Economic Theory (2023) 75:1039–1069
https://doi.org/10.1007/s00199-022-01436-2

RESEARCH ARTICLE

Bequests or education

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Received: 18 June 2021 / Accepted: 17 May 2022 / Published online: 9 July 2022

Abstract

This paper shows that, when parents can endow their offspring with bequests and human capital, markets cannot deliver (generically under laissez-faire) the planner's choice, if educational investments affect total factor productivity—as empirical evidence establishes. Moreover, for a human capital production function close enough to affine (around market and planner steady states with similar fertilities), the market steady state wage is higher than the marginal productivity of labor at the planner's steady state, so that the market steady state human capital is too low. In other words, the market misses the planner's allocation by leading households to transfer to their offspring more in bequests and less in education than would be optimal. These results obtain in spite of parents perfectly internalising (1) the value for their children of their bequests and educational investment, but not (2) the externality on total factor productivity—nor hence on factor prices. The planner's allocation can, nonetheless, be decentralised subsidising labor income through a lump-sum tax on saving returns that reduces bequests. An estimate of the subsidy needed—for standard functional forms and parameter values estimated from US data—suggests a sizeable market inefficiency.

Keywords Human capital · Bequests · Total factor productivity · Externalities · Overlapping generations

JEL Classification J24 · D64 · O40

Funding from the Social Policy Research Grant No. 0000038 and the FDCRGP Grant No. 11022021FD2908 from Nazarbayev University is gratefully acknowledged.

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1 Introduction

Parents can help their children bequeathing them wealth and investing in their human capital, the motives for each being possibly different. Regarding bequests, the literature points to several possible motives. In Barro (1974) bequests are assumed to be the result of intergenerational altruism, although they can be accidental too, as argued in Davies (1981)—uncertainty about the life-span explains the lack of the strong enough dissaving at old age that life-cycle models predict,[1] with even some saving still taking place prior to death. Bequests can even be regarded as terminal consumptions, as in Hurd (1989) where utility depends not only on the consumption path but also on the bequest itself, albeit with a small marginal utility.[2] Alternatively, Bernheim et al. (1985) test a model of strategic bequeathing in which parents try to influence children's behaviour through their will, with implications for capital accumulation and social security that differ substantially from those of other possible bequest motives—Amir (1996) extends its equilibrium existence results to the case of stochastic production. Finally, Gobbi and Goñi (2021) model bequests as a way to preserve the family lineage. In this paper, as the details of the model will show, bequests will be driven by altruism.[3]

As for parents' motives for educational investments, those shared with bequests—altruism, warm glow, parental consumption of children's utility ...—are compounded with the high returns to education. This makes the problem of educational investments reminiscent of the point made in Drazen (1978). Indeed, contrarily to what was argued in Barro (1974), Drazen (1978) establishes that government debt *is* net wealth—even for altruistic households with limited life-spans—as soon as the possibility of bequeathing through educational investments is added to the model.[4] The gist of the point in Drazen (1978) is that—since the implicit return to investments in human capital seems, by revealed preferences, to be empirically higher than that of physical capital (at least up to some threshold)—a liability passed on their children allows households to increase the return to their savings for retirement by investing in their children education and making them pay the taxes needed to repay the government debt, instead of investing in physical capital. From the assumption that the return to investments in children's human capital exceeds that of physical capital up to some threshold, Drazen (1978) concludes that bequests would be, as much as possible, in

[1] Brittain (1978), Mirer (1979) already pointed the observed saving behaviour of the elderly to be at odds with life cycle models.

[2] Hurd (1989), in spite of introducing bequests into the utility, finds evidence that most bequests seem to be accidental. Interestingly enough, the evidence put forward by Hurd (1989) contradicts Davies (1981): elderly average wealth holdings decrease with age.

[3] Whether the results might differ under alternative bequest motives would be interesting to check. Unfortunately, doing so would multiply several times the length of the paper—since all the developments next would need to be replicated for the different avatars of the model under each of the bequest motive assumptions—and is therefore left for future work.

[4] Government bonds expand the budget set of households by allowing for negative bequests capturing resources from future generations, by imposing on them the future taxes needed to pay for the interest and principal of the debt. Barro (1974) claimed that as long as households' choice is to make positive bequests—which seems to be the empirically relevant case—this possibility would not effectively change the equilibrium allocation, since even in their absence households can reduce their positive bequests but they choose not to.

human capital, the composition depending thus on whether the amount of the bequest exceeds or not the threshold. It remains, nonetheless, that the subsequent literature kept focusing on the Ricardian equivalence (or debt neutrality) debate—e.g. see Weil, (1987), Barnett et al. (2013)— instead of on the right mix of bequests and education.

More generally, although the literature on parental educational investment is abundant, as well as that on bequests, there is surprisingly little on, specifically, what is the right mix. Glomm and Ravikumar (1992) consider households that, effectively, consume income both as consumption good and as a bequest to their children—through an educational investment increasing the latter's income. In that framework, parents do not have actually the option to bequeath wealth and, as a consequence, the question of what is the right mix cannot be addressed. Pecchenino and Pollard (2007) introduce an *ad hoc* need of both public and household input into the human capital formation function, distorting effectively the households choice between education and bequests.[5] Additionally, households value in their framework their offspring's human capital as opposed to integrating in the objective their offspring's own value function, preventing thus to exploit the recursivity implicit in the problem. Finally, parents' human capital does not play there any role in the formation of their children's, against the available evidence.[6] Staffolani and Valentini (2007) share the same modelling choice of making parents value their offspring's human capital—rather than using their offspring's value function, preventing again to exploit recursivity—and, moreover, they consider financially constrained education expenditures, which necessarily distorts the households mix of education and bequests. On a related matter, Bénabou (2002) studies education subsidies financed through progressive income taxes, which surely would impact households' mix of educational investments and bequests but, moreover, he does it in a framework lacking any intertemporal choice—"clearly an oversimplified (but quite common) representation of asset market incompleteness [that] represents the main price of analytical tractability" (p.485)— which prevents to address the question at hand too.

At any rate, the fact is that households do choose the mix of wealth and human capital—the latter through education expenditures— they endow their offspring with for, say, altruistic reasons. Moreover, the effectiveness of households' education efforts arguably depends on the parents' own human capital—in such a way that any educational investment translates into a higher human capital for the offspring when compounded with a high human capital of the parents. Bearing this in mind, the question remains of whether households choose the optimal mix of bequests and education.

To answer this question, one would expect that—in the absence of any other interdependence—whenever parents take into account the value for their children of their chosen mix of bequests and education—as well as internalise the impact of their own human capital in the latter—they would be able to choose the right one, *i.e.* the planner's. Nevertheless, human capital has been shown to be a main determinant of total factor productivity (TFP), so that educational investments lead to increases

[5] Their modelling choice is nonetheless justified by their goal of addressing specifically the impact of ageing in funding public education and social security, which they take as a given.

[6] See Chevalier et al. (2005) for evidence that children's education (as well as income) is highly correlated to that of their parents—the correlation being stronger between maternal education and that of sons.

in factor prices through TFP that competitive households cannot internalise.[7] As a result, in spite of parents assessing correctly the value for their children of their mix of bequests and educational investment —as well as how their own education compounds with it—markets are shown below to be unable to deliver (generically under laissez-faire) a planner allocation.

Moreover, I show the market wage to be too high at a steady state—relative to the labor marginal productivity at the planner's steady state—whenever the human capital production function is close enough to affine in a neighbourhood of the market and planner steady states. The planner's steady state allocation can nonetheless be decentralised by a policy that steers households' choices towards it through the right incentives. Specifically, labor income needs to be subsidised through a second period lump-sum tax that reduces bequests, in order to give parents the incentive to both bequeath to their children and invest in their education the right amounts.

By focusing on the issue of the education-bequests mix, this paper goes beyond Drazen (1978) and others by establishing that—regardless a possible role for government bonds in expanding the budget set of altruistic households with limited life-spans in the presence of human capital—the mix of bequests and education provided by parents to children is, at a market equilibrium, inefficient as a result of the impact of human capital accumulation on total factor productivity. The paper provides too, without having to resort to any assumption on how the returns to human capital compare to those of physical capital, an assessment of the direction of the inefficiency—namely, parents provide less education and higher bequests than the planner would—as well as a policy to undo it.

It is worth mentioning that Caballé (1995) considers a similar model, but nonetheless significantly different in that children's human capital endowment increases there with the average investment too—instead of parents' human capital, as in this paper—which leads to inefficient endogenous growth. Moreover, the impact of educational investments on total factor productivity was crucially *not* taken into account in Caballé (1995), while a number of studies have established its empirical importance since

7 Among others, Benhabib and Spiegel (2005), Bronzini and Piselli (2009), Coe et al. (2009) show the elasticity of TFP with respect to years of schooling to be positive and statistically significant. Moreover, Erosa et al. (2010) show human capital accumulation to strongly amplify TFP differences across countries, and Wei and Hao (2011) show that improvements in education quality (measured by the teacher-student ratio and government expenditure on education) significantly enhance TFP growth. See also Dávila (2017) for the impact of consumption components of output—to which education expenditures belongs—on total factor productivity in general.

then.[8] As it is known to be the case for externalities similar to his,[9] in Caballé (1995) competitive households do not internalise the positive externality of their educational investments in their children's human capital on that of everybody else's children, resulting in a market allocation that delivers an inefficient underinvestment in human capital. Nevertheless, contrarily to what is generally argued, Caballé (1995) points that subsidising education might fail to increase the rate of growth in an economy of altruistic overlapping generations when young agents cannot borrow to make educational investment in their own human capital—which only their parents can do for them, as in the current paper—in the case in which the "physical bequest motive is not operative", that is to say when at equilibrium the non-negativity constraint on bequests is binding —so that households would have actually liked to bequeath liabilities rather than assets to their offspring. This result hinges, nonetheless, on the interplay of, on the one hand, the inefficiency resulting from the lack of internalisation of the positive externality from average education on human capital formation with, on the other hand, the usual inefficiency resulting from over-accumulation of physical capital when the latter is the only means of saving. Nevertheless, this problem is not present when households can save in some other asset—like fiat money or rolled-over government debt—or when there is some mechanism allowing, equivalently, to implement transfers from young to old —like, for instance, a pay-as-you-go pension scheme. In this paper, in particular, households can make contributions to a pension fund on top of lending to firms. All in all, since this paper differs from Caballé (1995) in a significant number of important ways, the analysis in the latter does not apply to the model considered here.

