ij} - v_{ji}) - r \sum_{i\in\mathcal{I}} C(X_i) \\
&= r \sum_{(ij)\in B} v_{ji} - r \sum_{i\in\mathcal{I}} C(X_i),
\end{aligned}$$

where the last equality follows from the balance of conflict valuations ($v_{ij} = v_{ji}$ for all $(ij) \in B$) at any balanced parametrisation. Since the cost functions are strictly concave

[38] This is quintessentially the "handshaking lemma."

for every $\mathbf{x} \in \mathbb{R}_+^{2b}$ and the first term is a constant, this function is strictly concave. Thus, any balanced parametrisation has a globally unique equilibrium. Applying the IFT to any such equilibrium, using the above arguments about F and the determinant of its Jacobian H, it follows that there exists a neighbourhood of parameters for which a globally unique equilibrium exists. □

Lemma 6 *The number of pure strategy Nash Equilibria is finite.*

Proof Conflict investments are bounded. Since this is true for every investment level in every equilibrium, the set of equilibria is compact (closed and bounded in $\mathbb{R}^{2b}$). By means of contradiction, suppose the number of equilibria is infinite. We could thus construct a (non-trivial) sequence of equilibria in this set. Since the sequence is bounded, there exists a convergent subsequence, the limit of which lies in the set due to compactness. Since we are operating in a metric space, $\forall \epsilon > 0, \exists N(\epsilon)$ s.th. $\forall m, n > N(\epsilon), d(x_m, x_n) < \epsilon$. But since the limit point is a locally unique equilibrium-as it is part of the set for which $\det(H) > 0$-we would have found an isolated point that is also a limit point. A contradiction. Thus, there cannot exist an infinite number of pure strategy Nash Equilibria. □

A.2 Proof of proposition 2

Assume a symmetric equilibrium, $x_{ij} = x^s > 0$ for all $(ij) \in B$ and denote $X^s = dx^s$ the total conflict investment of each agent in such an equilibrium. Consider the condition equivalent to the FOCs in Equation (1) at such an equilibrium with any $\bar{\bar{\boldsymbol{\omega}}}$,

$$\frac{\bar{a} f'(\bar{a}x^s)}{4 f(\bar{a}x^s)} \bar{v} = C'(dx^s), \quad \forall i, j \in \mathcal{I} \text{ and } (ij) \in B.$$

If $x^s \to 0$, the left-hand side is large since $f'(0)$ is strictly positive and $f(0) = \delta$ for some arbitrarily small $\delta > 0$. The right-hand side approaches zero. If $x^s \to \infty$, the left-hand side approaches zero since $f'(\cdot)$ is positive but bounded and $f(\bar{a}x^s) \to \infty$. The right-hand side approaches infinity. Since all involved functions are continuous, there must exist a finite $x^s > 0$ such that the above equality holds. The Jacobian of this system contains the following elements,

$$\begin{aligned}
\frac{\partial F_{ij}}{\partial x_{ij}} &= a_i^2 p_{ij}^{11} v_{ij} - C''(X_i) < 0 \\
\frac{\partial F_{ij}}{\partial x_{ji}} &= a_i a_j p_{ij}^{12} v_{ij} \lesseqgtr 0 \\
\frac{\partial F_{ij}}{\partial x_{iq}} &= -C''(X_i) < 0 \\
\frac{\partial F_{ij}}{\partial x_{qi}} &= 0.
\end{aligned}$$

Due to Proposition 1, we know that this equilibrium is globally unique for a neighbourhood around the baseline parametrisation. Since investments are symmetric in each conflict, we have $p_{lq}^{12} = 0$ for all $(lq) \in B$.[39] This results in a block-diagonal matrix $D_x(F) = \text{diag}(A_1, A_2, ..., A_n)$ with $A_i = B_i + E_i$ where

$$B_i = \begin{pmatrix} z_{i1} & 0 & \cdots\cdots & 0 \\ 0 & z_{i2} & 0 \ \cdots & 0 \\ \vdots & & \ddots & \\ \vdots & & & \ddots \\ 0 & & & z_{iN} \end{pmatrix},$$

with $z_{ij} := a_i^2 p_{ij}^{11} v_{ij}$ and $E_i = [e]_{ql} = -C''(X_i)$ for all (ql). Note that in the equilibrium at any baseline parametrisation in a d-regular network, we have $z_{ij} = z_{ql} = z = \overline{a}^2 p_{ij}^{11} \overline{v}$.

The inverse of this matrix can be obtained by applying the Sherman-Morrison formula,

$$A_i^{-1} = \frac{1}{z} I - \frac{\frac{1}{z^2} E}{1 - \frac{1}{z} d_i C''(X^s)}.$$

In a more compact way, this is

$$A^{-1} = G = [g]_{l,q} = \begin{cases} \frac{z-(d-1)C''(X^s)}{z-dC''(X^s)} z^{-1} & \text{if } l = q \\ \frac{C''(X^s)}{z-dC''(X^s)} z^{-1} & \text{else.} \end{cases}$$

