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Computing perfect stationary equilibria in stochastic games

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Abstract

The notion of stationary equilibrium is one of the most crucial solution concepts in stochastic games. However, a stochastic game can have multiple stationary equilibria, some of which may be unstable or counterintuitive. As a refinement of stationary equilibrium, we extend the concept of perfect equilibrium in strategic games to stochastic games and formulate the notion of perfect stationary equilibrium (PeSE). To further promote its applications, we develop a differentiable homotopy method to compute such an equilibrium. We incorporate vanishing logarithmic barrier terms into the payoff functions, thereby constituting a logarithmic-barrier stochastic game. As a result of this barrier game, we attain a continuously differentiable homotopy system. To reduce the number of variables in the homotopy system, we eliminate the Bellman equations through a replacement of variables and derive an equivalent system. We use the equivalent system to establish the existence of a smooth path, which starts from an arbitrary total mixed strategy profile and ends at a PeSE. Extensive numerical experiments, including relevant applications like dynamic oligopoly models and dynamic legislative voting, further affirm the effectiveness and efficiency of the method.

Keywords Stochastic games · Stationary equilibria · Perfectness · Logarithmic barrier differentiable homotopy method

JEL Classification C02 · C72 · C73

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1 Introduction

Stochastic games, dating back to the seminal paper by Shapley (1953), serve as a powerful mechanism for strategic interaction analysis in a dynamic environment with conflicts of interests. Stochastic games model the dynamic interaction between a finite number of players. A stochastic game consists of a sequence of stages, where the relevant part of the history at the beginning of each stage is summarized by a commonly known state variable. More explicitly, at the beginning of the first stage, the players are in some given initial state. They take their actions simultaneously and independently. Subsequently, they get their instantaneous payoffs, and each player is informed of the others' actions at this stage. The game then moves to the next stage. Based on the previous state and action profile, a new state is selected, potentially in a probabilistic way. This process is repeated over an infinite number of stages. A stochastic game therefore consists of a series of stochastically generated stage games. Extensive applications of stochastic games can be found in the literature such as Chatterjee et al. (1993), Amir et al. (2003), Goldlücke and Kranz (2018), Manea (2018), and Okada (2023) and the references therein.

Subgame perfect equilibrium in stationary strategies (SSPE) is one of the essential solution concepts in stochastic games. A stationary strategy only depends on the current state rather than the entire history of states and strategy profiles. A stationary strategy thereby satisfies the reasonable principle of "letting bygones be bygones" (Maskin and Tirole 2001; Herings and Peeters 2004). The existence of SSPEs was discussed in Fink (1964), Takahashi (1964), and Sobel (1971), which provided a solid theoretical foundation for the development of stochastic games. He (2022) proved that an SSPE exists for stochastic games with discontinuous payoffs under the condition that a player can identify another action at the current stage with the payoff not much worse than her current one. The computation of SSPEs has been substantially studied in the literature as well. Herings and Peeters (2004) developed the first globally convergent method to compute SSPEs. To do so, they extended the linear tracing procedure of Harsanyi (1975) from strategic games to stochastic games. Since then, there has been more and more interest in the computation of SSPEs, witnessing the development of a Gaussian iterative method in Doraszelski and Pakes (2007), a piecewise smooth homotopy method in Govindan and Wilson (2009), a logit homotopy path-following method in Eibelshäuser and Poensgen (2019), an arbitrary starting linear tracing procedure in Li and Dang (2020), and an interior-point homotopy method in Dang et al. (2022).

The notion of SSPE is based on the assumption that the decision-makers are rational and never make mistakes. As pointed out in Selten (1975) and Myerson (1978), a strategic game can have multiple Nash equilibria, some of which may be unstable and inconsistent with our intuitive notions about a reasonable outcome of the game. To eliminate some of these counterintuitive Nash equilibria, Selten (1975) introduced a refinement of Nash equilibrium called perfect equilibrium and proved the existence of perfect equilibria in normal-form games. In an extensive-form game, a perfect equilibrium is robust against the introduction of mistakes by which every player chooses each action with a small strictly positive probability. The equivalence between the perfect equilibria in an extensive-form game with perfect recall and its corresponding agent normal-form game was established in Selten (1975) as well. For the class of extensive two-person games with perfect recall, van den Elzen and Talman (1991) presented a complementary pivoting algorithm that traces a piecewise linear path, thereby inducing a normal-form perfect equilibrium if the starting vector is a completely mixed strategy profile. Aiming at the same problem, von Stengel et al. (2002) developed a much more efficient method that is based on the sequence form. This method was proven to be tractable for larger-scale games.

For similar reasons as in strategic games, a stochastic game can have a vast multiplicity of SSPEs, many of which are unreasonable. However, due to the extremely complicated structure of stochastic games, studies on the refinement of SSPEs are scarce, and their computation has been neglected so far in the literature. proposed the concept of Markov Trembling Hand Perfect Equilibrium (MTHPE) to get rid of some counterintuitive equilibria and proved the existence of MTHPE for dynamic voting games. In this paper, we extend Selten's perfectness concept for strategic games to stochastic games and formulate the notion of perfect stationary equilibrium (PeSE), which is defined as the limit of SSPEs for a sequence of perturbed stochastic games.[1] A PeSE extends the notion of perfect equilibrium for extensive-form games to the class of stochastic games.

Computational tools play an important role in the application of stochastic games, but the computation of PeSEs has not been addressed in the literature so far. An obvious idea would be as follows: Compute an SSPE using the existing methods and then determine whether this SSPE satisfies the perfectness criterion. Unfortunately, such an approach was proven to be an NP-hard problem by Hansen et al. (2010). Another idea to find a PeSE is to straightforward follow its definition and compute the limit of equilibrium points for a sequence of perturbed stochastic games. Nevertheless, the efficiency of this approach very much depends on the sequence and underlying methods for computing the equilibrium points, which may lead to a huge computational burden, especially when the problem is large. It was illustrated in Dang et al. (2022) that the equilibrium system of stochastic games can be rewritten as a mixed complementarity problem (MCP) and solved by a widely used software package for MCPs—the PATH solver, which employs Newton method.[2] However, the PATH solver fails to compute PeSEs as it is not designed to compute equilibria of suitably perturbed problems and then take limits of such equilibria.

It has been shown in the literature that homotopy methods have a compelling performance in solving fixed points problems. Moreover, these methods have been shown to be effective in the computation of perfect equilibria for strategic games. Chen and Dang (2019) developed a simplicial homotopy method to approximate perfect equilibria for small-scale strategic games. Later, a differentiable homotopy method was developed in Chen and Dang (2021) to compute perfect equilibria for larger-scale strategic games. The latter homotopy follows a smooth path of solutions and shows a performance which is both very stable and efficient.

Inspired by the above successes, we aim to design a differentiable homotopy method to compute PeSEs for stochastic games. To accomplish this objective, we exploit a

1 Note that a Nash equilibrium in stationary strategies of the perturbed game is a subgame perfect equilibrium and so is the limit of a sequence of such equilibria.

2 Interest readers are referred to Dirkse and Ferris (1995) for more details about the path solver.

continuously differentiable function $\theta : [0, 1] \to [0, 1]$ of the homotopy variable $t \in [0, 1]$ which remains zero as long as t is not larger than a given positive number $\zeta_0/2$. With this function, we incorporate a logarithmic barrier term into the original stochastic game and formulate a logarithmic-barrier stochastic game, which continuously deforms a trivial game to the perturbed stochastic game of interest as t varies from one to $\zeta_0/2$. As t descends further from $\zeta_0/2$ to zero, the perturbations vanish and the perturbed stochastic games eventually reduce to the unperturbed stochastic game of interest at $t = 0$. A well-chosen transformation of variables addresses the inherent conflict between the interiority requirement of differentiable homotopies and the perfectness criterion. As a result, we establish an everywhere smooth homotopy path, which starts from an arbitrarily chosen totally mixed strategy profile and ends at a perfect stationary equilibrium for the stochastic game of interest.

We call the resulting method a logarithmic-barrier differentiable homotopy (LB-DH) method. The employment of the logarithmic-barrier term in the method restricts the path to the interior of the strategy space before $\theta(t)$ vanishes, which is inspired by interior-point methods and expected to significantly enhance the numerical efficiency. For numerical comparisons, we develop a convex-quadratic-penalty differentiable homotopy (CQP-DH) method, which is a direct stochastic extension of the method developed in Chen and Dang (2021) for strategic games and can be regarded as an exterior-point differentiable homotopy method. We have implemented the LB-DH and CQP-DH methods to solve extensive randomly generated stochastic games. To further elicit the effectiveness of the LB-DH method for selecting a particular SSPE satisfying the perfectness criterion, we have also compared the LB-DH method with two powerful homotopy methods for computing SSPEs—the IPM developed in Dang et al. (2022) and the stochastic linear tracing procedure (SLTP) studied in Herings and Peeters (2004) and Li and Dang (2020). Moreover, we have exploited the LB-DH method to solve several applications like dynamic oligopoly models with entry and exit and dynamic legislative bargaining games. Numerical results further confirm the effectiveness and efficiency of the LB-DH method.

The remainder of the paper is organized as follows. In Sect. 2, we discuss stochastic games and define the concept of perfect stationary equilibrium (PeSE). In Sect. 3, we develop the LB-DH method to compute PeSEs and prove the global convergence of the method. For numerical comparisons, we present the CQP-DH method in Sect. 4. Extensive numerical results are reported in Sect. 5. The paper is concluded in Sect. 6.

2 Stationary equilibria and perfectness

2.1 Stationary equilibria in stochastic games

To further elicit the criterion of perfectness, we briefly review the notions of stochastic games and subgame perfect equilibria in stationary strategies (SSPE) in this subsection.[3] A finite discounted stochastic game with infinitely many stages is given by

[3] This subsection present a brief review and some details are omitted. Interested readers are referred to Dang et al. (2022) for more details about stochastic games and stationary equilibria.

$$\Gamma = \langle N, \Omega, \{S^i_\omega\}_{(i,\omega)\in N\times\Omega}, \{u^i\}_{i\in N}, \pi, \delta\rangle,$$

where

- $N = \{1, 2, \ldots, n\}$ is the set of players.
- $\Omega = \{\omega_1, \omega_2, \ldots, \omega_d\}$ is the set of states.
- $S^i_\omega = \{s^i_{\omega j} : j \in M^i_\omega\}$ is the set of actions for player $i \in N$ in state $\omega \in \Omega$ with $M^i_\omega = \{1, 2, \ldots, m^i_\omega\}$.
- $S_\omega = \prod_{i=1}^n S^i_\omega$ is the set of action profiles in state $\omega \in \Omega$.
- $u^i : D \to \mathbb{R}$ is a real-valued function, describing the instantaneous payoff function of player $i \in N$, where $D = \{(\omega, s_\omega) : \omega \in \Omega, s_\omega \in S_\omega\}$.
- For any state $\omega \in \Omega$ and any action profile $s_\omega \in S_\omega$,

$$\pi(\omega, s_\omega) = (\pi(\omega_1 : \omega, s_\omega), \pi(\omega_2 : \omega, s_\omega), \ldots, \pi(\omega_d : \omega, s_\omega)) \in \mathbb{R}^d,$$

  where, for $k = 1, \ldots, d$, $\pi(\omega_k : \omega, s_\omega)$ is the probability that the system jumps to state $\omega_k \in \Omega$ when the current state is $\omega \in \Omega$ and the action profile is s_ω. It holds that $\sum_{k=1}^d \pi(\omega_k : \omega, s_\omega) = 1$.
- $\Pi(s) \in \mathbb{R}^{d\times d}$ is a matrix with row k equal to the row vector $\pi(\omega_k, s_{\omega_k})$, that is, $\Pi(s) = (\pi(\omega_k, s_{\omega_k}))_{\omega_k\in\Omega}$.
- δ is the discount factor with $0 < \delta < 1$, which is used to discount future instantaneous payoffs.

For $i \in N$ and $\omega \in \Omega$, by taking the mixed extension of the action space S^i_ω, each player $i \in N$ uses a mixed strategy $x^i_\omega = (x^i_{\omega 1}, \ldots, x^i_{\omega m^i_\omega})$, where $x^i_{\omega j}$ is the probability assigned to action $s^i_{\omega j} \in S^i_\omega$. We denote by $X^i_\omega = \{x^i_\omega \in \mathbb{R}^{m^i_\omega}_+ : \sum_{j\in M^i_\omega} x^i_{\omega j} = 1\}$ the set of all mixed strategies for player i in state ω. Let $X^i = \prod_{\omega\in\Omega} X^i_\omega$ and $X = \prod_{i\in N} X^i_\omega$. Let $m = \sum_{i\in N}\sum_{\omega\in\Omega} m^i_\omega$ denote the total number of actions over players and states.

We restrict ourselves to stationary strategies in this paper. Given a stationary strategy profile $x \in X$, we let $\mu^i_\omega(x)$ denote the total expected payoff for player i starting from state ω. Then, a standard argument as for instance in Li and Dang (2020) shows that $\mu^i := \mu^i(x) = (\mu^i_\omega(x) : \omega \in \Omega)$ is the unique solution to the following linear system,

$$\mu^i_\omega = u^i(\omega, x_\omega) + \delta \sum_{\bar\omega\in\Omega} \pi(\bar\omega : \omega, x_\omega)\mu^i_{\bar\omega}, \quad \omega \in \Omega, \tag{1}$$

which is the so-called Bellman equation. To simplify our notation, we define

$$\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) = u^i(\omega, s^i_{\omega j}, x^{-i}_\omega) + \delta \sum_{\bar\omega\in\Omega} \pi(\bar\omega : \omega, s^i_{\omega j}, x^{-i}_\omega)\mu^i_{\bar\omega}. \tag{2}$$

Clearly, $\mu^i_\omega = \sum_{j\in M^i_\omega} x^i_{\omega j}\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) := \varphi^i(\omega, x_\omega, \mu^i)$. It was proved in Li and Dang (2020) that if (x, λ, μ) is a solution to (3), then x is an SSPE of Γ.

Conversely, any SSPE x of Γ corresponds with a unique solution (x, λ, μ) to (3).

$$\begin{array}{ll} \varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) + \lambda^i_{\omega j} - \mu^i_\omega = 0, & j \in M^i_\omega,\ \omega \in \Omega,\ i \in N, \\ x^i_{\omega j} \geq 0,\ \lambda^i_{\omega j} \geq 0,\ \lambda^i_{\omega j} x^i_{\omega j} = 0, & j \in M^i_\omega,\ \omega \in \Omega,\ i \in N, \\ \sum\limits_{j \in M^i_\omega} x^i_{\omega j} - 1 = 0, & \omega \in \Omega,\ i \in N. \end{array} \tag{3}$$

2.2 Perfectness

As mentioned in Sect. 1, some SSPEs of a stochastic game may be counterintuitive. Let us present an example to illustrate (Osborne and Rubinstein 1994).

Example 1 Consider a stochastic game with $N = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2\}$. For $i = 1, 2$, $S^i_{\omega_1} = \{s^i_{\omega_1 1}, s^i_{\omega_1 2}, s^i_{\omega_1 3}\}$ and $S^i_{\omega_2} = \{s^i_{\omega_2 1}\}$. The payoff matrices are given by

ω_1	$s^2_{\omega_1 1}$	$s^2_{\omega_1 2}$	$s^2_{\omega_1 3}$
$s^1_{\omega_1 1}$	(0, 0)	(0, 0)	(0, 0)
$s^1_{\omega_1 2}$	(0, 0)	(1, 1)	(2, 0)
$s^1_{\omega_1 3}$	(0, 0)	(0, 2)	(2, 2)

and

ω_2	$s^2_{\omega_2 1}$
$s^1_{\omega_2 1}$	(0, 0)

.

The transition probabilities are given by $\pi(\bar{\omega} : \omega, s_\omega) = 0.5$, for any $\bar{\omega}, \omega \in \Omega$.

