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RESEARCH ARTICLE

Contributing with private bundles to public goods

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Abstract
We extend to multiple private commodities the seminal model by Bergstrom et al. (J Public Econ 29:25–49, 1986) on the private provision of public goods. Considering the relative value of the aggregate donations, we define a notion of equilibrium and show its existence. We analyze the effects of resource redistributions on the equilibrium outcome, identifying conditions that guarantee neutrality. We study some further properties of the contribution equilibrium, and provide a strategic market game approach, defining a sequence of non-cooperative games whose equilibria converge to an equilibrium of the economy.

Keywords Public goods · Private provision · Voluntary contributions

JEL Classification D00 · D50 · D40 · H00

1 Introduction

Society faces a wide range of public and social issues for which it often becomes necessary to receive adequate funding and contributions. The difficulties in obtaining public support for the goals or an insufficient public budget make private donations worth considering. For instance, several health or environment-related institutions, among others, request contributions from citizens to provide goods and services that entail

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characteristics of public goods.[1] Individuals' contributions are also acknowledged for the preservation of historical buildings, museums, or cultural heritage.[2]

The issue of private contributions to social objectives has received much attention in the economics literature. Bergstrom et al. (1986), henceforth BBV, state a pioneering model of non-cooperative provision of public goods, giving rise to plenty of research papers.[3] The insightful BBV's approach explores a scenario in which a private good can either be consumed or donated for public aims. Thus, the authors analyze a game-theoretic model where the strategy of each player is their voluntary contribution to provide a public good.[4]

The BBV's work assumes just one private commodity and a public good, becoming a partial equilibrium model. Thus, one finds a variety of extensions for more general settings. Specifically, to analyze relative price modifications, a collection of papers considers several private goods that can be consumed or used as inputs by firms producing one public good, privately financed. See, for instance, Villanacci and Zenginobuz 2005; 2006; 2007; 2012, and Faias et al. (2014, 2015).[5]

The above references select a technology that transforms multiple private commodities into a public good. This detailing is crucial from both the production mechanism and the market viewpoints. Hence, the way to generalize BBV's model is not unique, and the consequences of relative price variations rely on the technology considered and, in particular, on their returns to scale.

This paper deviates from production requirements by proposing a different extension of BBV's model. Our approach is inspired by the fact that people often contribute bundles of goods to social causes that impact the lives of others. We find organizations that channel individual donations, pursuing goals that increase the welfare of society. Citizens donate food, clothes, shoes, furniture, books, toys, wheelchairs, and medicines, among other private commodities, to charity organizations that deliver them to people in need. Specifically, food banks play a role in providing food security to a portion of the disadvantaged population. In other situations, such as wars and natural disasters (wars, floods, earthquakes) material goods are also requested to help the victims. Our scheme focuses on these private contributions originating positive externalities understood as public goods.

In this way, we present a general equilibrium model, where individuals are endowed with multiple private commodities and decide both consumption plans and a part of their initial bundle to contribute to a public good. That is, as in BBV's paper, private resources can either be consumed or donated. A vector of prices defines the budget sets and the percentage donated of the total endowment worth. Indeed, the public good enters the utility function of every consumer as the relative value of the aggregate bundle of donations at the current prices. Then, taking into account the voluntary

[1] To mention a few: Cancer Associations in different countries, MSF international, Greenpeace, World Wildlife Fund,...

[2] Recent events highlighting the importance of such contributions are the Brazilian National Museum fire in 2018 or the Notre Dame fire in 2019.

[3] Based on this proposal, one also finds models studying voluntary public good provision in different frameworks where information issues play a role (see Barbieri 2023, and Lamprecht and Thum 2023).

[4] They show equilibrium existence, and present a comparative statics study about wealth distributions.

[5] Faias et al. (2020) recapitulate some basic results.

contributions of the others, all the information an agent needs to decide is in the prices of private commodities.[6] Consequently, we define a notion of contribution equilibrium showing its existence and interpreting it in monetary terms.

Furthermore, the analysis of individuals' contributions to public goods includes a remarkable result, referred to as the neutrality theorem, which was first established by Warr (1983), extended by BBV, and has inspired considerable literature. It states that any reallocation of income among contributors, whenever they do not lose more than their current donations, maintains the original private consumption and generates identical public goods. This neutrality result relies on the assumption of one private commodity and one public good, which allows a common measure in monetary units for all the variables, as different works highlight.[7]

We describe a counter-example showing that the repercussion of relative prices is also the key to non-neutrality. However, we find conditions on resource redistribution that guarantee neutrality. Explicitly, for each of the multiple private commodities, no contributor loses more than the amount given for the public good, which is a natural extension of the requirement in BBV's paper. The proof is constructive, stating how individual donations are modified to get the same outcome for every redistribution. Moreover, we prove that the previous condition is sufficient but not necessary for neutrality. We also confirm that a weaker assumption, requiring the value of the reallocation of endowments enough to purchase the original equilibrium private bundles, does not ensure neutrality. The example showing it allows us to identify further necessary conditions for the equilibrium to be neutral when facing resource reassignments.

Regarding Pareto optimality, the contribution equilibrium, in general, is not efficient. The inefficiency comes from the positive externality underlying the public good, which leads to market failure. Despite this, if we fix the value of the total donations, the corresponding equilibrium consumption allocation is efficient since, as we show, it is a Walrasian outcome of the resulting exchange economy.

We remark that the definition of the equilibrium solution, the existence result, and the studied properties do not explain the formation of the prices. To overcome this point, we present a game describing the mechanism leading to equilibria and focusing on price formation. We follow the adaptation proposed by Dubey and Geanakoplos (2003) from the seminal game by Shapley and Shubik (1977). Despite that, the market game we consider differs from theirs, provided the presence of donations requires reformulating the rules for price formation and the allocation of private commodities. The strategies are bids to buy consumption goods and to contribute for social purposes, and the payoff functions depend on the consumption bundles and the aggregate donations. Our market game allows us to show the convergence of a sequence of Nash equilibria to a contribution equilibrium of the economy.

The examples, the obtained results, and the game approach we state exhibit the contrasts, revealing the conceptual differences, between the extension of the BBV's

6 Hence, our proposal reduces the problem of equilibrium prices in as many dimensions as the number of public goods considered in the mentioned models with production.

7 When the model contemplates multiple private commodities, Villanacci and Zenginobuz (2007, 2012) show that the relative prices and the returns to scale properties of the technologies transforming private commodities into a public good are relevant for the impact of resources redistribution on the equilibrium.

model that we propose and those previously addressed in the related works, which consider private commodities as inputs for producing public goods.

The remainder of this paper is structured as follows. In Sect. 2, we define the contribution equilibrium concept stating its existence. In Sect. 3, we interpret our non-cooperative solution in monetary terms and state an example, for which we elaborate on a comparative statics study about redistributions of endowments. In Sect. 4, we analyze neutrality. In Sect. 5, we deduce other properties of the equilibrium. In particular, we define an n-types atomless economy that, as equilibria are concerned, is equivalent to the initial economy with n consumers. This coincidence is helpful in Sect. 6, where we associate a market game with the original economy obtaining a convergence result for the contribution equilibrium. In Sect. 7, we state some concluding remarks. A final Appendix contains all the proofs.

2 The model and contribution equilibrium

Let us consider an economy $\mathcal{E}$ with a finite number ℓ of private goods and a set N of n consumers. Each individual $i \in N = \{1, \ldots, n\}$ is endowed with a vector $\omega_i \in \mathbb{R}^{\ell}_{+}$ of private goods. The role of endowments in the economy is two-fold. Namely, resources can be used for private consumption and as donations for social and public purposes.

A price system $p = (p_h, h = 1, \ldots, \ell) \in \mathbb{R}^{\ell}_{+}$, specifies the prices for the ℓ private commodities. Given a price vector $p \in \mathbb{R}^{\ell}_{+}$, consumers choose private consumption bundles and voluntary contributions to provide a public good.[8] These private contributions are given as part of their endowments and each consumer decides her donation given the contributions of others. The public good enters in the consumers' preferences as the percentage value of the endowments that they decide to donate concerning the value of the original total resources. Thus, each consumer i has a preference relation represented by the utility function $U_i : \mathbb{R}^{\ell+1}_{+} \to \mathbb{R}$. The first ℓ coordinates of a vector in the domain $\mathbb{R}^{\ell+1}_{+}$ specify the private consumption bundle, whereas the last one indicates the relative value of the total donations at the current prices.[9]

To be precise, given a price system p and a vector $(e_j, j \in N, j \neq i)$ of voluntary contributions, each consumer i solves the problem:

$$\begin{aligned} \max_{(x,e)\in \mathbb{R}^{\ell}_{+}\times[0,\omega_i]} \quad & U_i(x, \hat{p}\cdot \bar{e}_{-i} + \hat{p}\cdot e) \\ s.t. \quad & p\cdot x + p\cdot e \leq p\cdot \omega_i \end{aligned}$$

where $\hat{p} = p/(p \cdot \bar{\omega})$, being $\bar{\omega} = \sum_{i=1}^{n} \omega_i$, and $\bar{e}_{-i} = \sum_{j\neq i} e_j$. This notation will be used along the paper.

[8] We can consider only one public good or that individuals contribute to multiple public goods without deciding the distribution of donations among them. Thus, a variant of the model could also contemplate the case in which each one decides a vector of contributions corresponding to a variety of social objectives.

[9] Addressing different scenarios, other works considering price-dependent preference include, for instance, Pollack (1977), Balasko (2003), Correia-da-Silva and Hervés-Beloso (2008), more recently, Cea-Echenique et al. (2017), Podczeck and Yannelis (2022), and Reck and Seibold (2022), among others.

