Economic Theory (2023) 75:949–981
https://doi.org/10.1007/s00199-022-01434-4

RESEARCH ARTICLE

Source and rank-dependent utility

Mohammed Abdellaoui[1] · **Horst Zank**[2]

Received: 26 February 2020 / Accepted: 27 April 2022 / Published online: 26 May 2022
© The Author(s) 2022

Abstract

Foundations are provided for rank-dependent preferences within the popular two-stage framework of Anscombe–Aumann, in which risk and ambiguity feature as distinct sources of uncertainty. We advance the study of attitudes towards ambiguity without imposing expected utility for risk. As a result, in our general model, ambiguity attitude can be captured by non-additive subjective probabilities as under Choquet expected utility or by a specific utility for ambiguity as in recursive expected utility or, if required, by both. The key property for preferences builds on (discrete) rates of substitution which are standardly applied in economics. By demanding consistency for these rates of substitution across events and within or across sources of uncertainty, we obtain a model that nests popular theories for risk and ambiguity. This way, new possibilities for theoretical and empirical analyses of these theories emerge.

Keywords Ambiguity · Recursive expected utility · Risk · Substitution consistency · Source-dependence · Source and rank-dependent utility

JEL Classification C78 · C91 · D81 · D90

1 Introduction

The standard model of rational choice under uncertainty in economics has independently been developed for two distinct sources of uncertainty. Expected utility for

We like to acknowledge the constructive comments of three anonymous referees and the handling editor, which have lead to several improvements. The earlier version titled "Recursive Rank-dependent Utility" has benefitted from discussions and remarks from Edi Karni, Peter Wakker and Chris Wallace. The paper was developed during Mohammed Abdellaoui's Hallsworth Visiting Professorship at the Univeristy of Manchester.

✉ Horst Zank
horst.zank@manchester.ac.uk

Mohammed Abdellaoui
abdellaoui@hec.fr

1 Department of Economics and Decision Sciences, HEC-Paris & GREGHEC-CNRS, Jouy-en-Josas, France

2 Department of Economics, University of Manchester, Manchester, UK

risk (EU; von Neumann and Morgenstern 1944) models situations where probabilities are exogenously known. For *ambiguity* subjective expected utility (SEU; Savage 1954) formalizes choice behavior when probabilities are unknown. Subsequently, a two-source derivation of SEU was provided by Anscombe and Aumann (1963; AA henceforth). As the objects of choice in the AA-setup are of a two-stage hybrid type, with ambiguous events leading to risky lotteries, subjective probabilities for events can be recovered from objective ones. Despite its normative appeal, SEU suffers from many descriptive shortcomings, such as non-linear probability weighting under risk (Tversky and Kahneman 1992) and ambiguity aversion (Ellsberg 1961). Popular generalizations of SEU within the AA-setup have maintained EU for risk (Schmeidler 1989; Gilboa and Schmeidler 1989; Klibanoff et al. 2005; Maccheroni 2004). This paper provides a generalization of SEU to ambiguity within the AA-setup which deviates from EU for risk.

Our model, called *source and rank-dependent utility* (SRU), supplements Schmeidlers' (1989) Choquet EU (CEU) with two descriptively desirable features. SRU takes rank-dependent utility (RDU) for risk as proposed by Quiggin (1982) and integrates this into Schmeidler's model in a source-dependent fashion. The accumulated experimental evidence shows that the restriction of the key normative axioms of the EU framework to subsets of one-stage objects of choice with a given rank-ordering of outcomes can account for the Ellsberg (1961) paradoxes under ambiguity as well as the Allais (1953) paradoxes under risk (Wakker 2010). According with the empirical evidence, SRU incorporates within-source rank-dependence for outcomes in the evaluation of two-stage acts. Moreover, SRU allows the utility of outcomes to be source-dependent as in recursive EU-type models (Nau 2006; He 2021). By using the AA-setup with a continuum of outcomes, we obtain an SRU preference representation in a continuous ordinal utility framework as in standard consumer theory (Debreu 1959). To this aim, we provide a general preference principle called substitution consistency. We show that, in the presence of standard axioms, a within-source application of substitution consistency characterizes SRU.

To our knowledge, substitution consistency is the first source-dependent extension to the AA-setup of the so-called tradeoff consistency property that was initially used in one-stage settings to obtain EU or RDU for risk (Wakker 1989; Chateauneuf and Wakker 1999) and SEU or RDU for ambiguity (Wakker and Tversky 1993; Köbberling and Wakker 2003). Original tradeoff consistency was also used to obtain other non-EU models such as regret theory (Diecidue and Somasundaram 2017) or purely subjective Maxmin EU (Alon and Schmeidler 2014). Other versions of tradeoff consistency have been applied to case-based decision making (Gilboa et al. 2002), extreme outcome separability (Alon 2014), probabilistic settings (Abdellaoui 2002; Werner and Zank 2019), probability dependent EU (Kübler et al. 2017), and to the general class of biseparable preferences (Ghirardato and Marinacci 2001; see Köbberling and Wakker 2003; or Chateauneuf et al. 2021). For SRU we provide a source-dependent extension of rank-dependent tradeoff consistency.

Putting aside the deviations from EU for risk, SRU combines two prominent yet hitherto distinct approaches to ambiguity. The first views ambiguity attitude as being driven by utility and this is accounted for by an ambiguity function. In recursive EU theories (REU; Dobbs 1991; Klibanoff et al. 2005; Nau 2006; Chew and Sagi 2008;

Ergin and Gul 2009; Neilson 2010; Cappelli et al. 2021; He 2021), ambiguity aversion is reflected as a concave transformation of the risky utility. The second approach is due to Schmeidler (1989). It accounts for ambiguity attitude through non-additive subjective probabilities. Specifically, events are assigned decisions weights inferred from a non-additive probability measure on the state space, i.e., a capacity. In SRU both the ambiguity function and the capacity are identified as potential channels through which ambiguity can be revealed. Consequently, SRU nests popular approaches to ambiguity and, thereby, allows for an improved descriptive analysis and empirical comparisons of these approaches within a common framework.

The replacement of probabilities by decision weights, in SRU for both risk and ambiguity, accords with the accumulated empirical findings of the last three decades. For risk, a long list of findings shows that individuals overweight small probabilities and underweight moderate and large probabilities (e.g., Bruhin et al. 2010; Wakker 2010). For ambiguity, empirical evidence points to ambiguity aversion for moderately likely and likely events paired with ambiguity seeking for unlikely events (e.g., Trautmann and Wakker 2018). The advantages of adopting RDU to empirically study ambiguity within a two-source framework has recently been demonstrated in Dimmock et al. (2016). Likewise, SRU's flexibility to study ambiguity as a utility driven attitude, echoes the approach taken in recent empirical studies that assume REU (Chakravarty and Roy 2009; Cubitt et al. 2018, 2020). Descriptively, SRU can serve as a model where deviations from EU caused by probability weighting for risk, non-additivity of subjective probabilities and utility-driven deviations attributed to source-dependence can be disentangled. For instance, SRU can be used to quantify the utility driven ambiguity in REU while factoring out the other deviations.

From a preference foundation perspective, substitution consistency is a standard utility identification tool, here applied to specific subsets of acts. The principle uses a preference-based comparison of outcome substitutions that is directly related to comparisons of utility differences. As observed in Baillon et al. (2012), comparisons of outcome substitutions can make marginal rates of substitution observable from choice. More recently, Baillon and l'Haridon (2021) invoke utility differences to demonstrate how the discrete version of the Arrow-Pratt relative and absolute indexes of risk aversion are be made observable directly from choices over specific acts. Similarly, in our continuous utility setup, when applied to specific subsets of acts, substitution consistency can be interpreted as the requirement for discrete rates of substitutions to be independent of the events or probabilities used to infer them. This is reminiscent of Gorman's 1968 conditions for additive separability that invoke independence of marginal rates of substitution across consumption time periods. Likewise, Werner (2005) shows that, for a risk averse agent whose preferences are represented by an additively separable utility, SEU requires marginal rates of substitution (conditional on different states) derived from first-order optimality conditions, to be proportional and, hence, independent of those states. As a result a cardinal utility function can be identified.

For empirical applications, the outcome substitution tool behind substitution consistency advances the identification of utility by circumventing deviations from EU. We incorporate this desirable feature by proposing a source-dependent version of this consistency principle which enables us to identify both the utility for risk and the ambiguity function of SRU. It turns out that, in the presence of standard axioms, a full-force application of this consistency principle results in SEU (Theorem 4). In other words, when combined with monotonicity, an unrestricted application of our consistency principle boils down to the requirement that the famous normative sure-thing principle holds in an "across-source" fashion, i.e., jointly for risk and ambiguity. When substitution consistency is demanded within sources, we identify a specific utility for ambiguity, in addition to the utility for risk; hence, we obtain REU (Theorem 7). Furthermore, when substitution consistency is applied to specific subsets of acts where the rank-ordering of outcomes matters for ambiguity and separately also for risk, we obtain SRU (Theorem 12). The across-source version of rank-dependent substitution consistency delivers SRU with a source-independent utility (Corollary 14). We note, without detailed elaborations, that our consistency principle is also applicable when probabilities for events are known from the outset (e.g., Segal 1987b).[1]

For comparative analyses, Baillon et al. (2012) proposed a technique to analyze relative concavity of utility for risk and for ambiguity using one-stage objects of choice under REU. Our work shows how to generalize their technique to two-stage acts and, thereby, extend their results from REU to SRU. These applications are presented in Sect. 6. The next section presents basic definitions. Section 3 formally introduces SRU. In Sect. 4 we introduce our consistency principle and derive preference foundations for SEU, REU, and SRU. Subsequently, in Sect. 5 we relate our results to the literature. We provide descriptive applications in Sect. 7 before we conclude; the "Appendix" contains proofs.

2 Preliminaries

This section presents notation and definitions for risk and ambiguity. Subsequently, traditional preference conditions that imply a continuously ordinal representation of preferences are invoked; in our framework they allow for the derivation of conditional certainty equivalents. The latter are used to generalize Schmeidler's (1989) notion of comonotonicity to include source-dependence.

2.1 Lotteries

Let $\mathcal{L}$ denote the set of all *lotteries* (finite probability distributions) over $\mathbb{R}$, the set of deterministic outcomes. A *lottery* that gives outcome x_j with nonnegative probability p_j, for $j = 1, \ldots, m$, is denoted by $\hat{x} = (p_1 : x_1, \ldots, p_m : x_m)$.[2] The subset of lotteries for which the m-tuple of probabilities $\mathbf{p} = (p_j)_{j=1}^m$ is commonly fixed is denoted $\mathcal{L}_{\mathbf{p}}$, i.e.,

$$\mathcal{L}_{\mathbf{p}} = \{(p_1 : x_1, \ldots, p_m : x_m) \in \mathcal{L} : x_j \in \mathbb{R}\}.$$

[1] Machina (2014, p. 3836) highlights Segal's model as belonging to the class of generalizations of SEU that can accommodate the Machina (2009; 2011; 2014) paradoxes for several ambiguity aversion models. This remark extends to two-stage models such as SRU and models where the outcomes of lotteries are (conditional) Savage-acts (He 2021).

[2] For binary lotteries we sometimes use $x_p y$ instead of $(p : x, (1 - p) : y)$, $p \in (0, 1)$.

The set $\mathcal{L}_{\mathbf{p}}^{\downarrow}$ denotes the subset of $\mathcal{L}_{\mathbf{p}}$ for which outcomes are ordered from best to worst, i.e., $x_1 \geq \cdots \geq x_m$. More generally, when the latter ordering of outcomes matters, we indicate this by denoting the set of lotteries as $\mathcal{L}^{\downarrow}$ instead of $\mathcal{L}$. For $\hat{x} \in \mathcal{L}_{\mathbf{p}}$ and outcome α we write

$$\alpha_j \hat{x} := (p_1 : x_1, \ldots, p_{j-1} : x_{j-1}, p_j : \alpha, p_{j+1} : x_{j+1}, \ldots, p_m : x_m)$$

for the lottery with the j-th outcome replaced by α; then $p_j > 0$ is implicit. In the sequel, when $\hat{x} \in \mathcal{L}' \subset \mathcal{L}$, the notation $\alpha_j \hat{x}$ means that $\alpha_j \hat{x} \in \mathcal{L}'$ (e.g., when $\mathcal{L}' = \mathcal{L}_{\mathbf{p}}^{\downarrow}$ the restriction $x_{j-1} \geq \alpha \geq x_{j+1}$ applies).

Under expected utility (EU) for risk, the value of a lottery $\hat{x} = (p_1 : x_1, \ldots, p_m : x_m) \in \mathcal{L}$ is given by $EU(\hat{x}) = \sum_{j=1}^{m} p_j u(x_j)$, where u is a cardinal utility (unique up to origin and unit), i.e., a strictly increasing real-valued function for outcomes. In our setup utility is continuous.

Rank-dependent utility (RDU) extends EU by allowing for non-linear probability weighting (Quiggin 1982; Segal 1987a; Wakker 1994). Under RDU, the value of a lottery $\hat{x} = (p_1 : x_1, \ldots, p_m : x_m) \in \mathcal{L}^{\downarrow}$ is given by

$$RDU(\hat{x}) = \sum_{j=1}^{m} \hat{\pi}_j u(x_j). \tag{1}$$

In Eq. (1), u is as in EU and $\hat{\pi}_j$ is a *decision weight* defined as $\hat{\pi}_1 = w(p_1)$ and $\hat{\pi}_j = w(\sum_{l=1}^{j} p_l) - w(\sum_{l=1}^{j-1} p_l)$ for $j = 2, \ldots, m$, where w is a *probability weighting function* on the unit interval (i.e., w is strictly increasing and satisfies $w(0) = 0$ and $w(1) = 1$). In our setup, w need not be continuous at 0 or at 1; such discontinuities have been supported empirically and their merits have been discussed in the literature.[3] If w is the identity function, then RDU reduces to EU.

2.2 Acts

We assume that $\mathcal{S}$ is a finite set of (at least two) *states* of the world and $\mathcal{A}$ an algebra of subsets, called *events*, of $\mathcal{S}$. We say that an event E occurs if any state in it occurs. A two-stage *act* is a finite-valued function from $\mathcal{S}$ to $\mathcal{L}$. We denote by $\mathbb{A}$ the set of all such acts which are measurable w.r.t. $\mathcal{A}$. Let $\mathcal{E} := (E_i)_{i=1}^{n}$ be a n-tuple of pairwise disjoint events partitioning $\mathcal{S}$ such that $E_i \in \mathcal{A}$ for $i = 1, \ldots, n$. In the sequel $\mathcal{E}$ is called an ordered partition (partition for short) of $\mathcal{S}$. An act that gives the lottery $\hat{x}_i$ if $E_i, i = 1, \ldots, n$, occurs is denoted by $\mathbf{x} = (E_1 : \hat{x}_1, \ldots, E_n : \hat{x}_n)$. When each lottery $\hat{x}_i$ in $\mathbf{x}$ is a deterministic outcome x_i, $i = 1, \ldots, n$, then $\mathbf{x}$ is called a *one-stage* act. For a given partition $\mathcal{E}$, $\mathbb{A}_{\mathcal{E}}$ denotes the set of acts based on that partition. Note that $\mathbb{A} = \cup_{\mathcal{E}} \mathbb{A}_{\mathcal{E}}$.