Last but not least, since I will allow for endogenous fertility, it is worth mentioning a link between bequests, education, *and fertility* that holds under some conditions. Indeed, Córdoba and Ripoll (2016) study, in a model similar to that of this paper, the role of inter-temporal transfers—specifically, the constraint requiring bequests to be non-negative—in explaining the negative correlation between fertility and income, since parents would want, in principle, to raise any child whose expected future income exceeds in present value the costs of raising (e.g. education costs). Nevertheless, if

8 Kim and Loayza (2019) provide a thorough account of the existing studies establishing the importance of educational investments in human capital as a main determinant of TFP. Specifically, Benhabib and Spiegel (1994) conclude from a study on 78 countries in the period 1965–1985 that the growth rate of TFP depends significantly on a nation's human capital stock. Miller and Upadhyay (2000) find that human capital contributes positively to TFP, through a study on 83 countries in 1960–1989. Barro (2001) establishes from an even bigger sample of countries (100) and a longer period of observations (1965–1995) that growth is significantly related to both quantity (years of schooling at the secondary and higher levels for males) and quality (students' test scores) of education. Griffith et al. (2004) focus on 12 OECD countries in 1974–1990, but find also that the percentage of population having attained higher school affects the rate of convergence of TFP growth. Benhabib and Spiegel (2005) find from 27 countries in the period 1960–1995 a positive elasticity of TFP with respect to years of schooling in the range 0.008–0.018 that is statistically significant. Coe et al. (2009) find this elasticity to be much higher (0.513–0.756) and statistically significant too for another sample of 24 countries at a later period (1971–2004). Bronzini and Piselli (2009) find an elasticity falling within the two previous extremes (0.379) from Italian data in the period 1985–2001. Erosa et al. (2010) use a model calibrated on US micro evidence in 1990–1995 to obtain that human capital accumulation strongly amplifies TFP differences across countries. Wei and Hao (2011) show from data across Chinese provinces in the period 1985–2004 that both school enrolment and education quality (measured by the teacher-student ratio and government expenditure on education) has a statistically significant, positive effect on TFP growth.

9 For instance, in Arrow (1962), Romer (1986), Lucas (1988).

(education) costs cannot be met by the parents, then the child will not be raised, since parents cannot borrow against the future income of the child. As a consequence, whenever a child's marginal cost increases with wages faster than the marginal benefit of the child's expected future income —something for which the authors provide possible channels—a decrease of fertility with income will ensue. Altruism will only reinforce this mechanism. As in Córdoba and Ripoll (2016), bequests are constrained to be non-negative in this paper, so that their analysis should apply as far as the link between fertility and income is concerned.

It might be worth to stress once more that the results below follow from the planner's ability to internalise the impact of educational investments on factor prices through TFP—something that households cannot do in a competitive equilibrium given their price-taking behavior. This happens to be the case regardless the level of capital—the only measure of wealth, and hence of 'development', in this model—so that this key mechanism (the internalisation of a pecuniary externality working its way through the total factor productivity) and the ensuing results do not depend on whether capital accumulates or grows, nor therefore on stages of development in the economy. It is reasonable, nonetheless, to expect the mechanism to have a stronger impact for developed economies whose factor prices are more responsive to increases in TFP. The same applies if land was separated from capital as a distinct factor since the externality works its way through the total factor productivity and hence does not depend on the number of factors of production.

As a final remark, since households could in principle choose to increase both education expenditures and bequests at the expense of the parents' consumption, one might wonder whether the empirical evidence supports the idea that there is actually a trade-off for parents between bequeathing to their children and investing in their education. Its existence might arguably depend on the specific income and wealth characteristics of the household but, in aggregate terms, one can address the question looking at the relationship in the US between the aggregate estate tax returns[10]—i.e. the amounts declared as bequeathed—and the consumption of education services,[11] both in per capita terms.[12] Figure 1 below shows (for the US during 1995–2020) a negative correlation between the two indeed, which supports the intuition of at least a seizable subset of households actually facing a choice between bequeathing wealth or educating their children.[13]

The rest of the paper is organised as follows. Section 2 presents the economy, namely its demographics and production possibilities. Section 3 characterises the allocations—whether non-stationary or stationary—that a planner would choose for such an economy. Section 4 then characterises the competitive equilibrium allocations when households can save by means of both lending to firms and contributing

10 https://www.irs.gov/statistics/soi-tax-stats-estate-tax-statistics-filing-year-table-1 (IRS tax statistics).

11 https://fred.stlouisfed.org/graph/?id=DTEDRC1A027NBEA (US Bureau of Economic Analysis).

12 Using https://fred.stlouisfed.org/series/POPTHM (US Bureau of Economic Analysis).

13 Specifically, $\hat{Y} = 932.1078 - 0.5099X$ with an correlation R equal to -0.4006 indicating a moderate inverse relationship between X (per capita estate tax return) and Y (per capita consumption expenditures in education services). Since the significance is acceptable with a p-value = 0.04256, it supports a negative correlation indeed. A similar exercise for economies with education services less expensive than in the US would be desirable, but is somewhat hampered by a readily availability of data.

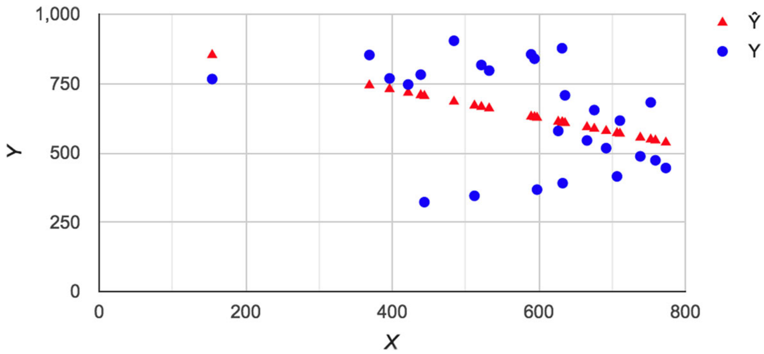

Fig. 1 **Relationship between bequests and education in the US.** X: per capita estate tax return in USD; Y: per capita consumption expenditures in education services in USD; $\hat{Y}$: linear regression of Y over X (annually 1995–2020, sources IRS and Bureau of Economic Analysis, details in footnotes 10–12)

to a pension fund,[14] as well as educate and leave bequests to their children.[15] By means of these characterisations, I show that a planner's allocation cannot be a market allocation, for any generic economy with altruistic households making education investments and bequests for their children. In Sect. 5, a policy is shown to decentralise the planner allocations—whether non-stationary or stationary—as competitive equilibria. Specifically, I identify a balanced labor income subsidy and old age lump-sum tax that decentralises the planner's allocations. Finally, Sect. 6 compares the planner and the market steady states and shows how the market steady state misses the planner's. This section provides also a quantitative assessment of the inefficiency through the size of the subsidy needed to undo it at the steady state. Section 7 concludes.

2 The economy

Consider an economy of identical 2-period lived overlapping generations of households reproducing by a factor $n^t \geq 0$ from period t into period $t+1$. The representative household born at t derives a direct utility $u(c_0^t, c_1^t)$ from its consumption —c_0^t when young and c_1^t when old.[16] Output is produced each period out of the (*per* young) previously unconsumed output used as physical capital k^{t-1}/n^{t-1}, and the young

[14] When capital is the only means of saving, it cannot generically achieve simultaneously the two goals of equalising the marginal return to capital to both the representative agent's inter-temporal rate of substitution —necessary for the optimality of the household savings—and the population growth factor at every period—necessary for the maximisation of the output net of investment.

[15] Strictly speaking, a bequest in the standard 2-period lived agents OLG model is an *inter vivos* transfer, since it decreases the old age wealth of the parent household and increases the young age wealth of the *contemporaneous* children. There is no room in such a model to distinguish between donations and bequests, the life span is too short—for a meaningful distinction generations need to possibly overlap at least two periods and death after the first of them needs to be random, see e.g. Nishiyama (2002).

[16] With u being differentiably strictly increasing and differentiably strictly quasi-concave—i.e. $Du(c_0^t, c_1^t) \in \mathbb{R}^2_{++}$ and $D^2u(c_0^t, c_1^t)$ is negative definite in the orthogonal space to $Du(c_0^t, c_1^t)$, for all $(c_0^t, c_1^t) \in \mathbb{R}^2_+$—and well-behaved at the boundary—i.e. $\lim_{(c_0^t, c_1^t)\to\partial\mathbb{R}^2_+\setminus\{0\}} Du(c_0^t, c_1^t)\cdot(c_0^t, c_1^t) = 0$— so that consumption demands will be interior for positive prices.

household's human capital h^t.[17] Technology is represented by a neoclassical production function F delivering a (*per* young) output $A(e^{t-1})F\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right)$ at t.[18] The total factor productivity $A(e^{t-1})$ increases with the educational investment in human capital e^{t-1} of period $t-1$ household—or the planner, in its behalf—on each of its children household's human capital h^t, so that $A' > 0$.[19] This investment results from the fact that each household born at, say, t derives utility from that of its n^t children households too, discounted by an altruism factor $\gamma \in (0, 1)$. The human capital h^t a household born at t is endowed with when young increases, therefore, with (i) the *per* child educational investment e^{t-1} made at $t-1$ by—or on behalf of, if chosen by a planner—its parent household, and with (ii) the parent household's own human capital h^{t-1}, through a human capital production function

$$h^t = H(e^{t-1}, h^{t-1}) \tag{1}$$

such that

$$\begin{aligned} H_e(e^{t-1}, h^{t-1}) > 0 < H_h(e^{t-1}, h^{t-1}) \\ H(0, h^{t-1}) = 0 = H(e^{t-1}, 0) \end{aligned} \tag{2}$$

—the last condition conveying the need of non-zero inputs of both educational investment and parent household's human capital for the production of its children's.[20]

Footnote 16 continued
The use of a representative household is justified if households are supposed to have preferences leading to an indirect utility function (or an expenditure function) of the Gorman form—*cf.* Gorman (1961) — since then the aggregate demand of all households behaves exactly like the demand of a representative household of the same type. Preferences with this property include all homothetic ones, among which those represented by popular choices in applied work like CES utility functions and, in particular, Cobb-Douglas utility functions. Quasilinear utilities are of the Gorman form too. On the other hand, the aggregability of preferences with the Gorman property is obtained at the expense of the distribution of income or wealth being inessential, which precludes to use any such model for issues, like inequality, where these distributions matter—the existence of a representative firm does not require any particular assumption on production sets, the reason being that firms have no endowments the distribution of which could matter.