The partial effects $\frac{\partial \mathbf{x}}{\partial \boldsymbol{\omega}} = -\left[D_x(F)\right]^{-1} D_{\boldsymbol{\omega}}(F)$ are eventually given by

$$\frac{\partial x_{ij}}{\partial v_{ij}} = -\frac{z-(d-1)C''(X^s)}{z-dC''(X)} \frac{p^1}{\overline{a} p^{11} \overline{v}} > 0 \tag{6}$$

$$\frac{\partial x_{iq}}{\partial v_{ij}} = -\frac{C''(X^s)}{z-dC''(X^s)} \frac{p^1}{\overline{a} p^{11} \overline{v}} < 0 \qquad \text{for } q \neq j \tag{7}$$

$$\frac{\partial x_{ij}}{\partial a_i} = -\frac{1+z}{z-dC''(X^s)} \left(\frac{p^1}{\overline{a}^2 p^{11}} + \frac{x^s}{\overline{a}} \right) \gtrless 0. \tag{8}$$

Thus, at and close to $\overline{\overline{\boldsymbol{\omega}}}$, an increase in v_{ij} leads Agent i to increase the conflict investment against Agent j and to decrease the conflict investment against all Agents $q \neq j$. Conversely, a decrease in v_{ij} leads Agent i to decrease the conflict investment against Agent j and to increase the conflict investment against all Agents $q \neq j$. To consider a discrete change in v_{ij}, we apply second-order Taylor approximations to each type of investment. We denote each best response function as a nested function of the strategies that constitute the shortest path through the graph to a nonzero derivative.

39 The cross-derivatives are given by $p_{lq}^{12} = \frac{a_i a_j f'(a_i x_{ij}) f'(a_j x_{ji})(f(a_i x_{ij}) - f(a_j x_{ji}))}{(f(a_i x_{ij}) + f(a_j x_{ji}))^3}$.

In a slight abuse of notation, let us denote agent i's best response function on conflict (ij) as $x_{ij}(x_{ji}(v_{ji}))$.

Note that the change induced in a nested function $f(g(x))$ is less than that induced in $g(x)$, due to a change in x, whenever

$$\begin{aligned} & |Df(g(x))g'(x)| < |g'(x)| \\ \Leftrightarrow \quad & |Df(g(x))||g'(x)| < |g'(x)| \\ \Leftrightarrow \quad & |Df(g(x))| < 1. \end{aligned}$$

The changes we consider are either those affecting the rival in a conflict, $\frac{\partial x_{ij}}{\partial x_{ji}}$, which is zero at any baseline parametrisation and close to zero near it, and those that affect the decision in another conflict due to the cost function. We need the following lemma.

Lemma 7 *At any symmetric pure strategy Nash Equilibrium, the derivative* $\frac{\partial x_{ij}}{\partial x_{ik}}$ *for any* $i \in \mathcal{I}$ *and any* $j, k \in N_i$ *is given by* $-\frac{C''(X^s)}{(d-1)C''(X^s)-z}$.

Proof Fix some $i \in \mathcal{I}$ and some $k \in N_i$. Let $\tilde{F}_i = \{F_{il} | l \in N_i \wedge l \neq k\}$, $\tilde{\omega} = \{\omega, x_{ik}\}$ and $\tilde{F} = \{F_i, \{F_j | j \in \mathcal{I} \wedge j \neq i\}\}$. The Hessian $D_x(\tilde{F})$ has the same structure $D_x(F)$. Thus, $\tilde{A}^{-1}$ is the same as A^{-1} after deleting the principal minor associated with x_{ik}. Thus, in addition to the partial effects mentioned before for all x_{kl} such that $(kl) \neq (ik)$, we obtain from $\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{\boldsymbol{\omega}}} = -\left[D_{\tilde{x}}(\tilde{F})\right]^{-1} D_{\tilde{\omega}}(\tilde{F})$ for each $j \in N_i$ such that $j \neq k$

$$\begin{aligned} \frac{\partial x_{ij}}{\partial x_{ik}} &= -\left[-C''(X^s)\frac{z-(d-2)C''(X^s)}{z-(d-1)C''(X^s)}z^{-1} - (d-2)\frac{C''(X^s)^2}{z-(d-1)C''(X^s)}z^{-1}\right] \\ &= -\left[\frac{C''(X^s)}{z-(d-1)C''(X^s)} + \frac{(d-2)C''(X^s)^2}{z-(d-1)C''(X^s)}z^{-1}\right. \\ &\quad \left. - \frac{(d-2)C''(X^s)^2}{z-(d-1)C''(X^s)}z^{-1}\right] \\ &= -\frac{C''(X^s)}{(d-1)C''(X^s)-z}. \end{aligned}$$

□

Thus, we have

$$\left|\frac{\partial x_{ij}}{\partial x_{ik}}\right| = \left|\frac{C''(X_i)}{(d-1)C''(X_i) - a_1^2 p^{11} v_{ij}}\right| < 1.$$

This implies that any effect of a sufficiently small change in parameters from $\overline{\boldsymbol{\omega}}$ diminishes with increasing length of a path. Thus, for any effect of order $\mathfrak{k}$, there exists a neighbourhood around $\overline{\boldsymbol{\omega}}$ such that the effect dominates any effects of order higher than $\mathfrak{k}$.