For event E from partition $\mathcal{E}$ and act $\mathbf{x}$, $\mathbf{x}_E$ represents the restriction of $\mathbf{x}$ to event E. For an act $\mathbf{x} \in \mathbb{A}_{\mathcal{E}}$, event E from $\mathcal{E}$ and lottery $\hat{y}$ we write $\hat{y}_E \mathbf{x}$ for the act that

[3] See, e.g. Kahneman and Tversky (1979), Birnbaum and Stegner (1981), Bell (1985), Lopes (1986), Cohen and Jaffray (1988), Gilboa (1988), Cohen (1992), Chateauneuf et al. (2007), Webb and Zank (2011).

gives lottery $\hat{y}$ if event E occurs and otherwise coincides with $\mathbf{x}$, that is, is identical to $\mathbf{x}_{E^c}$. We also employ the notation $\alpha_E\mathbf{x}$ instead of $\hat{y}_E\mathbf{x}$ if $\hat{y}$ is the degenerate lottery that gives outcome α with probability 1 . In the sequel, when we specify the set of acts such that $\mathbf{x} \in \mathbb{A}' \subset \mathbb{A}$, the notation $\hat{y}_E\mathbf{x}$ implicitly means that $\hat{y}_E\mathbf{x} \in \mathbb{A}'$. For act $\mathbf{x}$ from $\mathbb{A}' \subset \mathbb{A}$, outcome α, lottery $\hat{x} \in \mathcal{L}_{\mathbf{p}}$, and probability p_j, $j \in \{1, \ldots, m\}$, we write $(\alpha_j\hat{x})_E\mathbf{x}$ for the act that gives the lottery $\alpha_j\hat{x}$ if event E occurs and coincides with $\mathbf{x}$ otherwise. It is implicit in this notation that $(\alpha_j\hat{x})_E\mathbf{x} \in \mathbb{A}' \subset \mathbb{A}$.

Subjective expected utility (SEU) evaluates a two-stage act $\mathbf{x} \in \mathbb{A}_{\mathcal{E}}$, $\mathcal{E} = (E_i)_{i=1}^n$, as follows:

$$SEU(\mathbf{x}) = \sum_{i=1}^{n} \pi_i EU(\hat{x}_i), \tag{2}$$

where π_i and $EU(\hat{x}_i)$ stand, respectively, for the subjective probability of event E_i and the EU-value of lottery $\hat{x}_i$, $i = 1, \ldots, n$. We state explicitly that, for $\hat{x} \in \mathcal{L}_{\mathbf{p}}$ with $\mathbf{p} = (p_1, \ldots, p_m)$, in SEU we have $EU(\hat{x}) = \sum_{j=1}^{m} p_j u(x_j)$. In SEU, u is as in EU and the subjective probabilities are uniquely determined.

2.3 Standard preference conditions

A *preference* relation is a binary relation $\succcurlyeq$ on $\mathbb{A}$, with $\succ$ (strict preference) and $\sim$ (indifference) as usual. An event E is *null* if for all lotteries $\hat{x}, \hat{y}$ and acts $\mathbf{z}$ we have $\hat{x}_E\mathbf{z} \sim \hat{y}_E\mathbf{z}$; otherwise E is *non-null*. Unless otherwise specified, henceforth we assume that $\mathcal{A}$ contains at least two disjoint non-null events. A function $\mathcal{V} : \mathbb{A} \to \mathbb{R}$ *represents* $\succcurlyeq$ on $\mathbb{A}$ if $\mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow \mathcal{V}(\mathbf{x}) \geq \mathcal{V}(\mathbf{y})$ for all acts $\mathbf{x}, \mathbf{y} \in \mathbb{A}$. If a representing function exists then $\succcurlyeq$ is a *weak order*, that is, $\succcurlyeq$ is *complete* ($\mathbf{x} \succcurlyeq \mathbf{y}$ or $\mathbf{x} \preccurlyeq \mathbf{y}$ for all acts $\mathbf{x}, \mathbf{y}$) and transitive.

Next, we present a *monotonicity*. It requires that, for an act $\hat{x}_E\mathbf{x}$ where the event E is non-null, improving a non-null probability outcome within the lottery $\hat{x}$ results in a preferred act. For all partitions $\mathcal{E}$, each non-null event E from $\mathcal{E}$, all tuples of probabilities $\mathbf{p} = (p_1, \ldots, p_m)$ and each $p_j > 0$, $j \in \{1, \ldots, m\}$, all acts $\mathbf{x} \in \mathbb{A}_{\mathcal{E}}$ such that $\hat{x} \in \mathcal{L}_{\mathbf{p}}$, and all outcomes α, β, we have

$$\alpha > \beta \Rightarrow (\alpha_j\hat{x})_E\mathbf{x} \succ (\beta_j\hat{x})_E\mathbf{x}.$$

Monotonicity is defined in the strong sense, which ensures that utility in our models is strictly increasing. Moreover, monotonicity implies that the probability weighting function under RDU for risk is strictly increasing. Further, while monotonicity is mute for null events, the condition ensures that decision weights and subjective probabilities for (non-null) events are positive.

Given the assumption of a continuum of outcomes in our framework, we invoke the following (finite-dimensional) continuity property. For a partition $\mathcal{E} = (E_i)_{i=1}^n$ and probability tuples $\mathbf{p}^i = (p_1^i, \ldots, p_{m_i}^i)$, $i = 1, \ldots, n$, let

$$\mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)} := \{\mathbf{x} \in \mathbb{A} : \hat{x}_i \in \mathcal{L}_{\mathbf{p}^i}, i = 1, \ldots, n\}.$$

Clearly, $\mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)}$ is isomorphic to $\prod_{i=1}^n \mathbb{R}^{m_i}$. *Continuity* for $\succcurlyeq$ holds if for every partition $\mathcal{E} = (E_i)_{i=1}^n$, every collection of probability tuples $\mathbf{p}^i = (p_1^i, \ldots, p_{m_i}^i)$, $i = 1, \ldots, n$, and each act $\mathbf{x} \in \mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)}$, the sets

$$\{\mathbf{y} \in \mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)} : \mathbf{x} \succcurlyeq \mathbf{y}\}$$
and
$$\{\mathbf{y} \in \mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)} : \mathbf{y} \succcurlyeq \mathbf{x}\}$$

are closed in $\mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)}$.

Given that in the definition of continuity, $\mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)}$ being isomorphic to $\prod_{i=1}^n \mathbb{R}^{m_i}$, which is a connected subset of an Euclidean space endowed with the Euclidean topology, we invoke a finite-dimensional continuity property. When restricted to $\mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)}$, the weak order and continuity properties allow for a continuously ordinal representation of the preference on that domain; the result was derived by Debreu (1954). Because $\mathcal{S}$ is finite, by considering appropriate refinements of partitions and re-writing the lotteries conditional on events within that refined partition such as to have a common probability-tuple within the relevant event, different continuous representations (say the one on $\mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)}$ and that on $\mathbb{A}_{\mathcal{E}',(\mathbf{q}^1,\ldots,\mathbf{q}^{n'})}$)) can be shown to be restrictions of a common general representation over all acts. In the proof of our theorems, where this argument is used, the relevant refinements will be made precise. That said, the continuity property employed here ensures that the considered representations are only continuous in outcomes. We do not impose any continuity in probabilities; the latter could be added, whenever needed, as an auxiliary assumption without affecting any other aspect of our results or the subsequent analysis.

For a non-null event E and act $\hat{x}_E\mathbf{x}$, the *conditional certainty equivalent* (CE) of lottery $\hat{x}$ is defined as the deterministic outcome, $CE(\hat{x}) \in \mathbb{R}$, such that $\hat{x}_E\mathbf{x} \sim CE(\hat{x})_E\mathbf{x}$.[4] It can be shown that, under weak order, continuity and monotonicity of the preference $\succcurlyeq$, such CEs always exist; in general the CEs depend on the event E and $\mathbf{x}_{E^c}$. We exclude such dependence: the preference relation $\succcurlyeq$ satisfies *independence of conditional CEs* if for all non-null events E, F, acts $\mathbf{x}$ and lotteries $\hat{x}$ we have $\hat{x}_E\mathbf{x} \sim CE(\hat{x})_E\mathbf{x} \Leftrightarrow \hat{x}_F\mathbf{x} \sim CE(\hat{x})_F\mathbf{x}$.

The independence property demands that, for a given act, the CE of a lottery conditional on an event is independent of that event and, by changing the event on which the lottery obtains, the CE is also independent of the act outside the original event. We summarize the implication for existence of conditional CE's based on the properties mentioned thus far.

Lemma 1 *Let $\succcurlyeq$ be a preference relation on $\mathbb{A}$ that satisfies weak ordering, monotonicity, and continuity. For each lottery $\hat{x}$, each act $\mathbf{x}$ and each non-null event E, there exists a unique outcome $CE(\hat{x})$ such that $\hat{x}_E\mathbf{x} \sim CE(\hat{x})_E\mathbf{x}$. Further, if additionally independence of conditional CEs holds, we have $\hat{x}_F\mathbf{y} \sim CE(\hat{x})_F\mathbf{y}$ for all acts $\mathbf{y}$ and all non-null events F.* □

[4] For the sake of simplicity, we used notation $CE(\hat{x})$ instead of the cumbersome $CE_{E,\mathbf{x}}(\hat{x})$ that makes it explicit that the conditional certainty equivalent may depend on both event E and act $\mathbf{x}$.

A consequence of Lemma 1 is that a constant act can be identified with the constant one-stage act that replaces in each event the lottery by its conditional CE. Hence, one can compare unconditional lotteries (identified with constant acts) using their conditional CEs. In particular, one can use these CEs to rank-order lotteries within an act, which is key in the models presented next. Henceforth, we assume that Lemma 1 holds.

3 Source and rank-dependence

This section presents two popular models of ambiguity: recursive expected utility (REU) and Schmeidler's (1989) Choquet expected utility (CEU). They are the benchmarks for our approach to ambiguity. Subsequently, *source and rank-dependent utility* (SRU) is introduced. SRU accounts for ambiguity in a utility-driven fashion as is done in REU; additionally non-linear probability weighting for risk is invoked.

3.1 Recursive expected utility

To account for non-neutral attitude towards ambiguity, e.g. ambiguity aversion (Ellsberg 1961), SEU was generalized in different directions. A popular approach augmented SEU with a transformation ϕ of the EU-value of a lottery and, hence, of the risky utility, u. This gives a general *recursive expected utility* (REU) theory of ambiguity, where the act $\mathbf{x} \in \mathbb{A}_{\mathcal{E}}$, $\mathcal{E} = (E_i)_{i=1}^n$, is evaluated by

$$REU(\mathbf{x}) = \sum_{i=1}^{n} \pi_i \phi[EU(\hat{x}_i)]. \tag{3}$$

When lotteries within an act $\mathbf{x}$ are replaced by their conditional CEs, the value of the resulting one-stage act is $\sum_{i=1}^{n} \pi_i [\phi \circ u](CE(\hat{x}_i))$. This makes it transparent that the utility in REU is source-dependent: u captures attitudes towards risk and ϕ captures attitudes towards ambiguity. In particular, aversion to ambiguity is manifested through a concave function ϕ, which is entirely independent of events; hence, the term "utility-driven" ambiguity has emerged (e.g., Trautmann and Wakker 2018). REU-like representations include Klibanoff et al. (2005), Nau (2006), Chew and Sagi (2008), Ergin and Gul (2009), Neilson (2010), and Grant et al. (2009).

3.2 Choquet expected utility

In Schmeidler's (1989) *Choquet expected utility* (CEU) within the AA-setup, non-neutral attitude towards ambiguity is captured by means of non-additive subjective probabilities. The later result from the replacement of the subjective probability measure on the state space by a capacity.

Formally, a *capacity* is a measure $\nu : \mathcal{A} \to [0, 1]$, such that $\nu(\varnothing) = 0$, $\nu(\mathcal{S}) = 1$ and $\nu(E) \geq \nu(F)$ whenever $E \supseteq F$; a capacity is *strictly monotone if* $\nu(E) > \nu(F)$ whenever $E \supseteq F$ and $E \backslash F$ is non-null. If a capacity is additive (i.e., $\nu(E \cup F) =$

$\nu(E) + \nu(F)$ for all disjoint E and F), then it is a *probability measure* on $\mathcal{A}$ (Gilboa 1987; Wakker 2001).

A key concept in Schmeidler's CEU is comonotonicity, which comes down to a state-independent rank-ordering of lotteries. In our setup, acts $\mathbf{x}, \mathbf{y} \in \mathbb{A}_{\mathcal{E}}$ are *comonotonic* if there do not exist non-null events $E_k, E_{k'}$ in $\mathcal{E}$ such that both $CE(\hat{x}_k) > CE(\hat{x}_{k'})$ and $CE(\hat{y}_k) < CE(\hat{y}_{k'})$ hold. If preferences over lotteries are in agreement with EU, this notion of comonotonicity is equivalent to that of Schmeidler (1989). Given our Lemma 1, the subsets of acts denoted by

$$\mathbb{A}_{\mathcal{E}}^{\downarrow} := \{\mathbf{x} \in \mathbb{A}_{\mathcal{E}} : \hat{x}_i \in \mathcal{L}^{\downarrow}, i = 1, \dots, n, \text{ and } CE(\hat{x}_1) \geq \cdots \geq CE(\hat{x}_n)\}$$

contain only comonotonic acts, where the ordering of conditional CEs from best to worst, implicitly means that the partition $\mathcal{E} = (E_i)_{i=1}^n$ is ordered. The superscript arrow in $\mathbb{A}_{\mathcal{E}}^{\downarrow}$ indicates that rank-ordering for of outcomes matters and has implications for both the ordering of events (for ambiguity in the first stage) and the ordering of probabilities (for risk in the second stage). Note that the set of acts $\mathbb{A}_{\mathcal{E}}$ is the union all subsets $\mathbb{A}_{\mathcal{E}'}^{\downarrow}$, where $\mathcal{E}'$ is a different ordering of the events in the partition $\mathcal{E}$.

In our setup, CEU evaluates an act $\mathbf{x} \in \mathbb{A}_{\mathcal{E}}^{\downarrow}$ as follows

$$CEU(\mathbf{x}) = \sum_{i=1}^{n} \pi_i EU(\hat{x}_i), \tag{4}$$

where, relative to SEU, *decision weights* $\pi_i = \nu(\cup_{k=1}^{i} E_k) - \nu(\cup_{k=1}^{i-1} E_k)$, $i = 1, \dots, n$, replace subjective probabilities, and ν is a capacity. Source and Rank-dependent Utility

Next, we formally introduce source and rank-dependent utility (SRU) where two-stage acts are evaluated using non-additive probabilities for events as in CEU, source-dependent utility as in REU, and RDU applies to lotteries.