17 Assuming that both capital and labor are needed for production, output at t is positive only if h^t—and hence n^{t-1} too—are positive. Also, without loss of generality, physical capital fully depreciates in one period, for the sake of simplicity.

18 That is to say, a linearly homogeneous, concave function, satisfying Inada—i.e. marginal productivities with respect to any factor increase without bound as the latter converges to zero—and the condition that output from no capital or no labor is nil. Note also that for the sake of simplicity we do away with productivity shocks and, more generally, uncertainty. This is a price to pay for the aggregability of different households into a representative one if (quite realistically) the representative household is not going to be able to trade in complete markets of contingent claims, which is the case when all uncertainty might come only from uninsurable idiosyncratic shocks to effective human capital.

19 It can be argued that TFP will be impacted (mainly) by the educational effort made on the labor force that actually works with the capital at hand—the impact of past educational investments on previous generations being already captured by the current generation's level of human capital, according to the human capital production function in (1). Alternatively, making A depend on h^t instead amounts, given (1), to make it effectively depend on the whole infinite past history of educational investments on previous generations, a certainly cumbersome feature with no obvious modelling advantage.

20 Concavity can be assumed too for H for the planner's steady state to be unique when fertility is exogenous.

Moreover, H will also be assumed, when needed, to be close enough to be affine in a neighbourhood containing the planner's and the market steady states.

3 The planner's allocations

A planner would choose feasible sequences of consumptions c_0^t, c_1^t, physical capital k^t, education e^t, human capital h^t, and fertility n^t that maximise the representative household's overall utility, which comprises the utility the household derives from its own consumption $u(c_0^t, c_1^t)$ plus the overall utility—discounted by an altruism factor γ—of each of its n^t children households under the feasibility constraints imposed by the production of output and human capital.[21]

A planner's allocation is, therefore, a profile of sequences $c_0^t, c_1^t, k^t, e^t, h^t, n^t$ solution to the problem

$$\begin{aligned} &\max_{0\leq\{c_0^t,c_1^t,k^t,e^t,h^t,n^t\}_{t\in\mathbb{N}}} \sum_{t=1}^{+\infty}\prod_{\tau=1}^{t-1} n^\tau \gamma^{t-1} u(c_0^t, c_1^t) \\ &c_0^t + \frac{c_1^{t-1}}{n^{t-1}} + k^t + e^t n^t \leq A(e^{t-1}) F\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right) \\ &h^t \leq H(e^{t-1}, h^{t-1}) \\ &\text{for all} t \geq 1, \text{given } c_1^0, k^0, e^0, h^0, n^0 > 0 \end{aligned} \tag{3}$$

where the feasibility constraint for output is written in *per* young terms.[22]

Equivalently, the planner's problem value function V^p satisfies

$$\begin{aligned} V^p(c_1^{t-1}, k^{t-1}, e^{t-1}, h^{t-1}, n^{t-1}) = &\max_{0\leq c_0^t,c_1^t,k^t,e^t,h^t,n^t} u(c_0^t, c_1^t) + n^t\gamma V^p(c_1^t, k^t, e^t, h^t, n^t) \\ &c_0^t + \frac{c_1^{t-1}}{n^{t-1}} + k^t + n^t e^t \leq A(e^{t-1}) F\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right) \\ &h^t \leq H(e^{t-1}, h^{t-1}) \\ &\text{given } c_1^{t-1}, k^{t-1}, e^{t-1}, h^{t-1}, n^{t-1} \end{aligned} \tag{4}$$

[21] The overall utility of each of its children households comprises, in turn, the utility they derive from their own consumption profile $u(c_0^{t+1}, c_1^{t+1})$, plus the overall utility—discounted by the altruism factor γ—of each of their own n^{t+1} children households, which comprises the utility they derive from their own consumption profile $u(c_0^{t+2}, c_1^{t+2})$, plus the overall utility—discounted by an altruism factor γ—of each of its n^{t+2} children households... and so on.

[22] Note that the solution necessarily has $n^t > 0$ for all $t \geq 1$—so that the feasibility constraint can be written in *per* young terms indeed—since should $n^t = 0$ hold for some t, then output would collapse at $t+1$, and with it investment, so that output would be nil for all periods onwards, leading to zero consumption from $t+1$ onwards, which is suboptimal. On the other hand, the maximum exists if n^t remains bounded above by, and away from, $\frac{1}{\gamma}$—that is to say, if $\sup n^t < \frac{1}{\gamma}$—which actually allows for an altruism factor arbitrarily close to 1 for any economy with a total fertility rate below the replacement rate—the situation of, essentially, any developed economy, and many developing economies in recent years.

for all $t \in \mathbb{N}$, so that the truncation of the solution to (3) starting at any given period t is also the solution to the planner's problem of maximising the utility of the representative household born at that period t, given $c_1^{t-1}, k^{t-1}, e^{t-1}, h^{t-1}$, and n^{t-1}. As a consequence, by maximising the first generation's utility in (3), the planner maximises the utility of all future generations too.[23]

It follows from its necessary first-order conditions (see the appendix) that the solution to the planner's problem is characterised by the proposition next.[24]

Proposition 3.1 *The* **planner's allocation** *for the economy characterised by preferences, production of output, and production of human capital represented by* u, A, F *and* H*—under the assumptions stated in* Sect. 2*—and the initial conditions* $c_0^0, c_1^0, k^0, e^0, h^0, n^0$ *is a profile of positive sequences* $c_0^t, c_1^t, k^t, e^t, h^t, n^t$ *such that, if* $\sup n^t < \frac{1}{\gamma}$*, then*

$$\begin{aligned}
\frac{u_0(c_0^{t+1}, c_1^{t+1})}{u_1(c_0^t, c_1^t)} &= \frac{1}{\gamma} \\
\frac{u_0(c_0^t, c_1^t)}{u_1(c_0^t, c_1^t)} &= A(e^t) F_K\left(\frac{k^t}{n^t}, h^{t+1}\right) \\
\frac{\frac{1}{\gamma}}{H_e(e^{t-1}, h^{t-1})}\left[\frac{u_0(c_0^{t-1}, c_1^{t-1})}{u_0(c_0^t, c_1^t)} - \frac{u_1(c_0^{t-1}, c_1^{t-1})}{u_0(c_0^t, c_1^t)} A'(e^{t-1}) F\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right)\right] & \\
-n^t \frac{H_h(e^t, h^t)}{H_e(e^t, h^t)}\left[1 - \frac{u_1(c_0^t, c_1^t)}{u_0(c_0^t, c_1^t)} A'(e^t) F\left(\frac{k^t}{n^t}, h^{t+1}\right)\right] &= A(e^{t-1}) F_L\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right) \\
c_0^t + \frac{c_1^{t-1}}{n^{t-1}} + k^t + n^t e^t &= F\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right) \\
h^t &= H(e^{t-1}, h^{t-1})
\end{aligned} \tag{5}$$

23 Bernheim (1989) adresses the issue of the difficulty of defining, more generally, the planner's objective in the case of altruistic agents: since each altruistic generation takes into account—through its offspring's—the utility of *all* future descendants, then maximising a sum of all generations' utilities—weighted by a positive sequence ρ_t such that $\sum_t \rho_t = 1$, that is to say (for the constant population growth factor case), maximising $\sum_{t=1}^{+\infty} \rho_t \left[\sum_{t'=t}^{+\infty} (\gamma n)^{t'-t} u(c_0^{t'}, c_1^{t'})\right]$—leads to a double-counting of future consumption utilities that *de facto* weights generations t's utility from consumption $u(c_0^t, c_1^t)$ by a factor $\sum_{t'=1}^{t} \rho_{t'} (\gamma n)^{t-t'}$. In the current setup, maximising the utility of the first generation avoids this double-counting problem since it amounts to maximising the utility of all generations, due to the recursivity of the planner's problem exhibited in (4).

24 With endogenous fertility, a solution to the first-order conditions is not necessarily the solution to the problem—due to the non-convexity introduced by the term $n^t e^t$ in the constraints. Nevertheless, the solution to the problem is necessarily a solution to its first-order conditions and is therefore characterised by Proposition 3.1. If fertility was exogenous, the optimisation problem would be and its first-order conditions sufficient too.

for all integer $t \geq 1$. [25]

In the next section, the market equilibrium allocations are characterised, in order to be able to study how they compare to the planner's. Note that the assumptions on H in the second line in (2) imply that, at a planner's allocation, it holds $e^t > 0$ and $h^t > 0$, for all integer $t \geq 1$.[26] Accordingly, we will focus below on market equilibrium allocations in which $e^t > 0$ and $h^t > 0$ hold.

4 The market allocations

When interacting through markets, the altruistic households choose their consumption profiles, fertility, educational investment in their children's human capital, and bequests that maximise their utilities, given their labor income when young and the returns to their savings when old. Savings can be carried to the old age as loans to firms for a return, or as contributions to a pension fund. Resources can therefore flow inter-generationally in both directions, as old households receive pensions paid out of contemporaneous young households' contributions, as well as young households receive resources from parent households in the form bequests and paid education.

Specifically, households born at t can transfer resources to their offspring by (i) bequeathing to each of the n^t of them some amount b^t of physical capital when old,[27] or (ii) investing when young some amount e^t into the labor endowment of each of their children households. At the same time, young households at t can make a contribution m^t to a pension fund that entitles them to be paid a pension $\rho_{t+1}m^t$—with an endogenous gross return ρ_{t+1} on the contribution. Finally, households born at t can also save by means of lending some amount k^t of physical capital to firms at a gross rental rate or return factor r_{t+1} to be paid next period $t + 1$.

The representative household born at t makes therefore bequest, education, saving portfolio, fertility, and consumption choices —i.e. b^t, e^t, k^t, m^t, n^t, c_0^t and c_1^t when young and old respectively—from which it derives a direct utility $u(c_0^t, c_1^t)$ that adds to the overall utility obtained by each of its n^t children, weighted by an altruism factor $\gamma \in (0, 1)$, in order to solve the problem defined in the next section.