Definition A *contribution equilibrium* for the economy $\mathcal{E}$ is a price system p, a private consumption allocation $x = (x_i, i \in N)$, and a collection of contributions $e = (e_i, i \in N)$ such that, $\sum_{i=1}^n x_i + \bar{e} \leq \sum_{i=1}^n \omega_i$, with $\bar{e} = \sum_{i=1}^n e_i$, and (x_i, e_i) solves the problem of consumer i, for every $i \in N$.

In our model, decisions on both consumption and contributions are not modified as long as the relative prices remain the same. The percentage adjustment of the values of the private contributions provides a consistent measure in *real* terms, which takes values in the unit interval [0, 1]. In this way, the relative prices guide the allocations of private consumption and donations, and all the information agents need to decide is in prices.

To show the existence of equilibrium we state the following standard assumptions for every $i \in N$:

(A.1) *Interiority of endowments.* $\omega_i \in \mathbb{R}^\ell_{++}$.

(A.2) *Continuity, monotonicity, and convexity of preferences.* U_i is a continuous, non-decreasing, and quasi-concave function.[10]

Theorem 2.1 *Under assumptions (A.1) and (A.2), a contribution equilibrium exists.*

To prove this theorem, we define a *pseudo-game*, a tool used to obtain existence results in various contexts.[11] The participants are the consumers and a Walrasian auctioneer. The strategies and payoffs of the players lead to a correspondence whose fixed point is an equilibrium of the economy. The Maximum Theorem and Kakutani's fixed point theorem are applied. For it, we use assumptions (A.1) and (A.2).

Let $B_i(p) = \{(x_i, e_i) \in \mathbb{R}^\ell_+ \times [0, \omega_i]$ such that $p \cdot x_i + p \cdot e_i \leq p \cdot \omega_i\}$ be the individual i's budget set at prices p. Moreover, given a price system p and a vector of contributions $e = (e_i, i \in N)$, let $\hat{G}(p, e) = \hat{p} \cdot \sum_{i \in N} e_i$. We stress that the individual budget restrictions and $\hat{G}(\cdot, e)$ are homogeneous of degree zero. That is, $B_i(\lambda p) = B_i(p)$ and $\hat{G}(\lambda p, e) = \hat{G}(p, e)$ for every $\lambda > 0$. Therefore, if (p^*, x^*, e^*) is an equilibrium, then $(\lambda p^*, x^*, e^*)$ is also an equilibrium. It implies that, regarding prices, only relative prices matter, as in general equilibrium models. Then, without loss of generality, we can consider normalized prices. In particular, in the proof of the existence result, we look for equilibrium prices in the simplex of $\mathbb{R}^\ell_+$.

3 Equilibrium in monetary terms

We can rewrite the notion of equilibrium by considering that consumers decide to donate part of the value of their endowments to contribute to supporting a public good. That is, given a price system $p \in \mathbb{R}^\ell_+$, each consumer i chooses an amount $m_i \in [0, p \cdot \omega_i]$ to donate, given the value of the contributions of other individuals. Formally, given a price p and a vector $(m_j, j \in N, j \neq i)$ of monetary contributions, each consumer i solves the problem:

[10] The quasi-concavity property required is as follows: If $U_i(A) \neq U_i(B)$, then $U_i(\lambda A + (1 - \lambda)B) > \min\{U(A), U(B)\}$.

[11] See, for instance, Ichiisi (1983) and Hervés-Beloso et al. (2012), among others.

$$\begin{aligned} \max_{(x,m)\in \mathbb{R}_+^{\ell+1}} \quad & U_i\left(x, \hat{m}_{-i} + \hat{m}\right) \\ s.t. \quad & p \cdot x + m \le p \cdot \omega_i \end{aligned} \tag{M}$$

where $\hat{m}_{-i} = \dfrac{\sum_{j\neq i} m_j}{p \cdot \bar{\omega}}$, and $\hat{m} = \dfrac{m}{p \cdot \bar{\omega}}$.

Definition A *monetary equilibrium* for the economy $\mathcal{E}$ is a price system p, a private consumption allocation $x = (x_i, i \in N)$, and a vector $m = (m_i, i \in N)$ of monetary contributions to public goods, such that, there is a collection of bundles $(e_i, i \in N)$ for which the following conditions hold:

(i) $e_i \in [0, \omega_i]$ for every i, and $p \cdot e_i = m_i$.
(ii) (x_i, m_i) solves the problem (M) for every consumer $i \in N$.
(iii) $\sum_{i=1}^n x_i + \bar{e} \le \sum_{i=1}^n \omega_i$, with $\bar{e} = \sum_{i=1}^n e_i$.

Note that the monetary equilibrium is equivalent to the contribution equilibrium notion, in the following sense:

- If (p, x, m) is a monetary equilibrium, then (p, x, e) is a contribution equilibrium for any e decentralizing m at prices p.
- If (p, x, e) is a contribution equilibrium, then (p, x, m), with $m_i = p \cdot e_i$, is a monetary equilibrium.

The following example illustrates the contribution equilibrium and its relationship with the concept in monetary terms as we have specified. Moreover, we present a comparative statics study that analyses the impact of redistributions of endowments.

An example. Let us consider an economy with two consumers, 1 and 2, and two private commodities. Both individuals have the same utility function $U(x, y, G) = \ln x + \ln y + \ln G$, where G is the variable corresponding to the public good that is obtained from private contributions of resources. Agent 1 has endowments $\omega_1 = (\omega_1^x, \omega_1^y) = (1, 2)$ while agent 2 is endowed with $\omega_2 = (\omega_2^x, \omega_2^y) = (2, 1)$. An equilibrium is given by $p^* = (1, 1)$, the private consumption bundle $\alpha^* = (6/5, 6/5)$ for both consumers and any private donation vectors $e_1^* = (a, b)$, $e_2^* = (b, a)$ such that $e_1^* + e_2^* = (3/5, 3/5)$. That is, the value of the aggregate contribution is $1/5$ of $p^* \cdot (3, 3)$, i.e., 20% of the value of the total resources. The equivalent monetary equilibrium is given by the previous prices and consumption bundles, and contributions $m_1 = m_2 = 3/5$. We remark that this equilibrium can be decentralized by a continuum of private donations, as previously stated.

Let Ω be the set of redistributions of endowments that allow both individuals to consume the equilibrium bundle at p^*. That is,

$$\begin{aligned} \Omega = \{\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \mid \hat{\omega}_1 + \hat{\omega}_2 = (3, 3) \text{ and} \\ p^* \cdot \hat{\omega}_i = \hat{\omega}_i^x + \hat{\omega}_i^y \ge 12/5, i = 1, 2\}. \end{aligned}$$

For any redistribution of endowments $\hat{\omega} \in \Omega$, we have that $p^* = (1, 1)$ and $\alpha^* = (6/5, 6/5)$ remain as the equilibrium price and private bundle for both agents, respectively, and the equilibrium contributions are the collection of vectors $e(\hat{\omega}) = (e_1(\hat{\omega}), e_2(\hat{\omega})) \in [0, \hat{\omega}_1] \times [0, \hat{\omega}_2]$ given by:

$$e(\hat{\omega}) = \Big\{(\hat{e}_1, \hat{e}_2) \in [0, \hat{\omega}_1] \times [0, \hat{\omega}_2] \mid \hat{e}_1 + \hat{e}_2 = e_1^* + e_2^* = (3/5, 3/5),\\ \hat{e}_1^x + \hat{e}_1^y = p^* \cdot \hat{\omega}_1 - \tfrac{12}{5},\ \text{and}\ \hat{e}_2^x + \hat{e}_2^y = p^* \cdot \hat{\omega}_2 - \tfrac{12}{5}\Big\}.$$

Therefore, any redistribution in Ω leads to the same equilibrium outcome. To illustrate this point graphically, we partition Ω into three subsets, as represented in Fig. 1.

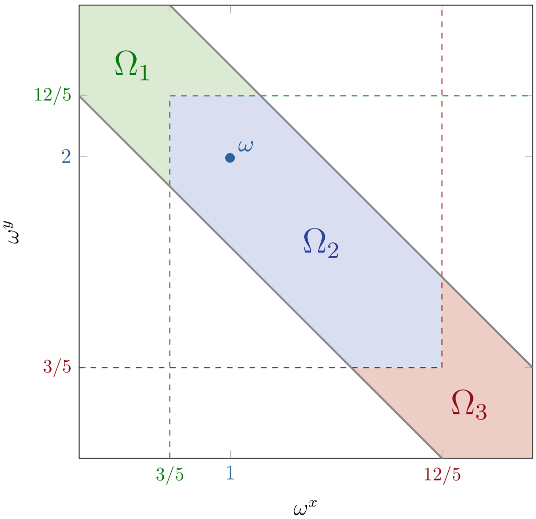

Fig. 1 Resources redistributions in $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. $\Omega_1 = \left\{\hat{\omega} \in \Omega \mid \hat{\omega}_1^x < 3/5 \text{ or } \hat{\omega}_2^y < 3/5\right\}$, $\Omega_2 = \left\{\hat{\omega} \in \Omega \mid \hat{\omega}_1^x, \hat{\omega}_1^y \in [3/5, 12/5]\right\}$, and $\Omega_3 = \left\{\hat{\omega} \in \Omega \mid \hat{\omega}_1^y < 3/5 \text{ or } \hat{\omega}_2^x < 3/5\right\}$

We remark that for redistributions in both Ω_1 and Ω_3, unlike Ω_2, the new resource allocation restricts the potential individual contributions compatible with the original equilibrium. The following graphs represent the implications of the reallocations of resources on the equilibrium donations.