The attacked agent decreases investment against i and increases it against all other agents $k \neq i, j$,

$$x_{ji}(x_{ij}(v_{ij})) \tilde{=} x^s + \frac{1}{2}\frac{\partial^2 x_{ji}}{(\partial x_{ij})^2}\left(\frac{\partial x_{ij}}{\partial v_{ij}}\right)^2 (v_{ij} - \overline{v})^2 < x^s$$

$$x_{jk}(x_{ji}(x_{ij}(v_{ij}))) \tilde{=} x^s + \frac{1}{2}\frac{\partial x_{jk}}{\partial x_{ji}}\frac{\partial^2 x_{ji}}{(\partial x_{ij})^2}\left(\frac{\partial x_{ij}}{\partial v_{ij}}\right)^2 (v_{ij} - \overline{v})^2 > x^s.$$

The other weak agents $k \neq i, j$ also decrease their investments against i and increase them against j and each other (for each k against all $l \neq k, i, j$),

$$x_{ki}(x_{ik}(v_{ij})) \tilde{=} x^s + \frac{1}{2}\frac{\partial^2 x_{ki}}{(\partial x_{ik})^2}\left(\frac{\partial x_{ik}}{\partial v_{ij}}\right)^2 (v_{ij} - \overline{v})^2 < x^s$$

$$x_{kj}(x_{ki}(x_{ik}(v_{ij}))) \tilde{=} x^s + \frac{1}{2}\frac{\partial x_{kj}}{\partial x_{ki}}\frac{\partial^2 x_{ki}}{(\partial x_{ik})^2}\left(\frac{\partial x_{ik}}{\partial v_{ij}}\right)^2 (v_{ij} - \overline{v})^2 > x^s$$

$$x_{kl}(x_{ik}(v_{ij})) \tilde{=} x^s + \frac{1}{2}\frac{\partial x_{kl}}{\partial x_{ki}}\frac{\partial^2 x_{ki}}{(\partial x_{ik})^2}\left(\frac{\partial x_{ik}}{\partial v_{ij}}\right)^2 (v_{ij} - \overline{v})^2 > x^s.$$

Note, that the sign of $(v_{ij} - \overline{v})^2$ is independent of whether $v_{ij} > \overline{v}$ or $v_{ij} < \overline{v}$. Thus, the results for all Agents $k, j \neq i$ apply both to an increase and a decrease in v_{ij}. □

A.3 Proof of proposition 3

For this proof, let us denote agent i's best response function on conflict (ij) as $x_{ij}(a_j x_{ji}(a_j))$. Note that $\frac{\partial x_{ij}}{\partial a_i} = 0$ implies $\frac{\partial(a_i x_{ij})}{\partial a_i} = \frac{\partial(a_i x_{ij})}{\partial a_i} + x_{ij} = x_{ij} > 0$. According to the partial derivative in (8), at $\overline{\overline{\boldsymbol{\omega}}}$, the derivative in a_i is generally ambiguous. Using a second-order Taylor approximation for a discrete step from $\overline{\overline{\boldsymbol{\omega}}}$, as in the previous proof, we see that i gets attacked less by all $j \neq i$,

$$\begin{aligned} x_{ji}(a_i x_{ij}(a_i)) &= x^s + \frac{1}{2}\frac{\partial^2 x_{ji}}{(\partial a_i x_{ij})^2}\left(\frac{\partial a_i x_{ij}}{\partial a_i}\right)^2 (a_i - \overline{a})^2 \\ &= x^s + \frac{1}{2}\frac{\partial^2 x_{ji}}{(\partial a_i x_{ij})^2}(x^s)^2 (a_i - \overline{a})^2 < x^s. \end{aligned}$$

Similarly, it follows that all other agents $j, k \neq i$ increase their investments against each other,

$$x_{jk}(x_{ji}(a_i x_{ij}(a_i))) = x^s + \frac{\partial x_{jk}}{\partial x_{ji}}\frac{\partial^2 x_{ji}}{(\partial a_i x_{ij})^2}(x^s)^2 (a_i - \overline{a})^2 > x^s.$$

Note, that the sign of $(a_i - \overline{a})^2$ is independent of whether $a_i > \overline{a}$ or $a_i < \overline{a}$. Thus, the results for all agents apply both to an increase and a decrease in a_i. □

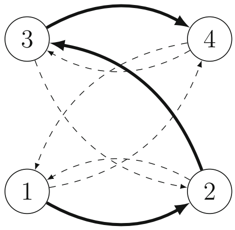


Fig. 6 Path from Agent 1 to Agent 4 in a bipartite network

A.4 Proof of proposition 4

Consider the path in the network of four.

If Agent 3's valuation of winning against Agent 4 v_{34} increases, there are two paths of length 2 leading to Agent 1. One such path can be described by the nesting

$$x_{12}(x_{21}(x_{23}(x_{32}(v_{34})))), \tag{9}$$

since we know from comparative statics that $\frac{\partial x_{32}}{\partial v_{34}}$ is nonzero and in fact negative. The other nesting is

$$x_{12}(x_{14}(x_{41}(x_{43}(x_{34}(v_{34}))))). \tag{10}$$

For ease of notation, let us rewrite function (11) as

$$f(g(h(k(x)))) := x_{12}(x_{21}(x_{23}(x_{32}(v_{34})))). \tag{11}$$

We will show that up to the fourth derivative of this function, all derivatives vanish. Then, we show that the fourth derivative is indeed negative. Applying the same technique using Taylor approximations as in the previous results yields the proof.
The first derivative of this function w.r.t. x is given by

$$f'g'h'k'(x) = 0,$$

since $f' = h' = 0$ because $f' = \frac{\partial x_{12}}{\partial x_{21}} = 0$ and $f' = \frac{\partial x_{23}}{\partial x_{32}} = 0$.
The second derivative w.r.t. x is given by

$$f''\left(g'\right)^2\left(h'\right)^2k'(x)^2 + f'g''\left(h'\right)^2k'(x)^2 + f'g'h''k'(x)^2 + f'g'h'k''(x) = 0,$$