[25] Moreover, the planner's value function must satisfy, at the solution,

$$V^p(c_1^t, k^t, e^t, h^t, n^t) = \frac{1}{\gamma}\Big[u_0(c_0^t, c_1^t)\big(e^t - \frac{k^t}{n^t}\big) - u_1(c_0^t, c_1^t)\frac{c_1^t}{n^t}\Big]$$

[26] Should $e^t = 0$ or $h^t = 0$ hold for some t, then human capital at $t + 1$ would collapse, and as a consequence output too, and with it investment, and hence output for all periods onwards, leading to zero consumption from $t + 1$ onwards, which is suboptimal.

[27] All the children of a household are supposed to receive the same share of the inheritance. Indeed, nowadays most jurisdictions establish a reserved portion of the estate—to be split equally among children—that limits the ability of the parent to favour some heirs over others, as opposed to other inheritance laws and customs, among which primogeniture, that were more prevalent in the past. This is intended to capture the historical trend towards more egalitarian successions. Indeed, in the US, primogeniture had been abolished by the end of the 18th century. In France, egalitarian inheritance had been enshrined by the Napoleonian civil code in 1804 —and thus exported across Europe with the Napoleonic wars. In the UK primogeniture was still the default rule in the absence of a will until its abolition by the parliament in 1925.

4.1 Household's optimal choice

Given the physical capital bequest b^{t-1} received, and the human capital h^t it is endowed with as a result of its parents education investment in combination with their own—that is to say, e^{t-1} and h^{t-1} respectively—the period t representative household maximises—with respect to its consumption profile c_0^t, c_1^t, saving choices (in capital and pension fund contributions) k^t, m^t, fertility n^t, educational effort e^t, and bequest b^t, and under the budget constraints determined by the pension fund returns and factors prices $x_t \equiv (\rho_{t+1}, w_t, r_{t+1})$, as well as under the human capital formation technology constraint—its overall utility $V^m(e^{t-1}, h^{t-1}, b^{t-1}; \mathbf{x}_t)$—where $\mathbf{x}_t \equiv \{x_\tau\}_{\tau \geq t}$ is all future pension fund returns and factor prices—comprising the utility it derives from its consumption profile $u(c_0^t, c_1^t)$, plus the maximum overall utility $V^m(e^t, h^t, b^t; \mathbf{x}_{t+1})$ of each of its n^t children, weighted by the altruism factor γ. That is to say , the period t representative households solves

$$
\begin{aligned}
V^m(e^{t-1}, h^{t-1}, b^{t-1}; \mathbf{x}_t) = \max_{0 \leq c_0^t, c_1^t, k^t, m^t, e^t, h^t, b^t, n^t} \quad & u(c_0^t, c_1^t) + n^t \gamma V^m(e^t, h^t, b^t; \mathbf{x}_{t+1}) \\
& c_0^t + k^t + m^t + n^t e^t \leq w_t h^t + b^{t-1} \\
& c_1^t + n^t b^t \leq r_{t+1} k^t + \rho_{t+1} m^t \\
& h^t \leq H(e^{t-1}, h^{t-1})
\end{aligned} \tag{6}
$$

for given pension fund returns and factor prices $\mathbf{x}_t \equiv \{x_\tau\}_{\tau \geq t} \equiv \{(\rho_{\tau+1}, w_\tau, r_{\tau+1})\}_{\tau \geq t}$, parents' choices $e^{t-1}, h^{t-1}, b^{t-1}$, and an altruism factor $\gamma \in (0, 1)$. [28]

It is worth noting that from the multiplication of fertility and educational effort in the first period budget constraint, households face a quantity-quality trade-off. Also, although it is obvious from the first period budget constraint that, at the solution, the third constraint in (6) is always binding, the recursive way in which human capital is formed requires h^t to be included in t's problem *as if* it was a variable of choice—which it is actually none, since it is pre-determined for the household by e^{t-1} and h^{t-1}, that is to say by $e^{t-1}, e^{t-2}, e^{t-3}, \ldots$. Superscript m (for market) in V^m distinguishes it from the value function V^p delivering the households' maximum utility in the planner's problem. Future pension fund returns and factor prices are assumed to be known with perfect foresight.

It follows from the necessary first-order conditions of the problem (see the Appendix) that period t household's optimal choice $c_0^t, c_1^t, k^t, m^t, e^t, b^t, n^t$, and human capital endowment h^t are necessarily characterised[29] —whenever $e^t > 0$,

28 The model being a single-commodity model in which all goods are aggregated into output, bequests comprise any type of asset bequeathed, making abstraction of the different roles of land, structures, equipment or financial capital. Nevertheless, land—the historically most important asset when it comes to bequests in pre-industrial times—contributes now, in the US, only 5% to total income (*cf.* Vanlentinyi and Herrendorf (2008)) compared to the 28% produced by all other types of capital. The contribution of land to GDP is comparable to that of the rest of capital only for agriculture, but farms only produced 0.6% of GDP in the US in 2019, hence its relatively marginal role in the production of income and the reason for making abstraction of it in the model. Having said so, in applied work on developing countries where agriculture still plays an important role, a distinctive role for land should be considered.

29 Ignoring the (at a solution, non-binding) non-negativity constraints for c_0^t, c_1^t (because of u being well-behaved at the boundary), for k^t (because, at equilibrium, the returns to k^t and m^t will be positive and equal,

and $n^t < \frac{1}{\gamma}$—by

$$
\begin{aligned}
\frac{u_0(c_0^{t+1}, c_1^{t+1})}{u_1(c_0^t, c_1^t)} &\leq \frac{1}{\gamma}(=, \text{ if } b^t > 0) \\
(\text{ if } m^t > 0, =)\rho_{t+1} \leq \frac{u_0(c_0^t, c_1^t)}{u_1(c_0^t, c_1^t)} &= r_{t+1} \\
\frac{\frac{1}{\gamma}}{H_e(e^{t-1}, h^{t-1})} \frac{u_0(c_0^{t-1}, c_1^{t-1})}{u_0(c_0^t, c_1^t)} - n^t \frac{H_h(e^t, h^t)}{H_e(e^t, h^t)} &= w_t \ (\text{ if } n^t > 0) \\
c_0^t + k^t + m^t + n^t e^t &= w_t h^t + b^{t-1} \\
c_1^t + n^t b^t &= r_{t+1} k^t + \rho_{t+1} m^t \\
h^t &= H(e^{t-1}, h^{t-1})
\end{aligned}
\tag{7}
$$

for all integer $t \geq 1$.[30]

4.2 Market equilibria

At a market equilibrium, capital and labor are remunerated by their marginal productivities—from the profit maximising behaviour of firms—so that

$$
\begin{aligned}
w_t &= A(e^{t-1}) F_L \left(\frac{k^{t-1}}{n^{t-1}}, h^t \right) \\
r_{t+1} &= A(e^t) F_K \left(\frac{k^t}{n^t}, h^{t+1} \right)
\end{aligned}
\tag{8}
$$

and the allocation is feasible, *i.e.*

$$
c_0^t + \frac{c_1^{t-1}}{n^{t-1}} + k^t + n^t e^t = A(e^{t-1}) F \left(\frac{k^{t-1}}{n^{t-1}}, h^t \right) \tag{9}
$$

and hence the composition of the optimal savings *portfolio* is indeterminate in the household's choice and necessarily positive—since u is well-behaved at the boundary—so that *one* of the non-negativity constraints on k^t and m^t can be dropped), and for h^t (because of the differentiably strictly increasing assumption on u, unless $H(e^{t-1}, h^{t-1}) = 0$ itself).

30 Moreover, the planner's value function must satisfy, at the solution,

$$
V^m(e^t, h^t, b^t; \mathbf{x}_{t+1}) \leq \frac{1}{\gamma}\left[u_0(c_0^t, c_1^t) e^t + u_1(c_0^t, c_1^t) b^t\right] (=, \text{ if } n^t > 0)
$$

with $\mathbf{x}_{t+1} \equiv \{x_\tau\}_{\tau \geq t+1} \equiv \{(\rho_{\tau+1}, w_\tau, r_{\tau+1})\}_{\tau \geq t+1}$. I focus on household optimal choices with positive educational investment—as in the planner's allocations and steady state—since the goal is to verify the market decentralizability of the latter. Finally, the remark in footnote 24 on convexity applies to (6) too.

which, as usual, follows from collapsing the budget constraints of the agents alive at any given period t, i.e.

$$\begin{aligned} c_0^t + k^t + m^t + n^t e^t &= w_t h^t + b^{t-1} \\ \frac{c_1^{t-1}}{n^{t-1}} + b^{t-1} &= r_t \frac{k^{t-1}}{n^{t-1}} + \rho_t \frac{m^{t-1}}{n^{t-1}} \end{aligned} \tag{10}$$

whenever the pension fund solvency condition next

$$m^t = \rho_t \frac{m^{t-1}}{n^{t-1}} \tag{11}$$

holds at equilibrium.

Thus, taking into account the households' optimal behaviour characterised in the previous section, the conditions necessarily characterising a market equilibrium allocation in which households' educational investment is positive are those provided by Proposition 4.1 next.