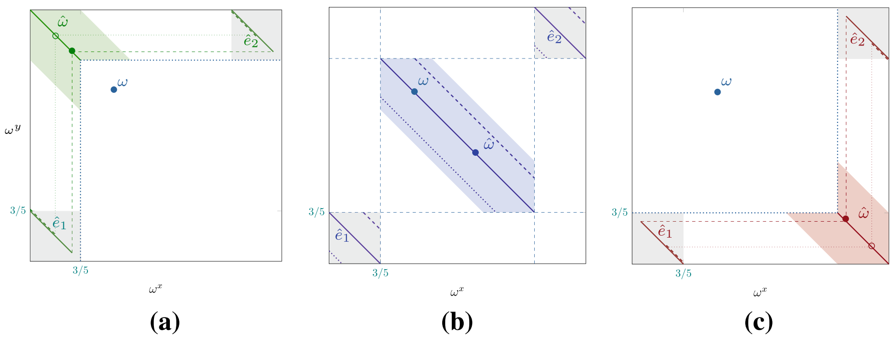

Fig. 2 Effects on contributions of redistributions in **a** Ω_1, **b** Ω_2, and **c** Ω_3

Figure 2b illustrates the effect of redistributions in Ω_2. Along the solid line, the value of the endowment remains the same for both consumers, i.e., $p^*\hat{\omega}_i = p^*\omega_i = 3$, for $i = 1, 2$. Then, $e(\hat{\omega}) = e(\omega)$ for every $\hat{\omega} \in \Omega_2$. We deduce that the equilibrium contributions are not modified and are given by any $(\hat{e}_1, \hat{e}_2)$ such that $\hat{e}_1 + \hat{e}_2 = (3/5, 3/5)$, and $\hat{e}_1^x + \hat{e}_1^y = \hat{e}_2^x + \hat{e}_2^y = 3/5$. If the redistribution lies to the right (resp. left) of the solid line, the wealth of individual 1 increases (resp. decreases) while the wealth of individual 2 decreases (resp. increases). Although the value of donations increments (resp. becomes smaller) for individual 1 and reduces (resp. becomes larger) for agent 2, the total contribution is not altered. The reasoning for redistributions in either Ω_1 or Ω_3 is analogous to the previous one, taking into account that for reallocation in $\Omega_1 \cup \Omega_3$ both agents have less than 3/5 of some good. In this case, the restrictions on possible contributions given by the new endowments become binding, as represented in Fig. 2a, c, respectively.

Therefore, in this example, when the redistributions of endowments allow consuming the equilibrium bundle at the equilibrium prices, the set of admissible contributions can be represented in an Edgeworth box, considering 3/5 as the upper bound for both commodities. The Figs. 3 and 4 illustrate the equilibrium contributions obtained from each block of the Ω partition.

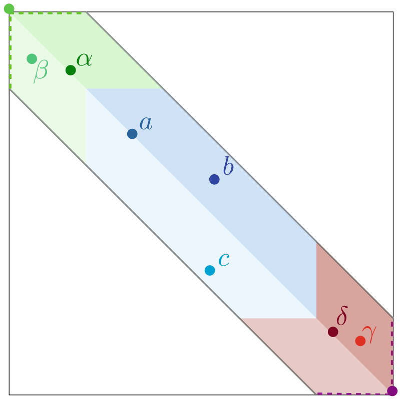

Fig. 3 Redistributions in Ω. $\alpha, \beta \in \Omega_1$, $a, b, c \in \Omega_2$, $\delta, \gamma \in \Omega_3$ Edgeworth box: 3×3

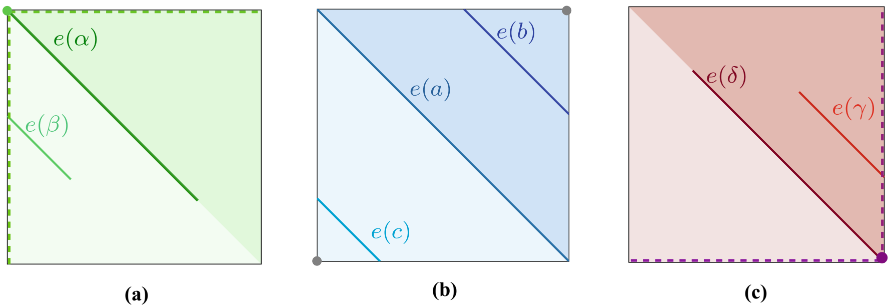

Fig. 4 Equilibrium donations $e = (e_1, e_2)$ for redistributions in Ω_1, Ω_2, and Ω_3. Figure **a–c**, respectively. Three Edgeworth boxes: $3/5 \times 3/5$

For every $\hat{\omega} \in \Omega$, the value of the aggregate equilibrium contribution is 20% of the value of the total resources, regardless of the underlying continuum of donations. However, if a reallocation $\hat{\omega}$ is in $\Omega_1 \cup \Omega_3$, then the collection $e(\hat{\omega})$ of equilibrium contributions shrinks because some coordinate of $\hat{\omega}$ is smaller than $3/5$. Note also that when one moves from the diagonal to the boundary of the set Ω, the length of the segment that determines the equilibrium contributions becomes smaller, and if the redistribution is in the boundary of Ω, then the equilibrium donation is just a point.

To summarize, in this example we find a family of redistributions of endowments that do not alter the equilibrium outcome. The approach in monetary terms is a help to identify the reallocations of resources that lead to this neutrality feature. On the other hand, the comparative statics analysis specifies how the modifications of the initial allocation affect the set of contributions that decentralize the equilibrium. The fact that both consumption commodities have the same price becomes crucial so that the equilibrium outcome is not modified. We elaborate on this issue in the next section, where we show that when prices are different, the previous redistributions of endowments do not guarantee the same equilibrium. This supports the intuition that relative prices matter for the resulting private contributions.

4 Neutrality

In the particular case of one private commodity and one public good, Bergstrom et al. (1986) obtain the so-called neutrality theorem that generalizes the invariance result of Warr (1983). Under convexity of preferences, this result states that income redistributions where no consumer loses more than her original contribution will induce a new equilibrium with the same total public good provision and where each consumer has precisely the same individual consumption as she had before.

In this section, we extend the neutrality result to our framework. For it, the redistributions considered are those where, for each private commodity, every consumer cannot lose more than their original contribution. This requirement is a natural extension of the one in BBV.

Let $\mathcal{E}(\tilde{\omega})$ denote the economy which coincides with the original one $\mathcal{E}$ except for the endowments that are $\tilde{\omega}$ instead of ω.

Theorem 4.1 ***(Neutrality).*** *Let (p^*, x^*, e^*) be a contribution equilibrium for the economy $\mathcal{E}$. Let $\tilde{\omega}$ be a redistribution of endowments such that $\tilde{\omega}_i \geq \omega_i - e_i^*$ for every i and $\tilde{\omega}_i = \omega_i$ for all non-contributing consumers. Then, there exists a vector of voluntary contributions to the public good $(\tilde{e}_i, i = 1 \ldots, n)$ such that $(p^*, x^*, \tilde{e})$ is a contribution equilibrium for the economy $\mathcal{E}(\tilde{\omega})$ and $p^* \cdot \sum_{i=1}^n e_i^* = p^* \cdot \sum_{i=1}^n \tilde{e}_i$.*

The proof we provide is constructive. Indeed, we show that $(p^*, x^*, \tilde{e})$ is a contribution equilibrium for the economy $\mathcal{E}(\tilde{\omega})$, being $\tilde{e}_i = e_i^* + \tilde{\omega}_i - \omega_i$, for each consumer $i \in N$.

In what follows, we present a non-neutrality example that allows deepening the study of the assumptions about the redistributions that ensure neutrality. For instance, we show that the condition that, for each private commodity, every consumer cannot lose more than their original contribution to the public good, cannot be weakened by requiring that the values of reallocated endowments allow each individual to consume their original equilibrium bundle.

A non-neutrality example. Let us consider an economy with two consumers, 1 and 2, two private commodities, x and y, and a public good G obtained from private contributions. Each consumer i has the utility function U_i given by $U_1(x, y, G) = \ln x + \ln y + \ln G$ and $U_2(x, y, G) = 2 \ln x + \ln y + \ln G$, respectively. The endowments are $\omega_1 = (1, 2)$ and $\omega_2 = (2, 2)$. A contribution equilibrium is given by the price vector $p^* = (1, 3/4)$, the private consumption bundles $\alpha_1^* = (1, 4/3)$, $\alpha_2^* = (2, 4/3)$, and contributions $e_1^* = e_2^* = (0, 2/3)$. Thus, $G^* = 1/6$ at equilibrium.

Let Ω be the set of all redistributions such that at prices p^* allow both individuals to attain their equilibrium private consumption bundle, which is defined as follows

$$\Omega = \left\{ \tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2) \in \mathrm{IR}_+^2 \times \mathrm{IR}_+^2 \mid \tilde{\omega}_1 + \tilde{\omega}_2 = (3, 4),\ \ p^* \cdot \tilde{\omega}_i \geq p^* \cdot \alpha_i^*, i = 1, 2 \right\}.$$

For every $\tilde{\omega} \in \Omega$, one deduces that p^* is an equilibrium price for the economy $\mathcal{E}(\tilde{\omega})$. However, not all the redistributions in Ω lead to the same equilibrium consumption allocation and contributions. To prove it, let us consider the partition of Ω given by $\Omega^L = \{\tilde{\omega} \in \Omega \mid \tilde{\omega}_1^x \leq 2\}$ and $\Omega_R = \{\tilde{\omega} \in \Omega \mid \tilde{\omega}_1^x > 2\}$, as represented in the Fig. 5 below.