again because because $f' = \frac{\partial x_{12}}{\partial x_{21}} = 0$ and $h' = \frac{\partial x_{23}}{\partial x_{32}} = 0$.
The third derivative w.r.t. x is then given by

$$f'''\left(g'\right)^3\left(h'\right)^3k'(x)^3 + f''2g'g''\left(h'\right)^3k'(x)^3 + f''\left(g'\right)^2 2h'h''k'(x)^3$$
$$+ f''\left(g'\right)^2\left(h'\right)^2 2k'(x)k''(x) +$$

$$\begin{aligned}&f''g'g''\left(h'\right)^3k'(x)^3+f'g'''\left(h'\right)^3k'(x)^3+f'g''2h'h''k'(x)^3\\&\quad+f'g''\left(h'\right)^22k'(x)k''(x)+\\&f''\left(g'\right)^2h''h'k'(x)^3+f'g''h'h''k'(x)^3+f'g'h'''k'(x)^3+f'g'h''2k'(x)k''(x)\\&f''\left(g'\right)^2\left(h'\right)^2k'(x)k''(x)+f'g''\left(h'\right)^2k'(x)k''(x)+f'g'h''k'(x)k''(x)\\&\quad+f'g'h'k'''(x)=0.\end{aligned}$$

We spare the reader from a sum with more than 64 elements by only considering elements from that third derivative, which do neither contain f' nor h' nor squares or cubics of those derivatives in the product since taking a derivative of these would only lead to elements containing f', h' or both. This leaves us with

$$f''\left(g'\right)^2 2h'h''k'(x)^3+f''\left(g'\right)^2h''h'k'(x)^3=3f''\left(g'\right)^2h'h''k'(x)^3.$$

The derivative of this element is thus identical to the fourth derivative of the nested function in Equation (11). It is given by

$$\begin{aligned}&3f'''\left(g'\right)^3\left(h'\right)^2h''k'(x)^4+3f''2g'g''\left(h'\right)^2h''k'(x)^4+3f''\left(g'\right)^2\left(h''\right)^2k'(x)^4\\&\quad+3f''\left(g'\right)^2h'h'''k'(x)^4+3f''\left(g'\right)^2h'h''3k'(x)^2k''(x)=3f''\left(g'\right)^2\left(h''\right)^2k'(x)^4.\end{aligned}$$

Plugging the original functions back in using Equation (11), we get

$$3\frac{\partial^2x_{12}}{(\partial x_{21})^2}\left(\frac{\partial x_{21}}{\partial x_{23}}\right)^2\left(\frac{\partial^2x_{23}}{(\partial x_{32})^2}\right)^2\left(\frac{\partial x_{32}}{\partial v_{34}}\right)^4<0,$$

which is the only nonzero element that would show up in a fourth-order Taylor approximation of $x_{12}(x_{21}(x_{23}(x_{32}(v_{34}))))$ around the symmetric equilibrium.
Repeating these steps for (10), we get

$$3\frac{\partial x_{12}}{\partial x_{14}}\frac{\partial^2x_{14}}{(\partial x_{41})^2}\left(\frac{\partial x_{41}}{\partial x_{43}}\right)^2\left(\frac{\partial^2x_{43}}{(\partial x_{34})^2}\right)^2\left(\frac{\partial x_{34}}{\partial v_{34}}\right)^4>0.$$

The fourth-order Taylor approximation of $x_{12}(v_{34})$ would thus be[40]

$$\begin{aligned}x_{12}(v_{34})\cong x_{12}(\bar v)+\frac{3}{4!}\Bigg(&\frac{\partial^2x_{12}}{(\partial x_{21})^2}\left(\frac{\partial x_{21}}{\partial x_{23}}\right)^2\left(\frac{\partial^2x_{23}}{(\partial x_{32})^2}\right)^2\left(\frac{\partial x_{32}}{\partial v_{34}}\right)^4\\&+\frac{\partial x_{12}}{\partial x_{14}}\frac{\partial^2x_{14}}{(\partial x_{41})^2}\left(\frac{\partial x_{41}}{\partial x_{43}}\right)^2\left(\frac{\partial^2x_{43}}{(\partial x_{34})^2}\right)^2\left(\frac{\partial x_{34}}{\partial v_{34}}\right)^4\Bigg)(v_{34}-\bar v)^4.\end{aligned}$$

40 The cross-derivatives of the two nestings are neglected as they all originate from $\frac{\partial x_{12}}{\partial x_{21}}$, which is zero at any symmetric equilibrium.

Since $(v_{34} - \bar{v})^4 > 0$, irrespective of whether we increase or decrease v_{34}, the sign of the resulting change is the same. Comparing the preceding two expressions in the parentheses, we see that x_{12} increases if and only if

$$\left|3\frac{\partial x_{12}}{\partial x_{14}}\frac{\partial^2 x_{14}}{(\partial x_{41})^2}\left(\frac{\partial x_{41}}{\partial x_{43}}\right)^2\left(\frac{\partial^2 x_{43}}{(\partial x_{34})^2}\right)^2\left(\frac{\partial x_{34}}{\partial v_{34}}\right)^4\right| > \left|3\frac{\partial^2 x_{12}}{(\partial x_{21})^2}\left(\frac{\partial x_{21}}{\partial x_{23}}\right)^2\left(\frac{\partial^2 x_{23}}{(\partial x_{32})^2}\right)^2\left(\frac{\partial x_{32}}{\partial v_{34}}\right)^4\right|.$$