As shown in the matrices above, the stochastic game in this example has three SSPEs, $(s^1_{\omega_1 1}, s^2_{\omega_1 1}, s^1_{\omega_2 1}, s^2_{\omega_2 1})$, $(s^1_{\omega_1 2}, s^2_{\omega_1 2}, s^1_{\omega_2 1}, s^2_{\omega_2 1})$, and $(s^1_{\omega_1 3}, s^2_{\omega_1 3}, s^1_{\omega_2 1}, s^2_{\omega_2 1})$. Nonetheless, the SSPEs corresponding to the top-left and bottom-right cells are unattractive, since both the first and the last actions for both players are dominated by their second action. Indeed, if players tremble and play all their actions with strictly positive probability, then their second action yields a strictly higher payoff than both their first and their last action. Therefore, only $(s^1_{\omega_1 2}, s^2_{\omega_1 2}, s^1_{\omega_2 1}, s^2_{\omega_2 1})$ survives as a reasonable SSPE.

To address the above issue and eliminate some less plausible SSPEs, we extend the perfectness criterion for strategic games to stochastic games and formulate the notion of perfect stationary equilibrium, which is a strict refinement of SSPE.

Definition 1 For $\varepsilon > 0$, a totally mixed strategy profile $x \in X$ is an ε-perfect stationary equilibrium of Γ if for all$\omega \in \Omega$, $i \in N$ and $j, k \in M^i_\omega$, $\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i(x)) < \varphi^i(\omega, s^i_{\omega k}, x^{-i}_\omega, \mu^i(x))$ implies $x^i_{\omega j} \leq \varepsilon$. A strategy profile $x^* \in X$ is a **perfect stationary equilibrium (PeSE)** if there is a convergent sequence of ε_k-perfect stationary equilibria, $x(\varepsilon_k)$, $k = 1, 2, \ldots$, such that $\lim_{k\to\infty} x(\varepsilon_k) = x^*$ and $\lim_{k\to\infty} \varepsilon_k = 0$.

To establish the existence of a PeSE, we first define a perturbed stochastic game $\Gamma(\varepsilon)$ where all players choose each action with probability greater than or equal to ε. More formally, we have that

$$\Gamma(\varepsilon) = \langle N, \Omega, \{X^i_\omega(\varepsilon)\}_{(i,\omega)\in N\times\Omega}, \{u^i\}_{i\in N}, \pi, \delta\rangle,$$

where $X^i_\omega(\varepsilon) = \{x^i_\omega \in X^i_\omega : \text{for all } j \in M^i_\omega,\ x^i_{\omega j} \geq \varepsilon\}$. For notational convenience, we define $X(\varepsilon) = \prod_{i\in N}\prod_{\omega\in\Omega} X^i_\omega(\varepsilon)$. Notice that $\Gamma(0) = \Gamma$. We establish the following theorem.

Theorem 1 *Each SSPE of* $\Gamma(\varepsilon)$ *is an* ε*-perfect stationary equilibrium of* Γ.

Proof In $\Gamma(\varepsilon)$, for any strategy profile $\hat{x} \in X(\varepsilon)$, the optimal strategy of player $i \in N$ in state ω can be found as a solution to the following linear optimization problem,

$$\begin{aligned} \max_{x^i_\omega \in X^i_\omega} \quad & \sum_{j\in M^i_\omega} x^i_{\omega j}\varphi^i(\omega, s^i_{\omega j}, \hat{x}^{-i}_\omega, \hat{\mu}^i) \\ \text{s.t.} \quad & x^i_{\omega j} \geq \varepsilon, \quad j \in M^i_\omega, \\ & \sum_{j\in M^i_\omega} x^i_{\omega j} = 1. \end{aligned} \tag{4}$$

From a similar discussion as for the original game Γ, we obtain the equilibrium system for $\Gamma(\varepsilon)$ as follows:

$$\begin{aligned} & \varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) + \lambda^i_{\omega j} - \beta^i_\omega = 0, && j \in M^i_\omega,\ \omega\in\Omega,\ i \in N, \\ & x^i_{\omega j} \geq \varepsilon,\ \lambda^i_{\omega j} \geq 0,\ \lambda^i_{\omega j}(x^i_{\omega j} - \varepsilon) = 0, && j \in M^i_\omega,\ \omega\in\Omega,\ i \in N, \\ & \sum_{j\in M^i_\omega} x^i_{\omega j} - 1 = 0, && \omega\in\Omega,\ i \in N, \\ & \mu^i_\omega - \varphi^i(\omega, x_\omega, \mu^i) = 0, && \omega\in\Omega,\ i \in N. \end{aligned} \tag{5}$$

Any $x \in \mathbb{R}^m$ satisfying system (5) is an SSPE of the perturbed stochastic game $\Gamma(\varepsilon)$. Suppose that $\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) < \varphi^i(\omega, s^i_{\omega k}, x^{-i}_\omega, \mu^i)$. From the first group of equations of system (5), we know that $\lambda^i_{\omega j} > \lambda^i_{\omega k}$. From the condition that $\lambda^i_{\omega k} \geq 0$, we have that $\lambda^i_{\omega j}$ is strictly positive. It follows from the second group of equations in (5) that $x^i_{\omega j} = \varepsilon$, which shows that x is an ε-perfect stationary equilibrium of Γ. This completes the proof. □

The existence of stationary equilibria of $\Gamma(\varepsilon)$ implies the existence of ε-perfect equilibria of Γ by virtue of Theorem 1. Together with Definition 1, which defines a PeSE as a limit of a sequence of ε-perfect stationary equilibria of Γ, this ensures the existence of PeSEs for the stochastic game Γ. We obtain the following corollary.

Corollary 1 *The game* Γ *has a PeSE.*

In the next section, we exploit system (5) to develop an effective differentiable homotopy method, called the logarithmic barrier differentiable homotopy (LB-DH) method, and compute a PeSE for the stochastic game Γ. With a homotopy variable $t \in [0, 1]$, we formulate a continuously differentiable homotopy system, whose solution set contains an everywhere smooth path starting from an arbitrary interior point x^0 at $t = 1$. As t varies from a given positive number $\zeta_0/2 \in (0, 1)$ to zero, the path

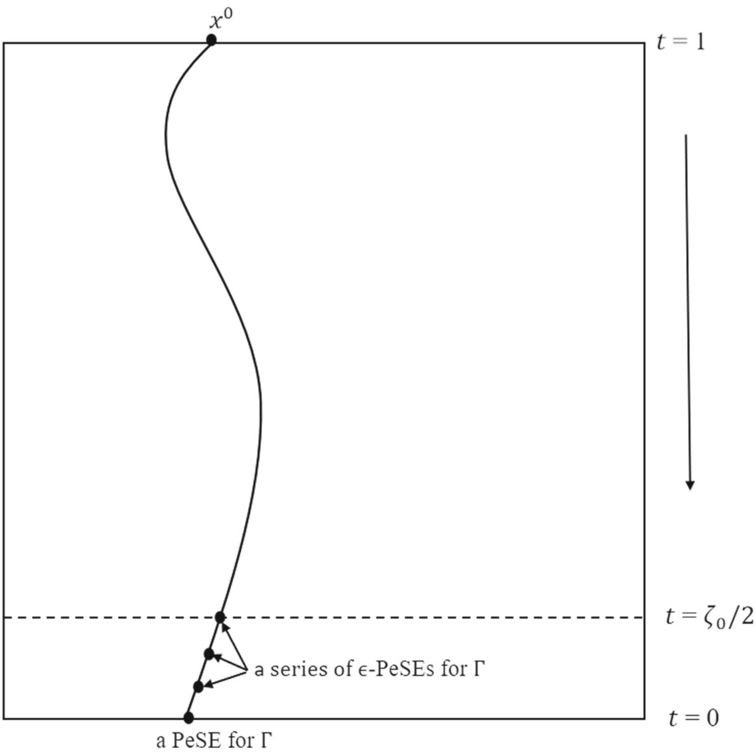

Fig. 1 A differentiable homotopy path

provides a series of $\varepsilon(t)$-perfect stationary equilibria for Γ. As t approaches zero, $\varepsilon(t)$ also goes to zero and according to Definition 1 the path eventually reaches a PeSE of Γ. Figure 1 illustrates how the homotopy works.

3 A logarithmic barrier differentiable homotopy method

As illustrated in the previous section, the homotopy variable t will descend from one to zero and generate an $\varepsilon(t)$-perfect stationary equilibrium for Γ when t is sufficiently small. Moreover, it holds that $\lim_{t\to 0}\varepsilon(t) = 0$. It is therefore convenient to let $\varepsilon(t) = t\eta_0$ in problem (4) with η_0 a given positive number satisfying $0 < \eta_0 < 1/\max_{\omega\in\Omega, i\in N} m^i_\omega$. Then system (5) becomes

$$\begin{array}{ll} \varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) + \lambda^i_{\omega j} - \beta^i_\omega = 0, & j \in M^i_\omega,\ \omega \in \Omega,\ i \in N, \\ x^i_{\omega j} \geq t\eta_0,\ \lambda^i_{\omega j} \geq 0,\ \lambda^i_{\omega j}(x^i_{\omega j} - t\eta_0) = 0, & j \in M^i_\omega,\ \omega \in \Omega,\ i \in N, \\ \sum_{j\in M^i_\omega} x^i_{\omega j} - 1 = 0, & \omega \in \Omega,\ i \in N, \\ \mu^i_\omega - \varphi^i(\omega, x_\omega, \mu^i) = 0, & \omega \in \Omega,\ i \in N. \end{array} \tag{6}$$

Now we want to eliminate the group of Bellman equations $\mu^i_\omega - \varphi^i(\omega, x_\omega, \mu^i) = 0$. It is obvious that the system (6) is equivalent to the following system,

$$\begin{aligned}
&\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) + \lambda^i_{\omega j} - (\nu^i_\omega + t\eta_0 \sum_{k\in M^i_\omega} \lambda^i_{\omega k}) = 0, \\
&\qquad\qquad\qquad\qquad j \in M^i_\omega,\ \omega \in \Omega,\ i \in N, \\
&x^i_{\omega j} \geq t\eta_0,\ \lambda^i_{\omega j} \geq 0,\ \lambda^i_{\omega j}(x^i_{\omega j} - t\eta_0) = 0,\ j \in M^i_\omega,\ \omega \in \Omega,\ i \in N, \\
&\sum_{j\in M^i_\omega} x^i_{\omega j} - 1 = 0,\ \omega \in \Omega,\ i \in N, \\
&\mu^i_\omega - \varphi^i(\omega, x_\omega, \mu^i) = 0,\ \omega \in \Omega,\ i \in N.
\end{aligned} \tag{7}$$

Multiplying the first group of equations by $x^i_{\omega j}$ and summing over $j \in M^i_\omega$ in system (7), we have that $\nu^i_\omega = \varphi^i(\omega, x_\omega, \mu^i)$, which implies that $\nu^i_\omega = \mu^i_\omega$. Consequently, system (6) is equivalent to the following system,

$$\begin{aligned}
&\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) + \lambda^i_{\omega j} - \mu^i_\omega - t\eta_0 \sum_{k\in M^i_\omega} \lambda^i_{\omega k} = 0, \\
&\qquad\qquad\qquad\qquad j \in M^i_\omega,\ \omega \in \Omega,\ i \in N, \\
&x^i_{\omega j} \geq t\eta_0,\ \lambda^i_{\omega j} \geq 0,\ \lambda^i_{\omega j}(x^i_{\omega j} - t\eta_0) = 0,\ j \in M^i_\omega,\ \omega \in \Omega,\ i \in N, \\
&\sum_{j\in M^i_\omega} x^i_{\omega j} - 1 = 0,\ \omega \in \Omega,\ i \in N.
\end{aligned} \tag{8}$$

Clearly, the perturbed stochastic game $\Gamma(t)$ coincides with the original stochastic game of interest Γ at $t = 0$.

Let

$$X^i_\omega(t) = \{x^i_\omega \in X^i_\omega : \text{ for every } j \in M^i_\omega,\ x^i_{\omega j} \geq t\eta_0\}$$

and $X(t) = \prod_{i\in N} \prod_{\omega\in\Omega} X^i_\omega(t)$. Clearly, the relative interior of $X(t)$ is non-empty. For further development, we make use of the following continuously differentiable function $\theta : [0, 1] \to [0, 1]$,

$$\theta(t) = \begin{cases} 0, & \text{if } t \leq \zeta_0/2, \\ \dfrac{1}{4}\dfrac{(2t-1)^2}{1-\zeta_0} + \dfrac{1}{2}(2t-1) + \dfrac{1}{4}(1-\zeta_0), & \text{if } \zeta_0/2 < t \leq 1 - \zeta_0/2, \\ 2t - 1, & \text{otherwise,} \end{cases} \tag{9}$$

where $\zeta_0 \in (0, 1)$. Obviously, $\theta(1) = 1$ and $\theta(t)$ remains equal to zero as soon as t is smaller than the given small positive number $\zeta_0/2$.[4]

[4] The formulation of the continuously differentiable function θ is not uniquely determined. We compared several possible formulations and find that the one proposed here achieves the highest numerical efficiency.

For $i \in N$, let $\mu^i = (\mu^i_\omega : \omega \in \Omega)$ be the unique solution to the linear system

$$\mu^i_\omega = (1-\theta(t))\varphi^i(\omega, x_\omega, \mu^i) + \theta(t)(1-\eta_0 m^i_\omega), \quad \omega \in \Omega. \tag{10}$$

Clearly, when $\theta(t) = 0$, (10) reduces to the Bellman equation (1). We use the function θ to incorporate a logarithmic barrier term into the objective function of the problem (4) and define an artificial stochastic game, in which for any strategy profile $\hat{x} \in X$, each player $i \in N$ in state $\omega \in \Omega$ solves the following strictly convex optimization problem,

$$\begin{aligned} \max_{x^i_\omega \in X^i_\omega(t)} \quad & (1-\theta(t)) \sum_{j \in M^i_\omega} x^i_{\omega j} \varphi^i(\omega, s^i_{\omega j}, \hat{x}^{-i}_\omega, \hat{\mu}^i) - \frac{1}{2} \sum_{j \in M^i_\omega} (x^i_{\omega j} - \hat{x}^i_{\omega j})^2 \\ & + \theta(t) \sum_{j \in M^i_\omega} (x^{0,i}_{\omega j} - \eta_0) \ln(x^i_{\omega j} - t\eta_0) \\ \text{s.t.} \quad & \sum_{j \in M^i_\omega} x^i_{\omega j} - 1 = 0, \end{aligned} \tag{11}$$

where $x^0 \in \text{Int}(X(1))$ is an arbitrarily given totally mixed strategy profile. The logarithmic term $\ln(x^i_{\omega j} - t\eta_0)$ enforces that $x^i_{\omega j} > t\eta_0$, that is, x is an interior point of the perturbed strategy space $X(t)$ before $\theta(t)$ vanishes. Note that the quadratic term $-(1/2)\sum_{j \in M^i_\omega} (x^i_{\omega j} - \hat{x}^i_{\omega j})^2$ in the objective function assures the strict concavity of the problem for any $t \in [0, 1]$.[5] The optimality conditions of the problem (11) are given by

$$\begin{aligned} & (1-\theta(t))\varphi^i(\omega, s^i_{\omega j}, \hat{x}^{-i}_\omega, \hat{\mu}^i) + \lambda^i_{\omega j} - \beta^i_\omega - (x^i_{\omega j} - \hat{x}^i_{\omega j}) = 0, \;\; j \in M^i_\omega, \\ & \lambda^i_{\omega j}(x^i_{\omega j} - t\eta_0) - \theta(t)(x^{0,i}_{\omega j} - \eta_0) = 0, \; \lambda^i_{\omega j} \geq 0, \; x^i_{\omega j} \geq t\eta_0, \quad j \in M^i_\omega, \\ & \sum_{j \in M^i_\omega} x^i_{\omega j} - 1 = 0. \end{aligned} \tag{12}$$

An application of the one-shot deviation principle together with $\hat{x} = x$ yields the equilibrium system for the artificial stochastic game,

$$\begin{aligned} & (1-\theta(t))\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) + \lambda^i_{\omega j} - \beta^i_\omega = 0, \; j \in M^i_\omega, \; \omega \in \Omega, \; i \in N, \\ & \lambda^i_{\omega j}(x^i_{\omega j} - t\eta_0) - \theta(t)(x^{0,i}_{\omega j} - \eta_0) = 0, \; \lambda^i_{\omega j} \geq 0, \; x^i_{\omega j} \geq t\eta_0, \\ & \qquad\qquad j \in M^i_\omega, \; \omega \in \Omega, \; i \in N, \\ & \sum_{j \in M^i_\omega} x^i_{\omega j} - 1 = 0, \; \omega \in \Omega, \; i \in N, \\ & \mu^i_\omega = (1-\theta(t))\varphi^i(\omega, x_\omega, \mu^i) + \theta(t)(1-\eta_0 m^i_\omega), \; \omega \in \Omega, \; i \in N. \end{aligned} \tag{13}$$

[5] With the extra term $-(1/2)\sum_{j \in M^i_\omega} (x^i_{\omega j} - \hat{x}^i_{\omega j})^2$, the mapping from the strategy space to the optimal solution set of the optimization problem (11) is a point-to-point continuous mapping. This extra term vanishes at a fixed point $x = \hat{x}$ in the equilibrium system.