Definition 2 *Source and rank-dependent utility* holds if the preference $\succcurlyeq$ on $\mathbb{A}$ is represented by

$$SRU(\mathbf{x}) = \sum_{i=1}^{n} \pi_i \phi[RDU(\hat{x}_i)], \tag{5}$$

whenever the act $\mathbf{x}$ is from $\mathbb{A}_{\mathcal{E}}^{\downarrow}$, $\mathcal{E} = (E_i)_{i=1}^n$, where the *decision weights* $\pi_i = \nu(\cup_{k=1}^{i} E_k) - \nu(\cup_{k=1}^{i-1} E_k)$, $i = 1, \dots, n$, are determined by a unique and strictly monotone capacity ν, RDU is as in Eq.(1), and $\phi : u(\mathbb{R}) \to \mathbb{R}$ is a strictly increasing and continuous transformation function. Further, for fixed utility u the transformation function ϕ is cardinal. Moreover, the utility function u can be replaced by $au + b$, $a > 0, b \in \mathbb{R}$, if ϕ is replaced by $\hat{\phi}(\cdot) = \phi((\cdot - b)/a)$. □

SRU does account for ambiguity attitude through both non-additive probabilities and ϕ. If one takes the view that ambiguity attitude is not utility-driven, then the transformation ϕ in SRU must be assumed linear; it can be dropped from Eq. (5). The resulting model, abbreviated SRU*, is an extension of Schmeidler's (1989) model as it evaluates lotteries by RDU. Schmeidler's CEU corresponds to a special case of SRU

Table 1 SRU and related preference representations considered

Models		Component(s) accounting for attitudes towards		
		Ambiguity		Probabilities
		ν: additive?	ϕ: linear?	w: identity?
SRU	Theorem 12	no	no	no
SRU*	Corollary 14	no	yes	no
REU	Theorem 7	yes	no	yes
SEU	Theorem 4	yes	yes	yes

SRU* is a generalization of Schmeidler's CEU

where both the transformation ϕ and the probability weighting function w are identity functions. If one takes the view that ambiguity attitude is not "event-driven" then the capacity ν in SRU must be assumed linear. The resulting model is an extension of REU that was alluded to in Klibanoff, et al. (2005, p. 1859) in which RDU instead of EU is applied to lotteries.

4 Preference foundations

This section first introduces the tool of *equivalent substitutions* given the continuum of outcomes. In our setup, such substitutions are defined in terms of indifferences of acts taken from specific subsets of $\mathbb{A}$. These substitutions echo the notion of (marginal) rates of substitution in traditional consumer theory. Equivalent substitution of outcomes are made meaningful for preference representations in our AA-setup by requiring an appropriate *substitution consistency* principle. Section 4.1 shows that "full-force" substitution consistency captures the normative content of the sure-thing principle for both risk and ambiguity, such that in the presence of the other standard axioms SEU is derived. Section 4.2 shows that, when substitution consistency is required within source, it supplements the classical SEU with a source-dependent utility, that is, REU is obtained. Then, Sect. 4.3 shows that restricting substitution consistency to acts with rank-ordered outcomes for both sources (i.e., for comonotonic acts over lotteries with commonly rank-ordered outcomes) results in SRU.

4.1 Substitution consistency for SEU

Our general preference condition, called *substitution consistency*, fundamentally requires a consistent measurement of utility. It demands that utility differences inferred from choice, which are derived from what we term "equivalent substitutions of outcomes," are independent of the specified stimuli used to obtain them.

To introduce the intuition behind substitution consistency, we first invoke *equivalent substitutions of outcomes* using acts from

$$\mathbb{A}_{\mathcal{E}|E,\mathbf{p}} := \{\hat{x}_E\mathbf{y} : \mathbf{y} \in \mathbb{A}_{\mathcal{E}}, \hat{x} \in \mathcal{L}_{\mathbf{p}}\},$$

where $\mathcal{E} := (E_i)_{i=1}^n$ is a partition of the state space, E is a non-null "gauge" event from it, and $\mathbf{p} = (p_1, \ldots, p_m)$ is a given m-tuple of probabilities.

We say that the substitution of outcome α for β is equivalent to the substitution of outcome γ for δ if the indifferences

$$\begin{cases} (\alpha_j \hat{x})_E \mathbf{x} \sim (\beta_j \hat{y})_E \mathbf{y}, \\ (\gamma_j \hat{x})_E \mathbf{x} \sim (\delta_j \hat{y})_E \mathbf{y}, \end{cases} \tag{6}$$

hold for acts $(\alpha_j \hat{x})_E \mathbf{x}$, $(\beta_j \hat{y})_E \mathbf{y}$, $(\gamma_j \hat{x})_E \mathbf{x}$, $(\delta_j \hat{y})_E \mathbf{y}$ in some subset $\mathbb{A}_{\mathcal{E}|E,\mathbf{p}}$.

To further clarify the implications of equivalent substitutions, we present the following derivations in the main text. Under SEU, assuming $E = E_k$, indifferences (6) imply

$$\begin{cases} \pi_k EU(\alpha_j \hat{x}) + \sum_{i \neq k} \pi_i EU(\hat{x}_i) = \pi_k EU(\beta_j \hat{y}) + \sum_{i \neq k} \pi_i EU(\hat{y}_i), \\ \pi_k EU(\gamma_j \hat{x}) + \sum_{i \neq k} \pi_i EU(\hat{x}_i) = \pi_k EU(\delta_j \hat{y}) + \sum_{i \neq k} \pi_i EU(\hat{y}_i), \end{cases} \tag{7}$$

where subjective probability $\pi_k \neq 0$ because E_k is non-null.

Taking the difference of equations (7) simplifies to

$$EU(\alpha_j \hat{x}) - EU(\beta_j \hat{x}) = EU(\gamma_j \hat{y}) - EU(\delta_j \hat{y}), \tag{8}$$

where $EU(\cdot_j \hat{z}) = p_j u(\cdot) + \sum_{l \neq j} p_l u(z_l)$ for $\hat{z} = \hat{x}, \hat{y}$ in $\mathcal{L}_\mathbf{p}$. Consequently,

$$u(\alpha) - u(\beta) = u(\gamma) - u(\delta) \tag{9}$$

is obtained. This equation conveys the information that, under SEU, the substitution of outcome α for β is equivalent in terms of utility difference to the substitution of outcome γ for δ.

Suppose that there is some inconsistency in equivalent substitutions, for instance that the following indifferences also hold

$$\begin{cases} (\alpha_{j'} \hat{x}')_{E'} \mathbf{x}' \sim (\beta_{j'} \hat{y}')_{E'} \mathbf{y}', \\ (\gamma_{j'} \hat{x}')_{E'} \mathbf{x}' \sim (\delta^*_{j'} \hat{y}')_{E'} \mathbf{y}', \end{cases} \tag{10}$$

with acts from some $\mathbb{A}_{\mathcal{E}'|E',\mathbf{p}'}$ for some $\delta^* \neq \delta$. Assuming SEU, we can repeat the preceding analysis to obtain $u(\alpha) - u(\beta) = u(\gamma) - u(\delta^*)$, which contradicts the preceding equivalent substitutions as u is strictly increasing. It is therefore clear that a preference principle that avoids such preference-based inconsistencies is not only desirable but necessary for the identification of utility in SEU.

The following definition formalizes this consistency requirement for equivalent substitutions by adding independence of equivalent substitutions from stimuli used to derive the former.

Definition 3 *Substitution consistency* holds if for all outcomes $\alpha, \beta, \gamma, \delta$, all partitions $\mathcal{E}, \mathcal{E}'$, non-null events E from $\mathcal{E}$ and E' from $\mathcal{E}'$, probability vectors $\mathbf{p} = (p_1, \ldots, p_m)$, $\mathbf{p}' = (p'_1, \ldots, p'_{m'})$, and $j \in \{1, \ldots, m\}$, $j' \in \{1, \ldots, m'\}$ we have

$$(\alpha_j \hat{x})_E \mathbf{x} \sim (\beta_j \hat{y})_E \mathbf{y} \;\&\; (\alpha_{j'} \hat{x}')_{E'} \mathbf{x}' \sim (\beta_{j'} \hat{y}')_{E'} \mathbf{y}',$$
$$\&\; (\gamma_j \hat{x})_E \mathbf{x} \sim (\delta_j \hat{y})_E \mathbf{y} \Rightarrow (\gamma_{j'} \hat{x}')_{E'} \mathbf{x}' \sim (\delta_{j'} \hat{y}')_{E'} \mathbf{y}', \tag{11}$$

whenever all acts in the left pair of indifferences are from $\mathbb{A}_{\mathcal{E}|E,\mathbf{p}}$ and all acts in the right pair of indifferences are from $\mathbb{A}_{\mathcal{E}'|E',\mathbf{p}'}$. □

In the presence of the standard assumptions of Lemma 1, SEU is satisfied if and only if equivalent substitutions of outcomes derived from indifferences (6) are consistent with those revealed from indifferences (10). We obtain the following result.

Theorem 4 *Assume that $\succcurlyeq$ is a preference relation on the set of acts $\mathbb{A}$. Then, the following statements are equivalent:*

(i) Subjective Expected Utility holds.
(ii) The preference relation $\succcurlyeq$ satisfies weak ordering, continuity, monotonicity, independence of conditional CEs, and substitution consistency.

The probability measure for events is unique and the utility u is cardinal. □

The requirement of substitution consistency for SEU in Theorem 4 parallels how the classical normative axiom for EU under risk, i.e., independence, is sometimes presented. In particular, under EU for risk, independence can be split into a simple dominance condition and a "substitution" axiom that allows for the replacement of a lottery in a mixture by another equally good lottery. This normative principle can be interpreted in terms of consistent measurement of utility (see Wakker 2010, section 2.6).[5]

Likewise, assuming standard axioms of a continuous weak order satisfying monotonicity and independence of conditional CEs, Theorem 4 shows that the conjoint compliance of substitution consistency and monotonicity for all two-stage acts, results in the normative SEU model. That is, together with Lemma 1, substitution consistency boils down to requiring the normative sure-thing principle to hold for both ambiguity and risk through consistency of revealed utility differences.

We have presented our derivation of SEU first in order to introduce the notion of equivalent substitution in a familiar theory. Next we provide derivations for both REU and SRU by restricting the domains of acts where the substitution consistency is invoked.

4.2 Substitution consistency for REU

In contrast to SEU, REU uses a specific utility for ambiguity, $\phi \circ u$, in addition to utility for risk, u. In our setup, the transformation function ϕ is a simple consequence of EU

[5] To illustrate, assume that $\alpha \sim X_p x$, where $\alpha \in [x, X]$ is a sure outcome, and $X_p x$ is the lottery that gives X with probability p , and x otherwise. Under EU with a normalized utility, this means that $u(\alpha) = p$. A mild version of substitution requires that the observed utility should not be affected if one considers the indifference $\alpha_\lambda \hat{x} \sim (X_p x)_\lambda \hat{x}$ obtained from probability mixing of the former indifference for any probability λ and any lottery $\hat{x}$, i.e., again we should obtain $u(\alpha) = p$. This shows that the main normative condition in EU essentially requires that utility measurement should not be impacted by a probability mixing that involves other common stimuli.

holding separately for ambiguity on the one hand, and for risk in the other hand. We implement this by requiring our substitution consistency of definition 3 to hold within each source of uncertainty, i.e., for risk and separately ambiguity, but not necessarily across these sources. In terms of utility, substitution consistency for risk speaks to consistent utility differences in terms of the u-scale, while substitution consistency for ambiguity speaks to consistent utility differences in terms of the $(\phi \circ u)$ -scale.

The following definition restricts the application of substitution consistency to risk. It states that, if acts $\mathbf{x}$ and $\mathbf{y}$ in condition (11) coincide outside the "gauge" events E and E', then consistent substitutions under risk are meaningful for choice behavior.

Definition 5 *Substitution consistency for risk* holds if condition (11) in Definition 3 is satisfied whenever $\mathbf{x}_{E^c} = \mathbf{y}_{E^c}$ and $\mathbf{x}'_{E'^c} = \mathbf{y}'_{E'^c}$. □

We noted earlier that, for SEU, condition (11) is necessary for the identification of source independent cardinal utility. Although the technical arguments used next are similar, we think it is important to elaborate that, for REU, substitution consistency for risk is necessary for the identification of u. To illustrate how ambiguity captured by ϕ is circumvented, assume that the left hand indifferences in condition (11) hold together with $\mathbf{x}_{E^c} = \mathbf{y}_{E^c}$ where $E = E_k$. Under REU, this implies

$$\begin{cases} \pi_k\phi[EU(\alpha_j\hat{x})] + A = \pi_k\phi[EU(\beta_j\hat{y})] + B, \\ \pi_k\phi[EU(\gamma_j\hat{x})] + A = \pi_k\phi[EU(\delta_j\hat{y})] + B, \end{cases} \tag{12}$$

where $A = \sum_{i\neq k}\pi_i\phi[EU(\hat{x}_i)]$ and $B = \sum_{i\neq k}\pi_i\phi[EU(\hat{y}_i)]$ are equal due to $\mathbf{x}_{E^c} = \mathbf{y}_{E^c}$. As $\pi_k \neq 0$ and ϕ is strictly increasing, equations (12) simplify to $EU(\alpha_j\hat{x}) = EU(\beta_j\hat{y})$ and $EU(\gamma_j\hat{x}) = EU(\delta_j\hat{y})$, where $EU(\cdot_j\hat{z}) = p_j u(\cdot) + \sum_{l\neq j} p_l u(z_l)$ for $\hat{z} = \hat{x}, \hat{y}$ in $\mathcal{L}_\mathbf{p}$. Therefore $u(\alpha) - u(\beta) = u(\gamma) - u(\delta)$ follows. Further, if it were the case that for some feasible $\delta^* \neq \delta$ both $(\alpha_{j'}\hat{x}')_{E'}\mathbf{x}' \sim (\beta_{j'}\hat{y}')_{E'}\mathbf{y}'$ and $(\gamma_{j'}\hat{x}')_{E'}\mathbf{x}' \sim (\delta^*_{j'}\hat{y}')_{E'}\mathbf{y}'$ hold within some $\mathbb{A}_{\mathcal{E}'|E',\mathbf{p}'}$ and $\mathbf{x}'_{E'^c} = \mathbf{y}'_{E'^c}$, we would obtain a violation of the former equivalent substitutions statement (i.e., we obtain $u(\alpha) - u(\beta) = u(\gamma) - u(\delta^*)$). This means that substitution consistency for risk is necessary for REU.

Next we restrict the application of substitution consistency to ambiguity. The corresponding property demands that equivalent substitutions with acts from $\mathbb{A}_{\mathcal{E}|E,\mathbf{p}}$ that are risk-free conditional on gauge event E ($p_j = 1$), i.e., $\alpha_E\mathbf{x} \sim \beta_E\mathbf{y}$ and $\gamma_E\mathbf{x} \sim \delta_E\mathbf{y}$, must not be contradicted.