Since at symmetry we have $\frac{\partial^2 x_{ij}}{(\partial x_{ji})^2} = \frac{\partial^2 x_{lq}}{(\partial x_{ql})^2}$ and $\frac{\partial x_{ij}}{(\partial x_{iq})^2} = \frac{\partial^2 x_{lk}}{(\partial x_{lh})^2}$ for any $(ij), (ql), (iq), (lk), (lh) \in B$, this reduces to

$$\left|\frac{\partial x_{12}}{\partial x_{14}}\left(\frac{\partial x_{34}}{\partial v_{34}}\right)^4\right| > \left|\left(\frac{\partial x_{32}}{\partial v_{34}}\right)^4\right|.$$

Using the explicit expressions from Proposition 1, this amounts to

$$\begin{aligned}
&\left|-\frac{C''(X^s)}{(d-1)C''(X^s)-z}\left(-\frac{z-(d-1)C''(X^s)}{z-dC''(X)}\frac{p^1}{\overline{a}p^{11}\overline{v}}\right)^4\right| \\
&> \left|\left(-\frac{C''(X^s)}{z-dC''(X^s)}\frac{p^1}{\overline{a}p^{11}\overline{v}}\right)^4\right| \\
&\Leftrightarrow \left|C''(X^s)\frac{\left(z-(d-1)C''(X^s)\right)^3}{(z-dC''(X))^4}\right| \\
&> \left|\left(\frac{C''(X^s)}{z-dC''(X^s)}\right)^4\right| \\
&\Leftrightarrow \left|\left(z-(d-1)C''(X^s)\right)^3\right| \\
&> \left|C''(X^s)^3\right| \\
&\Leftrightarrow \max\left\{\left(z-(d-1)C''(X^s)\right)^3, \left((d-1)C''(X^s)-z\right)^3\right\} \\
&> \max\left\{-C''(X^s)^3, C''(X^s)^3\right\} \\
&\Leftrightarrow (d-1)C''(X^s)-z > C''(X^s),
\end{aligned}$$

which is true. Thus, Agent 1 increases the investment against Agent 2 if v_{34} increases by some discrete but small amount starting from symmetry. Mirroring this procedure, we would arrive at the same condition for a decrease in x_{14} following a discrete but small increase in v_{34}.

Since the path length does not change when increasing the size of groups, the result applies to any complete bipartite network with two groups of equal size. □

A.5 Proof of proposition 5

The effect on x_{12}, resulting from a knock-on effect caused by an change in a_3 is given by the two terms

$$3\frac{\partial x_{12}}{\partial x_{14}}\frac{\partial^2 x_{14}}{(\partial x_{41})^2}\left(\frac{\partial x_{41}}{\partial x_{43}}\right)^2\left(\frac{\partial^2 x_{43}}{(\partial a_3 x_{34})^2}\right)^2\left(\frac{\partial a_3 x_{34}}{\partial a_3}\right)^4 > 0,$$

and

$$3\frac{\partial^2 x_{12}}{(\partial x_{21})^2}\left(\frac{\partial x_{21}}{\partial x_{23}}\right)^2\left(\frac{\partial^2 x_{23}}{(\partial a_3 x_{32})^2}\right)^2\left(\frac{\partial a_3 x_{32}}{\partial a_3}\right)^4 < 0.$$

Thus, the fourth-order Taylor approximation of $x_{12}(a_3)$ around $\bar{\bar{\omega}}$, is given by

$$x_{12}(a_3) \cong x_{12}(\bar{a}) + \frac{3}{4!}\left(\frac{\partial x_{12}}{\partial x_{14}}\frac{\partial^2 x_{14}}{(\partial x_{41})^2}\left(\frac{\partial x_{41}}{\partial x_{43}}\right)^2\left(\frac{\partial^2 x_{43}}{(\partial a_3 x_{34})^2}\right)^2\left(\frac{\partial a_3 x_{34}}{\partial a_3}\right)^4\right.$$
$$\left.+\frac{\partial^2 x_{12}}{(\partial x_{21})^2}\left(\frac{\partial x_{21}}{\partial x_{23}}\right)^2\left(\frac{\partial^2 x_{23}}{(\partial a_3 x_{32})^2}\right)^2\left(\frac{\partial a_3 x_{32}}{\partial a_3}\right)^4\right)(a_3 - \bar{a})^4.$$

As in the previous proof, as $(a_3 - \bar{a})^4 > 0$, irrespective of whether we increase or decrease a_3, the sign of the resulting change is the same. Comparing the summands in the parentheses preceding $(a_3 - \bar{a})^4 > 0$, we see that the negative effect is greater in magnitude than the positive effect if and only if

$$\left|-\frac{C''(X^s)}{(d-1)C''(X^s) - z}\right| < 1,$$

which is true.

In this case, the change in parameter a_3 affects all conflicts of Agent 3 symmetrically, so the effect also applies to x_{14}.