Like before, we eliminate the Bellman equation in homotopy system (13). Replacing β^i_ω by $\nu^i_\omega + t\eta_0 \sum_{k\in M^i_\omega} \lambda^i_{\omega k}$ in system (13), we have

$$
\begin{aligned}
&(1-\theta(t))\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) + \lambda^i_{\omega j} - (\nu^i_\omega + t\eta_0 \sum_{k\in M^i_\omega} \lambda^i_{\omega k}) = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j \in M^i_\omega,\ \omega\in\Omega,\ i\in N, \\
&\lambda^i_{\omega j}(x^i_{\omega j} - t\eta_0) - \theta(t)(x^{0,i}_{\omega j} - \eta_0) = 0,\ \lambda^i_{\omega j} \ge 0,\ x^i_{\omega j} \ge t\eta_0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j \in M^i_\omega,\ \omega\in\Omega,\ i\in N, \\
&\sum_{j\in M^i_\omega} x^i_{\omega j} - 1 = 0,\ \omega\in\Omega,\ i\in N, \\
&\mu^i_\omega = (1-\theta(t))\varphi^i(\omega, x_\omega, \mu^i) + \theta(t)(1-\eta_0 m^i_\omega),\ \omega\in\Omega,\ i\in N.
\end{aligned}
$$

Multiplying the first group of equations by $x^i_{\omega j}$ and summing over j in the system above, one obtains that

$$\nu^i_\omega = (1-\theta(t))\varphi^i(\omega, x_\omega, \mu^i) + \theta(t)(1-\eta_0 m^i_\omega).$$

That is, $\nu^i_\omega = \mu^i_\omega$. The equilibrium system (13) is therefore equivalent to the following system,

$$
\begin{aligned}
&(1-\theta(t))\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) + \lambda^i_{\omega j} - \mu^i_\omega \\
&\qquad\qquad\qquad -t\eta_0 \sum_{k\in M^i_\omega} \lambda^i_{\omega k} = 0,\ j \in M^i_\omega,\ \omega\in\Omega,\ i\in N, \\
&\lambda^i_{\omega j}(x^i_{\omega j} - t\eta_0) - \theta(t)(x^{0,i}_{\omega j} - \eta_0) = 0, \qquad j \in M^i_\omega,\ \omega\in\Omega,\ i\in N, \\
&\lambda^i_{\omega j} \ge 0,\ x^i_{\omega j} \ge t\eta_0, \qquad j \in M^i_\omega,\ \omega\in\Omega,\ i\in N, \\
&\sum_{j\in M^i_\omega} x^i_{\omega j} - 1 = 0, \qquad \omega\in\Omega,\ i\in N,
\end{aligned}
\tag{14}
$$

which is a continuously differentiable system in $(x, \lambda, \mu, t) \in X \times \mathbb{R}^m \times \mathbb{R}^{nd} \times [0, 1]$. The elimination of the Bellman equation in the homotopy system has two advantages. On the one hand, it significantly reduces the number of variables. On the other hand, it substantially alleviates the non-linearity of the homotopy function. Therefore, it can improve the numerical efficiency of the proposed method. This improvement in efficiency becomes clear in the numerical part.

We observe from the second group of equations in system (14) that when $t \in (\zeta_0/2, 1]$, $\theta(t) > 0$ and $\lambda^i_{\omega j}(x^i_{\omega j} - t\eta_0) = \theta(t)(x^{0,i}_{\omega j} - \eta_0) > 0$, which indicates that the solutions to the system (14) always stay in the interior of the feasible set, that is, $x \in \mathrm{Int}(X(t))$ and $\lambda \in \mathbb{R}^m_{++}$. Note that when $t \le \zeta_0/2$, $\theta(t)$ becomes equal to zero and the system (14) becomes identical to the equilibrium system (8) for the perturbed stochastic game $\Gamma(t)$. We show that the set of solutions to system (14) identifies a series of $t\eta_0$-perfect stationary equilibria as t varies from η_0 to zero and yields a PeSE of Γ at $t = 0$.

The next lemma states that our system has a unique starting point at $t = 1$.

Lemma 1 *At $t = 1$, the system (14) has a unique solution.*

Proof Let $t = 1$. It follows that $\theta(t) = 1$, so system (14) reduces to

$$\begin{aligned}
&\lambda^i_{\omega j} - \mu^i_\omega - \eta_0 \sum_{k\in M^i_\omega} \lambda^i_{\omega k} = 0,\ \ j \in M^i_\omega,\ \omega \in \Omega,\ i \in N,\\
&\lambda^i_{\omega j}(x^i_{\omega j} - \eta_0) - (x^{0,i}_{\omega j} - \eta_0) = 0,\ \lambda^i_{\omega j} \ge 0,\ x^i_{\omega j} \ge \eta_0,\ \ j \in M^i_\omega,\ \omega \in \Omega,\ i \in N,\\
&\sum_{j\in M^i_\omega} x^i_{\omega j} - 1 = 0,\ \omega \in \Omega,\ i \in N.
\end{aligned} \tag{15}$$

It follows from (10) that $\mu^i_\omega = 1 - \eta_0 m^i_\omega$. Then the first group of equations becomes

$$\lambda^i_{\omega j} - 1 + \eta_0 m^i_\omega - \eta_0 \sum_{k\in M^i_\omega} \lambda^i_{\omega k} = 0, \quad j \in M^i_\omega,\ \omega \in \Omega,\ i \in N.$$

Summing over j in the system above, one obtains that $\sum_{k\in M^i_\omega} \lambda^i_{\omega k} = m^i_\omega$. By substituting the expressions for μ^i_ω and $\sum_{k\in M^i_\omega} \lambda^i_{\omega k}$ in the first group of equations in (14), we find that, for all $i \in N$, $\omega \in \Omega$, $j \in M^i_\omega$, $\lambda^i_{\omega j} = 1$. Substituting $\lambda^i_{\omega j} = 1$ into the second group of equations, we have that, for all $i \in N$, $\omega \in \Omega$, $j \in M^i_\omega$, $x^i_{\omega j} = x^{0,i}_{\omega j}$. □

Let $\sigma^i_\omega : X \times [0, 1] \to X^i_\omega$ be the unique solution to the strictly convex optimization problem (11) and let $\phi : X \times [0, 1] \to X$ be the product of σ^i_ω over all $i \in N$ and $\omega \in \Omega$, so $\phi(x, t)$ satisfies the optimality conditions of problem (11) for all players in all states. The function ϕ is obviously a continuous mapping on $X \times [0, 1]$. For what comes next, we need the following fixed point theorem (Browder 1960; Herings 2000).

Theorem 2 (Browder's fixed point theorem) *Let S be a non-empty, compact and convex subset of $\mathbb{R}^m$ and let $f : S \times [0, 1] \to S$ be a continuous function. Then the set $F = \{(x, t) \in S \times [0, 1] : f(x, t) = x\}$ contains a connected set F^c such that $F^c \bigcap (S \times \{0\}) \neq \emptyset$ and $F^c \bigcap (S \times \{1\}) \neq \emptyset$.*

We denote by $\widetilde{P}^{-1}$ the set of all $(x, t) \in X \times [0, 1]$ satisfying system (14). It follows from Brouwer's fixed point theorem that, for every $t \in [0, 1]$, $\phi(\cdot, t)$ has a fixed point in the non-empty compact convex set X. Clearly, as $\hat{x} = x$ at a fixed point, the two systems (12) and (14) have precisely the same solutions and therefore $\widetilde{P}^{-1}$ can be rewritten as

$$\widetilde{P}^{-1} = \{(x, t) \in X \times [0, 1] : x = \phi(x, t)\}.$$

Then, a direct application of Browder's fixed point theorem results in the following corollary.

Corollary 2 *The set* $\widetilde{P}^{-1}$ *contains a connected component that intersects both sets* $X \times \{0\}$ *and* $X \times \{1\}$.

Corollary 2 assures the global convergence of the LB-DH method. Since all equations in system (14) are polynomial, $\widetilde{P}^{-1}$ is a semi-algebraic set. Hence, the component in this corollary is actually path-connected. That is, any two points in the component can be joined by a path (Schanuel et al. 1991). This establishes the following corollary.

Corollary 3 *The set* $\widetilde{P}^{-1}$ *contains a path-connected component that intersects both sets* $X \times \{0\}$ *and* $X \times \{1\}$.

To design an effective and efficient method for computing a PeSE for the original stochastic game Γ, we need to construct an everywhere smooth path, where some regularity conditions are required to hold. Recall that when $t \in (\zeta_0/2, 1]$, $\theta(t) > 0$, and it is possible to verify that zero is a regular value of (14). However, when $t \in [0, \zeta_0/2]$, this regularity disappears, and a natural conflict occurs between the interior requirement of differentiable homotopies and the perfectness criterion. More specifically, when $t \in [0, \zeta_0/2]$, $\theta(t) = 0$, and the second group of equations to system (14) becomes a group of complementarity constraints, which are needed to establish ε-perfectness. Precisely because of these constraints, the Jacobian matrix of the equilibrium system (14) may become singular. To address this conflict, we make the following transformation of variables.[6] For $i \in N$, $\omega \in \Omega$, and $j \in M^i_\omega$, we write $x^i_{\omega j}$ and $\lambda^i_{\omega j}$ as functions of a new variable $z^i_{\omega j}$ and the homotopy variable t,

$$\begin{aligned} x^i_{\omega j}(z,t) &= t\eta_0 + \left(\frac{q^i_{\omega j}(z,t) + z^i_{\omega j}}{2}\right)^{\kappa}, \ j \in M^i_\omega, \ \omega \in \Omega, \ i \in N, \\ \lambda^i_{\omega j}(z,t) &= \left(\frac{q^i_{\omega j}(z,t) - z^i_{\omega j}}{2}\right)^{\kappa}, \qquad j \in M^i_\omega, \ \omega \in \Omega, \ i \in N, \end{aligned} \tag{16}$$

where

$$q^i_{\omega j}(z,t) = \sqrt{(z^i_{\omega j})^2 + 4(\theta(t)(x^{0,i}_{\omega j} - \eta_0))^{1/\kappa}}$$

and $\kappa > 2$. This ensures the differentiability of system (16). The definitions in (16) guarantee that the second group of equations in system (14) automatically hold. By substituting (16) into system (14), one obtains the following system,

$$\begin{aligned} &(1-\theta(t))\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega(z,t), \mu^i) + \lambda^i_{\omega j}(z,t) - \mu^i_\omega \\ &\qquad\qquad -t\eta_0 \sum_{k \in M^i_\omega} \lambda^i_{\omega k}(z,t) = 0, \ j \in M^i_\omega, \ \omega \in \Omega, \ i \in N, \\ &\sum_{j \in M^i_\omega} x^i_{\omega j}(z,t) - 1 = 0, \ \omega \in \Omega, \ i \in N. \end{aligned} \tag{17}$$

[6] Related transformations of variables have been frequently used in the literature such as Herings and Peeters (2001), Herings and Schmedders (2006), and Chen and Dang (2021).

For any $t \in [0, 1]$, let $p(z, \mu, t)$ denote the left-hand side of system (17). The set of solutions to system (17) is given by

$$P^{-1} = \{(z, \mu, t) \in \mathbb{R}^m \times \mathbb{R}^{nd} \times [0, 1] : \ p(z, \mu, t) = 0\}.$$

The next lemma demonstrates that zero is a regular value of the homotopy system (17) at the starting level $t = 1$.

Lemma 2 *At $t = 1$, system (17) has a unique solution. Moreover, zero is a regular value of p on $\mathbb{R}^m \times \mathbb{R}^{nd} \times \{1\}$.*

Proof At $t = 1$, the unique solution to system (14) pins down a unique value of $z^i_{\omega j}$ for any $i \in N$, $\omega \in \Omega$, and $j \in M^i_\omega$, which is strictly positive and given by $(x^{0,i}_{\omega j} - \eta_0)^{1/\kappa} - 1$. We prove in "Appendix II" that the Jacobian matrix of p at $(z, \mu, 1) \in \mathbb{R}^m \times \mathbb{R}^{nd} \times \{1\}$ such that $p(z, \mu, 1) = 0$ is of full rank. Therefore, zero is a regular value of p on $\mathbb{R}^m \times \mathbb{R}^{nd} \times \{1\}$. □

The following theorem provides conditions such that the set of solutions to system (17) contains a smooth path leading to a perfect stationary equilibrium of the stochastic game Γ.

Theorem 3 *Suppose zero is a regular value of p on $\mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1)$. Then $P^{-1} \cap \mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1]$ is a smooth one-dimensional manifold with boundary. Moreover, P^{-1} connects the unique solution at $t = 1$ to a perfect stationary equilibrium of the stochastic game Γ at $t = 0$.*

Proof We first prove that the variables z and μ are uniquely determined for a given value of x. From the first group of equations in system (16), for any $i \in N$, $\omega \in \Omega$, and $j \in M^i_\omega$, $x^i_{\omega j}(z, t)$ is a strictly increasing function of $z^i_{\omega j}$, since the derivative of $x^i_{\omega j}$ with respect to $z^i_{\omega j}$ is positive. That is, any given $x^i_{\omega j}$ determines a unique value of $z^i_{\omega j}$. The second group of equations in (16) pins down a unique value of $\lambda^i_{\omega j}$ for any value of $z^i_{\omega j}$. Next, the first group of equations in (17) determines μ^i_ω uniquely given any value of $z^i_{\omega j}$. All the above results together with the compactness of the strategy space X lead to the compactness of the solution set P^{-1}. From a discussion similar to the one preceding Corollary 3, we find that P^{-1} has a path-connected component that intersects both sets $\mathbb{R}^m \times \mathbb{R}^{nd} \times \{1\}$ and $\mathbb{R}^m \times \mathbb{R}^{nd} \times \{0\}$. We have proved in Lemma 1 that system (14) has a unique solution at $t = 1$. Therefore, system (17) also has a unique solution at $t = 1$. Lemma 2 and the assumption that "zero is a regular value of p on $\mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1)$" ensure that $P^{-1} \cap \mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1]$ is a smooth one-dimensional manifold with boundary. The path-connected component in P^{-1} which starts from the unique point on the level of $t = 1$ and ends at a point on the level of $t = 0$. We derive from Definition 1 and Theorem 1 that the first point reached by the path at $t = 0$ is a perfect stationary equilibrium of the stochastic game Γ. □

Now we want to get rid of the assumption that "zero is a regular value of p on $\mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1)$" in Theorem 3. A general approach is to add a perturbation term

$-t(1-t)\gamma$ to system (17), where $\gamma \in \mathbb{R}^m$ with $\|\gamma\|$ sufficiently small. In this way we obtain a slightly modified homotopy system,

$$\begin{aligned}&(1-\theta(t))\varphi^i(\omega, s^i_{\omega j}, x^{-i}_{\omega}(z,t), \mu^i) + \lambda^i_{\omega j}(z,t) - \mu^i_{\omega}\\ &\quad -t\eta_0 \sum_{k\in M^i_{\omega}} \lambda^i_{\omega k}(z,t) - t(1-t)\gamma^i_{\omega j} = 0,\ \ j \in M^i_{\omega},\ \omega \in \Omega,\ i \in N,\\ &\sum_{j\in M^i_{\omega}} x^i_{\omega j}(z,t) - 1 = 0,\ \omega \in \Omega,\ i \in N.\end{aligned} \tag{18}$$

Clearly, the two systems (17) and (18) are identical when $t = 1$ or $t = 0$.[7] Let $p(z, \mu, t; \gamma)$ denote the left-hand side of system (18). For any fixed $\gamma \in \mathbb{R}^m$, we let $p_\gamma(z, \mu, t) = p(z, \mu, t; \gamma)$ and denote the set of solutions to system (18) by

$$P^{-1}_{\gamma} = \{(z, \mu, t) \in \mathbb{R}^m \times \mathbb{R}^{nd} \times [0, 1] :\ p_\gamma(z, \mu, t) = 0\}.$$

Clearly, $p(z, \mu, t; \gamma)$ is continuously differentiable and

$$\lim_{\|\gamma\|\to 0} p(z, \mu, t; \gamma) = p(z, \mu, t; 0) = p_0(z, \mu, t) = p(z, \mu, t).$$

The set P^{-1}_{γ} also contains a path-connected component connecting the unique starting point at $t = 1$ to an SSPE at $t = 0$. For a generic choice of γ, the regularity condition of Theorem 3 is satisfied and we obtain the following theorem.