Definition 6 *Substitution consistency for ambiguity* holds if condition (11) in Definition 3 is satisfied whenever $p_j = p'_{j'} = 1$. □

For REU the property in Definition 6 is a necessary identification tool for the function $\phi \circ u$. It is important to note that the ambiguity function ϕ cannot be identified separately from u, as the following analysis demonstrates. Starting with the left hand indifferences in condition (11) together with $p_j = 1$ (instead of $\mathbf{x}_{E^c} = \mathbf{y}_{E^c}$), under REU we obtain the equations (12), which imply $\phi[u(\alpha)] - \phi[u(\beta)] = \phi[u(\gamma)] - \phi[u(\delta)]$. A violation of this equation can be derived if $\alpha_{E'}\mathbf{x}' \sim \beta_{E'}\mathbf{y}'$ and $\gamma_{E'}\mathbf{x}' \sim {\delta^*}_{E'}\mathbf{y}'$ for some $\delta^* \neq \delta$. Therefore, substitution consistency for ambiguity is necessary for REU.

The following theorem proves that, in the presence of standard axioms in our AA-setup, invoking within-source substitution consistency properties characterizes REU.

Theorem 7 *Assume that $\succcurlyeq$ is a preference relation on the set of acts $\mathbb{A}$. Then, the following statements are equivalent:*

(i) Recursive Expected Utility holds.
(ii) The preference relation $\succcurlyeq$ satisfies weak ordering, continuity, monotonicity, independence of conditional CEs, substitution consistency for risk, and substitution consistency for ambiguity.

Uniqueness results for ϕ and u hold as in Definition 2. Further, the probability measure for events is unique. □

As explained earlier, REU preserves the normative appeal of EU within each source. Across sources, REU allows to capture discrepancies from SEU through the curvature of ϕ. For instance, a concave ϕ accords with the ambiguity aversion revealed in the famous two-color Ellsberg's (1961) paradox. The following subsection aims at more flexibility to accommodate source-dependence. Additional descriptive power is added to REU by restricting substitution consistency to acts within specific subsets of $\mathbb{A}_{\mathcal{E}|E,\mathbf{p}}$.

4.3 Substitution consistency for SRU

In this section we incorporate rank-dependence as proposed by Schmeidler (1989) and Quiggin (1982) into substitution consistency for ambiguity and for risk, respectively. This way, well-documented discrepancies from due to non-linear probability weighting (risk) and non-additive subjective probabilities (ambiguity) are accounted for. Those discrepancies hinder an unbiased identification of u and $\phi \circ u$ in REU. By restricting REU's source-dependent substitution consistency properties to subsets of acts where rank-dependence is controlled for, we obtain smaller domains of acts where the normative content of the substitution principle still holds. The corresponding properties deliver SRU. Specifically, the rank-dependence considerations are reflected in nonlinear probability weighting under risk and non-additive subjective probabilities under ambiguity.

As SRU assumes RDU for risk, for each partition $\mathcal{E}$, a non-null event E in it, and act $\mathbf{z} \in \mathbb{A}^{\downarrow}_{\mathcal{E}}$, we use equivalent substitutions for acts from

$$\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}} := \{\hat{x}_E\mathbf{y} : \mathbf{y} \in \mathbb{A}^{\downarrow}_{\mathcal{E}}, \hat{x} \in \mathcal{L}^{\downarrow}_{\mathbf{p}}\}$$

that coincide outside of E. Under SRU, comparing an act $\hat{x}_E\mathbf{z}$ to an act $\hat{y}_E\mathbf{z}$ in $\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}}$ boils down to the comparison of RDU-values of lotteries $\hat{x}$, $\hat{y}$ in $\mathcal{L}^{\downarrow}_{\mathbf{p}} \subset \mathcal{L}_{\mathbf{p}}$ for which outcomes are ordered from the best to the worst. The preference condition in the following definition reduces the normative demand of EU under REU through the restriction of substitution consistency under risk to lotteries in $\mathcal{L}^{\downarrow}_{\mathbf{p}}$.

Definition 8 *Rank-dependent substitution consistency* for risk holds if condition (11) in Definition 3 is satisfied whenever all acts in the left pair of indifferences are from

$\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}}$ with $\mathbf{x}_{E^c} = \mathbf{y}_{E^c}$, and all acts in the right pair of indifferences are from $\mathbb{A}^{\downarrow}_{\mathcal{E}'|E',\mathbf{p}'}$ with $\mathbf{x}'_{E'^c} = \mathbf{y}'_{E'^c}$. □

To illustrate how the effect of ambiguity captured by ϕ is circumvented in the presence of rank-dependence, assume that the left hand indifferences in condition (11) hold together with $\mathbf{x}_{E_k^c} = \mathbf{y}_{E_k^c}$ and all acts are in $\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}}$. Under SRU, this implies

$$\begin{cases} \pi_k \phi[RDU(\alpha_j \hat{x})] + A = \pi_k \phi[RDU(\beta_j \hat{y})] + B, \\ \pi_k \phi[RDU(\gamma_j \hat{x})] + A = \pi_k \phi[RDU(\delta_j \hat{y})] + B, \end{cases} \tag{13}$$

where $A = \sum_{i \neq k} \pi_i \phi[RDU(\hat{x}_i)]$ and $B = \sum_{i \neq k} \pi_i \phi[RDU(\hat{y}_i)]$ are equal due to $\mathbf{x}_{E^c} = \mathbf{y}_{E^c}$. Further, as the decision weight $\pi_k \neq 0$ cancels out and ϕ is strictly increasing, equations (13) simplify to $RDU(\alpha_j \hat{x}) = RDU(\beta_j \hat{y})$ and $RDU(\gamma_j \hat{x}) = RDU(\delta_j \hat{y})$, or equivalently

$$\begin{cases} \hat{\pi}_j u(\alpha) + \sum_{l \neq j} \hat{\pi}_l u(x_l) = \hat{\pi}_j u(\beta) + \sum_{l \neq j} \hat{\pi}_l u(y_l), \\ \hat{\pi}_j u(\gamma) + \sum_{l \neq j} \hat{\pi}_l u(x_l) = \hat{\pi}_j u(\delta) + \sum_{l \neq j} \hat{\pi}_l u(y_l), \end{cases} \tag{14}$$

where $\hat{\pi}_l$, $l = 1, \ldots, m$ stand for decision weights under RDU for risk (see Section 2.1, equation (1)). Taking differences between the latter equations gives $u(\alpha) - u(\beta) = u(\gamma) - u(\delta)$, meaning that, under SRU, the substitution of outcome α for β is equivalent to the substitution of outcome γ for δ in terms of utility under risk.

Remark 9 Note that equations (14) also hold under REU (with $\hat{\pi}_j = p_j, j = 1, \ldots, m$), meaning that, for indifferences between acts in $\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}}$ that coincide outside the gauge event E, equivalent substitutions for risk in REU agree with those for risk under SRU (i.e., in terms of u-differences).

Clearly, under SRU, for $\delta^* \neq \delta$, if $(\alpha_{j'} \hat{x}')_{E'} \mathbf{x}' \sim (\beta_{j'} \hat{y}')_{E'} \mathbf{y}'$ and $(\gamma_{j'} \hat{x}')_{E'} \mathbf{x}' \sim (\delta^*_{j'} \hat{y}')_{E'} \mathbf{y}'$ hold for acts in $\mathbb{A}^{\downarrow}_{\mathcal{E}'|E',\mathbf{p}'}$ with $\mathbf{x}'_{E'^c} = \mathbf{y}'_{E'^c}$, we obtain $u(\alpha) - u(\beta) = u(\gamma) - u(\delta^*)$, which contradicts the preceding equivalent substitutions as u is strictly increasing. Therefore, rank-dependent substitution consistency for risk is necessary for SRU.

The following definition applies substitution consistency for ambiguity by tailoring the normative demand of REU's substitution consistency for ambiguity to sets of rank-ordered acts. Specifically, we require that equivalent substitutions derived from acts in $\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}}$ that are risk-free ($p_j = 1$) conditional on the gauge event E, i.e., $\alpha_E \mathbf{x} \sim \beta_E \mathbf{y}$ and $\gamma_E \mathbf{x} \sim \delta_E \mathbf{y}$, are not contradicted when derived from acts in $\mathbb{A}^{\downarrow}_{\mathcal{E}'|E',\mathbf{p}'}$ that are risk-free ($p'_j = 1$) conditional on event E', i.e., $\alpha_{E'} \mathbf{x}' \sim \beta_{E'} \mathbf{y}'$ and $\gamma_{E'} \mathbf{x}' \sim \delta_{E'} \mathbf{y}'$ must also hold.

Definition 10 *Rank-dependent substitution consistency for ambiguity* holds if condition (11) in Definition 3 is satisfied whenever all acts in the left pair of indifferences are from $\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}}$ with $p_j = 1$, and all acts in the right pair of indifferences are from $\mathbb{A}^{\downarrow}_{\mathcal{E}'|E',\mathbf{p}'}$ with $p'_{j'} = 1$. □

Similarly to the derivations for REU, under SRU, if the left hand indifferences in condition (11) hold along with $p_j = 1$ with all acts from $\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}}$ we obtain $\phi[u(\alpha)] - \phi[u(\beta)] = \phi[u(\gamma)] - \phi[u(\delta)]$.

Remark 11 For REU and SRU, equivalent substitutions for acts in $\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}}$ that are risk-free conditional on E agree. That is, on these specific subsets of rank-ordered acts, REU and SRU coincide in terms of $\phi \circ u$.

Returning to Definition 10, if for $\delta^* \neq \delta$, both $\alpha_{E'}\mathbf{x}' \sim \beta_{E'}\mathbf{y}'$ and $\gamma_E\mathbf{x}' \sim \delta^*{}_E\mathbf{y}'$ hold and all acts are in $\mathbb{A}^{\downarrow}_{\mathcal{E}'|E',\mathbf{p}'}$ then SRU implies $\phi[u(\alpha)] - \phi[u(\beta)] = \phi[u(\gamma)] - \phi[u(\delta^*)]$, which contradicts the preceding equation $\phi[u(\alpha)] - \phi[u(\beta)] = \phi[u(\gamma)] - \phi[u(\delta)]$ given that $\phi \circ u$ is strictly increasing. Consequently, rank-dependent substitution consistency for ambiguity is necessary for SRU.

The following result shows that, in the presence of standard axioms in our AA-setup, rank-dependent substitution consistency conditions characterize SRU.

Theorem 12 *Assume that $\succcurlyeq$ is a preference relation on the set of acts $\mathbb{A}$. Then, the following statements are equivalent:*

(i) Source and Rank-dependent Utility holds.
(ii) The preference relation $\succcurlyeq$ satisfies weak ordering, continuity, monotonicity, independence of conditional CEs, rank-dependent substitution consistency for risk, and rank-dependent substitution consistency for ambiguity.

Uniqueness results hold as in Definition 2. □

One may now ask the question of what type of behavior is compatible with a source independent but rank-dependent application of the substitution consistency principle. The property that we formalize next characterizes a class of choice behavior that still includes Schmeidler's (1989) CEU as special case.

Definition 13 *Rank-dependent substitution consistency* holds if condition (11) of Definition 3 is satisfied, whenever all acts in the left pair of indifferences are from $\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}}$ and all acts in the right pair of indifferences are from $\mathbb{A}^{\downarrow}_{\mathcal{E}'|E',\mathbf{p}'}$.

Rank-dependent substitution consistency implies rank-dependent substitution consistency for risk and, separately, rank-dependent substitution consistency for ambiguity. However, by requiring additionally that equivalent substitution for risk are also equivalent substitutions for ambiguity, the effect of ϕ in Theorem 12 is neutralized. That is, now equivalent substitutions derived from different sources are consistent and, hence, they are source independent; this model is labeled SRU *. We obtain the following result.

Corollary 14 *Assume that $\succcurlyeq$ is a preference relation on the set of acts $\mathbb{A}$. Then, the following statements are equivalent:*

(i) Source and Rank-dependent Utility holds with a linear function ϕ.
(ii) The preference relation $\succcurlyeq$ satisfies weak ordering, continuity, monotonicity, independence of conditional CEs, rank-dependent substitution consistency.

Uniqueness results for the capacity and probability weighting function hold as in Definition 2; further, the source independent utility u is cardinal. □

The fact that the SRU* model comes down to Schmeidler's (1989) model when RDU for risk is replaced by EU, indicates the flexibility of SRU* to accommodate descriptive behavior under risk. Moreover SRU* can be interpreted as a theory that allows for a descriptive analysis of source-dependent probability weighting. To illustrate, assume that, in addition to SRU*, probabilistic sophistication holds, i.e., each event E can be assigned a subjective probability $P(E)$, so two-stage acts can be replaced by corresponding two-stage lotteries, and that the latter objects of choice are not necessarily evaluated by their EU (e.g., Machina and Schmeidler 1992). The empirical question that results is whether the weighting function f that transforms subjective probabilities for events, i.e., $\nu(E) = f(P(E))$ is identical with the probability weighting function for risk w.[6] If this hypothesis is confirmed, then we have a justification for models like Segal's (1987b) recursive RDU theory. This is a further example that shows how flexible SRU is and that it delivers on our objective to provide a unified setup in which many models of ambiguity, including the popular REU, can jointly be analyzed and compared.

5 Related literature

In contrast to our general SRU model, the literature has addressed descriptive shortcomings of SEU related to ambiguity considerations separately from those related to EU for risk. A first step to improve the descriptive power of preference representations in the AA-setup would be to extend the global ambiguity aversion as suggested by the two-color examples of Ellsberg (1961) while also addressing the preference for certainty under risk captured by the certainty effect phenomenon in the examples of Allais (1953). Such a model was recently proposed by Dean and Ortoleva (2017). It invokes a multiple RDU model for risk with convex weighting functions (i.e., pessimism; Wakker 1994) within the ambiguity-averse multiple prior model of Gilboa and Schmeidler (1989). Dean and Ortoleva propose a principle of preference for hedging (their Axiom 5) and use it to simultaneously capture pessimism under risk and ambiguity aversion. For risk, their model identifies a set of convex probability weighting functions from which the most pessimistic one is picked to evaluate lotteries (see also Gumen et al. 2014).[7] Universal ambiguity aversion is preserved in their setup by invoking a preference for objective risk property (their Axiom 7). Structurally, in addition to assuming a continuum of outcomes, as we do, Dean and Ortoleva also maintain probabilistic mixing of acts as a tool, thus they also build on the richness of probabilities. Cheridito et al. (2015) developed a related model by supplementing a weak ordering satisfying upper semicontinuity on AA-acts (axiom A4) and convexity (axiom A3) with monotonicity w.r.t. first-order stochastic dominance (axiom A2). In

[6] The weighting function $f : [0, 1] \to [0, 1]$ is strictly increasing with $f(0) = 0$ and $f(1) = 1$.