A.6 Proof of proposition 6

First, note that at any $\bar{\bar{\boldsymbol{\omega}}}$, in equilibrium, the partial first-order and second-order derivatives of a conflict investment with respect to a direct rival's (effective) investment are the same in each conflict. Thus, the indices can be used interchangeably.
From Proposition 3, we know that for all $m \neq i$ with $m \in S$

$$\Delta x_{mi} = x_{mi}(a_i) - x^s = \frac{1}{2}\frac{\partial^2 x_{mi}}{(\partial a_i x_{im})^2}\left(\frac{\partial a_i x_{im}}{\partial a_i}\right)^2(a_i - \overline{a})^2$$
$$= \frac{1}{2}\frac{\partial^2 x_{mi}}{(\partial x_{im})^2}(x^s)^2(a_i - \overline{a})^2.$$

This type of change (a direct rival of i reacting to i's increased strength) occurs d times.[41] Similarly, all d agents $m \neq i$ change their behaviour towards their $d-1$ rivals $q \neq i, m$ by

$$\Delta x_{mq} = x_{mq}(a_i) - x_{mq}(\overline{a}) = \frac{1}{2}\frac{\partial x_{mq}}{\partial x_{mi}}\frac{\partial^2 x_{mi}}{(\partial a_i x_{im})^2}(x^s)^2(a_i - \overline{a})^2.$$

Finally, agent i does not change conflict investments as previously shown. Thus, denoting $\Delta a_i = a_i - \overline{a}$, we have

$$\begin{aligned}\Delta X^s = &\frac{d}{2}\frac{\partial^2 x_{mi}}{(\partial a_i x_{im})^2}(x^s)^2\Delta a_i^2 \\ &+ \frac{d(d-1)}{2}\frac{\partial x_{mq}}{\partial x_{mi}}\frac{\partial^2 x_{mi}}{(\partial a_i x_{im})^2}(x^s)^2\Delta a_i^2.\end{aligned}$$

After factorising, we see that

$$\Delta X^s < 0 \Leftrightarrow \frac{d}{2}\frac{\partial^2 x_{mi}}{(\partial x_{im})^2}(x^s)^2\Delta a_i^2\left(1 + (d-1)\frac{\partial x_{mq}}{\partial x_{mi}}\right) < 0.$$

Since $\frac{\partial^2 x_{mi}}{(\partial x_{im})^2} < 0$, these inequalities hold whenever

$$\begin{aligned}&1 + (d-1)\frac{\partial x_{mq}}{\partial x_{mi}} \\ &\quad = 1 - (d-1)\frac{C''(X^s)}{(d-1)C(X^s) - a^2 p^{11} v} > 0 \\ &\quad \Rightarrow (d-1)C''(X^s) - a^2 p^{11} v - (d-1)C''(X^s) > 0 \\ &\quad \Leftrightarrow -a^2 p^{11} v > 0.\end{aligned}$$

This is a true statement, as the CSF is strictly concave. Note that, as long as the change in a_i is such that Proposition 3 holds, this result is independent of the magnitude and sign of Δa_i. □

A.7 Proof of proposition 7

Analogously to the previous proof, the total effect of an increase in v_{ij} on aggregate investments is thus given by

$$\underbrace{\left(\frac{\partial x_{ij}}{\partial v_{ij}} + (d-1)\frac{\partial x_{im}}{\partial v_{ij}}\right)\Delta v_{ij}}_{i'\text{s investment changes}}$$

[41] We could restrict this result to the clique S only, which would imply that this change only occurs s times. But since all agents connected to i are identical at $\overline{\overline{\omega}}$, this change also applies to the $d-s$ agents i and each $j \in S\backslash\{i\}$ are connected to beyond S.

$$+\underbrace{\frac{1}{2}\frac{\partial^2 x_{ji}}{(\partial x_{ij})^2}\left(\frac{\partial x_{ij}}{\partial v_{ij}}\right)^2\Delta v_{ij}^2+\frac{(d-1)}{2}\frac{\partial x_{jm}}{\partial x_{ji}}\frac{\partial^2 x_{ji}}{(\partial x_{ij})^2}\left(\frac{\partial x_{ij}}{\partial v_{ij}}\right)^2\Delta v_{ij}^2}_{j\text{'s investment changes}}$$
$$+\underbrace{\frac{d-2}{2}\frac{\partial^2 x_{mi}}{(\partial x_{im})^2}\left(\frac{\partial x_{im}}{\partial v_{ij}}\right)^2\Delta v_{ij}^2+\frac{(d-1)(d-2)}{2}\frac{\partial x_{mq}}{\partial x_{mi}}\frac{\partial^2 x_{mi}}{(\partial x_{im})^2}\left(\frac{\partial x_{im}}{\partial v_{ij}}\right)^2\Delta v_{ij}^2}_{\text{investment changes of all } m\neq i,j}. \tag{12}$$

The first row is positive.[42] We see that the third row is $d-2$ times the second row. The second row is proportional to

$$1+(d-1)\frac{\partial x_{jm}}{\partial x_{ji}},$$

which we know to be negative from the proof of Proposition 6. □

A.8 Proof of proposition 8

The comparative static in (8) provides us with the change in x_{ij} due to a unit increase in a_i. If this quantity is smaller than one, i.e.,

$$\left|\frac{\partial x_{ij}}{\partial a_i}\right| = \left|-\frac{1+z}{z-dC''(X^s)}\left(\frac{p^1}{\bar{a}^2 p^{11}}+\frac{x^s}{\bar{a}}\right)\right| < 1,$$

an increase in efficiencies increases the effective conflict investments of Agent i.[43] This is true for all impact functions of the type $f(ax)=(kax)^r$ for $k>0$ and $r\in(0,2)$. To see this, note that for $f(ax)=(kax)^r$ at symmetry, i.e., $a_j x_{ji}=a_i x_{ij}=\bar{a}x^s$, we have