Proposition 4.1 *A* **market equilibrium allocation** *with positive educational investments for an economy characterised by preferences, production of output, and production of human capital represented by* u, A, F *and* H*—under the assumptions stated in* Sect. 2*—and the initial conditions* $c_0^0, c_1^0, k^0, m^0, e^0, h^0, b^0, n^0$, *is a profile of sequence* $c_0^t, c_1^t, k^t, e^t, h^t, n^t$ *and* b^t, m^t, ρ_t *such that, if* $\sup n^t < \frac{1}{\gamma}$, *then*

$$\begin{aligned} \frac{u_0(c_0^{t+1}, c_1^{t+1})}{u_1(c_0^t, c_1^t)} &\le \frac{1}{\gamma} \ (=,\ \textit{if}\, b^t > 0) \\ (\textit{ if} m^t > 0, =)\rho_{t+1} \le \frac{u_0(c_0^t, c_1^t)}{u_1(c_0^t, c_1^t)} &= A(e^t) F_K\left(\frac{k^t}{n^t}, h^{t+1}\right) \\ \frac{\frac{1}{\gamma}}{H_e(e^{t-1}, h^{t-1})} \frac{u_0(c_0^{t-1}, c_1^{t-1})}{u_0(c_0^t, c_1^t)} - n^t \frac{H_h(e^t, h^t)}{H_e(e^t, h^t)} &= A(e^{t-1}) F_L\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right) \ (\textit{ if}\, n^t > 0) \\ c_0^t + k^t + m^t + n^t e^t &= A(e^{t-1}) F_L\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right) h^t + b^{t-1} \\ c_1^t + n^t b^t &= A(e^t) F_K\left(\frac{k^t}{n^t}, h^{t+1}\right) k^t + \rho_{t+1} m^t \\ h^t &= H(e^{t-1}, h^{t-1}) \\ m^t &= \rho_t \frac{m^{t-1}}{n^{t-1}} \end{aligned} \tag{12}$$

for all integer $t \ge 1$.[31]

[31] Moreover, the household's value function satisfies, at the solution,

$$V^m(e^t, h^t, b^t; \mathbf{x}_{t+1}) \le \frac{1}{\gamma}\left[u_0(c_0^t, c_1^t) e^t + u_1(c_0^t, c_1^t) b^t\right] (=, \text{if}\, n^t > 0)$$

It follows from the characterisations in Propositions 3.1 and 4.1 the next proposition stating the generic impossibility for the market, under laissez-faire, to decentralise any planner allocation and, hence, the need to look for a policy achieving that. Note that Proposition 4.2 in fact establishes the inefficiency of *all* market equilibrium allocations for any generic economy as described in Sect. 2.

Proposition 4.2 *No planner allocation of an economy characterised by preferences, production of output, and production of human capital represented by generic u, A, F and H—under the assumptions stated in* Sect. 2*—can be a market equilibrium allocation.*

Proof From the third equations in (5) and (12), a necessary condition for a planner allocation to be a market one is, whenever $n^t > 0$, that

$$\begin{aligned} &\frac{\frac{1}{\gamma}}{H_e(e^{t-1}, h^{t-1})} \frac{u_1(c_0^{t-1}, c_1^{t-1})}{u_0(c_0^t, c_1^t)} A'(e^{t-1}) F\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right) \\ &= n^t \frac{H_h(e^t, h^t)}{H_e(e^t, h^t)} \frac{u_1(c_0^t, c_1^t)}{u_0(c_0^t, c_1^t)} A'(e^t) F\left(\frac{k^t}{n^t}, h^{t+1}\right) \end{aligned} \tag{13}$$

or equivalently—after rearrangement, comparison with the third equation in (12), and replacement of the inter-temporal marginal rate of substitution by the RHS of the second equation in (12)—

$$n^t \frac{H_h(e^t, h^t)}{H_e(e^t, h^t)} = \frac{A(e^{t-1}) F_L\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right) A(e^t) F_K\left(\frac{k^t}{n^t}, h^{t+1}\right) A'(e^{t-1}) F\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right)}{A'(e^t) F\left(\frac{k^t}{n^t}, h^{t+1}\right) - A(e^t) F_K\left(\frac{k^t}{n^t}, h^{t+1}\right) A'(e^{t-1}) F\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right)} \tag{14}$$

for all t, which imposes a non-generic constraint on the functions A, F, and H. □

The intuition for the previous result is the following. Since households have a negligible individual weight in competitive markets, they are price-takers as a consequence. Thus, households cannot anticipate the impact that their own educational investment decisions have—through the total factor productivity—on the next period remuneration to the factors that they (capital) and their children (labor) supply. The planner, on the contrary, can. That the planner does take into account that impact can be seen in the terms multiplied by the derivative of the total factor productivity, A', in the LHS of the third equation in (5). Indeed, should this derivative be zero, then the two systems (5) and (12) would coincide.

The next section compares, in particular, the allocations that the market can deliver with those that the planner would choose, in order to identify a policy that would make them coincide.

with $\mathbf{x}_{t+1} \equiv \{x_\tau\}_{\tau \geq t+1} \equiv \{(\rho_{\tau+1}, w_\tau, r_{\tau+1})\}_{\tau \geq t+1}$.

5 Decentralisation of planner allocations

The decentralisation of a planner's allocation requires the introduction of a tax or subsidy (depending on its sign) that distorts the household's educational effort in order to make its impact—both direct, through the budget constraint when young, and indirect, through the children's human capital formation— replicate that of the planner. In order to do so, it is necessary to distort the rate at which labor is remunerated.[32]

Since the point of the intervention is just the decentralisation of the planner's allocation—no public spending needs to be funded— a balanced fiscal policy requires the compensation of the impact of the distortionary subsidy or tax on labor income through a non-distortionary lump-sum tax or transfer, which can be implemented both on the young or the old period budget constraints. I will present the details now for the second case, and then comment on the first case and an equivalent interpretation of the latter in terms of tax-funded public education.

Consider thus a policy consisting of (i) subsidising or taxing —depending on the sign of the rate—household t's labor income at a rate τ_t, while (ii) taxing or transferring respectively a lump-sum amount T_{t+1} when old. The representative household faces then the problem

$$\begin{aligned} V^m(e^{t-1}, h^{t-1}, b^{t-1}; \mathbf{x}_t) = & \max_{0 \le c_0^t, c_1^t, k^t, m^t, e^t, h^t, b^t, n^t} u(c_0^t, c_1^t) + n^t \gamma V^m(e^t, h^t, b^t; \mathbf{x}_{t+1}) \\ & c_0^t + k^t + m^t + n^t e^t \le (1+\tau_t) w_t h^t + b^{t-1} \\ & c_1^t + n^t b^t \le r_{t+1} k^t + \rho_{t+1} m^t + T_{t+1} \\ & h^t \le H(e^{t-1}, h^{t-1}) \end{aligned} \tag{15}$$

for a given policy, pension fund returns and factor prices $\mathbf{x}_t \equiv \{x_\tau\}_{\tau \ge t} \equiv \{(\tau_t, T_{t+1}, \rho_{\tau+1}, \ w_\tau, r_{\tau+1})\}_{\tau \ge t}$, parent choices $e^{t-1}, h^{t-1}, b^{t-1}$, and an altruism weight $\gamma < 1$.

The period t representative household's optimal choice for c_0^t, c_1^t, k^t, m^t, e^t, h^t, b^t, and n^t is—whenever $e^t > 0$ and $n^t < \frac{1}{\gamma}$—necessarily characterised by

$$\frac{u_0(c_0^{t+1}, c_1^{t+1})}{u_1(c_0^t, c_1^t)} \le \frac{1}{\gamma} \ (=, \ \text{if } b^t > 0)$$

$$(\text{if } m^t > 0, =) \ \rho_{t+1} \le \frac{u_0(c_0^t, c_1^t)}{u_1(c_0^t, c_1^t)} = r_{t+1}$$

$$\frac{\frac{1}{\gamma}}{H_e(e^{t-1}, h^{t-1})} \frac{u_0(c_0^{t-1}, c_1^{t-1})}{u_0(c_0^t, c_1^t)} - n^t \frac{H_h(e^t, h^t)}{H_e(e^t, h^t)} = (1+\tau_t) w_t \ (=, \text{if } n^t > 0)$$

$$c_0^t + k^t + m^t + n^t e^t = (1+\tau_t) w_t h^t + b^{t-1}$$

$$c_1^t + n^t b^t = r_{t+1} k^t + \rho_{t+1} m^t + T_{t+1}$$

[32] A tax or subsidy on capital income, consumption, pension fund return, or bequests would distort households trade-offs that are already in line with those of the planner in (5) and would therefore fail to implement the latter. Rewriting the corresponding household first-order conditions and subsequent equilibrium systems for each of these cases—not provided here since it follows closely the developments in appendix—suffices to verify this.

$$h^t = H(e^{t-1}, h^{t-1}) \tag{16}$$

for all integer $t \geq 1$, given $c_0^0, c_1^0, k^0, m^0, e^0, h^0, b^0, n^0$.[33]

At a market equilibrium, capital and labor are remunerated by their marginal productivities—from the profit maximising behaviour of firms—and the allocation is feasible if, and only if, the pension fund contributions exactly match the pensions paid, that is to say

$$m^t = \rho_t \frac{m^{t-1}}{n^{t-1}} \tag{17}$$

and the policy is balanced, i.e. τ_t, T_t are such that

$$0 = \tau_t w_t h^t + \frac{T_t}{n^{t-1}} \tag{18}$$

for every integer $t \geq 1$.

A market equilibrium with positive educational investments under the tax and transfer policy $\{\tau_t, T_t\}_{t\in\mathbb{N}}$ is therefore any collection of sequences for $c_0^t, c_1^t, k^t, m^t, e^t, h^t, n^t$ and ρ_t satisfying, for all integer $t \geq 1$, (16), (17), and (18) with the factor prices replaced by the corresponding marginal productivities, from where—by comparison with the planner's system in (5)—the following policy supporting the planner's steady state as a competitive equilibrium follows.

Proposition 5.1 *The planner's allocation in* (5) *is decentralised by the policy* $\{\tau_t, T_t\}_{t\in\mathbb{N}}$ *subsidising or taxing—whenever* τ_t *is positive or negative, respectively—labor income at a rate*

$$\begin{aligned}\tau_t =& \frac{\frac{1}{\gamma}}{H_e(e^{t-1}, h^{t-1})} \frac{u_1(c_0^{t-1}, c_1^{t-1})}{u_0(c_0^t, c_1^t)} \cdot \frac{A'(e^{t-1}) F\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right)}{A(e^{t-1}) F_L\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right)} \\ & - n^t \frac{H_h(e^t, h^t)}{H_e(e^t, h^t)} \cdot \frac{u_1(c_0^t, c_1^t)}{u_0(c_0^t, c_1^t)} \frac{A'(e^t) F\left(\frac{k^t}{n^t}, h^{t+1}\right)}{A(e^{t-1}) F_L\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right)}\end{aligned} \tag{19}$$

with a second-period lump-sum tax or transfer—whenever T_t *is negative or positive, respectively—*

$$T_t = -n^{t-1} \tau_t A(e^{t-1}) F_L\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right) h^t \tag{20}$$

[33] Moreover, the household's value function satisfies, at the solution,

$$V^m(e^t, h^t, b^t; \mathbf{x}_{t+1}) \leq \frac{1}{\gamma}\left[u_0(c_0^t, c_1^t)e^t + u_1(c_0^t, c_1^t)b^t\right] (=, \text{ if } n^t > 0)$$

with $\mathbf{x}_{t+1} \equiv \{x_\tau\}_{\tau \geq t+1} \equiv \{(\tau_t, T_{t+1}, \rho_{\tau+1}, w_\tau, r_{\tau+1})\}_{\tau \geq t+1}$.