[7] Dean and Ortoleva's (2017) multi-RDU for risk is conceptually related to Maccheroni's (2004) model in which Yaari's (1987) dual theory is endowed with a set of convex probability weighting functions (see also Safra and Segal 1998; Lemma 4).

contrast to standard monotonicity, the latter axiom does not involve any implicit separability or state-independence. In a subsequent approach, monotonicity with respect to second-order stochastic dominance is invoked to account for risk averse preferences in addition to ambiguity aversion. Overall, this provides a utility-driven account for the Ellsberg and Allais examples.

Although the role of substitution consistency in SRU is similar to the role of preference for hedging in Dean and Ortoleva's (2017) model, the former can accommodate more discrepancies from SEU and EU than the ambiguity aversion and certainty effect. These discrepancies from EU are accounted for in SRU by imposing convexity for both the capacity (Karni and Safra 1990; Wakker 2001) and for the probability weighting function and, further, by assuming a source independent utility. However, the substitution consistency principle is flexible and can be formulated to accommodate further empirical regularities beyond just the Ellsberg and Allais-type behaviors. In particular, SRU can account for the frequent empirical finding that people exhibit risk seeking for low probability gains by overweighting small probabilities (Tversky and Kahneman 1992; Tversky and Wakker 1995; Prelec 1998; Bruhin et al. 2010). Similarly, speaking to ambiguity, SRU can account for the observed tendency of decision makers to be ambiguity prone over unlikely events (e.g., Kilka and Weber 2001; Abdellaoui et al. 2005; Dimmock et al. 2016).

The SRU model provides foundations for non-additive probabilities, source-dependent utilities and probability weighting without constraining the shape of these components. Special cases of SRU-models, such as SRU* of Corollary 14 with source independent utility, are still general. Indeed, SRU* extends Schmeidler's (1989) CEU-model by accounting for non-linear treatment of probabilities through an RDU-evaluation of lotteries. In a Savagean-like two-source setup, using the richness of the state space instead of a continuum of outcomes, Ergin and Gul (2009) derived RDU for ambiguity with EU for risk as in Schmeidler (1989). In their Theorem 4, they used a comonotonic sure-thing principle (Axiom 6c), which can be considered a dual analog of our rank-dependent substitution consistency for ambiguity.

In the presence of an exogenously given probability distribution on the state space, the SRU* representation in Corollary 14 provides a basis for obtaining a representation of preferences for two-stage lotteries as in Segal's (1987b) recursive model.[8] By assuming RDU with a common probability weighting functions at both stages, Segal accounts for Ellsberg-type behavior, identified with the decision maker's attitude towards second-order risk, along with Allais-type behavior in a fully objective two-stage "decomposition" of acts. Our substitution consistency principle can be modified (through an adequate replacement of states by objective probabilities) to obtain the exact (objective) representation of preferences required in Segal's model similar to Corollary 14, or a generalization that additionally accounts for a specific utility function for ambiguity similar to the result in our Theorem 12. From an empirical point of view, Chew et al. (2015) report results in favor of the descriptive power of Segal's recursive model. Indirectly, these results also support the more general SRU. As indicated above, the substitution consistency principle is flexible and enables aspects of

[8] One can interpret Segal's (1987b) model as "equating" reduction of compound lotteries and ambiguity neutrality (Chew et al. 2015; Abdellaoui et al. 2015).

it to be turned on and off for both ambiguity and risk as desired. This means that many existing single-stage models for risk or ambiguity, such as those of Köbberling and Wakker (2003), are nested as special cases of SRU.[9] Popular two-stage models developed in the AA-framework can alternatively be obtained as special cases of our outcome continuum assumption through the addition of a linearity for probability mixtures condition.

The preference foundations for REU are structurally close to the main representation theorem in Nau (2006, Theorem 2). Like the latter, we adopt a dual approach to Ergin and Gul (2009) and exploit the richness of a continuum of outcomes rather than a rich set of states. Nau considers two finite collections of events $\mathcal{A}$ and $\mathcal{B}$ interpreted as two logically independent sources of uncertainty with subjective probabilities in source $\mathcal{B}$ in general being dependent on events from $\mathcal{A}$ (see also Mongin and Pivato 2015 for a recent contribution). Compared to the Savagean-like approach of Ergin and Gul (2009, Theorem 3), our within-source substitution conditions intersect with their comparative probability axioms (Axioms 5a, 5b, 5c) and a sure-thing principle applied to sources (i.e., issues *a* and *b* in their Axiom 6).

The representation result in Theorem 4 provides an alternative approach to the derivation of Anscombe and Aumann's (1963) result. Relative to the latter, we do not use any probability mixing and, therefore, we weaken the classical independence axiom based on probability mixing of acts.[10] Our structural changes to the AA-setup mean that we deliver SEU with a continuous utility function. Our "full-force" substitution consistency allows for additive separability as well as the existence of a state-independent utility function in one stroke. By focusing on the derivation of utility, substitution consistency identifies, as a by-product, the subjective probabilities for events. Clearly, a fully subjective version of our substitution consistency (replacing objective probabilities by a finite second-stage collection of events as in Nau 2006) can deliver the respective two-stage SEU representation with a single continuous utility function.

A different approach was considered by Bommier (2017), where the richness of the probability interval was exploited. Bommier adopts a standard AA-setting and advocates for a dual approach to aggregation. Initially, aggregation of (cumulated) objective probabilities for outcomes is demanded to obtain "equivalent unambiguous beliefs," which are then aggregated over the outcomes of an act in a rank-dependent fashion. The representation in Bommier's Theorem 1 resembles a feature of Chew and Wakker's (1996) outcome-dependent capacity model, as it involves an "outcome-dependent probability weighting function." Additional separability properties are then required to obtain a separation into a stand-alone cardinal utility for outcomes and a function over the unambiguous beliefs. As Bommier (Section 3) indicates in Table 1,

[9] The preference techniques invoking "probability substitutions," as used in Abdellaoui (2002) and Zank (2010), can also be extended in a similar fashion to obtain recursive RDU with objective probabilities.

[10] In Wakker and Zank (1999) it was shown that the independence property of Anscombe and Aumann (1963) implies jointly a substitution consistency for ambiguity as in our Corollary 10 and EU over lotteries. See also Borah and Kops (2016) for a recent behavioral foundation for subjective EU within the AA framework in which it is shown that one does not need the independence axiom to hold over the entire domain of AA-acts.

such separability conditions can lead to various models that have an interpretation dual to existing ambiguity theories, hence, dual to RRU.

Finally, we highlight potential extensions. In the two-stage model of He (2021) the objects of choice are lotteries that give one-stage Savage-acts. So, relative to SRU where the risky source succeeds ambiguity, He proposes to have the risky source first and ambiguity at the second stage. The tools we have developed for SRU can be adapted to He's setting if a continuum of outcomes is assumed. Thus, further possibilities to apply versions of our substitution consistency principle are feasible here. A more general world of sources is discussed in Cappelli et al. (2021). Instead of just risk and uncertainty, Capelli, et al. assume that a finite set of sources is identified. Then they discuss the importance of choice behavior which justifies the analysis of attitudes into "intra-source" versus "inter-source" components as we do for SRU. Like us, Capelli, et al. assume a continuum of outcomes, a weak order, monotonicity and continuity properties that ensure the existence of within-source certainty equivalents. Subsequently, they focus on aggregators of these conditional CE's, where they invoke a preference-based tradeoff consistency property, similar to our indifference-based substitution consistency for ambiguity, to quantify across-source attitudes.

6 Relative concavity of utility

This section shows how the equivalent substitutions defined in Sect. 4.1, the common tool that underlies our representation results, can be used to compare the concavity of the utilities for risk and for ambiguity in SRU. Our technique generalizes the tools of Baillon et al. (2012), who use a single-stage analysis of relative concavity of utility in a version of REU for binary acts, to our models with two-stage acts. Given our Remarks 9 and 11, which clarifies that REU and SRU coincide in terms of utility when specific subsets of acts are considered, and given the nature of our substitution consistency principle, the proofs of Baillon, et al. apply to the results of this section without further modification. Hence, below we present our propositions without proofs.

Assume that SRU holds. Outcome β is a *midpoint* between outcomes α and γ for the utility under ambiguity, or ($\phi \circ u$)- *midpoint*, whenever we have $\alpha_E \mathbf{x} \sim \beta_E \mathbf{y}$ and $\beta_E \mathbf{x} \sim \gamma_E \mathbf{y}$ with all acts in $\mathbb{A}^{\downarrow}_{\mathcal{E}}$ and that E is non-null. As demonstrated in Sect. 4.3, this means that

$$\phi(u(\beta)) = \frac{\phi(u(\alpha)) + \phi(u(\gamma))}{2}.$$

The shape of $\phi \circ u$ (concave, convex, linear) can be characterized through the comparison of $(\phi \circ u)$-midpoints β with the corresponding algebraic midpoints $(\alpha + \gamma)/2$ for all outcomes α and γ.

Similarly, outcome β^* is a *midpoint* between α and γ for the utility under risk, or u-*midpoint*, whenever we have $(\alpha_j \hat{x})_E \mathbf{x} \sim (\beta^*_j \hat{y})_E \mathbf{y}$ and $(\beta^*_j \hat{x})_E \mathbf{x} \sim (\gamma_j \hat{y})_E \mathbf{y}$, with $\mathbf{x}$ and $\mathbf{y}$ from $\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}}$, E is non-null, $p_j > 0$, and $\mathbf{x}_{E^c} = \mathbf{y}_{E^c}$ (to control for ambiguity captured by ϕ or ν). This immediately results in

$$u(\beta^*) = \frac{u(\alpha) + u(\gamma)}{2}.$$

When the u-midpoints β^* are systematically smaller than the corresponding outcome midpoints $(\alpha + \gamma)/2$, for all outcomes α and γ, one can infer that utility u is concave. The following proposition is a generalization of Theorem 2.2 in Baillon et al. (2012) from REU to SRU.

Proposition 15 *Under SRU, the following two statements are equivalent:*

(i) u *is concave* ($\phi \circ u$ *is concave);*
(ii) u*-midpoints* ($\phi \circ u$*-midpoints) are below the corresponding outcome midpoints.*

□

Next, we provide a characterization of within-source relative concavity. Assume that SRU holds for $\succcurlyeq^1$ with utility u^1, and transformation ϕ^1, and SRU holds for $\succcurlyeq^2$ with corresponding functions. For outcomes α and γ let β^s be the $(\phi^s \circ u^s)$-midpoint for individual $s = 1, 2$. The $(\phi^1 \circ u^1)$-midpoints are below the $(\phi^2 \circ u^2)$ -midpoints if, for all outcomes α and γ, we have

$$\beta^1 \leq \beta^2,$$

in which case $\phi^1 \circ u^1$ is more concave than $\phi^2 \circ u^2$. A similar reasoning can be used for risk to infer that u^1 is more concave than u^2. These results are formally stated in our next proposition.

Proposition 16 *Assume SRU for* $\succcurlyeq^s$, *with* u^s *and* ϕ^s, $s = 1, 2$. *Then, the following two statements are equivalent:*

(i) u^1 *is more concave than* u^2 ($\phi^1 \circ u^1$ *is more concave than* $\phi^2 \circ u^2$*);*
(ii) u^1*-midpoints* (($\phi^1 \circ u^1$)*-midpoints) are below* u^2 *-midpoints* (($\phi^2 \circ u^2$)*-midpoints).*

□

In concluding this section, we characterize the case where one decision maker's transformation ϕ^1 is more concave than a second decision maker's transformation ϕ^2. As observed by Baillon et al. (2012), the use of midpoints allows for comparative ambiguity aversion without assuming that the two agents share the same first stage beliefs under REU or Klibanoff et al.'s (2005) smooth ambiguity model. The following proposition generalizes Baillon, et al.'s Theorem 2.3 by assuming SRU instead of REU.

Proposition 17 *Assume SRU for* $\succcurlyeq^1$ *and for* $\succcurlyeq^2$. *Then, the following two statements are equivalent:*

(i) $u^1 = u^2$ *and* ϕ^1 *is more concave than* ϕ^2*;*
(ii) u^1*-midpoints are the same as* u^2*-midpoints, and* $(\phi^1 \circ u^1)$*-midpoints are below* $(\phi^2 \circ u^2)$ *-midpoints.*

□

The interpretation of the above proposition is model-dependent. Under REU with $u^1 = u^2$, a more concave ϕ^1 than ϕ^2 means that agent 1 exhibits more ambiguity aversion than agent 2. Under SRU such a statement is no longer valid in general as ambiguity attitudes can be reflected through utility as well as the possibly non-additive probabilities assigned to events.

7 Descriptive applications of SRU

As observed in Baillon et al. (2018), while ambiguity theories are nowadays widely applied in economics, measurements of ambiguity for applied purposes have lagged behind. This section shows that one can use SRU to measure ambiguity through two popular approaches. The utility-driven approach of REU, which maintains the normative appeal of EU within each source, can be made operational by measuring utility on specific subsets of acts. In this case SRU is used to control and factor out discrepancies from EU that originate from nonlinear probability weighting under risk and non-additivity of subjective probabilities. Alternatively, SRU can be used to measure ambiguity through willingness to bet as measured by decision weights assigned to events while factoring out nonlinear probability weighting.

7.1 Capturing ambiguity via utility in REU

Under REU, measuring ambiguity comes down to the elicitation of two source-dependent utility functions, one for risk and one for ambiguity. This is descriptively problematic unless one can circumvent non-additivity of subjective probabilities under ambiguity (Ellsberg 1961; Fox and Tversky 1995; Abdellaoui et al. 2005) and probability weighting resulting from violations of EU for risk (Starmer 2000; Wakker 2010). To address this difficulty one can elicit utility on subsets of acts where REU and SRU coincide in terms of utility (as observed in Remarks 9 and 11), where kinks on indifference curves in the preference domain caused by decision weights are avoided. Next, we illustrate how this idea can be implemented.