Theorem 4 *For a generic choice of* $\gamma \in \mathbb{R}^m$, $P^{-1}_{\gamma} \cap \mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1]$ *is a smooth one-dimensional manifold with boundary. Moreover,* P^{-1}_{γ} *connects the unique solution at* $t = 1$ *to an SSPE of the stochastic game* Γ *at* $t = 0$.

Proof Using the same argument as before, one can show that P^{-1}_{γ} contains a path-connected component that intersects both sets $\mathbb{R}^m \times \mathbb{R}^{nd} \times \{1\}$ and $\mathbb{R}^m \times \mathbb{R}^{nd} \times \{0\}$. At both $t = 0$ and $t = 1$, the perturbation term $t(1-t)\gamma$ vanishes and $p_\gamma(z, \mu, t) = p(z, \mu, t)$. Any point in P^{-1}_{γ} with $t = 0$ is therefore an SSPE of Γ. It has been proved in Lemma 2 that the solution to $p(z, \mu, 1) = 0$ is unique and that 0 is a regular value of p on $\mathbb{R}^m \times \mathbb{R}^{nd} \times \{1\}$. Hence, the path-connected component in P^{-1}_{γ} intersecting $t = 1$ also starts from this unique solution. We prove in "Appendix III" that zero is a regular value of $p(z, \mu, t; \gamma)$ on $\mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1) \times \mathbb{R}^m$. From the well-known transversality theorem, together with the result of Lemma 2, we obtain that zero is also a regular value of $p_\gamma(z, \mu, t)$ on $\mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1]$ for almost all $\gamma \in \mathbb{R}^m$.[8] It follows that for almost all $\gamma \in \mathbb{R}^m$, $P^{-1}_{\gamma} \cap \mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1]$ is a smooth one-dimensional manifold with boundary. □

Thus far, we have proved that the solution set to (18) contains an everywhere smooth path starting at $t = 1$. If system (18) would resume to system (17) when $t \leq \zeta_0$, then

[7] The perturbation term is used to generically rule out degeneracies and is always set to zero in numerical implementations.

[8] The transversality theorem is presented in "Appendix I".

this path yields a sequence of $t\eta_0$-perfect stationary equilibria for Γ, which has a PeSE as its limit for $t \to 0$. Nonetheless, the perturbation term $-t(1-t)\gamma$ will not completely vanish before t is equal to zero, which yields a concern that the end point of the path is not a perfect stationary equilibrium. Theorem 5 addresses this concern.

For every $t \in (0, 1]$, we define $\Xi_t = \{(z, \mu, t') \in P^{-1} : t' = t\}$ and, for every $\gamma \in \mathbb{R}^m$, $\Xi_{\gamma,t} = \{(z, \mu, t') \in P_\gamma^{-1} : t' = t\}$. For every $t \in (0, 1]$, for every $\iota_{\gamma,t} \in \Xi_{\gamma,t}$, the distance between the point $\iota_{\gamma,t}$ and the set Ξ_t is denoted by

$$d(\iota_{\gamma,t}, \Xi_t) = \min_{\iota_t \in \Xi_t} \|\iota_t - \iota_{\gamma,t}\|.$$

With the above notations, we present Theorem 5.

Theorem 5 *For every $t \in (0, 1]$, for any $\epsilon > 0$, there exists a $\delta_0 > 0$ such that, for every $\gamma \in \mathbb{R}^m$ with $\|\gamma\| < \delta_0$, for every $\iota_{\gamma,t} \in \Xi_{\gamma,t}$, $d(\iota_{\gamma,t}, \Xi_t) < \epsilon$.*

Proof Let $t \in (0, 1]$. We prove the theorem by contradiction. Suppose there exists an $\epsilon_0 > 0$, and a convergent sequence $\{\gamma^k\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \gamma^k = 0$ and a sequence $\{\iota_{\gamma^k,t}\}_{k\in\mathbb{N}}$ in $\Xi_{\gamma^k,t}$ such that, for every $k \in \mathbb{N}$, $d(\iota_{\gamma^k,t}, \Xi_t) \geq \epsilon_0$. Since the sequence $\{\iota_{\gamma^k,t}\}_{k\in\mathbb{N}}$ is bounded, without loss of generality, one can assume it is convergent with the limit, say, ι_t^*. It follows from the continuity of p that

$$0 = \lim_{k\to\infty} p(\iota_{\gamma^k,t}; \gamma^k) = p(\iota_t^*; 0),$$

so $\iota_t^* \in \Xi_t$. We therefore have

$$0 < \epsilon \leq \lim_{k\to\infty} d(\iota_{\gamma^k,t}, \Xi_t) = d(\iota_t^*, \Xi_t) = 0,$$

a contradiction. This completes the proof. □

Theorem 5 confirms that for every $t \in (0, 1]$, the perturbed path in P_γ^{-1} is arbitrarily close to the path in P^{-1} that leads to a PeSE for the stochastic game of interest. Therefore, the perturbed path in P_γ^{-1} leads to an approximate perfect stationary equilibrium for the original stochastic game Γ. With the above results, we establish the following corollary.

Corollary 4 *For a generic choice of $\gamma \in \mathbb{R}^m$ with $\|\gamma\|$ sufficiently small, there exists an everywhere smooth path in P_γ^{-1}, which starts from an arbitrary point at $t = 1$ and provides an approximate perfect stationary equilibrium for the stochastic game Γ as t approaches zero.*

4 A convex quadratic penalty homotopy method

For numerical comparisons, we develop a convex-quadratic-penalty differentiable homotopy (CQP-DH) method in this section. Let the perturbed strategy space $X(t)$ and continuously differentiable function $\theta(t)$ be defined as in Sect. 3. For any stationary strategy profile $\hat{x} \in X(t)$, we incorporate with $\theta(t)$ a convex quadratic penalty term into the perturbed stochastic game and construct an artificial penalty stochastic game, in which any player $i \in N$ solves the following optimization problem in state $\omega \in \Omega$.

$$\begin{aligned}\max_{x^i_\omega \in X^i_\omega} \; & (1-\theta(t)) \sum_{j \in M^i_\omega} x^i_{\omega j} \varphi^i(\omega, s^i_{\omega j}, \hat{x}^{-i}_\omega, \hat{\mu}^i) \\ & -\frac{\theta(t)}{2} \sum_{j \in M^i_\omega} (x^i_{\omega j} - x^{0,i}_{\omega j})^2 - \frac{1}{2} \sum_{j \in M^i_\omega} (x^i_{\omega j} - \hat{x}^i_{\omega j})^2 \\ \text{s.t.} \; & x^i_{\omega j} \geq t\eta_0, \quad j \in M^i_\omega \\ & \sum_{j \in M^i_\omega} x^i_{\omega j} - 1 = 0,\end{aligned} \tag{19}$$

where $x^0 \in \text{Int}(X(1))$ is an arbitrarily given totally mixed strategy profile, and $\hat{\mu}^i = (\hat{\mu}^i_\omega : \omega \in \Omega)$ is the unique solution to the linear system

$$\hat{\mu}^i_\omega = (1-\theta(t))\varphi^i(\omega, \hat{x}_\omega, \hat{\mu}^i) - \theta(t) \sum_{j \in M^i_\omega} \hat{x}^i_{\omega j}(\hat{x}^i_{\omega j} - x^{0,i}_{\omega j}), \quad \omega \in \Omega. \tag{20}$$

Then, from a similar discussion as in Sect. 3, one can formulate the equilibrium system for this stochastic game with quadratic penalty terms, which is given by

$$\begin{aligned}& (1-\theta(t))\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega, \mu^i) - \theta(t)(x^i_{\omega j} - x^{0,i}_{\omega j}) + \lambda^i_{\omega j} \\ & \qquad -\mu^i_\omega - t\eta_0 \sum_{k \in M^i_\omega} \lambda^i_{\omega k} = 0, \; j \in M^i_\omega, \; \omega \in \Omega, \; i \in N, \\ & x^i_{\omega j} \geq t\eta_0, \; \lambda^i_{\omega j} \geq 0, \; \lambda^i_{\omega j}(x^i_{\omega j} - t\eta_0) = 0, \; j \in M^i_\omega, \; \omega \in \Omega, \; i \in N, \\ & \sum_{j \in M^i_\omega} x^i_{\omega j} - 1 = 0, \; \omega \in \Omega, \; i \in N.\end{aligned} \tag{21}$$

Lemma 3 *At $t = 1$, the system (21) has a unique solution.*

Proof When $t = 1$, $\theta(t) = 1$ and the system (21) reduces to

$$\begin{aligned}& -(x^i_{\omega j} - x^{0,i}_{\omega j}) + \lambda^i_{\omega j} - \mu^i_\omega - \eta_0 \sum_{k \in M^i_\omega} \lambda^i_{\omega k} = 0, \; j \in M^i_\omega, \; \omega \in \Omega, \; i \in N, \\ & x^i_{\omega j} \geq \eta_0, \; \lambda^i_{\omega j} \geq 0, \lambda^i_{\omega j}(x^i_{\omega j} - \eta_0) = 0, \qquad j \in M^i_\omega, \; \omega \in \Omega, \; i \in N, \\ & \sum_{j \in M^i_\omega} x^i_{\omega j} - 1 = 0, \qquad \omega \in \Omega, \; i \in N.\end{aligned} \tag{22}$$

Let $i \in N$ and $\omega \in \Omega$. We take the sum over $j \in M^i_\omega$ in the first group of equations in (22) to obtain that

$$\sum_{j \in M^i_\omega} \lambda^i_{\omega j} - m^i_\omega \mu^i_\omega - \eta_0 m^i_\omega \sum_{k \in M^i_\omega} \lambda^i_{\omega k} = 0,$$

which can be reorganized as

$$\mu^i_\omega = \left(\frac{1}{m^i_\omega} - \eta_0\right) \sum_{j \in M^i_\omega} \lambda^i_{\omega j}.$$

Substituting the above equation into the first group of equations in (22), we find that

$$x^i_{\omega j} - x^{0,i}_{\omega j} = \lambda^i_{\omega j} - \frac{1}{m^i_\omega} \sum_{k \in M^i_\omega} \lambda^i_{\omega k},\ j \in M^i_\omega, i \in N, \omega \in \Omega. \tag{23}$$

Next we prove that for all $j \in M^i_\omega, \lambda^i_{\omega j} = 0$. Suppose that, for some $j \in M^i_\omega, \lambda^i_{\omega j} > 0$. We define $\overline{M}^i_\omega = \{j \in M^i_\omega : \lambda^i_{\omega j} > 0\}$ and denote the cardinality of $\overline{M}^i_\omega$ by $\overline{m}^i_\omega$. It follows from the second group of equations in (22) that $x^i_{\omega j} = \eta_0$ for all $j \in \overline{M}^i_\omega$. From the choice of η_0, we find that $\overline{m}^i_\omega < m^i_\omega$. Since $x^0 \in \text{Int}(X(1)), x^{0,i}_{\omega j} > \eta_0$. Then, we derive from Eq. (23) that, for any $j \in \overline{M}^i_\omega$,

$$0 > \lambda^i_{\omega j} - \frac{1}{m^i_\omega} \sum_{k \in M^i_\omega} \lambda^i_{\omega k} = \lambda^i_{\omega j} - \frac{1}{m^i_\omega} \sum_{k \in \overline{M}^i_\omega} \lambda^i_{\omega k}.$$

Summing over $j \in \overline{M}^i_\omega$ in the above group of inequalities, we have that

$$0 > \left(1 - \frac{\overline{m}^i_\omega}{m^i_\omega}\right) \sum_{j \in \overline{M}^i_\omega} \lambda^i_{\omega j} > 0,$$

a contradiction. Therefore, $\lambda^i_{\omega j} = 0$ for any $j \in M^i_\omega$. Then we have that $x^i_{\omega j} = x^{0,i}_{\omega j}$ and $\mu^i_\omega = 0$. □

Lemma 3 shows that the continuously differentiable system (21) has a unique solution at $t = 1$. Note that when t is not larger than the positive number $\zeta_0/2$, $\theta(t)$ is equal to zero and the homotopy system (21) reduces to the equilibrium system (8) for the perturbed stochastic game $\Gamma(t)$. At $t = 0$, the system (21) becomes identical to the equilibrium system (3) for the original stochastic game Γ. Next, we prove that there exists a path-connected component in the set of solutions to the homotopy system (21), which intersects both the level $t = 1$ and the level $t = 0$. We denote by $\widetilde{H}^{-1}$ the set of all $(x, t) \in X \times [0, 1]$ satisfying the equilibrium system (21). From a similar discussion as in the LB-DH method, we attain the following theorem.

Theorem 6 *The set $\widetilde{H}^{-1}$ contains a path-connected component that intersects both $X \times \{0\}$ and $X \times \{1\}$.*

Theorem 6 ensures the global convergence of the CQP-DH method. For numerical implementation, we further need to construct an everywhere smooth path leading to a PeSE. That is, one must eliminate the complementarity conditions $\lambda^i_{\omega j}(x^i_{\omega j} - t\eta_0) = 0$ by making an appropriate transformation of variables in the system (21). For any $i \in N$, $\omega \in \Omega$ and $y^i_{\omega j} \in \mathbb{R}^m$, let

$$\lambda^i_{\omega j}(y) = \max\{0, -y^i_{\omega j}\}^\ell \quad \text{and} \quad x^i_{\omega j}(y, t) = t\eta_0 + \max\{0, y^i_{\omega j}\}^\ell, \tag{24}$$

where $\ell > 2$.[9] Moreover, we formulate the following CQP-DH homotopy system,

$$\begin{aligned} &(1-\theta(t))\varphi^i(\omega, s^i_{\omega j}, x^{-i}_\omega(y,t), \mu^i) - \theta(t)(x^i_{\omega j}(y,t) - x^{0,i}_{\omega j}) + \lambda^i_{\omega j}(y) \\ &\quad -\mu^i_\omega - t\eta_0 \sum_{k \in M^i_\omega} \lambda^i_{\omega k}(y) - t(1-t)\alpha^i_{\omega j} = 0, \;\; j \in M^i_\omega, \;\omega \in \Omega, \; i \in N, \\ &\sum_{j \in M^i_\omega} x^i_{\omega j}(y,t) - 1 = 0, \;\; \omega \in \Omega, \; i \in N, \end{aligned} \tag{25}$$

where $\alpha \in \mathbb{R}^m$ is a small perturbation. Let $h(y, \mu, t; \alpha)$ denote the left-hand side of the system (25), which is clearly a continuously differentiable function. The system (25) has a unique starting point at the level of $t = 1$. For any $\alpha \in \mathbb{R}^m$, let $h_\alpha(y, \mu, t) = h(y, \mu, t; \alpha)$ and let the solution set to the system (25) be denoted by H^{-1}_α. It follows from the two systems (24) and (25) that y and μ are uniquely determined for any given x. Therefore, H^{-1}_α contains a path-connected component that intersects both $\mathbb{R}^m \times \mathbb{R}^{nd} \times \{1\}$ and $\mathbb{R}^m \times \mathbb{R}^{nd} \times \{0\}$. The following theorem verifies that this path-connected component forms an everywhere smooth path, which eventually leads to a perfect stationary equilibrium for Γ.