$$p^1=\frac{f'(a_i x_{ij})f(a_j x_{ji})}{\big(f(a_i x_{ij})+f(a_j x_{ji})\big)^2}=\frac{f'(\bar{a}x^s)}{4f(\bar{a}x^s)}=\frac{r}{4\bar{a}x^s}$$

and

$$\begin{aligned}p^{11}&=\frac{f''(a_i x_{ij})f(a_j x_{ji})^2+f''(a_i x_{ij})f(a_i x_{ij})f(a_j x_{ji})-2f'(a_i x_{ij})^2 f(a_j x_{ji})}{\big(f(a_i x_{ij})+f(a_j x_{ji})\big)^3}\\&=\frac{f''(\bar{a}x^s)f(\bar{a}x^s)-f'(\bar{a}x^s)^2}{4f(\bar{a}x^s)^2}=\frac{(r-1)r(\bar{a}x^s)^{r-2}(\bar{a}x^s)^r-r^2(\bar{a}x^s)^{2(r-1)}}{4(\bar{a}x^s)^{2r}}\\&=\frac{-r}{4(\bar{a}x^s)^2}.\end{aligned}$$

[42] Note that our proposition is only a statement about the second and third lines of this expression (i.e., all agents other than i). We state the full sum here for completeness.

[43] You could also think of this as conflict investments being *inelastic* to changes in efficiency.

This implies $\frac{p^1}{a^2 p^{11}} = -\frac{4(x^s)^2}{4\bar{a}x^s} = -\frac{x^s}{\bar{a}}$ and thus

$$\left| -\frac{1+z}{z - dC''(X^s)} \left(\frac{p^1}{\bar{a}^2 p^{11}} + \frac{x^s}{\bar{a}} \right) \right| = 0.$$

Therefore, under the assumed impact function, Agent i's investments do not change following a change in a_i. Thus, Agent i's effective investment changes in each conflict with the same sign as the change in a_i. Also, we established, more generally, that all other agents reduce their investments against Agent i. However, this second-order change is small relative to that of Agent i since we are close to the symmetric equilibrium, where the slope of the best response functions of Agent i's rivals is close to zero. Since the CSF is increasing in own effective investment and decreasing in the rivals' effective investments, this means that all of Agent i's winning probabilities on all their conflicts in equilibrium change in the same direction as the change in a_i.

Since the problem is the same from the perspective of all Agents $k \neq i$, their change in investments is identical.[44] Thus, their increase in each conflict against all their rivals other than Agent i is the same, keeping winning probabilities constant at $\frac{1}{2}$. □

A.9 Proof of proposition 9

We know $x'_{ij} > \bar{\bar{x}}_{ij}$ and $x'_{ik} < \bar{\bar{x}}_{ik}$ for all $k \in N_i$ from Proposition 2. We have also established that second-order effects originate from the best responses to these changes in Agent i's conflict investments. Close to symmetry, the slope of the best response functions is zero, however. Thus, for small changes in x_{ij} and x_{ik} for all $k \in N_i$, the changes in x_{ji} and x_{ki} for all $k \in N_i$ are relatively small. Thus, $x'_{ij} > x'_{ji}$ and $x'_{ik} < x'_{ki}$ for all $k \in N_i$, which in turn implies $p'_{ij} > \bar{\bar{p}}_{ij}$ and $p'_{ik} < \bar{\bar{p}}_{ik}$. For a decrease in v_{ij}, the reverse argument holds.

To assess how p_{jk} changes, we need to compare the changes in x_{jk} and x_{kj}. From Proposition 3, we see that $x'_{jk} > x'_{kj}$ if

$$\begin{aligned}
& \frac{1}{2}\frac{\partial x_{jk}}{\partial x_{ji}} \frac{\partial^2 x_{ji}}{\left(\partial x_{ij}\right)^2} \left(\frac{\partial x_{ij}}{\partial v_{ij}}\right)^2 \Delta v_{ij} > \frac{1}{2}\frac{\partial x_{kj}}{\partial x_{ki}} \frac{\partial^2 x_{ki}}{(\partial x_{ik})^2} \left(\frac{\partial x_{ik}}{\partial v_{ij}}\right)^2 \Delta v_{ij} \\
\Leftrightarrow \quad & \left|\frac{\partial x_{ij}}{\partial v_{ij}}\right| > \left|\frac{\partial x_{ik}}{\partial v_{ij}}\right| \\
\Leftrightarrow \quad & \left| -\frac{z-(d-1)C''(X^s)}{z-dC''(X^s)} \frac{p^1}{\bar{a}p^{11}\bar{v}} \right| > \left| -\frac{C''(X^s)}{z-dC''(X^s)} \frac{p^1}{\bar{a}p^{11}\bar{v}} \right| \\
\Leftrightarrow \quad & (d-1)C''(X^s) - z > C''(X^s).
\end{aligned}$$

[44] More precisely, in case we had an asymmetric equilibrium with $d \geq 2$, it must be asymmetric among the agents other than Agent i, since we can pin down that agent's response due to Proposition 3. This would violate uniqueness, which we established close enough around any baseline parametrisation, however, because Agents $k \neq i$ are identical and thus interchangeable.

This holds for $d \geq 2$. Thus, $p'_{jk} > \bar{\bar{p}}_{jk}$. The reverse argument holds for a decrease in v_{ij}.