As a thought example, suppose that a coin is slightly bent, that is, we have an "unknown" coin, thus ambiguity as the source of uncertainty. A coin flip that comes up heads (H) gives a monetary gain x while tails (T) gives a smaller gain y. Under REU, ϕ summarizes attitude towards ambiguity over the events H and T. Measuring ϕ requires the successive elicitation of first $\phi \circ u$ (using one-stage acts) and then u (using lotteries). To elicit $\phi \circ u$, fix an initial outcomes α_0 and two gauge outcomes, g^* and g_*, such that $\alpha_0 > g^* > g_* > 0$. Next, determine an increasing sequence of outcomes α_i, $i = 1, \ldots, l$ such that

$$(\alpha_{i-1})_H g^* \sim (\alpha_i)_H g_*. \tag{15}$$

Such indifferences are depicted in Fig. 1 (top left panel, ambiguity). A common feature of these indifferences is that they are all obtained using comonotonic acts where SRU and REU coincide in terms of $\phi \circ u$ (in Fig. 1 this is the domain below the 45-degree line). Formally, assuming either SRU or REU in Eq. (15), results in

$$\phi(u(\alpha_i)) - \phi(u(\alpha_{i-1})) = \phi(u(\alpha_{i+1})) - \phi(u(\alpha_i)) \tag{16}$$

for all $i = 1, \ldots, l$, showing that all potential bias captured through non-additive probabilities is factored out. After normalization of the utility scale ($\phi(u(\alpha_0)) = 0$

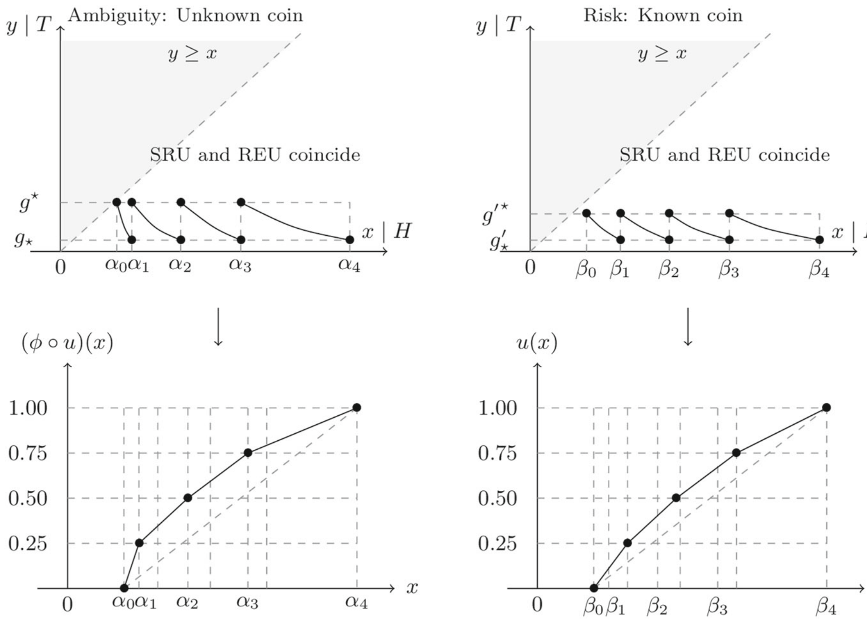

Fig. 1 Capturing ambiguity via utility under REU and SRU

and $\phi(u(\alpha_l)) = 1$), we obtain the utility curve, e.g., as illustrated in Fig. 1, bottom left panel.

The elicitation of utility u under risk proceeds in similar steps to the elicitation of $\phi \circ u$. Now one derives indifferences using a fair coin, i.e., the probability of either heads or tails is known to be equal to 1/2. We fix an outcome β_0 and two gauge outcomes, say $\beta_0 > g'^* > g'_* > 0$, and find an increasing sequence of outcomes $\beta_i, i = 1, \ldots, l$, such that $(\beta_{i-1})_H g'^* \sim (\beta_i)_H g'_*$. These indifferences are illustrated in Fig. 1 (top right panel, risk). Substitution of SRU or REU into these indifferences results in the equations

$$u(\beta_i) - u(\beta_{i-1}) = u(\beta_{i+1}) - u(\beta_i) \tag{17}$$

for $i = 1, \ldots, l$, where potential bias due to probability weighting is factored out. After normalization of the utility scale ($u(\beta_0) = 0$ and $u(\beta_l) = 1$), we obtain the utility curve illustrated in Fig. 1, bottom right panel.

The measurement method illustrated above can also be applied to Klibanoff, et al.'s (2005) smooth ambiguity model (see also Baillon et al. 2012). For instance, the Ellsberg-like experimental setups in Cubitt et al. (2018) and Chakravarty and Roy (2009) used to measure the smooth ambiguity model can also be used to measure ϕ through equivalent substitutions. While REU and the smooth ambiguity model have similar preference functionals, their approach to ambiguity is different. Under REU, ϕ summarizes attitudes towards ambiguity over the true state. In the smooth ambiguity model of Klibanoff, et al., however, ϕ reveals attitudes towards ambiguity over the right prior on the state space. Ambiguous beliefs are represented by a subjective probability distribution over the plausible priors. It was noted that, in general, such a probability distribution may not be directly observable from choices (see Klibanoff, et al. p. 1856).

7.2 Capturing ambiguity through non-additive probabilities

In addition to the utility-driven ambiguity component, SRU captures ambiguity via non-additive probabilities. For a simple illustration on how to quantify the latter component, assume SRU*, i.e., SRU with a linear ϕ as in Corollary 14. Let p_E be the *matching probability* of event E, defined through the indifference $x_E y \sim x_{p_E} y$. While under SEU this implies that the event E has subjective probability $P(E) = p_E$, substitution of SRU* into the above indifference gives

$$\nu(E) = w(p_E), \tag{18}$$

where ν is the capacity over events and w is the probability weighting function. Equation (18) can be used to empirically study ambiguity while using risk as a "base-line source of uncertainty." Similar to how probability distortions away from linearity are capturing attitude towards probabilistic risk, additional deviations from the probability weighting function is now attributed to ambiguity. For instance, Abdellaoui et al. (2005) elicited decision weights $\nu(E)$ with corresponding matching probabilities and derived w by fitting p_E values against the corresponding decision weights $\nu(E)$.

An alternative way of exploiting Eq. (18) is to infer properties of ν by eliciting matching probabilities and the weighting function w. Recently Dimmock et al. (2016) use matching probabilities to study ambiguity attitudes in a field study. They assume RDU for ambiguous one-stage acts with capacity ν and utility u and they separately assume RDU for lotteries with a weighting function w and the same utility u to obtain Eq. (18). Clearly, together with probabilistic sophistication, SRU* (Corollary 14) provides the natural framework to carry out their analysis. Dimmock, et al. adopt an Ellsberg-like setup in which probabilistic sophistication (Machina and Schmeidler 1992; Chew and Sagi 2006) can simply be inferred from symmetry arguments. This way they obtain a subjective probability, $P(E)$, over ambiguous events. Hence, one can write $\nu(E) = f(P(E))$, where the transformation function f is called a source function. By combining this decomposition of $\nu(E)$ with Eq. (18), Dimmock, et al., infer that the matching probability p_E captures ambiguity attitude (see Dimmock, et al., p. 10).[11] Specifically, ambiguity aversion (seeking) corresponds to the situation where p_E is below (above) $P(E)$. Dimmock et al. (2016) observe that ambiguity aversion is not the predominnt behavior for unlikely events, a finding in accordance with a number of recent studies (e.g., Baillon and Bleichrodt 2015; Chew et al. 2015).

Equation (18) can further facilitate the study of the properties of the capacity ν once the probability weighting function is known. As such, SRU provides a general foundation for Wakker's (2004) comparison of sensitivity towards ambiguity relative to sensitivity towards risk (Tversky and Fox 1995). To illustrate, assume additionally

[11] Clearly, we can write $p_E = w^{-1}(f(P(E)))$, thereby linking subjective to matching probabilities, where $w^{-1} \circ f$ is called the ambiguity function (Dimmock et al. 2016).

an event F disjoint from E with $\nu(E \cup F)$ being bounded away from 1 and $x_{E\cup F}\, y \sim x_{(p+q)}\, y$. Empirical findings suggest that people exhibit more sensitivity to increments from p to $p+q$ than they exhibit sensitivity to the improvement from E to $E \cup F$. Under SRU* this results in

$$\nu(E) = w(p) \ \& \ \nu(E \cup F) = w(p+q) \Rightarrow \nu(F) \geq w(q).$$

One interpretation of this implication is that the increment $\nu(E \cup F) - \nu(E)$, which is inferred equal to $w(p+q) - w(p)$, requires the addition of the event F that, on its own, is "heavier" in terms of ν than the weight $w(q)$. In terms of preferences this means $x_F\, y \succcurlyeq x_q\, y$ can be predicted under SRU*.

8 Conclusion

This paper offers a unified framework in which many popular models of risk and ambiguity can jointly be analyzed and compared. The theoretical and empirical study of risk and ambiguity is facilitated by invoking preference conditions based on a common substitution consistency principle. The latter has been used to derive our general SRU model, which can accommodate a large spectrum of risk and ambiguity attitudes beyond the famous EU and SEU paradoxes of Allais (1953) and Ellsberg (1961). Our general model encompasses many popular theories for risk and ambiguity as special cases. In particular, foundations for many existing utility-driven ambiguity models and event-driven ambiguity models can been obtained by appropriately tailoring our unifying substitution consistency property to corresponding domains of acts. We have demonstrated how SRU and its underlying preference principle can flexibly be used as tools for comparative analyses and for descriptive elicitation of ambiguity and risk attitudes, and have shown how the substitution consistency tools can unify the apparently competing utility-driven and event-driven approaches to the study of ambiguity.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Appendix: Proofs

Proof of Lemma 1 Fix a non-null event E, a lottery $\hat{y} = (p_1 : y_1, \ldots, p_m : y_m)$ with $y_1 \geq \cdots \geq y_m$, and an act $\mathbf{x} \in \mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)}$ for some partition $\mathcal{E} = (E_i)_{i=1}^n$ and probability tuples $\mathbf{p}^i = (p_1^i, \ldots, p_{m_i}^i)$, $i = 1, \ldots, n$. Consider the act $\hat{y}_E\mathbf{x}$ that gives lottery $\hat{y}$ if E obtains and $\mathbf{x}$ otherwise. Assume, without loss of generality, that both

p_1 and p_m are positive. Monotonicity implies that

$$(y_1)_E\mathbf{x} \succcurlyeq \hat{y}_E\mathbf{x} \succcurlyeq (y_m)_E\mathbf{x}.$$

Define $\tilde{\mathcal{E}} = (E, E_1 \cap E, \ldots, E_n \cap E)$ and let $\mathbf{p} = (p_1, \ldots, p_m)$ be the probability tuple for $\hat{y} \in \mathcal{L}_\mathbf{p}$. If $y_1 = y_m$, then set $CE(\hat{y}) := y_1$ and it follows, by reflexivity, that $\hat{y}_E\mathbf{x} \sim CE(\hat{y})_E\mathbf{x}$. Assume that $y_1 > y_m$ and set

$$\begin{cases} A := \{z \in \mathbb{R} : z_E\mathbf{x} \succcurlyeq \hat{y}_E\mathbf{x}\} = \{z \in \mathbb{R} : z_E\mathbf{x} \in O_A\} \\ B := \{z \in \mathbb{R} : z_E\mathbf{x} \preccurlyeq \hat{y}_E\mathbf{x}\} = \{z \in \mathbb{R} : z_E\mathbf{x} \in O_B\} \end{cases}$$

where $O_A = \{\mathbf{w} \in \mathbb{A}_{\tilde{\mathcal{E}},(\mathbf{p},\mathbf{p}^1,\ldots,\mathbf{p}^n)} : \mathbf{w} \succcurlyeq \hat{y}_E\mathbf{x}\}$ and $O_B = \{\mathbf{w} \in \mathbb{A}_{\tilde{\mathcal{E}},(\mathbf{p},\mathbf{p}^1,\ldots,\mathbf{p}^n)} : \mathbf{w} \preccurlyeq \hat{y}_E\mathbf{x}\}$ are closed sets in $\mathbb{A}_{\tilde{\mathcal{E}},(\mathbf{p},\mathbf{p}^1,\ldots,\mathbf{p}^n)}$, by continuity of $\succcurlyeq$. Further, it follows from the Lemma 0.2.1 in Wakker (1989, p. 12) that A and B are closed sets in $\mathbb{R}$. By monotonicity, A and B are non-empty (from $(y_1)_E\mathbf{x} \succcurlyeq \hat{y}_E\mathbf{x} \succcurlyeq (y_m)_E\mathbf{x}$ it follows that $(y_1)_E\mathbf{x} \in A$ and $(y_m)_E\mathbf{x} \in B$). We note that the union of A and B is $\mathbb{R}$. As the latter is a connected topological space, A and B have a non-empty intersection. Thus, there exists an outcome $CE(\hat{y}) \in A \cap B$ for which both $CE(\hat{y})_E\mathbf{x} \succcurlyeq \hat{y}_E\mathbf{x}$ and $\hat{y}_E\mathbf{x} \succcurlyeq CE(\hat{y})_E\mathbf{x}$ hold. This is equivalent to $\hat{y}_E\mathbf{x} \sim CE(\hat{y})_E\mathbf{x}$.

Next we show that $CE(\hat{y})$ is the unique outcome satisfying $\hat{y}_E\mathbf{x} \sim CE(\hat{y})_E\mathbf{x}$. Suppose, to the contrary, that there exists another outcome $CE(\hat{y})'$ such that $\hat{y}_E\mathbf{x} \sim CE(\hat{y})'_E\mathbf{x}$. Assume $CE(\hat{y})' > \bar{y}$ (the proof of the case when $CE(\hat{y})' < \bar{y}$ follows by a similar argument). Monotonicity implies that $CE(\hat{y})'_E\mathbf{x} \succ CE(\hat{y})_E\mathbf{x}$ and from $\hat{y}_E\mathbf{x} \sim CE(\hat{y})'_E\mathbf{x}$ and transitivity it follows that $\hat{y}_E\mathbf{x} \succ CE(\hat{y})_E\mathbf{x}$, a contradiction. It follows that $CE(\hat{y})$ is the unique outcome satisfying $\hat{y}_E\mathbf{x} \sim CE(\hat{y})_E\mathbf{x}$. In general, $CE(\hat{y})$ can depend on the event E and the act $\mathbf{x}$ outside this event, i.e. on $\mathbf{x}_{E^c}$. As the non-null event E was arbitrary chosen as well as the lottery $\hat{y}$ and the act $\mathbf{x}$, these conclusions apply to all non-null events, all lotteries and all acts. Next, we invoke independence of conditional CEs to exclude such dependencies.

Let E, F be arbitrary non-null events, $\hat{y}$ an arbitrary lottery and $CE(\hat{y})^E$ and $CE(\hat{y})^F$ the unique conditional CEs of $\hat{y}$ corresponding to $\hat{y}_E\mathbf{x}$ and $\hat{y}_F\mathbf{x}$, respectively. By independence of conditional CEs we have $\hat{y}_E\mathbf{x} \sim (CE(\hat{y})^E)_E\mathbf{x} \Leftrightarrow \hat{y}_F\mathbf{x} \sim (CE(\hat{y})^E)_F\mathbf{x}$ and, from the latter indifference, $\hat{y}_F\mathbf{x} \sim (CE(\hat{y})^F)_F\mathbf{x}$ and transitivity, we obtain $(CE(\hat{y})^E)_F\mathbf{x} \sim (CE(\hat{y})^F)_F\mathbf{x}$. Using the uniqueness of conditional CEs it follows that $CE(\hat{y})^E = CE(\hat{y})^F$. This means that the conditional CEs are independent of the (non-null) event used to derive them.