Theorem 7 *For a generic choice of $\alpha \in \mathbb{R}^m$ with $\|\alpha\|$ sufficiently small, there exists a smooth path in H^{-1}_α, which starts from an arbitrary point at $t = 1$ and ends at an approximate perfect stationary equilibrium for the stochastic game Γ as t approaches zero.*

Proof It has been proved in "Appendix IV" that zero is a regular value of $h(y, \mu, t; \alpha)$ on $\mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1] \times \mathbb{R}^m$. From the transversality theorem, for almost all $\alpha \in \mathbb{R}^m$, zero is also a regular value of $h_\alpha(y, \mu, t)$. Moreover, we derive from a highly similar discussion as the proof of Theorem 5 that, when $\|\alpha\|$ is sufficiently small, the smooth path contained in the set H^{-1}_α leads to an approximate perfect stationary equilibrium for Γ as t approaches zero. □

[9] The reason for choosing $\ell > 2$ is the use of the transversality theorem in the following analysis, which requires the homotopy function to be at least second-order continuously differentiable.

5 Numerical performance

In this section, we apply the proposed LB-DH method to solve various numerical examples, including several well-known stochastic games, randomly generated stochastic games, an application to dynamic legislative bargaining, and a dynamic oligopoly model with entry and exit. A predictor–corrector method has been adopted for numerically tracing the generated homotopy paths (Allgower and Georg 2012; Chen and Dang 2021; Eaves and Schmedders 1999). In our implementation, we set $\eta_0 = 1/(\max_{i\in N,\omega\in\Omega} m^i_\omega + 5)$, $\zeta_0 = 10^{-5}$, $\kappa = 3$, and $\delta = 0.95$. To reveal the effectiveness of the LB-DH method for selecting a PeSE, we have exploited the IPM and an arbitrary starting SLTP proposed in Li and Dang (2020) to solve several well-known examples. In this section, we have plot the development of the homotopy paths for several stochastic games to illustrate how the methods work. To demonstrate the numerical efficiency of the LB-DH method, we have also implemented the CQP-DH method and compared its computation time with that of the LB-DH method. All the methods are coded in MatLab (R2019a).

5.1 Several well-known stochastic games

We test the numerical effectiveness of the LB-DH method for computing a PeSE in Example 1. Recall that the stochastic game in Example 1 has three SSPEs, $(s^1_{\omega_1 1}, s^2_{\omega_1 1}, s^1_{\omega_2 1}, s^2_{\omega_2 1})$, $(s^1_{\omega_1 2}, s^2_{\omega_1 2}, s^1_{\omega_2 1}, s^2_{\omega_2 1})$ and $(s^1_{\omega_1 3}, s^2_{\omega_1 3}, s^1_{\omega_2 1}, s^2_{\omega_2 1})$, and only $(s^1_{\omega_1 2}, s^2_{\omega_1 2}, s^1_{\omega_2 1}, s^2_{\omega_2 1})$ is a PeSE. By applying the LB-DH method to this example, we find the unique PeSE. The method starts from a totally mixed strategy profile, $(x^{0,1}_{\omega_1 1}, x^{0,1}_{\omega_1 2}, x^{0,1}_{\omega_1 3}, x^{0,2}_{\omega_1 1}, x^{0,2}_{\omega_1 2}, x^{0,2}_{\omega_1 3}, x^{0,1}_{\omega_2 1}, x^{0,2}_{\omega_2 1}) = (0.2, 0.5, 0.3, 0.2, 0.5, 0.3, 1, 1)$. The development of different variables and the number of iterations is plotted in Fig. 2.

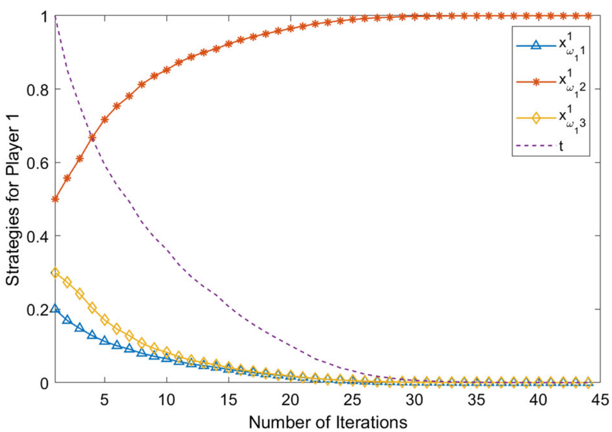

Fig. 2 The development of different variables in iterations

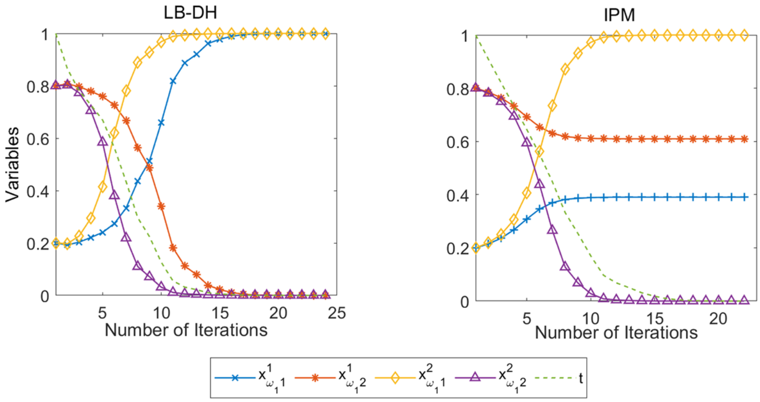

Fig. 3 Numerical comparisons

5.1.1 Comparisons with the IPM

In the sequel, we apply both the LB-DH and the IPM method to Examples 2 and 3 and plot the development of the homotopy paths generated by these methods.

Example 2 $N = \{1, 2\}$ and $\Omega \in \{\omega_1, \omega_2\}$. For $i = 1, 2$, $S^i_{\omega_1} = \{s^i_{\omega_1 1}, s^i_{\omega_1 2}\}$ and $S^i_{\omega_2} = \{s^i_{\omega_2 1}\}$. The payoff matrices for the players are given by

$$\begin{array}{c|cc} \omega_1 & s^2_{\omega_1 1} & s^2_{\omega_1 2} \\ \hline s^1_{\omega_1 1} & (0,0) & (0,-1) \\ s^1_{\omega_1 2} & (0,0) & (-1,-1) \end{array}, \quad \begin{array}{c|c} \omega_2 & s^2_{\omega_2 1} \\ \hline s^1_{\omega_2 1} & (1,1) \end{array}.$$

The transition probability is $\pi(\bar{\omega} : \omega, s_\omega) = 0.5$, for any $\bar{\omega}, \omega \in \Omega$.[10]

There are infinitely many SSPEs in this stochastic game, but only $(s^1_{\omega_1 1}, s^2_{\omega_1 1}, s^1_{\omega_2 1}, s^2_{\omega_2 1})$ is a PeSE. Both methods start from the same mixed strategy profile, $(x^{0,1}_{\omega_1 1}, x^{0,1}_{\omega_1 2}, x^{0,2}_{\omega_1 1}, x^{0,2}_{\omega_1 2}, x^{0,1}_{\omega_2 1}, x^{0,2}_{\omega_2 1}) = (0.2, 0.8, 0.2, 0.8, 1, 1)$. The development of the different variables in state ω_1 can be found in Fig. 3. It is easy to observe from Fig. 3 that IPM leads to an SSPE that is not perfect, $(x^1_{\omega_1 1}, x^1_{\omega_1 2}, x^2_{\omega_1 1}, x^2_{\omega_1 2}, x^1_{\omega_2 1}, x^2_{\omega_2 1}) = (0.6, 0.4, 1, 0, 1, 1)$. The LB-DH method successfully finds the unique PeSE $(x^1_{\omega_1 1}, x^1_{\omega_1 2}, x^2_{\omega_1 1}, x^2_{\omega_1 2}, x^1_{\omega_2 1}, x^2_{\omega_2 1}) = (1, 0, 1, 0, 1, 1)$.

[10] The game in this example is derived from the stochastic extension of a normal-form game in Mertens (1989).

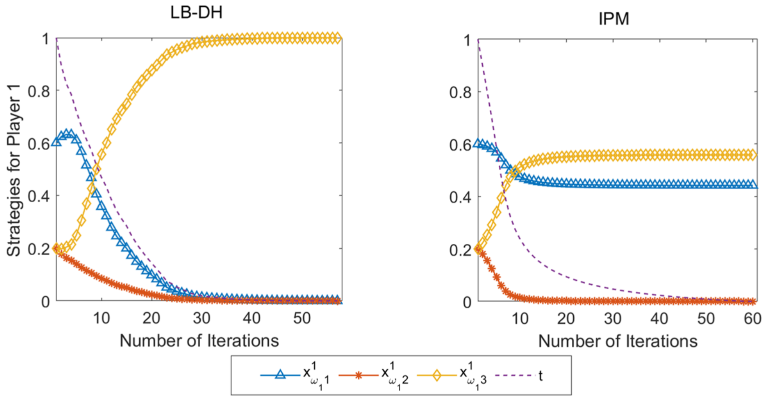

Fig. 4 Numerical comparisons

Example 3 We have $N = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2\}$, $S^1_{\omega_1} = \{s^1_{\omega_1 1}, s^1_{\omega_1 2}, s^1_{\omega_1 3}\}$, $S^2_{\omega_1} = \{s^2_{\omega_1 1}, s^2_{\omega_1 2}, s^2_{\omega_1 3}\}$, $S^1_{\omega_2} = \{s^1_{\omega_2 1}\}$, and $S^2_{\omega_2} = \{s^2_{\omega_2 1}\}$. The payoff matrices are given by

$$\begin{array}{cccc} \omega_1 & s^2_{\omega_1 1} & s^2_{\omega_1 2} & s^2_{\omega_1 3} \\ s^1_{\omega_1 1} & (1,1) & (0,0) & (1,1) \\ s^1_{\omega_1 2} & (0,0) & (0,0) & (0,10) \\ s^1_{\omega_1 3} & (1,1) & (5,0) & (1,1) \end{array} \quad \textit{and} \quad \begin{array}{cc} \omega_2 & s^2_{\omega_2 1} \\ s^1_{\omega_2 1} & (0,0) \end{array}.$$

The transition probability is $\pi(\bar{\omega} : \omega, s_\omega) = 0.5$, for any $\bar{\omega}, \omega \in \Omega$.[11]

There are infinitely many SSPEs in this stochastic game, which include all mixtures between the first and third actions for both players in state ω_1. Nevertheless, only $(s^1_{\omega_1 3}, s^2_{\omega_1 3}, s^1_{\omega_2 1}, s^2_{\omega_2 1})$ is a PeSE. Both methods start from the same point, $(x^{0,1}_{\omega_1 1}, x^{0,1}_{\omega_1 2}, x^{0,1}_{\omega_1 3}, x^{0,2}_{\omega_1 1}, x^{0,2}_{\omega_1 2}, x^{0,2}_{\omega_1 3}, x^{0,1}_{\omega_2 1}, x^{0,2}_{\omega_2 1}) = (0.6, 0.2, 0.2, 0.6, 0.2, 0.2, 1, 1)$. The development of $x^1_{\omega_1 1}$, $x^1_{\omega_1 2}$, and $x^1_{\omega_1 3}$ is plotted in Fig. 4. It can be seen from Fig. 4 that IPM fails to find the PeSE while the LB-DH method is successful in doing so.

The above examples illustrate the effectivenss of the LB-DH method for finding a PeSE. As we know, the starting point matters greatly for the development of the homotopy path and, in fact, different starting points may lead to different ending points. To further ensure the rigor of the results in the above experiments and confirm the effectiveness of the LB-DH method, we have repeatedly run both methods with various randomly generated starting strategy profiles x^0 and report the success rate of the methods in Tables 1, 2, where "S" (or "F") means the method succeeds (or fails) to compute a PeSE. It follows from the numerical results that the LB-DH method has achieved a 100% success rate in computing PeSEs regardless of the starting point, while the IPM might reach any possible SSPE and therefore fails to find a PeSE for stochastic games with a large number of SSPEs.

[11] The game in this example is derived from an extension of a normal-form game in McKelvey and Palfrey (1995).

Table 1 Numerical performance in Example 2

Test	$\left(x^{0,1}_{\omega_1 1}, x^{0,1}_{\omega_1 2}, x^{0,2}_{\omega_1 1}, x^{0,2}_{\omega_1 2}\right)$	LB-DH	IPM
1	(0.4826, 0.5174, 0.2528, 0.7472)	S	F
2	(0.6935, 0.3065, 0.6449, 0.3551)	S	F
3	(0.4621, 0.5379, 0.4752, 0.5248)	S	F
4	(0.4833, 0.5167, 0.4878, 0.5122)	S	F
5	(0.2757, 0.7243, 0.5145, 0.4855)	S	F
6	(0.6382, 0.3618, 0.2376, 0.7624)	S	F
7	(0.5962, 0.4038, 0.5800, 0.4200)	S	F
8	(0.1879, 0.8121, 0.6156, 0.3844)	S	F
9	(0.7977, 0.2023, 0.7686, 0.2314)	S	F
10	(0.5718, 0.4282, 0.2728, 0.7272)	S	F

Table 2 Numerical performance in Example 3

Test	$\left(x^{0,1}_{\omega_1 1}, x^{0,1}_{\omega_1 2}, x^{0,1}_{\omega_1 3}, x^{0,2}_{\omega_1 1}, x^{0,2}_{\omega_1 2}, x^{0,2}_{\omega_1 3}\right)$	LB-DH	IPM
1	(0.3874, 0.1816, 0.4310, 0.3253, 0.3564, 0.3183)	S	F
2	(0.4710, 0.2215, 0.3075, 0.4007, 0.3225, 0.2768)	S	F
3	(0.3309, 0.3033, 0.3033, 0.2059, 0.4541, 0.3400)	S	F
4	(0.4962, 0.1755, 0.3283, 0.2304, 0.4058, 0.3638)	S	F
5	(0.2609, 0.4470, 0.2921, 0.3404, 0.3966, 0.2630)	S	F
6	(0.5994, 0.2026, 0.1980, 0.3566, 0.4141, 0.2293)	S	F
7	(0.2896, 0.3760, 0.3344, 0.3686, 0.3912, 0.2402)	S	F
8	(0.3442, 0.3450, 0.3108, 0.3622, 0.4638, 0.1740)	S	F
9	(0.2407, 0.4949, 0.2644, 0.3800, 0.1734, 0.4466)	S	F
10	(0.1823, 0.3916, 0.4261, 0.3255, 0.2752, 0.3993)	S	F

5.1.2 Comparisons with the SLTP

Recall that the SLTP, extending a reasoning process in Harsanyi and Selten (1988) to the class of stochastic games, is a well-known effective approach to solve for an SSPE in stochastic games. However, the SSPE obtained by the SLTP cannot be guaranteed to be perfect. To further affirm this advantage of the LB-DH method over the SLTP, we have implemented both methods to solve the following example and compared their numerical performance for computing a PeSE.

Example 4 $N = \{1, 2, 3\}$ and $\Omega \in \{\omega_1, \omega_2\}$. For $i = 1, 2, 3$, $S^i_{\omega_1} = \{s^i_{\omega_1 1}, s^i_{\omega_1 2}\}$ and $S^i_{\omega_2} = \{s^i_{\omega_2 1}\}$. The payoff matrices for the players are given by

$$\begin{array}{ccc} \omega_1 & s^2_{\omega_1 1} & s^2_{\omega_1 2} \\ s^1_{\omega_1 1} & (1, 1, 1) & (1, 0, 1) \\ s^1_{\omega_1 2} & (1, 1, 1) & (0, 0, 1) \\ & s^3_{\omega_1 1} & \end{array}, \quad \begin{array}{ccc} \omega_1 & s^2_{\omega_1 1} & s^2_{\omega_1 2} \\ s^1_{\omega_1 1} & (1, 1, 0) & (0, 0, 0) \\ s^1_{\omega_1 2} & (0, 1, 0) & (1, 0, 0) \\ & s^3_{\omega_1 2} & \end{array}, \quad \begin{array}{cc} \omega_2 & s^2_{\omega_2 1} \\ s^1_{\omega_2 1} & (0, 0, 0) \\ & s^3_{\omega_2 1} \end{array}.$$

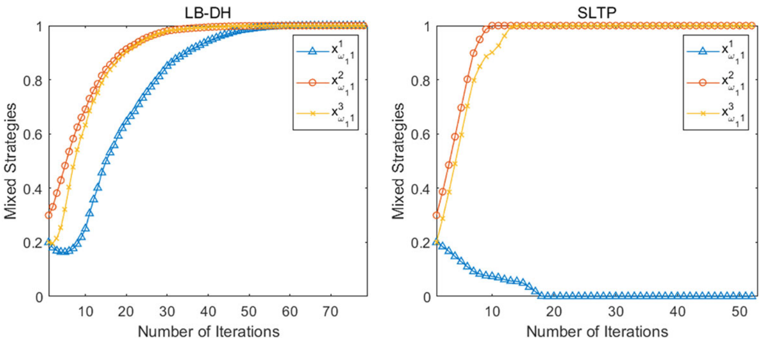

Fig. 5 The changes of different variables in the iterations for both methods

The transition probability is $\pi(\bar{\omega} : \omega, s_\omega) = 0.5$, for any $\bar{\omega}, \omega \in \Omega$.