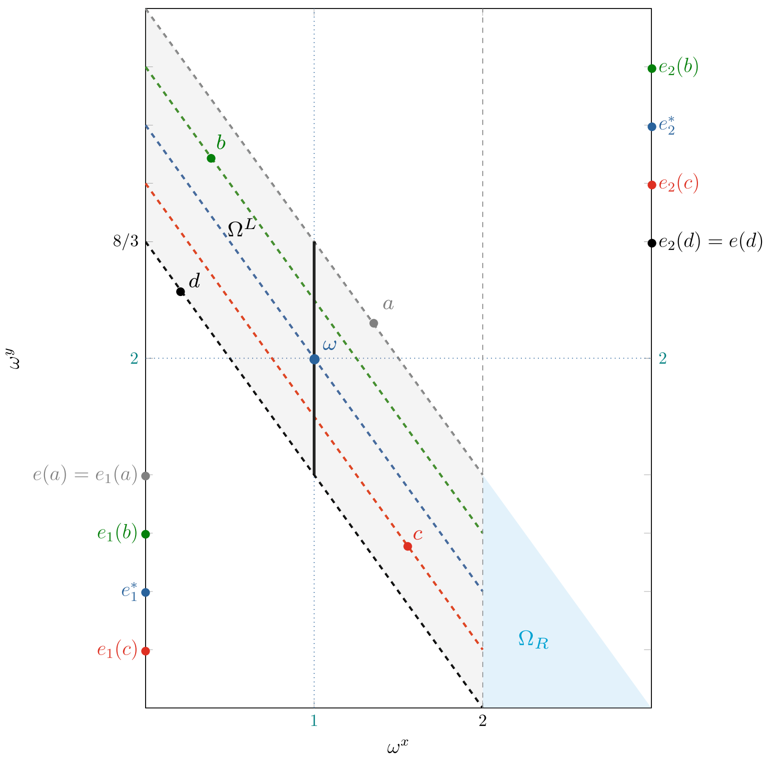

Fig. 5 Non-neutrality for redistributions in Ω_R

If $\tilde{\omega}$ belongs to Ω^L, then we obtain that $(p^*, \alpha^*, e(\tilde{\omega}))$ is an equilibrium for $\mathcal{E}(\tilde{\omega})$, being $e_1(\tilde{\omega}) = \left(0, \frac{4}{3}(p^* \cdot \tilde{\omega}_1 - 2)\right)$ and $e_2(\tilde{\omega}) = \left(0, \frac{4}{3}(p^* \cdot \tilde{\omega}_2 - 3)\right)$.[12] We have $e_1(\tilde{\omega}) + e_2(\tilde{\omega}) = (0, 4/3)$ and, in turn, the equilibrium result is the same as in the original economy.

If $\tilde{\omega}$ belongs to Ω_R, an equilibrium price in the economy $\mathcal{E}(\tilde{\omega})$ is $p^* = (1, 3/4)$ leading to the consumption allocations $\alpha_i(\tilde{\omega}) = (x_i(\tilde{\omega}), y_i(\tilde{\omega}))$ and private contributions $e_i(\tilde{\omega}) = \left(e_i^x(\tilde{\omega}), e_i^y(\tilde{\omega})\right)$, $i = 1, 2$, as functions of $\tilde{\omega}$, given by

$$\alpha_i(\tilde{\omega}) = \begin{cases} \left(\frac{p^* \cdot \tilde{\omega}_1}{2}, \frac{2p^* \cdot \tilde{\omega}_1}{3}\right) \text{ if } i = 1 \\ \left(\frac{p^* \cdot \tilde{\omega}_2}{2}, \frac{p^* \cdot \tilde{\omega}_2}{3}\right) \text{ if } i = 2 \end{cases} \qquad e_i(\tilde{\omega}) = \begin{cases} (0, 0) \text{ if } i = 1 \\ \left(0, \frac{p^* \cdot \tilde{\omega}_2}{3}\right) \text{ if } i = 2 \end{cases}$$

Note that at prices p^*, consumption clears the market of commodity x in the original economy. Thus, to obtain neutrality after a redistribution, the corresponding equilibrium contribution has to involve only the commodity y. When $\tilde{\omega} \in \Omega_R$, we have that there is no $c \in \left(0, \tilde{\omega}_1^y\right]$ such that $2 + \frac{3}{4}c = p^* \cdot \tilde{\omega}_1 = \tilde{\omega}_1^x + \frac{3}{4}\tilde{\omega}_1^y$. Precisely, the fact that the relative equilibrium prices differ from one states reasons why if $\tilde{\omega} \in \Omega_R$, then the endowments restrictions to contributions is binding for agent 1, avoiding the decentralization of the value $p^* \cdot (\tilde{\omega} - \alpha_1^*)$ via a possible non-null gift, resulting in non-neutrality.

Therefore, in the statement of the Theorem 4.1, the assumption $\tilde{\omega}_i \geq \omega_i - e_i^*$ for all i cannot be replaced by the weaker one requiring redistributions that the agents' consumption bundles be in their budget sets at p^*, which is a necessary but not sufficient condition to attain neutrality. Moreover, when there is no donation of a good at the original equilibrium, as in this particular scenario, neutrality requires that the resource reallocations lead to consuming the total endowment of such a good. It is also a necessary assumption that is not sufficient for obtaining neutrality, as this example shows through the redistributions in Ω_R.

On the other hand, Ω^L includes the set of redistributions verifying the hypothesis in our neutrality result, represented by the black segment in Fig. 5, where both agents have 1 unit of x and at least 4/3 of y. In these cases, applying Theorem 4.1 we obtain that $\tilde{e}_i^y = \tilde{\omega}_i^y - \frac{4}{3}$, $i = 1, 2$, defines equilibrium contributions for $\mathcal{E}(\tilde{\omega})$. Since not only the redistribution in the above-stated segment but also those in Ω^L lead to neutrality, we deduce that the assumption we find suffices to obtain neutrality nevertheless it is not a necessary condition.

5 Some other properties of the equilibrium

We deepen the study of our equilibrium concept following two different directions. The first way establishes links with the equilibria of associated pure exchange economies. The second one presents a characterization of the equilibrium notion by considering a continuum approach to the framework with a finite number of consumers. In

[12] Note that $p^* \cdot \tilde{\omega}_1 \in [2, 3]$ and $p^* \cdot \tilde{\omega}_2 \in [3, 4]$ for any $\tilde{\omega} \in \Omega$.

the following section, we elaborate on the atomless description from a game theory perspective.

In the model we address, every private contribution and price vector specifies the relative value for the donations. On the other hand, when the consumer i gives $e_i \in [0, \omega_i]$ for the public good, her resources become $\omega_i - e_i$. Thus, a vector of gifts $e = (e_1, \ldots, e_n)$ determines a pure exchange economy $\mathcal{E}^*(e)$, where each consumer $i \in N$ is characterized by endowments $\omega_i - e_i$, and a utility function V_i defined as $V_i(x, p) = U_i\left(x, \frac{p}{p\cdot\bar{\omega}} \cdot \sum_{i=1}^{n} e_i\right)$, for each price p.[13]

Proposition 5.1 *If (p^*, x^*, e^*) is a contribution equilibrium for the original economy $\mathcal{E}$, then (p^*, x^*) is a Walrasian equilibrium for the associated exchange economy $\mathcal{E}^*(e^*)$.*

The converse of the previous result is not true. To show this, consider the economy in the example presented in Section 3, with two consumers who have the same utility function $U(x, y, G) = xyG$ and endowments $\omega_1 = (1, 2)$, and $\omega_2 = (2, 1)$. Let e be the private contributions given by $e_1 = (0, 1)$, and $e_2 = (0, 0)$. The equilibrium for the associated exchange economy $\mathcal{E}^*(e)$ is given by the prices $p = (1, 3/2)$ and the allocation that assigns $(5/4, 5/6)$ to consumer 1 and $(7/4, 7/6)$ to consumer 2. However, the previous prices, private consumption bundles, and contributions do not define an equilibrium for the original economy. To prove it, note that the bundle $(4/3, 8/9)$ and any donation $(a, b) \in [0, \omega_1]$ such that $a + \frac{3b}{2} = 4/3$ belong to the budget set for individual 1 at prices $p = (1, 3/2)$ and lead to a better situation for such agent.

A remark on efficiency. When we consider just one private commodity, all the variables are measured in terms of such good, and our model becomes the one by BBV's. Thus, the equilibria may be inefficient. To see this, consider two consumers, each of them endowed with 1/2 unit of a private good. Preferences for consumers 1 and 2 are represented by the utility functions $U_1(x, G) = xG^2$ and $U_2(x, G) = xG$, respectively, where x denotes consumption of the private commodity and G is the amount of public good. The contributions $g_1 = 3/10$ and $g_2 = 1/10$ define a Nash equilibrium, that leads to utility levels $U_1^* = U_1(1/5, 2/5)$ and $U_2^* = U_2(2/5, 2/5)$ for each consumer, and it is not efficient. Both agents are better off by increasing their contributions in 1/20. To be precise, $U_1(3/20, 1/2) > U_1^*$ and $U_1(7/20, 1/2) > U_2^*$.

Despite this, we can obtain a *restricted efficiency* property as follows. Consider a contribution equilibrium (p^*, x^*, e^*) for the economy $\mathcal{E}$, and let us denote $G = \frac{p^*}{p^*\cdot\bar{\omega}} \cdot \sum_{i=1}^{n} e_i^*$. Then, applying the previous Proposition 5.1, we deduce that the allocation of private goods x^* is efficient for the exchange economy where each agent i is characterized by endowments $\omega_i - e_i^*$ and a utility function $V_i(\cdot) = U_i(\cdot, G)$.