If $d \geq 4$ then $\exists k, q \neq i, j$ such that $(kq) \in B$. As in the previous proof, their conflict investments must change by the same quantity in response to asymmetry.[45] Thus, $p'_{kq} = \bar{\bar{p}}_{kq} = \frac{1}{2}$. □

B Two-agent contest with a lottery contest success function

To draw parallels between existing models of asymmetries in contests and our model, we revisit the framework and results of Baik (1994) here. This allows us to compare the results when we either support or hinder one of the conflicting parties. Consider a contest with two risk-neutral agents competing against each other to win a prize $v = 1$. Following the notation from Baik (1994), we define x_1 and x_2 as the conflict investment levels selected by Agent 1 and Agent 2, respectively. These conflict investment levels determine the probability of winning the prize. Let $p(x_1, x_2)$ be the probability that Agent 1 wins the prize, such that

$$p(x_1, x_2) = \frac{\sigma h(x_1)}{\sigma h(x_1) + h(x_2)},$$

where the parameter $\sigma > 1$ represents the relative *strength* of Agent 1 against Agent 2. Exerting conflict investment is costly; each agent faces a cost of conflict investment determined by a cost function $c(x_i) = x_i$. For ease of exposition of the main insights from Baik (1994), we consider the impact function $h(x) = x$. Then, our expected payoffs for each agent are

$$\pi_1 = \frac{\sigma x_1}{\sigma x_1 + x_2} - x_1 \quad \text{for Agent 1 and}$$
$$\pi_2 = \frac{x_2}{\sigma x_1 + x_2} - x_2 \quad \text{for Agent 2.}$$

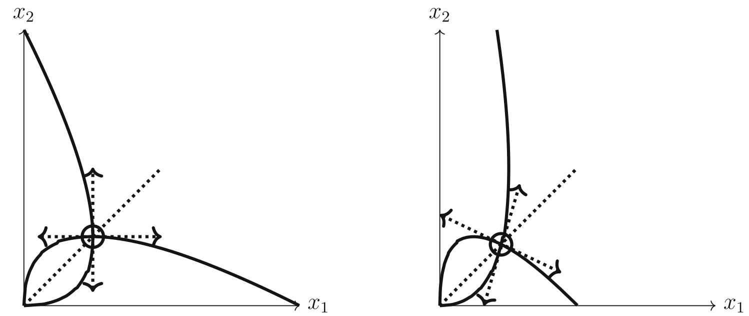
Each agent chooses a conflict investment level x_i such that their corresponding expected payoff is maximised. The first-order conditions obtain as

$$\frac{\sigma x_2}{(\sigma x_1 + x_2)^2} \overset{!}{=} 1 \quad \text{for Agent 1 and}$$
$$\frac{\sigma x_1}{(\sigma x_1 + x_2)^2} \overset{!}{=} 1 \quad \text{for Agent 2.}$$

Since the right-hand sides of these conditions are the same, this implies $x_1 = x_2$ in equilibrium.[46] By using these first-order conditions, we can also find the following

[45] Again, the existence of an asymmetric equilibrium would imply multiplicity of equilibria, which we ruled out earlier.

[46] Note that this symmetry in equilibrium investments is independent of σ. This is a discrepancy to our model that stems from the linear cost function used here for simplicity.



(a) Symmetric agents ($\sigma = 1$) where $(x_1^*, x_2^*) = (\frac{1}{4}, \frac{1}{4})$

(b) Asymmetric agents ($\sigma = 2$) where $(x_1^*, x_2^*) = (\frac{2}{9}, \frac{2}{9})$

Fig. 7 Best response functions in a two-agent contest

best response functions,

$$x_1 = \sqrt{\frac{x_2}{\sigma}} - \frac{x_2}{\sigma} \text{ and } x_2 = \sqrt{\sigma x_1} - \sigma x_1$$
$$\sigma x_1 = \sqrt{\sigma x_2} - x_2 \text{ and } x_2 = \sqrt{\sigma x_1} - \sigma x_1.$$

By taking a closer look at the best response functions when agents are either symmetric or asymmetric, we can observe some noticeable differences. Figure 7 depicts the best response functions when agents are symmetric in Panel (a) and when agents are asymmetric in Panel (b). In Panel (a), we observe that both best response functions intersect at the point where they attain their maximum value (i.e., the gradients are equal to zero in equilibrium; see the dotted lines with arrows in Panel (a)). At that point, both agents are best responding to each other, defining the Nash equilibrium strategies. Now, if we introduce asymmetry by making Agent 1 stronger (i.e., $\sigma = 2$), the best response functions do not intersect at their maximum anymore (i.e., the gradients are not equal to zero in equilibrium; see the dotted arrows in Panel (b)). The Nash equilibrium (x_1^*, x_2^*) efforts in this framework are

$$x_1^* = x_2^* = \frac{\sigma}{(1+\sigma)^2}.$$

In the two-agent contest with a lottery contest success function, the aggregate effort is

$$\mathcal{X} = x_1^* + x_2^* = \frac{2\sigma}{(1+\sigma)^2} \text{ where } \frac{\partial \mathcal{X}}{\partial \sigma} = \frac{2(1-\sigma^2)}{(1+\sigma)^4} < 0 \text{ as } \sigma > 1.$$

In this simple example, we observe that the introduction of asymmetries reduces the conflict investment level in the Nash equilibrium when compared to the symmetric case. This specific pattern was identified originally by Baik (1994) and has triggered subsequent studies examining the robustness of this result (see Dechenaux et al. (2015), for a comprehensive review from both a theoretical and experimental perspective).

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