Clearly, a strategy profile $(x^1_{\omega_1 1}, s^2_{\omega_1 1}, s^3_{\omega_1 1}, s^1_{\omega_2 1}, s^2_{\omega_2 1}, s^3_{\omega_2 1})$ for any $x^1_{\omega_1 1} \in [0, 1]$ is an SSPE, which indicates that the stochastic game has infinitely many SSPEs. However, $(s^1_{\omega_1 1}, s^2_{\omega_1 1}, s^3_{\omega_1 1}, s^1_{\omega_2 1}, s^2_{\omega_2 1}, s^3_{\omega_2 1})$ is the unique PeSE for this game, that is, every player implements her first action.

First, we have run the LB-DH and SLTP methods with the same starting point,

$$(x^{0,1}_{\omega_1 1}, x^{0,1}_{\omega_1 2}, x^{0,2}_{\omega_1 1}, x^{0,2}_{\omega_1 2}, x^{0,3}_{\omega_1 1}, x^{0,3}_{\omega_1 2}, x^{0,1}_{\omega_2 1} x^{0,2}_{\omega_2 1}, x^{0,3}_{\omega_2 1})$$
$$= (0.2, 0.8, 0.7, 0.3, 0.2, 0.8, 1, 1, 1).$$

Experimental results show that both methods find the unique PeSE. Next, we change the starting point to (0.2, 0.8, 0.3, 0.7, 0.2, 0.8, 1, 1, 1). Starting from the new given point, SLTP leads to the SSPE

$$(x^1_{\omega_1 1}, x^1_{\omega_1 2}, x^2_{\omega_1 1}, x^2_{\omega_1 2}, x^3_{\omega_1 1}, x^3_{\omega_1 2}, x^1_{\omega_2 1} x^2_{\omega_2 1}, x^3_{\omega_2 1}) = (0, 1, 1, 0, 1, 0, 1, 1, 1),$$

which is not perfect. Still, the LB-DH method finds the unique PeSE. The development of the different variables in the various iterations for both methods are plotted in Fig. 5.

Example 4 illustrates that the effectiveness of the SLTP for computing a perfect stationary equilibrium is sensitive to the starting point and cannot be guaranteed. Nevertheless, the LB-DH method always approaches a PeSE, which confirms the theoretical convergence of the proposed method numerically.

5.2 Randomly generated stochastic games

In addition to the above examples, we have generated extensive randomly generated stochastic games for varying n, d, and m_0, where m_0 denotes the number of actions for each player in each state. Payoffs are uniformly chosen from the interval [−10, 10] and set to be zero with probability "pd0", where "pd0" is a random value in [0, 0.8]. Clearly, "pd0" measures the sparseness of the payoff matrix; that is, a larger value of "pd0" leads to a sparser payoff matrix. For numerical comparisons, we have run the LB-DH and

CQP-DH methods to compute PeSEs for the randomly generated games.[12] Moreover, to verify that the LB-DH method gains from eliminating the Bellman equation (1), we have tested the efficiency of the LB-DH method without eliminating the Bellman equation (LB-DH-NR). Each experiment with the same triple of (n, d, m_0) has been run ten times.

5.2.1 Comparisons with the CQP-DH method

We let n be equal to 2, 3, and 4. For any given n, we take d and m_0 from 2 to 5, which induces several groups of stochastic games with different scales. The LB-DH and CQP-DH methods are used to solve those games, and the results are reported in Table 3, where "AVER" is the average computation time (in seconds) for each triple, "MAX" is the maximal computation time (in seconds), "MIN"is the minimal computation time (in seconds), "STDEV" is the standard deviation in the computation time, and "Ratio" equals $\dfrac{\text{AVER of LB-DH}}{\text{AVER of CQP-DH}}$, which we set in bold if it is smaller than one.

From the last column of Table 3, it can be seen that the percentage ratio of the computation time of the LB-DH and CQP-DH methods is around 10%, which implies that the LB-DH method significantly outperforms the CQP-DH method. The standard deviations of computation time show that the LB-DH method is much more stable than the CQP-DH method.

5.2.2 Comparisons with the LB-DH-NR

This section focuses on large-scale stochastic games, which are difficult to solve with the CQP-DH method in a reasonable time. The LB-DH and LB-DH-NR have been implemented to compute PeSEs for these games, where the homotopy system for the LB-DH-NR is given by

$$\begin{aligned}
&(1-\theta(t))\varphi^i(\omega, s^i_{\omega j}, x^{-i}_{\omega}(z,t), \mu^i) + \lambda^i_{\omega j}(z,t) - \beta^i_{\omega} = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad j \in M^i_{\omega},\ \omega \in \Omega,\ i \in N, \\
&\sum_{j \in M^i_{\omega}} x^i_{\omega j}(z,t) - 1 = 0,\ \omega \in \Omega,\ i \in N, \\
&\mu^i_{\omega} = (1-\theta(t))\varphi^i(\omega, x_{\omega}(z,t), \mu^i) + \theta(t)(1-\eta_0 m^i_{\omega})\ \omega \in \Omega, i \in N,
\end{aligned} \tag{26}$$

with $x(z,t)$ and $\lambda(z,t)$ the same as in (16). We report the average computation time (in seconds) in Table 4. The improvement in efficiency brought by the elimination is also shown in Table 4, which reads as "ImRatio"=$1 - \dfrac{\text{AVER of LB-DH}}{\text{AVER of LB-DH-NR}}$. We make this column bold to highlight the improvement of the numerical efficiency brought by this elimination.

[12] To shuffle the deck even more against us and illustrate the numerical efficiency of the LB-DH method, we have implemented the CQP-DH method with $\ell = 2$ in numerical experiments, which on average leads to shorter computational times than the case with $\ell > 2$.

Table 3 Numerical performance and comparisons

(d, m_0)	LB-DH				CQP-DH				Ratio (%)
	MAX	MIN	**AVER**	STDEV	MAX	MIN	**AVER**	STDEV	
$n = 2$									
(2, 2)	4.17	0.43	1.99	1.25	57.32	0.62	24.01	20.61	**8.29**
(2, 5)	13.51	0.86	5.29	3.77	140.23	1.26	59.18	46.62	**8.94**
(3, 4)	14.16	4.31	10.19	1.75	242.65	14.14	146.10	31.72	**6.97**
(4, 3)	20.24	6.46	13.16	4.65	226.32	66.47	178.47	49.37	**7.37**
(5, 2)	29.97	5.95	15.02	7.38	524.02	32.92	214.06	154.37	**7.01**
$n = 3$									
(2, 2)	17.19	2.83	5.57	4.34	282.87	7.85	81.17	82.04	**6.86**
(2, 5)	26.27	6.75	15.91	6.91	294.71	72.54	163.49	90.62	**9.73**
(3, 3)	36.48	15.61	24.08	7.13	467.12	127.00	270.75	114.91	**8.89**
(4, 2)	45.20	9.05	24.05	11.66	598.61	56.45	251.89	166.20	**9.54**
$n = 4$									
(2, 2)	20.42	3.30	9.47	5.21	299.77	12.08	100.13	85.84	**9.45**
(2, 4)	37.59	7.97	21.61	10.58	403.44	24.52	216.18	143.61	**9.99**
(3, 2)	39.23	13.73	27.46	2.74	279.94	36.54	128.49	83.71	**12.63**
(4, 2)	162.75	22.07	63.20	49.92	3505.53	45.20	1081.92	1378.93	**7.63**

Bold values indicate the advantage of our proposed LB-DH method over the CQP-DH method in computation time

Table 4 Average computation time and comparisons

	LB-DH	LB-DH-NR	ImRatio (%)
$n = 3/(d, m_0)$			
(3, 7)	149.58	179.25	**16.55**
(4, 6)	472.45	558.02	**15.33**
(5, 5)	299.36	371.59	**19.43**
(6, 6)	901.96	1181.59	**23.66**
(7, 3)	426.56	606.57	**29.67**
(7, 7)	4820.88	7240.56	**33.41**
$n = 4/(d, m_0)$			
(3, 7)	769.48	871.32	**11.69**
(4, 6)	711.73	1055.56	**32.57**
(4, 7)	1470.76	2237.81	**34.27**
(5, 5)	1379.07	1978.78	**30.31**
(6, 5)	2045.60	2878.29	**28.93**
(7, 4)	2384.97	2920.30	**18.33**
$n = 5/(d, m_0)$			
(3, 6)	2639.46	3674.06	**28.16**
(4, 5)	1585.87	1981.08	**19.94**
(5, 4)	2368.15	3281.12	**27.82**
(6, 3)	1553.67	2041.08	**23.88**
$n = 6/(d, m_0)$			
(3, 5)	2101.86	2732.08	**23.06**
(4, 4)	2209.21	2736.01	**19.25**
(5, 3)	2259.47	2746.82	**17.74**
$n = 7/(d, m_0)$			
(3, 4)	2033.90	2552.62	**20.26**
(4, 3)	1462.38	1804.41	**18.95**
(4, 4)	6018.24	7570.19	**20.50**

Table 4 confirms the effectiveness of the LB-DH method to compute PeSEs for stochastic games with scales up to $n = 7$, $d = 7$, and $m_0 = 7$. It can be seen that the average computation time increases in n, d, and m_0. The last column of Table 4 affirms that the elimination of the Bellman equation enhances the numerical efficiency of the LB-DH method. Moreover, among the three parameters n, d, and m_0, n is the most influential factor for the computational cost, which aligns with the observations made for the computation of SSPEs in Herings and Peeters (2004) and Li and Dang (2020).

5.3 An application to voting problems

Consider a voting model carried out by three voters for two options. In any stage t, the voters simultaneously and independently vote, a or b. If they choose the same

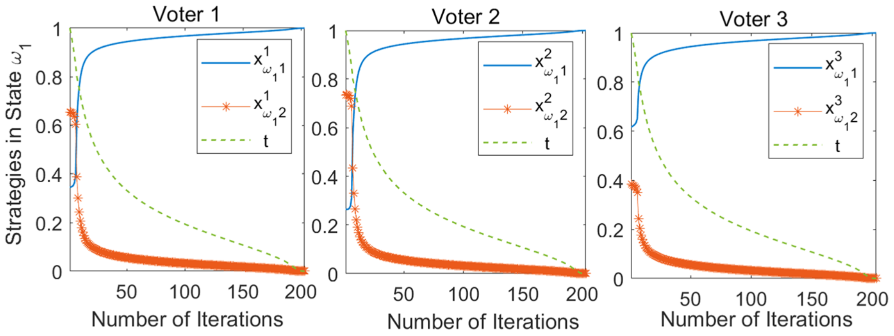

Fig. 6 Development of t and x_{ω_1} in the various iterations

option, the voting ends, and this option will be implemented in the subsequent stages. Otherwise, the voters pay a voting fee in stage t and start a new round of voting in stage $t+1$. This voting problem can be formulated into a stochastic game with infinitely many stages. More specifically, $N = \{1, 2, 3\}$ and $\Omega = \{\omega_1, \omega_2, \omega_3\}$, where $\omega_1 = \{\text{a new round of voting starts}\}$. The states ω_2 and ω_3 correspond to the states in which the voting has ended, where $\omega_2 = \{a\text{has been implemented}\}$ and $\omega_3 = \{b\text{has been implemented}\}$. In ω_1, the voters have two actions, which read as: $s^i_{\omega_1 1} = \{\text{vote for} a\}$ and $s^i_{\omega_1 2} = \{\text{vote for} b\}$ with $i = 1, 2, 3$. Moreover, the payoff matrices are given by

$$
\begin{array}{ccc}
\omega_1 & s^2_{\omega_1 1} & s^2_{\omega_1 2} \\
s^1_{\omega_1 1} & (1, 1, 1) & (-1, -1, -1) \\
s^1_{\omega_1 2} & (-1, -1, -1) & (-1, -1, -1) \\
 & s^3_{\omega_1 1} &
\end{array},
\quad
\begin{array}{cc}
\omega_2 & s^2_{\omega_2 1} \\
s^1_{\omega_2 1} & (1, 1, 1) \\
 & s^3_{\omega_2 1}
\end{array},
$$

$$
\begin{array}{ccc}
\omega_1 & s^2_{\omega_1 1} & s^2_{\omega_1 2} \\
s^1_{\omega_1 1} & (-1, -1, -1) & (-1, -1, -1) \\
s^1_{\omega_1 2} & (-1, -1, -1) & (-1, -1, -1) \\
 & s^3_{\omega_1 2} &
\end{array},
\quad
\begin{array}{cc}
\omega_3 & s^2_{\omega_3 1} \\
s^1_{\omega_3 1} & (0, 0, 0) \\
 & s^3_{\omega_3 1}
\end{array}.
$$

If the current state is ω_1 and unanimity is not achieved, the system will jump to ω_1 with probability 1. Otherwise, the system will jump to ω_2 or ω_3 with probability 1. Furthermore, states ω_2 and ω_3 are absorbing. That is, once the system reaches states ω_2 or ω_3, it will never leave them.

The strategy profile with all individuals voting for a in the state ω_1 is the unique PeSE in this game. However, there exists an SSPE with all individuals voting for b which is not perfect. Starting from a randomly generated strategy profile, the LB-DH method finds the unique PeSE, where $(x^1_{\omega_1 1}, x^2_{\omega_1 1}, x^3_{\omega_1 1}) = (1, 1, 1)$. The development of the different variables along the homotopy path is plotted in Fig. 6.

5.4 An application to dynamic legislative bargaining

To illustrate the importance of selecting a perfect stationary equilibrium, we present an example from the literature on dynamic legislative bargaining in this section (Gomes and Jehiel 2005; Duggan and Kalandrakis 2012; Eraslan et al. 2022). Consider n legislators, who bargain over the policy of investing in technology to reduce CO_2 emissions. The state variable in the related stochastic economy corresponds to the state of the climate. A proposal passes if a sufficiently large coalition of legislators chooses to accept the policy and fails otherwise. If decision making is governed by a quotum $q \in [0, N]$, then for a proposal to pass, the coalition has to belong to the collection $\{C \subseteq N : |C| \geq q\}$, where N is the set of all legislators. Among quota rules, the majority rule is one of the most common voting rules in practice, that is, $\{C \subseteq N : |C| > \frac{n}{2}\}$.

To model this situation, we consider a stochastic game with three players and six states, that is, $\omega_1 = \{\text{great}\}$, $\omega_2 = \{\text{good}\}$, $\omega_3 = \{\text{medium}\}$, $\omega_4 = \{\text{passable}\}$, $\omega_5 = \{\text{bad}\}$, and $\omega_6 = \{\text{terrible}\}$. Each player in each state has two choices, invest or not invest. State transitions occur with probabilities depending on the strategy profile in the current period and the payoffs of the legislators in each period rely on the current state of the climate. More specifically, if the investment proposal is accepted, which happens whenever two or three legislators opt for invest, the climate will go up one state with a probability of 1/2 and remain in the current state with probability 1/2 in the next period. If the proposal is rejected, which happens when none or only of the legislators wants to invest, the climate will go down one state with probability 1/2 and remain with probability 1/2. For simplicity, we assume that if the current state is ω_1 and the policy is accepted, or, if the current state is ω_6 and the policy is rejected, the state will remain in the next period. The utilities of the legislators are generated as follows: first we randomly generate the utilities of each legislator in the worst climate state ω_6, then we obtain the utilities for the state ω_5 by adding a randomly generated positive number to those utilities in $\omega_6 \ldots$, and we continue in this way to obtain the utilities of each legislator in each state. Moreover, the cost of investment is the same for all legislators and normalized to 1.