In the rest of this section, we present the second direction of the study of equilibrium that we refer to as a continuum approach. For this, given the finite economy $\mathcal{E}$, let us consider an associated economy $\mathcal{E}_c$ with a continuum of consumers represented by

[13] If preferences are separable, in the sense that $U_i(x, G) = \mathcal{U}_i(x) + H_i(G)$, then as far as the equilibrium of $\mathcal{E}^*(e)$ is concerned we can consider without loss of generality that $V_i(x, p) = \mathcal{U}_i(x)$. It is an analogous situation when the original preferences have the property that $U_i(a, G) \geq U_i(b, G)$ if and only if $U_i(a, \hat{G}) \geq U_i(b, \hat{G})$, whatever $G, \hat{G}$ may be.

the real interval $I = [0, n]$, endowed with the Lebesgue measure, denoted by μ. Each consumer i in the economy $\mathcal{E}$ is represented in $\mathcal{E}_c$ by the real interval $I_i = [i - 1, i)$ if $i \neq n$, and consumer n is represented by $I_n = [n - 1, n]$. Each consumer $t \in I_i$ has endowment $\omega_t = \omega_i$ and preference relation to private consumption, and the value of contributions to the provision of public goods represented by the utility function V_t defined below.

To define utility functions V_t, one requires some notation. Let $e : [0, n] \to \mathbb{R}^{\ell}_{+}$ be a function which specifies a private contribution to public goods $e(t) \in [0, \omega_t]$ for every consumer t. We write $e_i = \int_{I_i} e(t) d\mu(t)$ and $e_{-i} = \int_{I \setminus I_i} e(t) d\mu(t)$. Thus, for a price system p, the utility function of an agent $t \in I_i$ is given by $V_t(x, e, p) = U_i(x, \hat{p} \cdot (e_{-i} + e(t)))$, with $\hat{p} = \frac{p}{p \cdot \bar{\omega}}$. That is, in the continuum each individual is unaffected by the contributions of others of her type and acts as a representative member of her type.

The definition of a contribution equilibrium for the continuum economy is straightforward – almost all agents maximize given prices, and feasibility must be satisfied – so we do not provide a formal statement. Next, we show that, concerning the equilibrium solution, the continuum and the discrete treatment can be considered equivalent.

Proposition 5.2 *A contribution equilibrium for the finite economy induces an equilibrium for the continuum economy and the converse.*

(i) If (p^*, x^*, e^*) is an equilibrium for the economy $\mathcal{E}$, then (p^*, x, e) is a contribution equilibrium for the economy $\mathcal{E}_c$, where $x(t) = x_i^*$ and $e(t) = e_i^*$ for every $t \in I_i$.

(ii) Reciprocally, if (p, x, e), is an equilibrium for the economy $\mathcal{E}_c$, then $(p, x_i, e_i, i = 1, \ldots, n)$ is an equilibrium for the economy $\mathcal{E}$, where $x_i = \int_{I_i} x(t) d\mu(t)$ and $e_i = \int_{I_i} e(t) d\mu(t)$ for every $i = 1, \ldots, n$.

6 A market game approach

In this section, we provide strategic foundations for the contribution equilibrium concept. We follow a variant of the originally considered by Shapley and Shubik (1977), where trade uses a commodity as a means of payment. Specifically, we adapt the games that Dubey and Geanakoplos (2003) state, with a continuum of players but n different types.

Given the economy $\mathcal{E}$ with contributions to public goods, we define an auxiliary game with a continuum of players, represented by the interval $I = [0, n]$, but n types. The set I, endowed with the Lebesgue measure, is divided into n disjoint intervals given by $I_i = [i - 1, i)$, $i = 1, \ldots, n - 1$, and $I_n = [n - 1, n]$. Each player $t \in I_i$ is characterized by an endowment $\omega_t = \omega_i$, and the utility function $U_t = U_i$. Note that $\bar{\omega} = \sum_{i=1}^{n} \omega_i = \int_I \omega_t d\mu(t)$.

There is a "trading-post" for each private commodity $h \in \{1, \ldots, \ell\}$. Each agent $t \in I$ puts up her endowment $\omega_{t,h}$ at post h. Besides, players can borrow from a bank a certain amount of money M, at zero interest for their bids at each post to purchase private commodities and contribute to the public good. Hence, the strategy set is the same for each player $t \in I$ and is defined as

$$\Theta = \left\{ (\rho, m) \in [0, M]^{2\ell}, \text{ such that } \sum_{h=1}^{\ell} \rho_h + \sum_{h=1}^{\ell} m_h \leq M \right\},$$

where ρ_h and m_h are the amounts a player would like to spend in the private good h and contribute with commodity h for public purposes.

The strategy sets of these games differ from Dubey and Geanakoplos (2003) since players bid not only to buy private goods but also to contribute for social purposes. It requires reformulating the rules for price formation and the allocation of private commodities. In this setting, the payoff functions depend on private consumption bundles and aggregate donations, as we precise below.

A strategy profile is a measurable function $\theta : I \longrightarrow \Theta$ such that $\theta_t = (\rho_t, m_t) \in \Theta$ for each $t \in I$. An external agent bids $\varepsilon > 0$ at each post to ensure that trade occurs. Thus, given a profile of strategies θ, the price of the private good h is defined as follows:

$$p_h(\theta) = \frac{\int_I (\rho_{t,h} + m_{t,h}) d\mu(t) + \varepsilon}{\int_I \omega_{t,h} d\mu(t)}.$$

Given a profile θ, let θ_{-t} denotes the vector of strategies of all players except t. Note that $p(\theta) = p(\theta_{-t}, s)$ for every $s \in \Theta$, that is, the strategy of a single player does not affect prices.

The amounts of each private good $h \in \{1, \ldots, \ell\}$ that end up being consumed and donated to the public good by each player t are determined by $x_{t,h}(\theta) = \rho_{t,h}/p_h(\theta)$ and $e_{t,h}(\theta) = \min\left\{m_{t,h}/p_h(\theta), \omega_{t,h}\right\}$, respectively. Since the parameter ε guarantees the strict positivity of every price, the consumption allocation and the donation rules are well-defined.

The payoff function for each player $t \in I_i$ is

$$\Pi_t(\theta) = U_t\left(x_t(\theta), G_{-i}(\theta) + \frac{p(\theta)}{p(\theta) \cdot \bar{\omega}} \cdot e_t(\theta)\right) - \max\{0, d_t(\theta)\},$$

where $G_{-i}(\theta) = \frac{p(\theta)}{p(\theta) \cdot \bar{\omega}} \cdot \int_{I \setminus I_i} e_t(\theta) d\mu(t)$, and $d_t(\theta) = \sum_{h=1}^{\ell} \left(\rho_{t,h} + m_{t,h} - p_h(\theta)\omega_{t,h}\right)$.

The term $\max\{0, d_t(\theta)\}$ in the payoff functions stands for a penalty that is effective in case of defaulting on the loans.[14]

In this way, for each integer M we have a game that we denote by $\mathcal{G}(M)$. Next, we first state an existence result for Nash equilibrium for every $\mathcal{G}(M)$, and then, we obtain a contribution equilibrium of the economy as the limit of a Nash equilibrium sequence, when M increases.

Theorem 6.1 *Assume that (A.1) and (A.2) hold. Then, there exists a Nash equilibrium for the game $\mathcal{G}(M)$.*

[14] This will guarantee that budget constraints hold when M increases.

Remark Note that if θ is a Nash equilibrium, then $\frac{m_{t,h}}{p_h(\theta)} \leq \omega_{t,h}$ for almost all player $t \in I$ and each private commodity h. To show our point, assume that there is h and $A \subset I$, with $\mu(A) > 0$, such that $\frac{m_{t,h}}{p_h(\theta)} > \omega_{t,h}$ for every $t \in A$. Then, each consumer $t \in A$ can increase her payoff by deviating unilaterally and choosing any strategy $\hat{\theta}_t = (\hat{\rho}_t, \hat{m}_t)$ such that $\hat{\rho}_{t,h} = \rho_t + \varepsilon$ and $\hat{m}_{t,h} = m_{t,h} - \varepsilon \geq \omega_{t,h} p_h(\hat{\theta}_t, \theta_{-t}) = \omega_{t,h} p_h(\theta)$, with $\varepsilon > 0$. In this case, $d_t(\theta) = d_t(\hat{\theta}_t, \theta_{-t})$, and all the other outcomes remain the same as well, except the consumption of commodity h for the consumer $t \in A$ that increases.

We also point out that the definition of the resource allocation mechanism leads to feasible assignments. Namely, for each strategy profile θ, with $\theta_t = (\rho_t, m_t) \in \Theta$, and every commodity h, we have $\int_I x_{t,h}(\theta) d\mu(t) + \int_I e_{t,h}(\theta) d\mu(t) = \frac{1}{p_h(\theta)} \left(\int_I \left(\rho_{t,h} + m_{t,h} \right) d\mu(t) \right) \leq \int_I \omega_{t,h} d\mu(t)$.

To move from Nash to contribution equilibrium, we follow the steps provided by Dubey and Geanakoplos (2003), DG, going from Nash to Walras. Despite the similarities with DG, the donations in our game affect strategies and payoffs, implying technical difficulties to overcome. Specifically, some required boundedness properties of the variables and the feasibility of the outcomes, among others, are not straightforward. In particular, the property of the Nash equilibrium stated in the previous remark plays a role.