Subsequently, we use independence of conditional CEs to show that the CEs conditional on some event E cannot depend on the act $\mathbf{x}$ outside that event, that is, conditional CEs are independent of $\mathbf{x}_E$. Take an arbitrary non-null event E such that E^c is also non-null (such exist as we have at least two disjoint non-null events in $\mathcal{A}$), an arbitrary lottery $\hat{y}$ and two arbitrary acts act $\mathbf{x}$ and $\mathbf{z}$. Let $\mathbf{y} := \mathbf{x}_E\mathbf{z}$. Obviously, by our notation we have that $\hat{y}_E\mathbf{y} = \hat{y}_E\mathbf{z}$ and $\hat{y}_{E^c}\mathbf{y} = \hat{y}_{E^c}\mathbf{x}$. With $F = E^c$, independence of conditional CEs implies that $\hat{y}_E\mathbf{z} \sim CE(\hat{y})_E\mathbf{z} \Leftrightarrow \hat{y}_F\mathbf{x} \sim CE(\hat{y})_F\mathbf{x}$, and a second application of independence of conditional CEs gives $\hat{y}_F\mathbf{x} \sim CE(\hat{y})_F\mathbf{x} \Leftrightarrow \hat{y}_E\mathbf{x} \sim CE(\hat{y})_E\mathbf{x}$. Thus $\hat{y}_E\mathbf{z} \sim CE(\hat{y})_E\mathbf{z} \Leftrightarrow \hat{y}_E\mathbf{x} \sim CE(\hat{y})_E\mathbf{x}$ follows. For the case that E^c is null the latter equivalence follows trivially. This shows that the conditional CE of a conditional lottery is, in addition to being independent of the (non-null) event on which it is conditioned, also independent of the restriction of the act outside the event on which the lottery was conditioned.

We conclude that each lottery $\hat{y}$ has a unique conditional CE, $CE(\hat{y})$, which can be derived using an arbitrary non-null event and an arbitrary act. This completes the proof of Lemma 1 . □

As indicated in the main text, all our theorems are based on the same substitution consistency principle, with the main difference in our models resulting from the application of the consistency principle to specific domains of acts. For SRU, the rank-dependent substitution consistency applies for risk and separately for ambiguity. For REU, no rank-ordering restrictions apply. It is, therefore, convenient to derive the preference foundations for REU first, followed by those of SRU. Subsequently, the derivations of SEU and SRU* are provided.

Proof of Theorem 7 First we assume statement (i) and prove statement (ii). Weak order follows from the fact that REU represents the preference on $\mathbb{A}$. Continuity and monotonicity of the preference relation follow from continuity and strict increasingness of ϕ and of u and the fact that each non-null event has a positive subjective probability. That REU implies substitution consistency for ambiguity and substitution consistency for risk follows from the analysis in the main text in Sect. 4.2. Thus, statement (ii) has been derived.

Next we assume statement (ii) and derive statement (i). The proof involves in several steps, which we briefly describe before proceeding. First, we show that the standard preference conditions of Lemma 1 imply the existence of a general representation $\mathcal{V}$ of $\succcurlyeq$ on $\mathbb{A}$. Second, we show that substitution consistency for ambiguity implies that $\mathcal{V}$ is an SEU functional for one-stage acts (i.e., acts that give a degenerate lottery in each event) that can be extended to represent preference over all acts. Third, using substitution consistency for risk we show that for each non-null event in $\mathcal{A}$, the restriction of $\succcurlyeq$ to lotteries conditional on that event is represented by the same EU functional, which then allows for the preference on $\mathbb{A}$ to be represented by REU. Finally, uniqueness results are established.

Step 1 Recall that $\mathcal{S}$ is a finite set of (at least two) *states* and that there exists at least two disjoint non-null events in $\mathcal{A}$. The preference $\succcurlyeq$ satisfies weak order, monotonicity and continuity . By repeated applications of Debreu (1954), it follows that there exist an ordinal representing functional $\mathcal{V}$ for $\succcurlyeq$ on $\mathbb{A}$ that is continuous and strongly monotonic. As mentioned in the main text, Debreu's result is applies to each "finite dimensional" subset of acts for a fixed partition and conditional lotteries using fixed probability tuples, i.e, for each set $\mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)}$, $\mathcal{E}=(E_i)_{i=1}^n$. As two distinct arbitrary sets of acts, $\mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)}$ and $\mathbb{A}_{\mathcal{E}',(\mathbf{q}^1,\ldots,\mathbf{q}^{n'})}$ are subsets of a refined partition where the probability tuples of conditional lotteries are also refined and where Debreu's result holds, the separate representations on $\mathbb{A}_{\mathcal{E},(\mathbf{p}^1,\ldots,\mathbf{p}^n)}$ and $\mathbb{A}_{\mathcal{E}',(\mathbf{q}^1,\ldots,\mathbf{q}^{n'})}$ are restrictions of a more general representation.[12] This means that a representation $\mathcal{V}$ of the preference over all

[12] In particular, this argument applies for $\mathcal{E}=(E_i)_{i=1}^n=\mathcal{E}'$, when for each event E_i the set of lotteries $\mathcal{L}_{\mathbf{p}^i}$ and $\mathcal{L}_{\mathbf{q}^i}$ are from a common subset $\mathcal{L}_{\mathbf{r}^i}$ (e.g., if $\mathbf{p}^i=(p_1^i,\ldots,p_{m^i}^i)$ and $\mathbf{q}^i=(q_1^i,\ldots,q_{l^i}^i)$, define $\mathbf{r}^i$

acts exists. It is well-known that this representation is continuously ordinal, that is, it is unique up to continuous strictly monotonic transformations.

Let $\hat{x}$ be an arbitrary lottery, $\mathbf{y} \in \mathbb{A}$ and $E \in \mathcal{A}$ an arbitrary non-null event. Consider the act $\hat{x}_E\mathbf{y}$. By independence of conditional CEs in conjunction with Lemma 1 we know that the conditional CE, $CE(\hat{x})$, of the lottery $\hat{x}$ is independent of the non-null event E and the act $\mathbf{y}$. This means that conditional CEs can be used to represent the preference over lotteries. By defining $\mathcal{V}(E_1 : \hat{x}, \ldots, E_n : \hat{x}) := \mathcal{V}(E_1 : CE(\hat{x}), \ldots, E_n : CE(\hat{x}))$, for an arbitrary partition $\mathcal{E} = (E_i)_{i=1}^n$, we obtain a representation of the preference over lotteries through $\mathcal{V}$.

Step 2 Let $\mathbb{A}^{one}$ denote the set of one-stage acts, and for $\mathcal{E} = (E_i)_{i=1}^n$, $\mathbb{A}_{\mathcal{E}}^{one} := \mathbb{A}^{one} \cap \mathbb{A}_{\mathcal{E}}$. On $\mathbb{A}^{one}$ the preference relation $\succcurlyeq$ satisfies weak order, monotonicity and continuity, inherited from the preference $\succcurlyeq$ on $\mathbb{A}$, and substitution consistency for ambiguity. The latter comes down to Köbberling and Wakker's (2003) tradeoff consistency for $\succcurlyeq$ on $\mathbb{A}^{one}$. As we assumed that there exist at least two disjoint non-null events, it follows from Corollary 10 of Köbberling and Wakker (2003) that on $\mathbb{A}^{one}$ the preference relation $\succcurlyeq$ is represented by SEU, i.e., there exists a unique probability measure $\nu : \mathcal{A} \to [0, 1]$ and a (cardinal) strictly increasing and continuous function $V : \mathbb{R} \to \mathbb{R}$ such that each one-stage act $\mathbf{x} \in \mathbb{A}_{\mathcal{E}}^{one}$, is evaluated by

$$\sum_{i=1}^{n} \pi_i V(x_i),$$

where $\pi_i = \nu(E_i)$, $i = 1, \ldots, n$.

Recall that, by the analysis in Step 1, each act $\mathbf{y} \in \mathbb{A}_{\mathcal{E}}$, is indifferent to a one-stage act $(E_1 : CE(\hat{y}_1), \ldots, E_n : CE(\hat{y}_n))$ obtained by replacing all lotteries $\hat{y}_i$ by their conditional CEs $CE(\hat{y}_i)$, $i = 1, \ldots, n$. This allows us to extend the SEU representation of $\succcurlyeq$ on $\mathbb{A}^{one}$ to a SEU-like representation of $\succcurlyeq$ on all acts through the value of one-stage acts determined by the corresponding conditional CEs, i.e., we can assume that the representation obtained in Step 1 can be written as an SEU-functional, as follows: act $\mathbf{y} \in \mathbb{A}_{\mathcal{E}}$, is first evaluated by

$$\mathcal{V}(\mathbf{y}) = \mathcal{V}(E_1 : CE(\hat{y}_1), \ldots, E_n : CE(\hat{y}_n)),$$

which follows from $\mathbf{y} \sim (E_1 : CE(\hat{y}_1), \ldots, E_n : CE(\hat{y}_n))$. Further, by the analysis presented above in this step, we obtain

$$\mathcal{V}(E_1 : CE(\hat{y}_1), \ldots, E_n : CE(\hat{y}_n)) = \sum_{i=1}^{n} \pi_i V(CE(\hat{y}_i)).$$

Hence, we have

$$\mathcal{V}(\mathbf{y}) = \sum_{i=1}^{n} \pi_i V(CE(\hat{y}_i)). \tag{19}$$

such that $r^i_{jk} = p^i_j q^i_k$, for $j = 1, \ldots, m^i$ and $k = 1, \ldots, l^i$) for each $i = 1 \ldots, n$, showing that a general representation for all acts in $\mathbb{A}_{\mathcal{E}}$exists.

Henceforth in this proof, we can assume that the functional in Eq. (19) is representing $\succcurlyeq$ on $\mathbb{A}$ with a cardinal function V that is independent of events in the partition $\mathcal{E} = (E_i)_{i=1}^n$ and also independent of the partition $\mathcal{E}$.

Step 3 Because the function V in Eq. (19) is independent the events, it represents, like $\mathcal{V}$ does, the preference over lotteries. Further, $\succcurlyeq$ satisfies substitution consistency for risk. Consider the restriction of the preference $\succcurlyeq$ to lotteries conditioned on an arbitrary non-null event E defined through

$$\hat{x} \succcurlyeq \hat{y} \Leftrightarrow \hat{x}_E\mathbf{z} \succcurlyeq \hat{y}_E\mathbf{z}$$

for all $\hat{x}, \hat{y} \in \mathcal{L}$ and some $\mathbf{z} \in \mathbb{A}$. By Step 1 this is well-defined as substitution of $\mathcal{V}(\cdot)$ and cancellation of common terms gives

$$\hat{x} \succcurlyeq \hat{y} \Leftrightarrow V(CE(\hat{x})) \geq V(CE(\hat{y}))$$

for all $\hat{x}, \hat{y} \in \mathcal{L}$. Further, this restricted preference satisfies weak order, monotonicity, continuity and substitution consistency for risk. The latter is equivalent to tradeoff consistency of Köbberling and Wakker (2003) for lotteries. Using the remarks in their Section 5.3 in combination with Corollary 10 of Köbberling and Wakker (2003) it follows that EU represents the preference restricted to lotteries. Therefore, we conclude that lotteries obtained on an arbitrary non-null event $E \in \mathcal{A}$ are represented by the same EU functional, $EU(\cdot)$, with utility u independent of E. Hence, V is a strictly increasing and continuous transformation of a common expected utility functional. That is, there exists a strictly increasing and continuous functions $\phi : u(\mathbb{R}) \to \mathbb{R}$ such that $V(CE(\hat{x})) = \phi[EU(\hat{x})]$ for each lottery $\hat{x} \in \mathcal{L}$.

Note that, by construction, the function ϕ is cardinal and if u is replaced by $au + b$ for $a > 0$ and real b (which is the only freedom for choosing u) we need to replace $\phi(\cdot)$ by $\phi((\cdot - b)/a)$ to ensure that the range of the latter two functions remains unchanged. There is no further flexibility in the choice of ϕ and u. Hence REU holds for $\succcurlyeq$ on $\mathbb{A}$.

Combining Steps 1– 3 completes the derivation of statement (ii) from statement (i) in Theorem 7. Uniqueness results follow from the construction in Steps 1– 3 as highlighted at the end of Step 3. This completes the proof of Theorem 7. □

Proof of Theorem 12 That statement (i) implies statement (ii) follows similarly to the corresponding derivation in the proof of Theorem 7, in conjunction with the analysis in the main text in Sects. 4.2 and 4.3; independence of conditional CE's follows from the fact that in SRU the functions ϕ and u are independent of events. Next we assume statement (ii) and derive statement (i). We proceed in several steps similar to the proof of Theorem 7.

Step 1 This step is identical to Step 1 in the proof of Theorem 7. As the preference $\succcurlyeq$ satisfies weak order, monotonicity and continuity, this implies the existence of a general representation $\mathcal{V}$ of $\succcurlyeq$ on $\mathbb{A}$ that is strictly monotone and continuous and represents the preference over lotteries.

Step 2 We show that rank-dependent substitution consistency for ambiguity implies that $\mathcal{V}$ is a SRU functional for one-stage acts. Recall that $\mathbb{A}^{one}$ denotes the set of one-stage acts. On $\mathbb{A}^{one}$ the preference relation $\succcurlyeq$ is a monotonic continuous weak order that satisfies the rank-dependent substitution consistency for ambiguity. The latter comes down to Köbberling and Wakker's (2003) comonotonic tradeoff consistency of $\succcurlyeq$ on $\mathbb{A}^{one}$. We can apply Corollary 10 of Köbberling and Wakker (2003) to derive an SRU-like representation for $\succcurlyeq$ on $\mathbb{A}^{one}$, that is, there exists a unique strictly monotone capacity, ν, for events and a (cardinal) strictly increasing and continuous function $V : \mathbb{R} \to \mathbb{R}$ such that a one-stage act $\mathbf{x} \in \mathbb{A}^{\downarrow}_{\mathcal{E}}$, $\mathcal{E} = (E_i)_{i=1}^n$, is evaluated by

$$\sum_{i=1}^{n} \pi_i V(x_i),$$

where $\pi_i = \nu(\cup_{k=1}^{i} E_k) - \nu(\cup_{k=1}^{i-1} E_k)$, $i = 1, \ldots, n$, and we set $\nu(\cup_{k=1}^{0} E_k) := \nu(\emptyset) = 0$, as usual. As ν is strictly monotone, it follows that the decision weights are positive whenever E_i is non-null and that they sum to one. Because this representation only applies to one-stage acts it is also referred to as RDU for ambiguity (Wakker 2010).

It now follows, similar to the proof of Step 2 in Theorem 7, that $\mathcal{V}$ can be extended to represent the preference on all acts through the replacement of lotteries by their conditional CE's.

Step 3 Similar to Step 3 in the proof of Theorem 7, now exploiting that rank-dependent substitution consistency for risk comes down to comonotonic tradeoff consistency of Köbberling and Wakker (2003) for lotteries, it follows that, first lotteries are evaluated by RDU for risk as in Eq. (1) with a uniquely defined weighting function w and cardinal utility u. Thus, for all (non-null) events the conditional lotteries are evaluated by a common RDU-representation over lotteries. That is, there exists a strictly increasing and continuous function $\phi : u(\mathbb{R}) \to \mathbb{R}$ such that $V(CE(\hat{x})) = \phi[RDU(\hat{x})]$ for all lotteries $\hat{x} \in \mathcal{L}^{\downarrow}$. Hence, SRU holds for $\succcurlyeq$ on $\mathbb{A}$.