There exists a plethora of stationary equilibria, many of which are unappealing. For instance, any strategy profile where in each state all legislators are in favor or all legislators are against is a stationary equilibrium. Indeed, with a unilateral deviation, there are still two players against or still two players in favor, and the voting outcome would not change when majority rule is in place. This generates 64 stationary equilibria, irrespective of the actual utilities of the legislators. To rule out such unappealing equilibria, we apply the LB-DH method to this problem and plot the development of the probabilities of choosing the "invest" action for each legislator in each state along the homotopy path in Fig. 7.

From the numerical results, we see that the LB-DH method does not end up at stationary equilibria which are not perfect and successfully finds the PeSE, where all the legislators accept the investment policy in each state. Keeping those utilities unchanged, we randomly generate several new starting points and implement the LB-DH method repeatedly. For those utilities, the paths will always lead to the same PeSE.

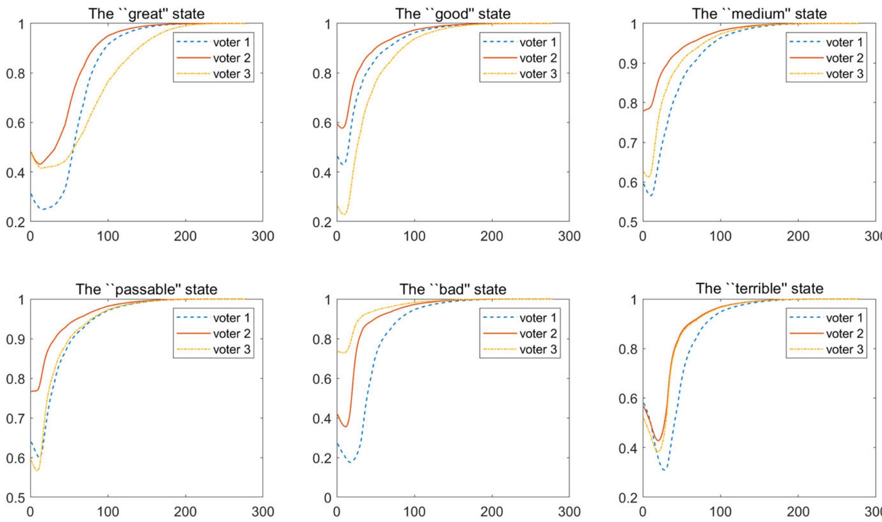

Fig. 7 Development of the investment strategies in different climate states in the various iterations

Moreover, we randomly generate another sets of utilities for the legislators using the same way as before and compute the equilibria for the new problems by exploiting the LB-DH method. The results illustrate that the obtained equilibria are always in pure strategies and that investment is least likely in state ω_1. The reason for this phenomenon is that the current climate is good enough and the legislators cannot benefit much from the investment. This is also the state where consensus among the legislators is least likely.

5.5 An application to a dynamic oligopoly model with entry and exit

In this example, we present a dynamic oligopoly model with entry and exit, based on Herings et al. (2005).[13] Suppose that n firms produce homogeneous goods to serve a market with a linear demand curve, so with firm $j \in N$ producing quantity q^j, leading to a total production quantity equal to $Q := \sum_{j \in N} q^j$ and market price $p = a - bQ$. Firms have constant marginal costs of production equal to c. Firms interact during infinitely many periods. In each period, a firm currently in the market decides about a production quantity and about whether or not to remain in the market, whereas firms out of the market decide whether or not to enter. Firms are forward looking and maximize discounted expected profits.

This problem can be modeled as a stochastic game with n players and 2^n states, which correspond to all possible industry structures, i.e., which firms are active in the current period. For any state ω, when there are k firms active in the market, it is easy to verify that each of them optimally produces $q^i_\omega = \dfrac{a-c}{k+1}b$, resulting in instantaneous

[13] This dynamic IO problem was also discussed in Ericson and Pakes (1995), Doraszelski and Satterthwaite (2010) and Abbring et al. (2018).

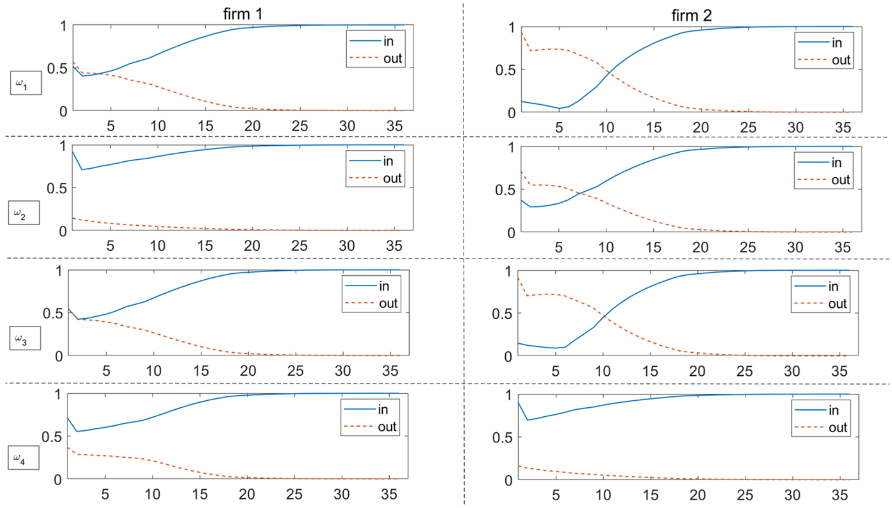

Fig. 8 Development of the strategies in state $\omega_1 : (o, o)$, $\omega_2 : (o, i)$, $\omega_3 : (i, o)$, and $\omega_4 : (i, i)$ in the homotopy path

payoff $u^i_\omega = \dfrac{(a-c)^2}{(k+1)^2 b}$. Inactive firms do not produce and obtain nothing. We can therefore model an active firm i in any state ω as having two actions, $S^i_\omega = \{i, o\}$, that is, either be in (active in) or out of (inactive in) the market next period.

Closed-form solutions for the two-firm case were studied in Herings et al. (2005) and revealed the possibility of an alternating monopoly as an equilibrium outcome. In states where both firms are active, each of them leaves the market with a probability strictly in between 0 and 1. In states where no firm is active, each of them enters with a probability strictly in between 0 and 1. In states where exactly one firm is active, the active firm leaves and the inactive firm enters.

In this section, we first consider the problem presented in Herings et al. (2005) with $n = 2$, that is, the set of players $N = \{1, 2\}$ and state space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, where $\omega_1 = (o, o)$, $\omega_2 = (o, i)$, $\omega_3 = \{i, o\}$, and $\omega_4 = (i, i)$. In each period, the action profile chosen by the players yields the industry structure in the next period. The discount factor is set to be 0.95. Without loss of generality, we normalize a, b, and c such that $(a-c)^2/b = 1$. With the above settings, in each period, any active firm gets a profit of 1/4 when only this firm is active in the market and gets 1/9 when both firms are in the market. We apply the LB-DH method to this problem. Starting from a randomly generated point, the path generated by our method leads to a PeSE, where both firms always choose to be active in the market. In each state, the development of the mixed strategies for each firm in the various iterations is plotted in Fig. 8, and the total expected profits of both firms by coordinating on this equilibrium in each state are reported in Table 5.

To verify the capability of the LB-DH method to find multiple equilibria, we start the method from a different starting point and eventually find another PeSE, which corresponds to the "alternating monopoly equilibrium"—when both firms are active

Table 5 Total expected profits of both firms in each state

Total profits	ω_1	ω_2	ω_3	ω_4
Firm 1	2.1111	2.1111	2.3611	2.2222
Firm 2	2.1111	2.3611	2.1111	2.2222

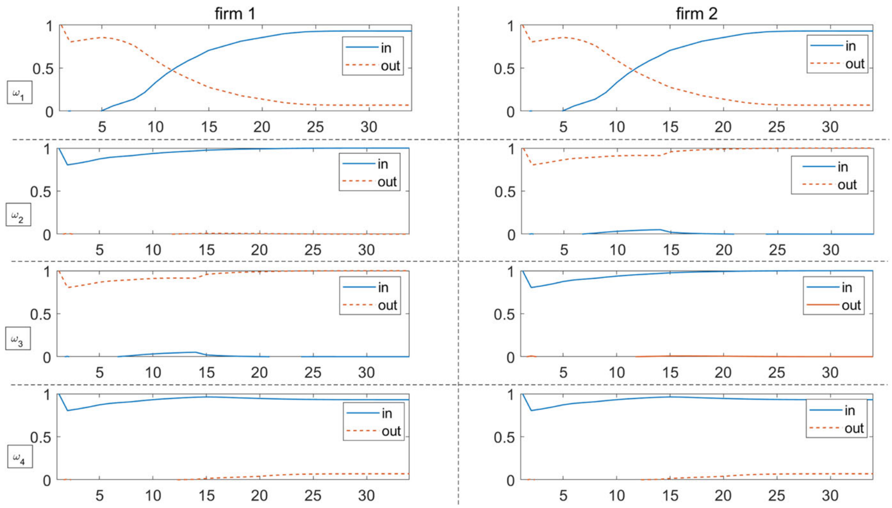

Fig. 9 Development of the strategies in all states in the homotopy path

Table 6 Total expected profits of both firms in each state

Total profits	ω_1	ω_2	ω_3	ω_4
Firm 1	2.3055	2.4359	2.5641	2.4166
Firm 2	2.3055	2.5641	2.4359	2.4166

or inactive in the current period, they will become active in the next period with probability 0.9306; when only one of the firms is active in the current period, the active firm will be inactive in the next period, whereas the inactive firm will be active for sure one period later, indicating that the market displays an alternating monopoly. We plot the development of the probabilities of choosing "in" and "out" actions for each firm in each state in Fig. 9 and report the total expected profits of both firms in this equilibrium in Table 6.

Additionally, we change the starting point of the LB-DH method again and obtain one more equilibrium. In this equilibrium, in state ω_1, firm 1 will be active with probability 0.7305 and firm 2 will be active for sure in the next period; in state ω_2, the inactive firm will become active for sure while the active firm will turn to be inactive with probability very close to 1 in the next period; in state ω_3, the inactive firm will become active for sure while the active firm will remain to be active with probability 0.1579 one period later; in state ω_4, firm 1 will remain to be active in the next period, whereas firm 2 will become inactive with a small probability of 0.0825. The transitions between any two states are illustrated graphically in Fig. 10, where we

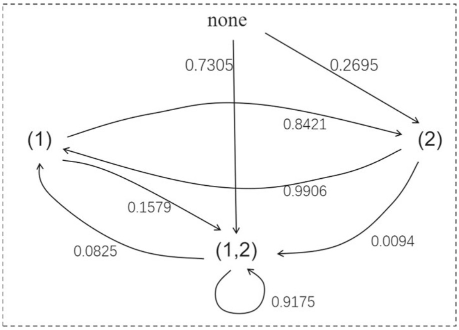

Fig. 10 State transitions in market structure

Table 7 Total expected profits of both firms in each state

Total profits	ω_1	ω_2	ω_3	ω_4
Firm 1	2.3029	2.4241	2.5529	2.4241
Firm 2	2.1466	2.3611	2.2222	2.2222

depict the states by the coalitions which are active (like "(1,2)" indicates the state with both firms being active, and "none" means the state ω_1—no firm chooses to be active) and indicate by arrows how one moves from one state to another under the equilibrium strategy profile. The total expected profits for the firms in this equilibrium are reported in Table 7. Furthermore, by changing the starting point, the LB-DH method is capable to find multiple equilibria for stochastic games.

Next we analyze the more complex scenario with $n = 3$. We therefore consider a player set $N = \{1, 2, 3\}$ and state space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_8\}$, where $\omega_1 = (o, o, o)$, $\omega_2 = (o, o, i), \ldots, \omega_8 = (i, i, i)$. As before, we take the discount factor equal to 0.95 and normalize $(a - c)^2/b = 1$. In each period, an active firm gets a profit of 1/4 when only this firm is active in the market; when there are two firms in the market, they both get profits of 1/9; when all three firms are active, each of them gets a profit of 1/16. To illustrate the efficiency of the LB-DH method, we compare it to the SLTP method for solving this game. By starting both methods from the same randomly generated point, we obtain the PeSE, where all firms are active in the next period whatever the current state is.

The LB-DH method turns out to be much more efficient than the SLTP method both in terms of number of iterations and computation time. More specifically, it takes SLTP 62 iterations and 48.19 s to find the PeSE, while the LB-DH method only needs 52 iterations and 25.98 s to find the same equilibrium. The development of the distance to the PeSE for both methods is plotted in Fig. 11.

Table 8 shows the total expected profits to the firms by coordinating on this equilibrium for each possible initial state.

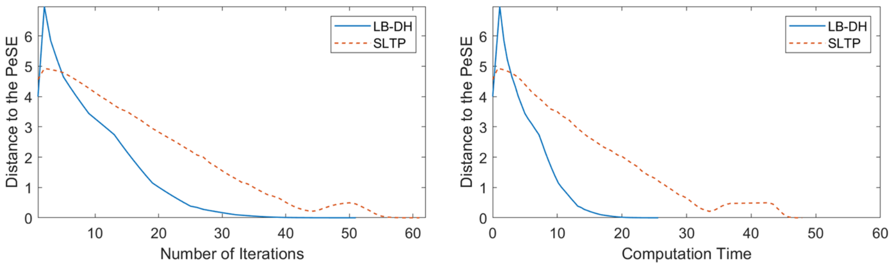

Fig. 11 Development of the distance to a PeSE in the various iterations and computation time

Table 8 Total expected profits of both firms in each state

Total profits	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
Firm 1	1.1875	1.1875	1.1875	1.4375	1.1875	1.2986	1.2986	1.2500
Firm 2	1.1875	1.1875	1.4375	1.1875	1.2986	1.1875	1.2986	1.2500
Firm 3	1.1875	1.4375	1.1875	1.1875	1.2986	1.2986	1.1875	1.2500

6 Conclusions and future research

In this paper, we have extended the concept of perfect equilibrium for strategic games to stochastic games and formulated the notion of perfect stationary equilibrium (PeSE), which can effectively eliminate some counterintuitive stationary equilibria in stochastic games. To find such an equilibrium, we have developed a logarithmic-barrier differentiable homotopy (LB-DH) method. The basic idea of the method is incorporating a logarithmic-barrier term into the objective functions of the original stochastic game and constituting a logarithmic-barrier stochastic game. We have been able to eliminate the Bellman equations in our homotopy system, which significantly reduces the number of variables in the equilibrium system of the logarithmic-barrier game. We have proved that the set of solutions to the resulting system contains a differentiable homotopy path, which starts from an arbitrary given point and ends at a PeSE for the stochastic game of interest.

In numerical experiments, we have applied the LB-DH method to extensive stochastic games. To elicit the effectiveness of the LB-DH method for selecting a particular SSPE satisfying the perfectness criterion, we have implemented our method and two well-known homotopy methods—the IPM and the SLTP, to solve the stochastic games with multiple SSPEs and a unique PeSE. Experimental results illustrate that the IPM and the SLTP may end at a non-perfect SSPE while the LB-DH method always leads to the unique PeSE. To illustrate the numerical efficiency of the LB-DH method, we have compared it with the stochastic extension of an existing method, called the convex-quadratic-penalty homotopy (CQP-DH) method, on extensive randomly generated stochastic games. Numerical results show that the LB-DH method significantly outperforms the CQP-DH method in terms of computation time. We have also confirmed by numerical comparisons that the LB-DH method benefits from the elimination of the Bellman equations. Furthermore, we have used our method to shed new light on several important economic applications: dynamic legislative bargaining and dynamic oligopoly with entry and exit. The perspective of the proposed method creates some opportunities to investigate several other refinements of stationary equilibria, such as proper stationary equilibria and perfect d-proper stationary equilibria.