Theorem 6.2 *Assume that (A.1) and (A.2) hold. For each natural number M, let θ_M be a symmetric Nash equilibrium for $\mathcal{G}(M)$. Let $(p_M/\|p_M\|, x_M, e_M)$ be the corresponding sequence of prices, consumption allocations and private contributions given by such sequence of Nash equilibria. Then, there exists a subsequence that converges to a price system p, an allocation x and private contributions e, such that (p, x, e) defines a contribution equilibrium for the economy $\mathcal{E}$.*

It is worth mentioning that the market game approach and the results in this section reflect the main differences between the extensions we propose of BBV's model to multiple private commodities and those addressed by considering a production system for public goods in the previous literature.

7 Some final remarks

This paper contains two proofs showing the existence of the contribution equilibrium we have defined. The first follows the standard procedure of obtaining equilibrium as a fixed point of a correspondence. The second one focuses on a game theoretical perspective, where the equilibrium is a limit of a sequence of Nash equilibria. The construction of the non-cooperative games illustrates the price formation mechanism and the rules for allocating commodities for consumption and donations to a public good.[15]

The payoff functions of the specified market games include a penalty that infers a cardinal character to the utility functions. However, except for the market game

[15] The problem of collective goods provision has also been analyzed within a general equilibrium model, using the cooperative solution of the core (see Basile et al. 2021, and the references therein).

approach, the rest of this work can be written by considering preference relations without requiring a cardinal property of their representation.

On the other hand, since only relative prices matter, any norm in $\mathbb{R}^\ell$ can be applied to normalize prices. To be precise, regarding the equilibrium, we can assume that prices belong to the simplex of $\mathbb{R}^\ell_+$, or take a private commodity as numeraire. Moreover, the normalization can be stated by a reference commodity bundle, for instance, the total endowment or any other selected vector of private goods.

Finally, we stress that both the reallocation of the initial resources and the relative prices play an important role when we attempt to study neutrality issues.

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Appendix

Proof of Theorem 2.1 Given the total endowments $\bar{\omega}$, let $B[0, \kappa]$ be the closed ball in $\mathbb{R}^\ell$ centered at the origin and with radius $\kappa = 2\|\bar{\omega}\|$. Let us consider the compact sets $K = B[0, \kappa] \cap \mathbb{R}^\ell_+$, $W_i = [0, \omega_i]$, $X = K^n$, and $W = \prod_{i \in N} W_i$. Let us also consider prices p for commodities in the simplex of $\mathbb{R}^\ell_+$ denoted by Δ. In this cases, we have that $\|p\| = \sum_{h=1}^{\ell} p_h = 1$, for every $p \in \Delta$. For each $i = 1, \ldots n$, define the following function F_i and correspondence B_i :

$$\begin{aligned} F_i :& X \times W \times \Delta \to \mathbb{R} \\ & (x, e, p) \to U_i(x_i, \hat{p} \cdot \bar{e}) \\ B_i :& X \times W \times \Delta \twoheadrightarrow K \times W_i \\ & (x, e, p) \twoheadrightarrow \{(z, g) \in K \times W_i \mid p \cdot z + p \cdot g \le p \cdot \omega_i\} \end{aligned}$$

From assumptions (A.1) and (A.2), we deduce that every F_i is continuous, and B_i takes non-empty compact values and is upper and lower hemicontinuous (uhc and lhc).

Let us also define the following functions and correspondences:

$$\begin{aligned} F_{n+1} :& X \times W \times \Delta \to \mathbb{R} \\ & (x, e, p) \to \sum_{i=1}^{n} p \cdot (x_i + e_i - \omega_i) \end{aligned} \qquad \begin{aligned} B_{n+1} :& X \times W \times \Delta \twoheadrightarrow \Delta \\ & (x, e, p) \twoheadrightarrow \Delta \end{aligned}$$

F_{n+1} is continuous. Since B_{n+1} is constant, it is upper hemicontinuous (uhc) and lower hemicontinuous (lhc). Applying the Maximum Theorem to F_i and B_i, $i = 1, \ldots, n$, we obtain that the following correspondences Γ_i, $i = 1, \ldots, n$, Γ_{n+1} are uhc and takes non-empty and compact values.

$$
\begin{aligned}
\Gamma_i :\ & X \times W \times \Delta \twoheadrightarrow K \times W_i \\
& (x, e, p) \twoheadrightarrow \operatorname{argmax} \{U_i(z, \hat{p} \cdot \bar{e}_{-i} + \hat{p} \cdot g), (z, g) \in B_i(x, e, p)\},
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{n+1} :\ & X \times W \times \Delta \twoheadrightarrow \Delta \\
& (x, e, p) \twoheadrightarrow \operatorname{argmax} \left\{ \sum_{i=1}^{n} q \cdot (x_i + e_i - \omega_i), q \in \Delta \right\}
\end{aligned}
,
$$

The quasi-concavity of U_i guarantees that Γ_i takes convex values for each $i \in N$. Moreover, F_{n+1} is linear in p and then it is concave. Thus, Γ_{n+1} also takes convex values.

Let the correspondence $\Gamma = \prod_{i=1}^{n+1} \Gamma_i : X \times W \times \Delta \twoheadrightarrow X \times W \times \Delta$, which is upper hemicontinuous and takes non-empty, compact and convex values. Applying Kakutani's theorem we have that Γ has a fixed point, that is, there exists $(x^*, e^*, p^*) \in \Gamma(x^*, e^*, p^*)$. To finish the proof, we will show that (p^*, x^*, e^*) is an equilibrium for the economy $\mathcal{E}$.

Summing up the budget constraints we have $\sum_{i=1}^{n} p^* \cdot (x_i^* + e_i^*) \leq \sum_{i=1}^{n} p^* \cdot \omega_i$. On the other hand, since $p^* \in \Gamma_{n+1}(x^*, e^*, p^*)$, we have $0 \geq \sum_{i=1}^{n} p^* \cdot (x_i^* + e_i^* - \omega_i) \geq p \cdot \sum_{i=1}^{n} (x_i^* + e_i^* - \omega_i)$, for all $p \in \Delta$. This implies, in particular, $0 \geq p^* \cdot \sum_{i=1}^{n} (x_i^* + e_i^* - \omega_i) \geq b_h \cdot \sum_{i=1}^{n} (x_i^* + e_i^* - \omega_i)$, where b_h is the vector in Δ with all the coordinates zero except the h^{th} that is 1. Thus, taking prices as the standard basis of $\mathbb{R}^\ell$ one has that $\sum_{i=1}^{n} x_i^* + \sum_{i=1}^{n} e_i^* \leq \sum_{i=1}^{n} \omega_i$.

It remains to prove that (x_i^*, e_i^*) maximizes $U_i(x_i, \hat{p}^* \cdot \bar{e}_{-i}^* + \hat{p}^* \cdot e_i)$ in the budget set $\mathcal{B}_i(p^*) = \{(x_i, e_i) \in \mathbb{R}_+^\ell \times W_i \mid p^* \cdot x_i + p^* \cdot e_i \leq p^* \cdot \omega_i\}$. Assume that for some consumer i there is $(x_i, e_i) \in \mathcal{B}_i(p^*)$ such that $U_i(x_i, \hat{p}^* \cdot \bar{e}_{-i}^* + \hat{p}^* \cdot e_i) > U_i(x_i^*, \hat{p}^* \cdot \bar{e}_{-i}^* + \hat{p}^* \cdot e_i^*) = U_i^*$. For each $\lambda \in (0, 1)$ let $x_\lambda = \lambda x_i^* + (1 - \lambda) x_i$, and $e_\lambda = \lambda e_i^* + (1 - \lambda) e_i$.[16] By the convexity of preferences in (A.2), we obtain $U_i(x_\lambda, \hat{p}^* \cdot \bar{e}_{-i}^* + \hat{p}^* \cdot e_\lambda) > U_i^*$, for every $\lambda \in (0, 1)$. We can take λ close enough to 1 such that $x_\lambda \in K$. This is in contradiction with $(x_i^*, e_i^*) \in \Gamma_i(x^*, e^*, p^*)$. □

Proof of Theorem 4.1 For each consumer i, let us define $\tilde{e}_i = e_i^* + \tilde{\omega}_i - \omega_i$. Note that $\tilde{e}_i \in [0, \tilde{\omega}_i]$ and $p^* \cdot x_i^* + p^* \cdot \tilde{e}_i = p^* \cdot \tilde{\omega}_i$. By construction, we have that $\sum_{i=1}^{n} \tilde{e}_i = \sum_{i=1}^{n} e_i^*$ provided that $\sum_{i=1}^{n} \tilde{\omega}_i = \sum_{i=1}^{n} \omega_i$. In which follows we show that $(p^*, x^*, \tilde{e})$ is a contribution equilibrium for the economy $\mathcal{E}(\tilde{\omega})$. It remains to check that for every consumer i the bundle $(x_i^*, \tilde{e}_i)$, solves the following individual problem:

$$
\begin{aligned}
\max_{(x,e) \in \mathbb{R}_+^\ell \times [0, \tilde{\omega}_i]} \quad & U_i(x, \tilde{M}_{-i} + \hat{p}^* \cdot e) \\
\text{such that} \quad & p^* \cdot x + p^* \cdot e \leq p^* \cdot \tilde{\omega}_i
\end{aligned}
$$

where $\tilde{M}_{-i} = \hat{p}^* \cdot \sum_{j \neq i} \tilde{e}_j$, and $\hat{p}^* = p^* / (p^* \cdot \bar{\omega})$.

[16] Note that W_i is a convex set.