Proof It follows from the comparison of the household first-order conditions under the policy above in (16) with those of the planner's allocation in (5) that, for the policy to decentralise the latter through the market, the product of the subsidy with the wage—or rather labor productivity at equilibrium—in the right-hand side must exactly match the terms on the planner's condition in (5) in which the latter takes into account the impact of educational investment on total factor productivity. The lump-sum (tax or transfer) T_t follows from the government balanced budget condition (18). □

The intuition of the previous result is the following. Since parents cannot realise in a competitive setup how their educational investments in their children human capital increase not only the latter but also the remuneration to capital and labor through an increase in total factor productivity, a rate τ_t distorting the remuneration to labor is needed for them to take into account this impact, and therefore adjust their educational effort at the expense of bequests—since whenever τ_t is positive it is funded by a non-distortionary lump-sum tax in the second period that reduces the amounts that can be bequeathed.

If the lump-sum was transferred to (or taxed from) young age income instead, the household's first order conditions would remain unchanged nonetheless—so that Proposition 5.1 still applies— and the only difference in (16) would appear in the budget constraints, which would become

$$\begin{aligned} c_0^t + k^t + m^t + n^t e^t &= (1+\tau_t) w_t h^t + b^{t-1} + T_t \\ c_1^t + n^t b^t &= r_{t+1} k^t + \rho_{t+1} m^t \end{aligned} \tag{21}$$

In this case, the feasibility of the allocation amounts—by adding up the budget constraints of the two agents alive in any given period t—to the usual pension fund solvency condition

$$m^t = \rho_t \frac{m^{t-1}}{n^{t-1}} \tag{22}$$

if the fiscal policy is balanced, i.e. if

$$T_t = -\tau_t w_t h^t \tag{23}$$

so that the budget constraint when young is actually the same as before. From the difference in the second period budget constraints it follows nevertheless that either the bequests or the pension fund contributions—depending on which is positive—have to vary across the two cases in order to make up for the difference.

It is worth noticing that this second case is equivalent to that of a policy of public education funded through taxes on parent households' labor income whenever $\tau_t < 0$.

Indeed, assume the household's problem becomes

$$
\begin{aligned}
V^m(e^{t-1}, h^{t-1}, b^{t-1}; \mathbf{x}_t) = \max_{0\le c_0^t, c_1^t, k^t, m^t, e^t, h^t, b^t, n^t} & \; u(c_0^t, c_1^t) + n^t \gamma V^m(e^t, h^t, b^t; \mathbf{x}_{t+1}) \\
& c_0^t + k^t + m^t + n^t e^t \le (1+\tau_t) w_t h^t + b^{t-1} \\
& c_1^t + n^t b^t \le r_{t+1} k^t + \rho_{t+1} m^t \\
& h^t \le H(e^{t-1} + e_{t-1}^p, h^{t-1})
\end{aligned}
\tag{24}
$$

given e_{t-1}^p—a publicly provided educational investment in the human capital of the household born at t funded out of the parents' labor income taxes—within $\mathbf{x}_t$ as well now. The household's first-order conditions would remain unchanged, and for a balanced fiscal policy such that

$$\tau_t w_t h^t = -n^t e_t^p \tag{25}$$

feasibility still amounts to

$$m^t = \rho_t \frac{m^{t-1}}{n^{t-1}}. \tag{26}$$

Then the proposition next follows.

Proposition 5.2 *The planner's allocation in* (5) *can be decentralised by a public education* $\{e_t^p\}_{t\in\mathbb{N}}$ *funded with the proceeds from the labor income tax* $\{\tau_t\}_{t\in\mathbb{N}}$ *in* Proposition 5.1*—whenever* $\tau_t < 0$*—while households' educational effort is crowded out by the same amount as a result.*

Proof Comparing the resulting equilibrium system with that of the planner's allocation in (5), whenever $m^t = 0$ and $b^t > 0$, the two systems coincide if[34]

$$\bar{e}^t = \tilde{e}^t + e_t^p \tag{27}$$

and τ_t is that of (19) in Proposition 5.1. In other words, the implementation of the planner's allocation through public education paid for by means of parents' labor income taxes crowds out households' educational efforts. □

It follows from Propositions 5.1 and 5.2 that public education can nonetheless lead to an inefficient allocation if (27) does not hold or labor income tax rates and transfers are not those given by (19) and (20)—whenever $m^t = 0$ and $b^t > 0$. Indeed, note that the household internalises—through the altruistic term in its objective—the consequences of its own choices on fertility and education, but not those of the publicly provided, which is a given for the household.

[34] The bar identifies the planner's level of educational effort, and the tilde that of the market allocation.

6 Planner and market steady states

In order to get a sense of how does the market miss the planner's choice, we focus in the next section on the steady states delivered by the market and chosen by the planner.

6.1 Planner steady state

From Proposition 3.1 follows the characterisation of a planner's steady state next.

Proposition 6.1 *A* **planner's steady state** *of the economy characterised by preferences, production of output, and production of human capital represented by* u, A, F *and* H*—under the assumptions stated in* Sect. 2*—is a profile of positive* c_0, c_1, k, e, h, n *such that, if* $n < \frac{1}{\gamma}$*, then*[35]

$$\begin{aligned} \frac{u_0(c_0,c_1)}{u_1(c_0,c_1)} &= A(e)F_K\left(\frac{k}{n},h\right) = \frac{1}{\gamma} \\ \left[1-\gamma A'(e)F\left(\frac{k}{n},h\right)\right]\frac{\frac{1}{\gamma}-nH_h(e,h)}{H_e(e,h)} &= A(e)F_L\left(\frac{k}{n},h\right) \\ c_0+\frac{c_1}{n}+k+ne &= F\left(\frac{k}{n},h\right) \\ h &= H(e,h) \end{aligned} \tag{28}$$

In the case in which fertility is exogenous, the planner's steady state is, moreover, unique, as stated in the next proposition —which follows from the (easily checked) convexity of the constrained set when the population growth factor is constant, the concavity of the human capital production function, and the strict quasi-concavity of the planner's objective.

Proposition 6.2 *If the population growth factor is constant, for an economy characterised by population dynamics preferences, production of output, and production of human, capital represented by* n, u, A, F *and* H*—under the assumptions stated in* Sect. 2*—the planner's steady state is unique.*

6.2 Market steady state

A specific instance of the market equilibrium allocations characterised in Proposition 4.1 is any stationary market equilibrium allocation that treats all households equally, as characterised in the proposition next.

Proposition 6.3 *A* **market steady state** *with positive educational investments for an economy characterised by preferences, production of output, and production of human*

[35] Moreover, the planner's value function must satisfy, at the steady state,

$$V^p(c_1,k,e,h,n) = \frac{1}{\gamma}\Big[u_0(c_0,c_1)\big(e-\frac{k}{n}\big) - u_1(c_0,c_1)\frac{c_1}{n}\Big]$$

capital represented by u, A, F *and* H*—under the assumptions stated in* Sect. 2*—is a profile* $c_0, c_1, k, m, e, h, b, n,$ *and* ρ *such that*[36]

$$\begin{aligned}
(\textit{ if } m > 0, =)\; n \le \frac{u_0(c_0, c_1)}{u_1(c_0, c_1)} &= A(e)F_K\left(\frac{k}{n}, h\right) \;\le \frac{1}{\gamma}(=, \textit{if } b > 0)\\
\frac{\frac{1}{\gamma} - nH_h(e,h)}{H_e(e,h)} &= A(e)F_L\left(\frac{k}{n}, h\right)\\
c_0 + k + m + ne &= A(e)F_L\left(\frac{k}{n}, h\right)h + b\\
\frac{c_1}{n} + b &= A(e)F_K\left(\frac{k}{n}, h\right)\frac{k}{n} + m\\
h &= H(e,h)\\
1 &= \rho\frac{1}{n}
\end{aligned} \tag{29}$$

From the characterisation in Proposition 6.3 follows that :

(i) a market steady state exists only if

$$n \le \frac{1}{\gamma} \tag{30}$$

which is implied by the condition needed for the household problem to be well defined, namely that $\sup n^t < \frac{1}{\gamma}$; it is guaranteed in case of population decrease and a decreasing altruism towards descendants increasingly distant into the future, but it is not guaranteed in case of an increasing population—in this case altruism has to decrease fast enough

(ii) at a market steady state the impact of increasing parents' human capital on their children's has to remain below the reciprocal of the altruism factor for all their children, i.e.

$$H_h(e,h) < \frac{1}{n\gamma} \tag{31}$$

—this is not necessarily the case for the planner's steady state, for which it could be otherwise if it satisfies

$$\frac{u_0(c_0, c_1)}{u_1(c_0, c_1)} = \frac{1}{\gamma} < A'(e)F\left(\frac{k}{n}, h\right) \tag{32}$$

[36] Moreover, the household's value function satisfies, at the solution,

$$V^m(e,h,b;\mathbf{x}) \le \frac{1}{\gamma}\left[u_0(c_0,c_1)e + u_1(c_0,c_1)b\right] \;(=, \text{if } n > 0)$$

where $\mathbf{x} \equiv \{x\}_{\tau\ge 1} \equiv \{(\rho, w, r)\}_{\tau \ge 1}$.

i.e. if at the planner's steady state the inter-temporal marginal rate of substitution remains below the impact of educational investments on output, through its total factor productivity

(iii) at a market steady state, intergenerational transfers of resources take place only in one direction, either from young to old through contributions to the pension fund, or from old to young through bequests, as established by the proposition next.

Proposition 6.4 *At a steady state market allocation either $m = 0$ or $b = 0$.*

Proof Should both $m > 0$ and $b > 0$ hold, then it would follow from (29) that

$$n = \frac{u_0(c_0, c_1)}{u_1(c_0, c_1)} = \frac{1}{\gamma} \tag{33}$$

which cannot be since $n\gamma \leq \sup n_t \gamma < 1$. □

The intuition of the previous result is that only the *net* amount of intergenerational transfers matters for the allocation of resources. The same holds true, close enough to the limit, for any equilibrium converging to a market steady state, as the following corollary of Proposition 6.4 states, with the same intuition.