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

Appendix I

Theorem 8 (Transversality Theorem, Mas-Colell (1989)) *Let* $f : S \times \mathbb{R}^l \to \mathbb{R}^s$ *be* C^r, *where* $S \subset \mathbb{R}^n$ *is an open set and* $r \geq 1 + \max\{0, n - s\}$. *If zero is a regular value of* f, *then zero is a regular value of* $f(\cdot, w) : S \to \mathbb{R}^s$ *for almost all* $w \in \mathbb{R}^l$.

Appendix II

This appendix shows that the Jacobian matrix of p at $(z, \mu, 1) \in \mathbb{R}^m \times \mathbb{R}^{nd} \times \{1\}$ such that $p(z, \mu, 1) = 0$ is of full rank. This result is used in the proof of Lemma 2. At $t = 1$, system (17) reduces to

$$\begin{aligned} &\lambda^i_{\omega j}(z, 1) - \mu^i_\omega - \eta_0 \sum_{k \in M^i_\omega} \lambda^i_{\omega k}(z, 1) = 0, \ j \in M^i_\omega, \ \omega \in \Omega, \ i \in N, \\ &\sum_{j \in M^i_\omega} x^i_{\omega j}(z, 1) - 1 = 0, \qquad \omega \in \Omega, \ i \in N. \end{aligned} \tag{27}$$

The Jacobian matrix of p at the starting point $(z, \mu, 1)$ reads as

$$Jp(z, \mu, 1) = \begin{pmatrix} A_0 & -\text{diag}(\mathbf{e}_{m^i_\omega}) \\ B_0 & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{(m+nd)\times(m+nd)},$$

where $\mathbf{e}_{m_\omega^i} \in \mathbb{R}^{m_\omega^i}$ is a column vector with all elements equal to one and $A_0 = \text{diag}(C_\omega^i : \omega \in \Omega, i \in N) \in \mathbb{R}^{m \times m}$ is a block diagonal matrix with

$$C_\omega^i = \begin{pmatrix} (1-\eta_0)\frac{\partial \lambda_{\omega 1}^i(z,1)}{\partial z_{\omega 1}^i} & -\eta_0 \frac{\partial \lambda_{\omega 2}^i(z,1)}{\partial z_{\omega 2}^i} & \cdots & -\eta_0 \frac{\partial \lambda_{\omega m_\omega^i}^i(z,1)}{\partial z_{\omega m_\omega^i}^i} \\ -\eta_0 \frac{\partial \lambda_{\omega 1}^i(z,1)}{\partial z_{\omega 1}^i} & (1-\eta_0)\frac{\partial \lambda_{\omega 2}^i(z,1)}{\partial z_{\omega 2}^i} & \cdots & -\eta_0 \frac{\partial \lambda_{\omega m_\omega^i}^i(z,1)}{\partial z_{\omega m_\omega^i}^i} \\ \vdots & \vdots & \ddots & \vdots \\ -\eta_0 \frac{\partial \lambda_{\omega 1}^i(z,1)}{\partial z_{\omega 1}^i} & -\eta_0 \frac{\partial \lambda_{\omega 2}^i(z,1)}{\partial z_{\omega 2}^i} & \cdots & (1-\eta_0)\frac{\partial \lambda_{\omega m_\omega^i}^i(z,1)}{\partial z_{\omega m_\omega^i}^i} \end{pmatrix}.$$

Moreover,

$$B_0 = \frac{\partial x(z,t)}{\partial z} = \text{diag}(\partial x_\omega^i) \in \mathbb{R}^{nd \times m}$$

with

$$\partial x_\omega^i = \left(\frac{\partial x_{\omega j}^i}{\partial z_{\omega j}^i}\right)_{j \in M_\omega^i} \in \mathbb{R}^{1 \times m_\omega^i}.$$

Now we prove that C_ω^i is of full rank. Suppose there exists a vector $v \in \mathbb{R}^{m_\omega^i}$ such that $C_\omega^i v = 0$. That is,

$$\begin{aligned} &(1-\eta_0)\frac{\partial \lambda_{\omega 1}^i(z,1)}{\partial z_{\omega 1}^i} v_1 - \eta_0 \frac{\partial \lambda_{\omega 2}^i(z,1)}{\partial z_{\omega 2}^i} v_2 - \ldots - \eta_0 \frac{\partial \lambda_{\omega m_\omega^i}^i(z,1)}{\partial z_{\omega m_\omega^i}^i} v_{m_\omega^i} = 0, \\ &-\eta_0 \frac{\partial \lambda_{\omega 1}^i(z,1)}{\partial z_{\omega 1}^i} v_1 + (1-\eta_0)\frac{\partial \lambda_{\omega 2}^i(z,1)}{\partial z_{\omega 2}^i} v_2 - \ldots - \eta_0 \frac{\partial \lambda_{\omega m_\omega^i}^i(z,1)}{\partial z_{\omega m_\omega^i}^i} v_{m_\omega^i} = 0, \\ &\quad \vdots \\ &-\eta_0 \frac{\partial \lambda_{\omega 1}^i(z,1)}{\partial z_{\omega 1}^i} v_1 - \eta_0 \frac{\partial \lambda_{\omega 2}^i(z,1)}{\partial z_{\omega 2}^i} v_2 \ldots + (1-\eta_0)\frac{\partial \lambda_{\omega m_\omega^i}^i(z,1)}{\partial z_{\omega m_\omega^i}^i} v_{m_\omega^i} = 0. \end{aligned} \tag{28}$$

Summing all the equations in the system above, we have that

$$(1 - m_\omega^i \eta_0)\left(\frac{\partial \lambda_{\omega 1}^i(z,1)}{\partial z_{\omega 1}^i} v_1 + \frac{\partial \lambda_{\omega 2}^i(z,1)}{\partial z_{\omega 2}^i} v_2 + \cdots + \frac{\partial \lambda_{\omega m_\omega^i}^i(z,1)}{\partial z_{\omega m_\omega^i}^i} v_{m_\omega^i}\right) = 0.$$

Recall that $\eta_0 < 1/\max_{\omega\in\Omega,i\in N} m^i_\omega$. Therefore it holds for all $i \in N$ and $\omega \in \Omega$ that $1 - m^i_\omega \eta_0 > 0$. It follows that

$$\frac{\partial \lambda^i_{\omega 1}(z,1)}{\partial z^i_{\omega 1}} v_1 + \frac{\partial \lambda^i_{\omega 2}(z,1)}{\partial z^i_{\omega 2}} v_2 + \cdots + \frac{\partial \lambda^i_{\omega m^i_\omega}(z,1)}{\partial z^i_{\omega m^i_\omega}} v_{m^i_\omega} = 0.$$

Multiplying both sides of the above equation by η_0 and adding the result to the first equation in system (28), we obtain that

$$\frac{\partial \lambda^i_{\omega 1}(z,1)}{\partial z^i_{\omega 1}} v_1 = 0.$$

Similarly, one can prove that for any $j \in M^i_\omega$,

$$\frac{\partial \lambda^i_{\omega j}(z,1)}{\partial z^i_{\omega j}} v_j = 0.$$

At the starting point $z^i_{\omega j} = (x^{0,i}_{\omega j} - \eta_0)^{1/\kappa} - 1$ it holds that

$$\frac{\partial \lambda^i_{\omega j}(z,1)}{\partial z^i_{\omega j}} = \frac{\kappa}{2}\left(\frac{z^i_{\omega j}}{z^i_{\omega j}+2} - 1\right),$$

which is obviously negative. Consequently, $v = 0$, which implies that C^i_ω is of full rank. Hence, A_0 is also of full rank.

Next, at the starting point $z^i_{\omega j} = (x^{0,i}_{\omega j} - \eta_0)^{1/\kappa} - 1$, it holds that

$$\frac{\partial x^i_{\omega j}(z,1)}{\partial z^i_{\omega j}} = \kappa \frac{z^i_{\omega j}+1}{z^i_{\omega j}+2},$$

which is strictly positive. Therefore, B_0 is clearly of full row rank.

By applying standard row operations to the Jacobian matrix $Jp(z, \mu, 1)$, one transforms this Jacobian matrix to the following matrix,

$$\begin{pmatrix} A_0 & -\text{diag}(\mathbf{e}_{m^i_\omega}) \\ \mathbf{0} & B_0 A_0^{-1} \text{diag}(\mathbf{e}_{m^i_\omega}) \end{pmatrix}.$$

Since B_0 and $A_0^{-1}\text{diag}(\mathbf{e}_{m^i_\omega})$ are both diagonal matrices, it follows that $B_0 A_0^{-1}\text{diag}(\mathbf{e}_{m^i_\omega})$ is a diagonal matrix. We now compute the diagonal element corresponding to $i \in N$

and $\omega \in \Omega$, which is equal to $\partial x^i_\omega (C^i_\omega)^{-1} \mathbf{e}_{m^i_\omega}$. We define

$$v = \frac{1}{1 - m^i_\omega \eta_0} \left(\frac{1}{\frac{\partial \lambda^i_{\omega 1}(z,1)}{\partial z^i_{\omega j}}}, \dots, \frac{1}{\frac{\partial \lambda^i_{\omega m^i_\omega}(z,1)}{\partial z^i_{\omega m^i_\omega}}} \right)^\top,$$

a strictly negative column vector in $\mathbb{R}^{m^i_\omega}$. It holds that $C^i_\omega v = \mathbf{e}_{m^i_\omega}$, so our designated diagonal element is equal to

$$\partial x^i_\omega (C^i_\omega)^{-1} \mathbf{e}_{m^i_\omega} = \partial x^i_\omega (C^i_\omega)^{-1} C^i_\omega v = \partial x^i_\omega v,$$

the product of a strictly positive and a strictly negative vector, so a strictly negative number. It follows that $B_0 A_0^{-1} \mathrm{diag}(\mathbf{e}_{m^i_\omega})$ is of full rank. As a result, $Jp(z, \mu, 1)$ is of full rank.

Appendix III

We prove in this appendix that the Jacobian matrix of p has full row rank if $t \in (0, 1)$. This result is used in the proof of Theorem 4. When $t \in (0, 1)$, the Jacobian matrix of $p(z, \mu, t; \gamma)$ reads as

$$Jp(z, \mu, t; \gamma) = \begin{pmatrix} \frac{\partial p_1}{\partial z} & \frac{\partial p_1}{\partial \mu} & \frac{\partial p_1}{\partial t} & -t(1-t)I_m \\ B_0 & \mathbf{0} & \frac{\partial p_2}{\partial t} & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{(m+nd)\times(2m+nd+1)},$$

where p_1 and p_2 represent the first and second groups of equations in system (18), respectively. The matrix B_0 has been defined in Appendix II and has full row rank. Obviously, $-t(1-t)I_m$ is of full rank. It follows immediately that the Jacobian matrix $Jp(z, \mu, t; \gamma)$ has full row rank and $\mathrm{Rank}[Jp(z, \mu, t; \gamma)] = m + nd$. This together with Lemma 2 establishes that zero is a regular value of p on $\mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1] \times \mathbb{R}^m$.

Appendix IV

We prove in this appendix that zero is a regular value of h on $\mathbb{R}^m \times \mathbb{R}^{nd} \times (0, 1] \times \mathbb{R}^m$. This result is used in the proof of Theorem 7.

First, let us consider the case that $t = 1$. System (25) becomes

$$
\begin{aligned}
&-(x^i_{\omega j}(y,1) - x^{0,i}_{\omega j}) + \lambda^i_{\omega j}(y) - \mu^i_\omega - \eta_0 \sum_{j \in M^i_\omega} \lambda^i_{\omega j}(y) = 0, \quad j \in M^i_\omega,\ \omega \in \Omega,\ i \in N,\\
&\sum_{j \in M^i_\omega} x^i_{\omega j}(y,1) - 1 = 0, \qquad \omega \in \Omega,\ i \in N.
\end{aligned}
\tag{29}
$$

We evaluate the Jacobian matrix of h at a point $(y, \mu, 1) \in \mathbb{R}^m \times \mathbb{R}^{nd} \times \{1\}$ such that $h(y, \mu, 1) = 0$. The matrix is given by

$$
Jh(y,\mu,1) = \begin{pmatrix} A & -\mathrm{diag}(\mathbf{e}_{m^i_\omega}) \\ B & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{(m+nd)\times(m+nd)},
$$

where $\mathbf{e}_{m^i_\omega} \in \mathbb{R}^{m^i_\omega}$ is a column vector with all elements equal to one, so $\mathrm{diag}(\mathbf{e}_{m^i_\omega}) \in \mathbb{R}^{m\times nd}$, and $A = \ell \cdot \mathrm{diag}(D^i_\omega : \omega \in \Omega, i \in N) \in \mathbb{R}^{m\times m}$ is a block diagonal matrix with

$$
D^i_\omega = \begin{pmatrix}
-\xi^i_{\omega 1} - (1-\eta_0) f^i_{\omega 1} & \eta_0 f^i_{\omega 2} & \cdots & \eta_0 f^i_{\omega m^i_\omega} \\
\eta_0 f^i_{\omega 1} & -\xi^i_{\omega 2} - (1-\eta_0) f^i_{\omega 2} & \cdots & \eta_0 f^i_{\omega m^i_\omega} \\
\vdots & \vdots & \ddots & \vdots \\
\eta_0 f^i_{\omega 1} & \eta_0 f^i_{\omega 2} & \cdots & -\xi^i_{\omega m^i_\omega} - (1-\eta_0) f^i_{\omega m^i_\omega}
\end{pmatrix},
$$

where $\xi^i_{\omega j} = \max\{0, y^i_{\omega j}\}^{\ell-1}$ and $f^i_{\omega j} = \max\{0, -y^i_{\omega j}\}^{\ell-1}$. Moreover, it holds that $B = \ell \cdot \mathrm{diag}(\xi^{i\top}_\omega) \in \mathbb{R}^{nd\times m}$, where $\xi^i_\omega = (\xi^i_{\omega j})_{j\in M^i_\omega}$ is a column vector with dimension m^i_ω.

Since $h(y, \mu, 1) = 0$, it holds that, for every $i \in N$, for every $\omega \in \Omega$, for every $j \in M^i_\omega$, $y^i_{\omega j} = (x^{0,i}_{\omega j} - \eta_0)^{1/\ell} > 0$, so the matrix A is a full-rank diagonal matrix and the matrix B is of full row rank. By row operations, the Jacobian matrix $Jh(y, \mu, 1)$ can be transformed to

$$
\begin{pmatrix} A & -\mathrm{diag}(\mathbf{e}_{m^i_\omega}) \\ \mathbf{0} & BA^{-1}\mathrm{diag}(\mathbf{e}_{m^i_\omega}) \end{pmatrix}
$$

The matrix $BA^{-1}\mathrm{diag}(\mathbf{e}_{m^i_\omega}) \in \mathbb{R}^{nd\times nd}$ is a diagonal matrix with rank nd. Therefore, the Jacobian matrix $Jh(y, \mu, 1)$ is of full rank, which shows that zero is a regular value of $h(y, \mu, 1)$.

Next, we consider the case that $t \in (0, 1)$. We evaluate the Jacobian matrix of h at a point $(y, \mu, t; \alpha) \in \mathbb{R}^m \times \mathbb{R}^{nd} \times (0,1) \times \mathbb{R}^m$ such that $h(y, \mu, t; \alpha) = 0$. It is given by

$$
Jh(y,\mu,t;\alpha) = \begin{pmatrix} E_1 & E_2 & E_3 & t(1-t)I_m \\ B & \mathbf{0} & \eta_0 \mathbf{e} & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{(m+nd)\times(2m+nd+1)},
$$

where B is defined as above and $\mathbf{e} \in \mathbb{R}^{nd}$. The matrices $E_1 \in \mathbb{R}^{m\times m}$, $E_2 \in \mathbb{R}^{m\times nd}$, and $E_3 \in \mathbb{R}^{m\times 1}$ are the derivatives of the first group of equations with respect to y, μ, and t, respectively. Clearly, $t(1-t)I_m$ is of rull rank m when $t \in (0, 1)$. It follows from the previous discussion that B is of full row rank nd. It follows that the rank of the Jacobian matrix $Jh(y, \mu, t;\alpha)$ is $(m+nd)$. Therefore, $Jh(y, \mu, t;\alpha)$ is of full row rank. This completes the proof.

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