Let $\Delta\omega_i = \tilde{\omega}_i - \omega_i$, $M^* = \hat{p}^* \cdot \sum_{i=1}^n e_i^*$, and $M_{-i}^* = \hat{p}^* \cdot \sum_{j \neq i} e_j^*$. There are two possible cases:

(i) Consider that $p^* \cdot \Delta\omega_i \leq 0$. Assume that $(x_i^*, \tilde{e}_i)$ does not solve i's optimization problem. Hence, there is $(x, e) \in \mathrm{IR}_+^\ell \times [0, \tilde{\omega}_i]$ such that $p^* \cdot x + p^* \cdot e = p^* \cdot \tilde{\omega}_i$ and $U_i(x, \tilde{M}_{-i} + \hat{p}^* \cdot e) > U_i(x_i^*, M^*)$. Let us take $\mu = p^* \cdot e - p^* \cdot \tilde{e}_i + p^* \cdot e_i^*$ and note that $p^* \cdot x + \mu = p^* \cdot \omega_i$. There is a unique $\beta \in [0, 1]$ such that $p^* \cdot \omega_i - p^* \cdot x = \beta p^* \cdot \omega_i$. Let $\gamma = \beta\omega_i \in [0, \omega_i]$. By construction, (x, γ) is in the budget set of consumer i in the original economy $\mathcal{E}$ and $p^* \cdot \bar{e}_{-i}^* + p^* \cdot \gamma = p^* \cdot \bar{\tilde{e}}_{-i} + p^* \cdot e$. This is in contradiction with the fact that (p^*, x^*, e^*) is a contribution equilibrium for the economy $\mathcal{E}$.

(ii) Consider the case $p^* \cdot \Delta\omega_i > 0$. As in (i), assume that there is $(x, e) \in \mathrm{IR}_+^\ell \times [0, \tilde{\omega}_i]$ such that $p^* \cdot x + p^* \cdot e = p^* \cdot \tilde{\omega}_i$ and $U_i(x, \tilde{M}_{-i} + \hat{p}^* \cdot e) > U_i(x_i^*, M^*)$. For each $\lambda \in (0, 1)$, let $x_\lambda = \lambda x_i^* + (1 - \lambda)x$ and $m_\lambda = \lambda p^* \cdot e_i^* + (1 - \lambda)p^* \cdot (e - \Delta\omega_i)$. Since i is a contributor, $m_\lambda > 0$ for λ close enough to 1. We remark that, by construction, we can deduce that $p^* \cdot x_\lambda + m_\lambda = p^* \cdot \omega_i$ and $M_{-i}^* + \hat{m}_\lambda = \lambda M^* + (1 - \lambda)\left(\tilde{M}_{-i} + \hat{p}^* \cdot e\right)$, being $\hat{m}_\lambda = m_\lambda/(p^* \cdot \bar{\omega})$. Moreover, for each $\lambda \in (0, 1)$ there is a unique β_λ such that $p^* \cdot (\omega_i - x_\lambda) = \beta_\lambda p^* \cdot \omega_i$. Now, take $e_\lambda = \beta_\lambda \omega_i$.
Therefore, for λ close enough to 1, consumer i can choose (x_λ, e_λ) in the economy $\mathcal{E}$ where, given M_{-i}^*, leads to the outcome $\left(x_\lambda, \lambda M^* + (1 - \lambda)\left(\tilde{M}_{-i} + \hat{p}^* \cdot e\right)\right)$. By convexity of preferences, agent i prefers this outcome rather than (x_i^*, M^*), which is in contradiction with the fact that (p^*, x^*, e^*) is a contribution equilibrium for the original economy $\mathcal{E}$. □

Proof of Proposition 5.1 Note that $\sum_{i=1}^n x_i^* + \sum_{i=1}^n e_i^* = \sum_{i=1}^n \omega_i$ implies that x^* is a feasible allocation in the economy $\mathcal{E}^*(e^*)$, where endowments are $\tilde{\omega}_i = \omega_i - e_i^*$, for each consumer $i \in N$. Moreover, $p^* \cdot x_i^* \leq p^* \cdot \tilde{\omega}_i$, for every $i \in N$. Assume that for some individual $j \in N$ there is y such that $p^* \cdot y \leq p^* \cdot \tilde{\omega}_j$, and $V_j(y, p^*) > V_j(x_j^*, p^*)$. This implies that (x_j^*, e_j^*) does not solve the agent j's problem at prices p^* in the original economy, which is a contradiction with the fact that (p^*, x^*, e^*) is an equilibrium for $\mathcal{E}$. □

Proof of Proposition 5.2 Let (p^*, x^*, e^*) be an equilibrium for economy $\mathcal{E}$. If (p^*, x, e) is not an equilibrium for $\mathcal{E}_c$, then there exists a positive measure subset of consumers A, with $A \subset I_k$ for some $k \in N$ and there exist $\hat{x}(t)$ and $\hat{e}(t) \in [0, \omega_k]$, such that $p^* \cdot (\hat{x}(t) + \hat{e}(t)) \leq p^* \cdot \omega_k$ and $U_k(\hat{x}(t), \hat{p}^* \cdot (e_{-i}^* + \hat{e}(t))) > U_k(x^*, \hat{p}^* \cdot e^*)$, for all $t \in A$, being $\hat{p}^* = p^*/(p^* \cdot \bar{\omega})$. Let $\hat{x}_k = \frac{1}{\mu(A)} \int_A \hat{x}(t) d\mu(t)$ and $\hat{e}_k = \frac{1}{\mu(A)} \int_A \hat{e}(t) d\mu(t)$. Then by convexity of preferences, $U_k(\hat{x}_k, \hat{p}^* \cdot (e_{-k}^* + \hat{e}_k)) > U_k(x_k^*, \hat{p}^* \cdot e^*)$, which is in contradiction with the fact that (p^*, x^*, e^*) is an equilibrium for $\mathcal{E}$.

To show the converse, suppose that $(p, x_i, e_i, i = 1, \ldots, n)$ is not an equilibrium for $\mathcal{E}$, and let $\hat{p} = p/(p \cdot \bar{\omega})$. Then, there is $(\hat{x}_k, \hat{e}_k) \in B_k(p)$, such that $U_k(\hat{x}_k, \hat{p} \cdot (e_{-k} + \hat{e}_k)) > U_k(x_k, \hat{p} \cdot e)$, for some $k \in N$. The convexity of preferences guarantees the existence of $A \subset I_k$ with $\mu(A) > 0$ such that $U_t(\hat{x}_k, \hat{p} \cdot (e_{-k} + \hat{e}_k)) > U_t(x(t), \hat{p} \cdot$

$(e_{-k}+e(t)))$, for all $t \in A$, which contradicts the fact that (p, x, e), is an equilibrium for $\mathcal{E}_c$.[17] □

Proof of Theorem 6.1 Given a type-symmetric profile of strategies, $\theta = (\theta_1, ..., \theta_n) \in \Theta^n$, consider the best reply correspondence for any player $t \in I_i$,

$$\Phi_t(\theta) = \underset{\theta_t \in \Theta}{\operatorname{argmax}}\, \Pi_t(\theta_t, \theta_{-t}) = \underset{(\rho_t, m_t)\in\Theta}{\operatorname{argmax}}\, \Pi_t((\rho_t, m_t), \theta_{-t})$$
$$= \underset{\theta_t \in \Theta}{\operatorname{argmax}} \left\{ U_i\left(x_t(\theta), G_{-i}(\theta) + \tfrac{p(\theta)}{p(\theta)\cdot\bar{\omega}}\cdot e_t(\theta)\right) - \max\{0, d_t(\theta)\}\right\}$$

where $\theta_t = (\rho_t, m_t)$, $x_{t,h}(\theta) = \frac{\rho_{t,h}}{p_h(\theta)}$, $e_{t,h}(\theta) = \min\left\{\frac{m_{t,h}}{p_h(\theta)}, \omega_{t,h}\right\}$ for each $h \in \{1, \ldots, \ell\}$, and $d_t(\theta) = \sum_{h=1}^{\ell} \left(\rho_{t,h} + m_{t,h} - p_h(\theta)\omega_{t,h}\right)$.

Symmetry implies that $\Phi_t = \Phi_i$ for all $t \in I_i$. Since $\theta_t = (\rho_t, m_t)$ does not modify $p(\theta)$ it follows that $x_t(\theta)$, and $d_t(\theta)$ are linear in θ_t. Moreover, $e_{t,h}(\cdot, \theta_{-t})$ is concave.[18] We obtain that $G_{-i}(\theta) + \frac{p(\theta)}{p(\theta)\cdot\bar{\omega}} \cdot e_t(\theta)$ is concave in θ_t.

Since the utility function U_i is non-decreasing and quasi-concave, we deduce that $\Pi_i(\cdot, \theta_{-t})$ is quasi-concave in $\theta_t = (\rho_t, m_t)$. This implies that $\Phi_i(\cdot, \theta_{-t})$ is convex-valued provided that Θ is convex. Since each payoff function Π_i is continuous and Θ is compact and convex, by the Maximum Theorem we have that Φ_i is upper semi-continuous and compact-valued. We get the same conclusions for the correspondence $\Phi = (\Phi_1, ..., \Phi_n)$. Kakutani's theorem guarantees that Φ has a fixed point, that is a Nash equilibrium of the game. □

Proof of Theorem 6.2 Since θ_M is a symmetric Nash equilibrium for the game $\mathcal{G}(M)$, we have $\theta_{M,t} = \theta_{M,i} = (\rho_{M,i}, m_{M,i},)$ for every $t \in I_i$. This equilibrium defines the price $p_M = p(\theta_M)$ which leads to the consumption allocations, contributions, and net deficits $(x_M, e_M, d_M) = (x(\theta_M), e(\theta_M), d(\theta_M)) = (x_{M,i}, e_{M,i}, d_{M,i}\ i \in N)$. By the remark preceding this Theorem 6.2, we have $e_{M,i,h}(\theta) = \frac{m_{M,i,h}}{p_{M,h}(\theta)} \le \omega_{i,h}$, and then $G_M = G(\theta_M) = \frac{1}{p_M\cdot\bar{\omega}} \sum_{i=1}^n \sum_{h=1}^{\ell} m_{M,i,h} = \frac{p_M}{p_M\cdot\bar{\omega}} \cdot \sum_{i=1}^n e_{M,i}$.