Proposition 6.5 *For a market allocation converging to a market steady state it holds that, from some period t onwards, either $m^t = 0$ or $b^t = 0$.*

Proof Should, for all t, both $m^t > 0$ and $b^t > 0$ hold, then it would follow from (12) that

$$\begin{aligned} 1 &= \lim_{t\to\infty} \frac{u_0(c_0^t, c_1^t)}{u_0(c_0^{t+1}, c_1^{t+1})} \\ &= \lim_{t\to\infty} \frac{u_0(c_0^t, c_1^t)}{u_1(c_0^t, c_1^t)} \cdot \lim_{t\to\infty} \frac{u_1(c_0^t, c_1^t)}{u_0(c_0^{t+1}, c_1^{t+1})} = \lim_{t\to\infty} \rho_{t+1} \cdot \gamma \\ &= \lim_{t\to\infty} n^t \cdot \gamma \\ &= n \cdot \gamma \end{aligned} \tag{34}$$

which cannot be since $n\gamma \leq \sup n^t \gamma < 1$. □

6.3 Planner's vs market steady states

From the characterisations of the the planner's steady state $\tilde{c}_0, \tilde{c}_1, \tilde{k}, \tilde{e}, \tilde{h}$, and $\tilde{n}$ in (28) and market equilibrium steady state $\bar{c}_0, \bar{c}_1, \bar{k}, \bar{m}, \bar{e}, \bar{h}, \bar{n}$, and $\bar{b}$ in (29) it follows that

(i) a planner steady state *cannot* be decentralised as a laissez-faire market equilibrium steady state, given that educational investments have an impact on total factor productivity since $A'(e) \neq 0$—for any economy, not just a generic subset of them[37]

[37] This result is hence not implied by Proposition 4.2.

(ii) it holds that

$$A(\bar{e})F_L\left(\frac{\bar{k}}{\bar{n}}, \bar{h}\right)\frac{H_e(\bar{e}, \bar{h})}{\frac{1}{\gamma} - \bar{n}H_h(\bar{e}, \bar{h})} > A(\tilde{e})F_L\left(\frac{\tilde{k}}{\tilde{n}}, \tilde{h}\right)\frac{H_e(\tilde{e}, \tilde{h})}{\frac{1}{\gamma} - \tilde{n}H_h(\tilde{e}, \tilde{h})} \tag{35}$$

or equivalently—whenever H is close enough to affine in a neighbourhood of the market and planner's steady states[38]—

$$A(\bar{e})F_L\left(\frac{\bar{k}}{\bar{n}}, \bar{h}\right) > A(\tilde{e})F_L\left(\frac{\tilde{k}}{\tilde{n}}, \tilde{h}\right) \tag{36}$$

i.e. the steady state market equilibrium wage exceeds the marginal productivity of capital that the planner would choose

which is summarised in the proposition next.

Proposition 6.6 *In the economy characterised by preferences, production of output, and production of human capital represented by* u, A, F *and* H*—under the assumptions stated in* Sect. 2—

(i) *there is no laissez-faire competitive equilibrium that decentralises the planner's steady state*
(ii) *whenever* H *is close enough to affine in a neighbourhood of the market and planner steady states, and the planner and market steady state fertility are close enough, then the steady state market wage per efficient unit of labor is too high compared to the marginal productivity of labor at the planner's steady state.*

The way the market and planner steady states marginal productivity of labor compare under the conditions of Proposition 6.6 implies that the market steady state human capital is too low if the market and planner levels of capital, educational investment, and fertility are not too far apart. The intuition for the second result—the first follows naturally from Proposition 4.2—is that whenever variations in education and parents' human capital have a constant impact on their children's, they can be ignored for comparisons since they affect equally the planner and the market allocations, and solely the impact of planner's steady state education on total factor productivity decreases the productivity of labor with respect to that of the market steady state.

This points to a need for subsidising labor income in order to incentivise educational investment in human capital as a means to undo the market inefficiency at the steady state. Indeed, the decentralising rate in (19) becomes at the steady state

$$\tau = \frac{1 - n\gamma H_h(e, h)}{H_e(e, h)} \cdot \frac{A'(e)F\left(\frac{k}{n}, h\right)}{A(e)F_L\left(\frac{k}{n}, h\right)} > 0 \tag{37}$$

[38] If fertility was exogenous, for an affine H, the impact of educational investment and parents' human capital on the formation of their children's is constant and does not depend on their levels. Hence, the ratios showing as second factors in the products on both the LHS and the RHS of (20) coincide. The same argument applies, by continuity, if H is sufficiently close to affine and $\bar{n} \approx \tilde{n}$.

where the first factor in the RHS is positive from (28), for a sufficiently low impact of educational investments on the total factor productivity.[39]

6.4 Quantitative assessment of the inefficiency

The inefficiency can be assessed indirectly through the size of the subsidy needed to undo it at, say, the steady state, for standard production functions and empirically supported values of their parameters. For instance, a choice of Cobb-Douglas for both F and H, and a TFP linear in e lead to a subsidy rate equal to

$$\tau = \frac{1 - n\gamma(1-\beta)}{(1-\alpha)\beta} \tag{38}$$

where $1-\alpha$ and $1-\beta$ are the elasticities of output and children households' human capital with respect to parent households' human capital.[40]

Given that the production function F is Cobb-Douglas, the human capital elasticity of output $1-\alpha$ can be measured by means of the share of labor compensation in GDP which, for US data, has been on average 61.06% for the period 1975–2019.[41]

As for the human capital Cobb-Douglas production function—expressed as

$$\ln h^t = \ln C + \beta_e \ln e^t + \beta_h \ln h^{t-1} \tag{39}$$

in log-linear form—we can use for e^t the series of personal consumption expenditures in education services from the FRED database during the same period 1975–2019,[42] expressed in per capita—using the US population series[43]—and real terms—dividing it by the price level of household consumption.[44] As for h^t, we can use the annual index for human capital per person for the US.[45]

In order to lag h^t—to consider the level of parents' human capital entering into their children's—a parenthood age needs to be assumed. Since a representative agent model is being used, as age gap between generations we should consider the age of parents —whether the mother's, the father's, or some weighted average of both—not at the birth of their first or any other particular child, but at that of their *representative* child, i.e. their average age at birth across all their children. Moreover, it can be argued

39 Specifically, if the derivative of the planner's steady state with respect to educational investment is below $\frac{1}{\gamma}$, which is an empirical matter. At any rate, the quantitative assessment provided in the next section establishes τ to be positive for empirically relevant values for the parameters of standard production functions.

40 It is worth noting that it follows from F and H being Cobb-Douglas and A being linear that the slope of the latter cancels out from the expression for τ in (37) and, therefore, needs not be known in (38).

41 https://fred.stlouisfed.org/series/LABSHPUSA156NRUG (U. of Groningen; UC Davis).

42 https://fred.stlouisfed.org/graph/?id=DTEDRC1A027NBEA (US Bureau of Economic Analysis).

43 https://fred.stlouisfed.org/series/POPTHM (US Bureau of Economic Analysis).

44 https://fred.stlouisfed.org/series/PLCCPPUSA670NRUG (U. of Groningen; UC Davis).

45 https://fred.stlouisfed.org/series/HCIYISUSA066NRUG (U. of Groningen; UC Davis).

that it is the age of the oldest parent that matters for human capital transmission, so that the human capital of the eldest parent, typically the father, is chosen as the one determining the main impact on children's human capital.[46] The mean age of first child fathers raised from 27.4 to 30.9 in the US between 1972 and 2015—*cf.* Yash et al. (2017)—so that we should consider a parenthood age of at the very least 30. While it turns out that only by assuming a generational lag in the late 30's-early 40's instead do the regressions exhibit a normality of residuals, homogeneity of variance, and lack of multicollinearity of the regressors, nevertheless the implied lower bound for the subsidy is shown below to remain anyway constant for all parental ages, which is enough for our purposes.

Indeed, parenthoods at, say, 30, 35 or even 40, deliver the estimations, respectively

$$\begin{aligned} \ln h^t &= 0.682779 + 0.0195992 \ln e^t + 0.296749 \ln h^{t-1} \\ \ln h^t &= 0.369138 + 0.0550896 \ln e^t + 0.159828 \ln h^{t-1} \\ \ln h^t &= 0.188454 + 0.0731847 \ln e^t + 0.107624 \ln h^{t-1} \end{aligned} \tag{40}$$

with all coefficients being significant at the .05 level.[47] Thus, evaluating (38) with values $1-\alpha = 0.6106$ and $1-\beta$ equal to any of the estimates for the elasticity of human capital with respect to parental human capita above—i.e. 0.296749, 0.159828, or 0.107624 respectively—it follows that the subsidy τ runs within the intervals—for parental ages 30, 35, and 40 respectively—

$$\begin{aligned} \tau &= \frac{1-n\gamma(1-\beta)}{(1-\alpha)\beta} = \frac{1-n\gamma \cdot 0.296749}{0.6106 \cdot 0.703251} \in (1.637733377, 2.328803481) \\ \tau &= \frac{1-n\gamma(1-\beta)}{(1-\alpha)\beta} = \frac{1-n\gamma \cdot 0.159828}{0.6106 \cdot 0.840172} \in (1.637733377, 1.949283453) \\ \tau &= \frac{1-n\gamma(1-\beta)}{(1-\alpha)\beta} = \frac{1-n\gamma \cdot 0.107624}{0.6106 \cdot 0.892376} \in (1.637733377, 1.835250362) \end{aligned} \tag{41}$$

varying *inversely* with γ as the latter runs in $(0, \frac{1}{n})$. In other words, the planner's steady state would only be decentralisable through a labor income subsidy of *at least* 163,77%, regardless the assumed first child parental age. The estimate—whose objective is not accuracy, but rather providing an order of magnitude—clearly points to a sizeable inefficiency.

[46] It should be noted that parental age difference has been decreasing anyway during that period—*cf.* Yash et al. (2017).

[47] Incidentally, the production of human capital shows decreasing returns to scale.

7 Concluding remarks

A final remark is in order. The comparison in Sect. 6 of two steady states—the market and the planner's—starting from *different* initial conditions should be considered under the light of the question it answers, namely can the best possible steady state (from the viewpoint of an egalitarian planner) ever be a market outcome under laissez-faire?

The fact that the answer to this question has been shown to be negative prompts the question: can the best possible steady state (from the viewpoint of an egalitarian planner) then be a market outcome under some policy? That the answer to this second question has been shown to be positive instead is of interest in itself, even if the implementation of the best steady state might require a shift of the initial condition to the planner's steady state that prevents it from being Pareto improving. Indeed, that the best steady state can be decentralised through the market is of interest, independently of whether—in a political economy expansion of the model—society is willing or not to (make some generation) pay the price necessary to move to it. Should it had turned out that the best steady state could not have been decentralised, there would not even be room to ask the political economy question.[48]

Finally, it should be noted that bequests play an important role in generating and perpetuating wealth inequality—as shown in, for instance, Boserup et al. (2016). Nevertheless, the representative agent assumption and the lack of uncertainty—exploited in the model for the sake of obtaining analytical results—prevents to address inequality issues. In order to do so, these assumptions need to be dropped, which is left for future work.

Appendix A

A.1 Proof of Proposition 3.1

The maximum is finite whenever the product $\prod_{\tau=1}^{t-1} n^{\tau} \gamma^{t-1}$ is bounded above by, and away from, 1 for all integer $t \geq 1$. This is guaranteed if households' fertility choices satisfy $\sup n^t < \frac{1}{\gamma}$. Since this inequality is strict, it serves no purpose to add it to the constraints to the maximisation in (4). The first-order conditions are, nonetheless, contingent to satisfying it.

48 If, alternatively, one had bound oneself to compare only market and planner allocations with the *same* initial conditions, then either the market or the planner's would not be stationary, so that the very question of how do market and planner steady states compare cannot even be posed under the common initial condition requirement.

The solution to the RHS of (4) is necessarily characterised by the first-order conditions

$$\begin{pmatrix} u_0(c_0^t, c_1^t) \\ u_1(c_0^t, c_1^t) + n^t \gamma V_c^p(c_1^t, k^t, e^t, h^t, n^t) \\ n^t \gamma V_k^p(c_1^t, k^t, e^t, h^t, n^t) \\ n^t \gamma V_e^p(c_1^t, k^t, e^t, h^t, n^t) \\ n^t \gamma V_h^p(c_1^t, k^t, e^t, h^t, n^t) \\ \gamma\left[V^p(c_1^t, k^t, e^t, h^t, n^t) + n^t V_n^p(c_1^t, k^t, e^t, h^t, n^t)\right] \end{pmatrix}$$
$$= \lambda_t \begin{pmatrix} 1 \\ 0 \\ 1 \\ n^t \\ -A(e^{t-1}) F_L\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right) \\ e^t \end{pmatrix} + \mu_t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{42}$$

where—from the enveloppe theorem[49]—

$$\begin{aligned} V_c^p(c_1^t, k^t, e^t, h^t, n^t) &= -\lambda_{t+1} \frac{1}{n^t} \\ V_k^p(c_1^t, k^t, e^t, h^t, n^t) &= \lambda_{t+1} A(e^t) F_K\left(\frac{k^t}{n^t}, h^{t+1}\right) \frac{1}{n^t} \\ V_e^p(c_1^t, k^t, e^t, h^t, n^t) &= \lambda_{t+1} A'(e^t) F\left(\frac{k^t}{n^t}, h^{t+1}\right) + \mu_{t+1} H_e(e^t, h^t) \\ V_h^p(c_1^t, k^t, e^t, h^t, n^t) &= \mu_{t+1} H_h(e^t, h^t) \\ V_n^p(c_1^t, k^t, e^t, h^t, n^t) &= \lambda_{t+1}\left[\frac{c_1^t}{n^t} + A(e^t) F_K\left(\frac{k^t}{n^t}, h^{t+1}\right) \frac{k^t}{n^t}\right] \frac{1}{n^t} \end{aligned} \tag{43}$$

49 The derivatives of the value $V^p(c_1^t, k^t, e^t, h^t, n^t)$ of the Lagrangian of the problem at $t+1$, that is to say,

$$\begin{aligned} &u(c_0^{t+1}, c_1^{t+1}) + n^{t+1} \gamma V^p(c_1^{t+1}, k^{t+1}, e^{t+1}, h^{t+1}, n^{t+1}) \\ &\quad - \lambda_{t+1}\left[c_0^{t+1} + \frac{c_1^t}{n^t} + k^{t+1} + n^{t+1} e^{t+1} - A(e^t) F\left(\frac{k^t}{n^t}, h^{t+1}\right)\right] \\ &\quad - \mu^{t+1}\left[h^{t+1} - H(e^t, h^t)\right] \end{aligned}$$

with respect to c_1^t, k^t, e^t, h^t, and n^t are indeed those provided in (43).

so that

$$
\begin{aligned}
u_0(c_0^t, c_1^t) &= \lambda_t \\
u_1(c_0^t, c_1^t) - n^t \gamma \lambda_{t+1} \frac{1}{n^t} &= 0 \\
n^t \gamma \lambda_{t+1} A(e^t) F_K\left(\frac{k^t}{n^t}, h^{t+1}\right) \frac{1}{n^t} &= \lambda_t \\
n^t \gamma \left[\lambda_{t+1} A'(e^t) F\left(\frac{k^t}{n^t}, h^{t+1}\right) + \mu_{t+1} H_e(e^t, h^t)\right] &= \lambda_t n^t \\
n^t \gamma \mu_{t+1} H_h(e^t, h^t) &= \\
\mu_t - \lambda_t A(e^{t-1}) F_L\left(\frac{k^{t-1}}{n^{t-1}}, h^t\right) & \\
\gamma\left[V^p(c_1^t, k^t, e^t, h^t, n^t) + \lambda_{t+1}\left(\frac{c_1^t}{n^t} + A(e^t) F_K\left(\frac{k^t}{n^t}, h^{t+1}\right)\frac{k^t}{n^t}\right)\right] &= \lambda_t e^t
\end{aligned}
\tag{44}
$$

from which, eliminating the multipliers, the characterisation (5) follows.

A.2 Household's optimal choice first-order conditions (7)

The first-order conditions necessarily characterising the household choice in (6) are[50]

$$
\begin{pmatrix} u_0(c_0^t, c_1^t) \\ u_1(c_0^t, c_1^t) \\ 0 \\ 0 \\ n^t \gamma V_e(e^t, h^t, b^t; \mathbf{x}_{t+1}) \\ n^t \gamma V_h(e^t, h^t, b^t; \mathbf{x}_{t+1}) \\ n^t \gamma V_b(e^t, h^t, b^t; \mathbf{x}_{t+1}) \\ \gamma V(e^t, h^t, b^t; \mathbf{x}_{t+1}) \end{pmatrix} = \lambda_0^t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ n^t \\ -w_t \\ 0 \\ e^t \end{pmatrix} + \lambda_1^t \begin{pmatrix} 0 \\ 1 \\ -r_{t+1} \\ -\rho_{t+1} \\ 0 \\ 0 \\ n^t \\ b^t \end{pmatrix} + \mu^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
$$
$$
+ \nu_m^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \nu_e^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \nu_b^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \nu_n^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}
\tag{45}
$$

[50] Ignoring the (at a solution, non-binding) non-negativity constraints for c_0^t, c_1^t —because of u being well-behaved at the boundary—for k^t —because the returns to k^t and m^t will be positive and equal at equilibrium, so that the household's optimal savings *portfolio* composition is indeterminate but positive and hence *one* of the non-negativity constraints on k^t and m^t can be dropped—and for h^t—because of u being strictly increasing, unless $H(e^{t-1}, h^{t-1})$ is 0 itself.

for some $\lambda_0^t, \lambda_0^t, \mu^t, \nu_m^t, \nu_e^t, \nu_b^t, \nu_n^t \geq 0$, along with the constraints binding, for all t, where—from the envelope theorem[51]—

$$\begin{aligned} V_e^m(e^t, h^t, b^t; \mathbf{x}_{t+1}) &= \mu^{t+1} H_e(e^t, h^t) \\ V_h^m(e^t, h^t, b^t; \mathbf{x}_{t+1}) &= \mu^{t+1} H_h(e^t, h^t) \\ V_b^m(e^t, h^t, b^t; \mathbf{x}_{t+1}) &= \lambda_0^{t+1} \end{aligned} \tag{46}$$

that is to say

$$\begin{aligned} u_0(c_0^t, c_1^t) &= \lambda_0^t \\ u_1(c_0^t, c_1^t) &= \lambda_1^t \\ 0 &= \lambda_0^t - \lambda_1^t r_{t+1} \\ 0 &= \lambda_0^t - \lambda_1^t \rho_{t+1} - \nu_m^t \\ n^t \gamma \mu^{t+1} H_e(e^t, h^t) &= \lambda_0^t n^t - \nu_e^t \\ n^t \gamma \mu^{t+1} H_h(e^t, h^t) &= \mu^t - \lambda_0^t w_t \\ n^t \gamma \lambda_0^{t+1} &= \lambda_1^t n^t - \nu_b^t \\ \gamma V^m(e^t, h^t, b^t; \mathbf{x}_{t+1}) &= \lambda_0^t e^t + \lambda_1^t b^t - \nu_n^t \end{aligned} \tag{47}$$

from which—whenever educational investment is positive, and after eliminating the nonnegative multipliers, and noting $\nu_e^t = 0$ for $e^t > 0$—the characterisation (7) follows.

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[51] The derivatives of the value $V^m(e^t, h^t, b^t; \mathbf{x}_{t+1})$ of the Lagrangian of the problem faced by generation $t+1$, that is to say,

$$\begin{aligned} & u(c_0^{t+1}, c_1^{t+1}) + n^t \gamma V^m(e^{t+1}, h^{t+1}, b^{t+1}; \mathbf{x}_{t+2}) \\ & \quad - \lambda_0^{t+1}\Big[c_0^{t+1} + k^{t+1} + m^{t+1} + n^{t+1} e^{t+1} - w_{t+1} h^{t+1} - b^t\Big] \\ & \quad - \lambda_1^{t+1}\Big[c_1^{t+1} + n^{t+1} b^{t+1} - r_{t+2} k^{t+1} - m^{t+1} \rho_{t+2}\Big] \\ & \quad - \mu^{t+1}\Big[h^{t+1} - H(e^t, h^t)\Big] \end{aligned}$$

with respect to e^t, h^t, and b^t are indeed those provided in (37).

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