Note that $G_M \le 1$ for every M. Moreover, the definition of the game ensures that $\int_I x_{M,t} d\mu(t) + \int_I e_{M,t} d\mu(t) = \sum_{i=1}^n x_{M,i} + \sum_{i=1}^n e_{M,i} \le \bar{\omega} = \sum_{i=1}^n \omega_i = \int_I \omega_t d\mu(t)$.[19] Thus, the value of equilibrium contributions G_M and the consumption bundles allocated to consumers x_M are bounded.

Note also that if a player $t \in I_i$ deviates from θ_M and selects the strategy $\theta = 0$, then her payoff becomes $U_i(0, G_{-i}(\theta_M)) \ge U_i(0, 0)$, with $G_{-i}(\theta_M) = \frac{p_M}{p_M\cdot\bar{\omega}} \sum_{j\neq i} e_{M,i}$. This implies that $U_i(\bar{\omega}, 1) - d_{M,i+} \ge \Pi_i(\theta_M) = U_i(x_{M,i}, G_M) - d_{M,i+} \ge U_i(0, 0)$

17 The contradictions are obtained by using the García-Cutrín and Hervés-Beloso (1993) Lemma, which is based on the mean value of the integral. When the utility functions are concave we can apply Jensen's inequality instead. Note that the concavity of the utility functions implies the required convexity of preferences.

18 To be precise, $e_{t,h}(\cdot, \theta_{-t})$ is constant for every $m_{t,h} \ge \omega_{t,h} p_h(\theta)$ and is linear otherwise. Then, $p(\theta) \cdot e_t(\cdot, \theta_t)$ is concave since it is a sum of concave functions.

19 Note that $x_{M,i,h} = \frac{\rho_{M,i,h}}{p_{M,h}}$, $e_{M,i,h} \le \frac{m_{M,i}}{p_{M,h}}$ and $p_{M,h} = \frac{\sum_{i=1}^n (\rho_{M,i,h} + m_{M,i,h}) + \varepsilon}{\sum_{i=1}^n \omega_{i,h}}$.

and, consequently, $d_{M,i+} = \max\{0, d_{M,i}\}$ is bounded from above by $U_i(\bar{\omega}, 1) - U_i(0, 0)$.

For each M, consider the subsets of types of consumers who are in deficit and those who are in surplus, defined as $D_M = \{i \in N | d_{M,i} > 0\}$ and $S_M = \{i \in N | d_{M,i} < 0\}$, respectively. The same argument as in DG allows us to obtain $\sum_{i \in S_M} -d_{M,i} = \varepsilon\ell + \sum_{i \in D_M} d_{M,i}$ and conclude that $d_{M,i}$ is bounded.

Then, the sequence $\left(\frac{p_M}{\|p_M\|}, x_M, e_M, d_M\right)_M$ has a convergent subsequence, when M goes to infinity, with limit (p, x, e, d), and we write $x_{M,i} \to x_i$, $e_{M,i} \to e_i$, $d_{M,i} \to d_i$, for each type i and $\frac{p_M}{\|p_M\|} \to p$. We will show that (p, x, e) is a contribution equilibrium.

Since $e_{M,i} \leq \omega_i$ for every i and M, we have $e_i \leq \omega_i$ for every $i \in N$, and G_M converges to $\frac{p}{p \cdot \bar{\omega}} \cdot \bar{e}$. Given that $\varepsilon\ell > 0$, reasoning as in DG, we deduce that the set S_M is nonempty, and $p_M \cdot (\omega_i - e_i) > M$ for $i \in S_M$ implies that $\|p_M\| \to \infty$ when $M \to \infty$. It follows that $\frac{d_{M,i}}{\|p_M\|} = \frac{p_M}{\|p_M\|} \cdot (x_{M,i} - \omega_i + e_{M,i}) \to 0$, and then $p \cdot x_i + p \cdot e_i = p \cdot \omega_i$, for every $i \in N$.

To finish the proof, we show that $U_i(z, \frac{p}{p \cdot \bar{\omega}} \cdot (\bar{e}_{-i} + g)) \leq U_i(x_i, \frac{p}{p \cdot \bar{\omega}} \cdot \bar{e})$ for any (z, g) in the budget set $B_i(p)$, for every $i \in N$. For it, take a bundle $(z, g) \in B_i(p)$, and for each $\lambda \in (0, 1)$, consider the strategy $\tilde{\theta}_{M,i}(\lambda) = (\tilde{\rho}_M(\lambda), \tilde{m}_M(\lambda)) \in \Theta$ given by $\tilde{\rho}_{M,h}(\lambda) = \lambda p_{M,h} z_h$ and $\tilde{m}_{M,h}(\lambda) = \lambda p_{M,h} g_h$, for each $h \in \{1, \ldots, \ell\}$. Since $\Pi_i(\theta_M) \geq \Pi_i(\theta_{M,-i}, \tilde{\theta}_{M,i})$ and θ_M incurs at least as much penalty, we have $U_i(x_{M,i}, G_M) \geq U_i(\lambda z, \frac{p}{p \cdot \bar{\omega}} \cdot (\bar{e}_{-i} + \lambda g))$, for every $\lambda \in (0, 1)$ and all M large enough. Passing to the limit, we conclude $U_i(x_i, \frac{p}{p \cdot \bar{\omega}} \cdot \bar{e}) \geq U_i(z, \frac{p}{p \cdot \bar{\omega}} \cdot (\bar{e}_{-i} + g))$. $\square$

References

Balasko, Y.: Economies with price-dependent preferences. J. Econ. Theory **109**(2), 333–359 (2003)

Barbieri, S.: Complementarity and information in collective action. Econ. Theory **75**, 167–206 (2023). https://doi.org/10.1007/s00199-021-01394-1

Basile, A., Gilles, R.P., Graziano, M.G., Pesce, M.: The Core of economies with collective goods and a social division of labour. Econ. Theory **71**, 1085–1119 (2021). https://doi.org/10.1007/s00199-020-01258-0

Bergstrom, T., Blume, L., Varian, H.: On the private provision of public goods. J. Public Econ. **29**, 25–49 (1986)

Cea-Echenique, S., Hervés-Beloso, C., Torres-Martínez, J.P.: Endogenous differential information. Econ. Theory **63**, 51–72 (2017). https://doi.org/10.1007/s00199-015-0924-5

Correia-da-Silva, J., Hervés-Beloso, C.: Subjective expectations equilibrium in economies with uncertain delivery. J. Math. Econ. **44**, 641–650 (2008)

Dubey, P., Geanakoplos, J.: From Nash to Walras via Shapley–Shubik. J. Math. Econ. **39**, 391–400 (2003)

Faias, M., Moreno-García, E., Wooders, M.: A strategic market game approach for the private provision of public goods. J. Dyn. Games **1**(2), 283–298 (2014)

Faias, M., Moreno-García, E., Wooders, M.: On neutrality with multiple private and public goods. Math. Soc. Sci. **76**, 103–106 (2015)

Faias, M., Moreno-García, E., Myles, G.D.: Bergstrom, Blume, and Varian: voluntary contributions and neutrality. J. Public Econ. Theory **22**, 285–301 (2020)

García-Cutrín, J., Hervés-Beloso, C.: A discrete approach to continuum economies. Econ. Theory **3**, 577–584 (1993). https://doi.org/10.1007/BF01209704

Hervés-Beloso, C., Martínez, F., Rivera, J.: Walrasian prices in markets with tradable rights. Econ. Theory **49**, 497–512 (2012). https://doi.org/10.1007/s00199-011-0635-5

Ichiishi, T.: Game Theory for Economic Analysis. Academic Press, New York (1983)

Lamprecht, J., Thum, M.: Opacity in bargaining over public good provision. J. Public Econ. Theory **1–27** (2023). https://doi.org/10.1111/jpet.12646
Podczeck, K., Yannelis, N.C.: Existence of Walrasian equilibria with discontinuous, non-ordered, interdependent and price-dependent preferences, without free disposal, and without compact consumption sets. Econ. Theory **73**, 413–420 (2022). https://doi.org/10.1007/s00199-021-01400-6
Pollak, R.A.: Price dependent preferences. Am. Econ. Rev. **67**(2), 64–75 (1977)
Reck, D., Seibold, A.: The welfare economics of reference dependence. CESifo Working Paper, no. 9999 (2022)
Shapley, L.S., Shubik, M.: Trade using one commodity as a means of payment. J. Polit. Econ. **85**, 937–968 (1977)
Villanaci, A., Zenginobuz, Ü.: Existence and regularity of equilibria in a general equilibrium model with private provision of a public good. J. Math. Econ. **41**, 617–636 (2005)
Villanacci, A., Zenginobuz, Ü.: Pareto improving interventions in a general equilibrium model with private provision of public goods. Rev. Econ. Des. **10**, 249–271 (2006)
Villanacci, A., Zenginobuz, Ü.: On the neutrality of redistribution in a general equilibrium model with public goods. J. Public Econ. Theory **9**(2), 183–200 (2007)
Villanacci, A., Zenginobuz, Ü.: Subscription equilibrium with production: nonneutrality and constrained suboptimality. J. Econ. Theory **147**, 407–425 (2012)
Warr, P.G.: The private provision of a public good is independent of the distribution of income. Econ. Lett. **13**, 207–211 (1983)