These steps complete the derivation of statement (ii) from statement (i) in Theorem 12. Uniqueness results for u and ϕ are as in Step 3 in the proof of Theorem 7. This concludes the proof of Theorem 12. □

Proof of Corollary 14 As noted in the main text, rank-dependent substitution consistency implies both rank-dependent substitution consistency for ambiguity and rank-dependent substitution consistency for risk. Hence, Theorem 12 holds and it only remains to show that ϕ is linear. Take an arbitrary non-null event $E \in \mathcal{A}$ included in an arbitrary partition $\mathcal{E}$. Locally, for arbitrary $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ ($\beta \neq \alpha$) there exists acts $\mathbf{x}, \mathbf{y}$ such that $\alpha_E\mathbf{x} \sim \beta_E\mathbf{y}$ and $\gamma_E\mathbf{x} \sim \delta_E\mathbf{y}$ and the acts in these indifferences are from the same set $\mathbb{A}^{\downarrow}_{\mathcal{E}}$. By weak order, continuity and monotonicity, one can always find such acts, whenever, $\alpha, \beta, \gamma, \delta$ are sufficiently close. Further, there exists lotteries $\hat{x}, \hat{y} \in \mathcal{L}^{\downarrow}_{\mathbf{p}}$, $0 < p_1 < 1$, and an act $\mathbf{z}$, such that $(\alpha_1\hat{x})_E\mathbf{z} \sim (\beta_1\hat{y})_E\mathbf{z}$ and $(\gamma_1\hat{x})_E\mathbf{z} \sim (\delta_1\hat{y})_E\mathbf{z}$ with all acts being from $\mathbb{A}^{\downarrow}_{\mathcal{E}|E,\mathbf{p}}$. Again, locally, the existence of such lotteries and acts are ensured by weak order, continuity and monotonicity. By rank-dependent substitution consistency, any of the former three indifferences imply the fourth, and, as shown in the main text, substitution of SRU into the first pair of indifferences implies $\phi[u(\alpha)] - \phi[u(\beta)] = \phi[u(\gamma)] - \phi[u(\delta)]$ and substitution of SRU into the second pair of indifferences implies $u(\alpha) - u(\beta) = u(\gamma) - u(\delta)$. Thus, locally, for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ the first equation holds if and only if the second one holds too. This means that, locally, ϕ must be linear. From local linearity, global linearity of ϕ follows. This concludes the proof of Corollary 14. □

Proof of Theorem 4 The proof is similar to that of Corollary 14, except that one need not account for the rank-ordering of outcomes or conditional CE's of lotteries and, therefore, invoke Theorem 7 instead of Theorem 12. This completes the proof of Corollary 4. □

References

Abdellaoui, M.: A genuine rank-dependent generalization of the von Neumann-Morgenstern expected utility theorem. Econometrica **70**(2), 717–736 (2002)

Abdellaoui, M., Vossmann, F., Weber, M.: Choice-based elicitation and decomposition of decision weights for gains and losses under uncertainty. Manag. Sci. **51**(9), 1384–1399 (2005)

Abdellaoui, M., Klibanoff, P., Placido, L.: Experiments on compound risk in relation to simple risk and to ambiguity. Manag. Sci. **61**(6), 1306–1322 (2015)

Allais, M.: Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'ecole américaine. Econometrica **21**(4), 503–546 (1953)

Alon, S.: Derivation of a cardinal utility through a weak trade-off consistency requirement. Math. Oper. Res. **39**(2), 290–300 (2014)

Alon, S., Schmeidler, D.: Purely subjective maxmin expected utility. J. Econ. Theory **152**, 382–412 (2014)

Anscombe, F.J., Aumann, R.J.: A definition of subjective probability. Ann. Math. Stat. **34**, 199–205 (1963)

Baillon, A., Bleichrodt, H.: Testing ambiguity models through the measurement of probabilities for gains and losses. Am. Econ. J. Microecon. **7**(2), 77–100 (2015)

Discrete Arrow-Pratt indexes for risk and uncertainty: Baillon, A., l'Haridon, O. Econ. Theory **72**, 1375–1393 (2021)

Baillon, A., Driesen, B., Wakker, P.P.: Relative concave utility for risk and ambiguity. Games Econ. Behav. **75**, 481–489 (2012)

Baillon, A., Huang, Z., Selim, A., Wakker, P.P.: Measuring ambiguity attitudes for all (natural) events. Econometrica **86**(5), 1839–1858 (2018)

Bell, D.E.: Disappointment in decision making under uncertainty. Oper. Res. **33**, 1–27 (1985)

Birnbaum, M.H., Stegner, S.E.: Measuring the importance of cues in judgment for individuals: subjective theories of IQ as a function of heredity and environment. J. Exp. Soc. Psychol. **17**, 159–182 (1981)

Bommier, A.: A dual approach to ambiguity aversion. J. Math. Econ. **71**, 104–118 (2017)

Borah, A., Kops, C.: The Anscombe-Aumann representation and the independence axiom: a reconsideration. Theor. Decis. **80**(2), 211–226 (2016)

Bruhin, A., Fehr-Duda, H., Epper, T.: Risk and rationality: uncovering heterogeneity in probability distortion. Econometrica **78**(4), 1375–1412 (2010)

Cappelli, V.R., Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., Minardi, S.: Sources of uncertainty and subjective prices. J. Europ. Econ. Assoc. **19**(2), 872–912 (2021)

Chakravarty, S., Roy, J.: Recursive expected utility and the separation of attitudes towards risk and ambiguity: an experimental study. Theor. Decis. **66**, 199–228 (2009)

Chateauneuf, A., Maccheroni, F., Zank, H.: On the separation of utility and beliefs. Mimeo (2021)

Chateauneuf, A., Eichberger, J., Grant, S.: Choice under uncertainty with the best and worst in mind: NEO-additive capacities. J. Econ. Theory **137**, 538–567 (2007)

Chateauneuf, A., Wakker, P.P.: An axiomatization of cumulative prospect theory for decision under risk. J Risk Uncertainty **18**, 137–145 (1999)

Cheridito, P., Delbaen, F., Drapeau, S., Kupper, M.: Stochastic order-monotone uncertainty-averse preferences.(2015). https://doi.org/10.2139/ssrn.2572745

Chew, S.H., Miao, B., Zhong, S.: Partial ambiguity. Working paper, National University of Singapore (2015)

Chew, S.H., Sagi, J.S.: Event exchangeability: probabilistic sophistication without continuity or monotonicity. Econometrica **74**(3), 771–786 (2006)

Chew, S.H., Sagi, J.S.: Small worlds: modeling attitudes toward sources of uncertainty. J. Econ. Theory **139**, 1–24 (2008)

Chew, S.H., Wakker, P.P.: The comonotonic sure-thing principle. J. Risk Uncertain. **12**, 5–27 (1996)

Cohen, M.: Security level, potential level, expected utility: a three criteria decision model under risk. Theory Decis. **33**, 101–134 (1992)
Cohen, M., Jaffray, J.-Y.: Certainty effect versus probability distortion: an experimental analysis of decision making under risk. J. Exp. Psychol. Hum. Percept. Perform. **14**, 554–560 (1988)
Cubitt, R., van de Kuilen, G., Mukerji, S.: The strength of sensitivity to ambiguity. Theory Decis. **85**, 275–302 (2018)
Cubitt, R., van de Kuilen, G., Mukerji, S.: Discriminating between models of ambiguity attitude: a qualitative test. J. Eur. Econ. Assoc. **18**(2), 708–749 (2020)
Dean, M., Ortoleva, P.: Allais, Ellsberg, and preferences for hedging. Theor. Econ. **12**, 377–424 (2017)
Debreu, G.: Representation of a preference ordering by a numerical function. In: Thrall, R.M., Coombs, C.H., Davis, R.L. (eds.) Decis. Process., pp. 159–165. Wiley, New York (1954)
Debreu, G.: Theory of Value. An Axiomatic Analysis of Economic Equilibrium. Wiley, New York (1959)
Diecidue, E., Somasundaram, J.: Regret theory: a new foundation. J. Econ. Theory **172**, 88–119 (2017)
Dimmock, S.G., Kouwenberg, R., Wakker, P.P.: "Ambiguity attitudes in a large representative sample: measurement and an application to the non-participation puzzle. Manag. Sci. **62**(5), 1363–1380 (2016)
Dobbs, I.M.: A Bayesian approach to decision-making under ambiguity. Economica **58**, 417–440 (1991)
Ellsberg, D.: Risk, ambiguity and the Savage axioms. Q. J. Econ. **75**(4), 643–669 (1961)
Ergin, H., Gul, F.: A theory of subjective compound lotteries. J. Econ. Theory **144**, 899–929 (2009)
Fox, C.R., Tversky, A.: Ambiguity aversion and comparative ignorance. Q. J. Econ. **110**(3), 585–603 (1995)
Ghirardato, P., Marinacci, M.: Risk, ambiguity, and the separation of utility and beliefs. Math. Oper. Res. **26**(4), 864–890 (2001)
Gilboa, I.: Expected utility with purely subjective non-additive probabilities. J. Math. Econ. **16**, 65–88 (1987)
Gilboa, I.: A combination of expected utility theory and maxmin decision criteria. J. Math. Psychol. **32**, 405–420 (1988)
Gilboa, I., Schmeidler, D.: Maxmin expected utility with a non-unique prior. J. Math. Econ. **18**, 141–153 (1989)
Gilboa, I., Schmeidler, D., Wakker, P.P.: Utility in case-based decision theory. J. Econ. Theory **105**(2), 483–502 (2002)
Gorman, W.M.: The structure of utility functions. Rev. Econ. Stud. **35**, 367–390 (1968)
Grant, S., Polak, B., Strzalecki, T.: Second-order expected utility. Mimeo (2009)
Gumen, A., Ok, E., Savochkin, A.: Decision-making under subjective risk: toward a general theory of pessimism. Mimeo (2014)
He, Y.: Revisiting Ellsberg's and Machina's paradoxes: a two-stage evaluation model under ambiguity. Manag. Sci. forthcoming (2021)
Kahneman, D., Tversky, A.: Prospect theory: an analysis of decision under risk. Econometrica **47**(2), 263–291 (1979)
Karni, E., Safra, Z.: Rank-dependent probabilities. Econ. J. **100**, 487–495 (1990)
Kilka, M., Weber, M.: What determines the shape of the probability weighting function under uncertainty. Manag. Sci. **47**(12), 1712–1726 (2001)
Klibanoff, P., Marinacci, M., Mukerji, S.: A smooth model of decision making under ambiguity. Econometrica **73**(6), 1849–1892 (2005)
Köbberling, V., Wakker, P.P.: Preference foundations for nonexpected utility: a generalized and simplified technique. Math. Oper. Res. **28**(3), 395–423 (2003)
Kübler, F., Selden, L., Wei, X.: What are asset demand tests of expected utility really testing? Econ. J. **127**, 784–808 (2017)
Lopes, L.L.: What naive decision makers can tell us about risk. In: Daboni, L., Montesano, A., Lines, M. (eds.) Recent Developments in the Foundations of Utility and Risk Theory, pp. 311–326. The Netherlands, Reidel, Dordrecht (1986)
Maccheroni, F.: Yaari's dual theory without the completeness axiom. Econ. Theory **23**, 701–714 (2004)
Machina, M.J.: Risk, ambiguity, and the rank-dependence axioms. Am. Econ. Rev. **99**(1), 385–392 (2009)
Machina, M.J.: Event-separability in the Ellsberg urn. Econ. Theor. **48**, 425–436 (2011)
Machina, M.J.: Ambiguity aversion with three or more outcomes. Am. Econ. Rev. **104**(12), 3814–3840 (2014)
Machina, M.J., Schmeidler, D.: A more robust definition of subjective probability. Econometrica **60**(4), 745–780 (1992)

Mongin, P., Pivato, M.: Ranking multidimentional alternatives and uncertain prospects. J. Econ. Theory **157**, 146–171 (2015)
Nau, R.F.: Uncertainty aversion with second-order utilities and probabilities. Manag. Sci. **52**(1), 136–145 (2006)
Neilson, W.S.: A simplified axiomatic approach to ambiguity aversion. J. Risk Uncertain. **41**, 113–124 (2010)
Prelec, D.: The probability weighting function. Econometrica **66**(3), 497–527 (1998)
Quiggin, J.: A theory of anticipated utility. J. Econ. Behav. Organ. **3**, 323–343 (1982)
Safra, Z., Segal, U.: Constant risk aversion. J. Econ. Theory **83**, 19–42 (1998)
Savage, L.J.: The Foundations of Statistics. Wiley, New York (1954)
Schmeidler, D.: Subjective probability and expected utility without additivity. Econometrica **57**(3), 571–587 (1989)
Segal, U.: Some remarks on Quiggin's anticipated utility. J. Econ. Behav. Organ. **8**, 145–154 (1987)
Segal, U.: The Ellsberg paradox and risk aversion: an anticipated utility approach. Int. Econ. Rev. **28**, 175–202 (1987)
Starmer, C.: Developments in non-expected utility theory: the hunt for a descriptive theory of choice under risk. J. Econ. Lit. **38**, 332–382 (2000)
Trautmann, S., Wakker, P.P.: Making the Anscombe-Aumann approach to ambiguity suitable for descriptive applications. J. Risk Uncertain. **56**, 83–116 (2018)
Tversky, A., Fox, C.R.: Weighing risk and uncertainty. Psychol. Rev. **102**, 269–283 (1995)
Tversky, A., Kahneman, D.: Advances in prospect theory: cumulative representation of uncertainty. J. Risk Uncertain. **5**, 297–323 (1992)
Tversky, A., Wakker, P.P.: Risk attitudes and decision weights. Econometrica **63**(6), 1255–1280 (1995)
von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior. Princeton University Press, Princeton NJ (1944, 1947, 1953)
Wakker, P.P.: Additive Representations of Preferences. A New Foundation of Decision Analysis. Kluwer Academic Publishers, Dordrecht (1989)
Wakker, P.P.: Separating marginal utility and probabilistic risk aversion. Theory Decis. **36**, 1–44 (1994)
Wakker, P.P.: Testing and characterizing proprieties of nonadditive measures through violations of the sure thing principle. Econometrica **69**(4), 1039–1059 (2001)
Wakker, P.P.: On the composition of risk preference and belief. Psychol. Rev. **111**, 236–241 (2004)
Wakker, P.P.: Prospect Theory: For Risk and Ambiguity. Cambridge University Press, Cambridge (2010)
Wakker, P.P., Tversky, A.: An axiomatization of cumulative prospect theory. J. Risk Uncertain. **7**, 147–176 (1993)
Wakker, P.P., Zank, H.: A unified derivation of classical subjective expected utility models through cardinal utility. J. Math. Econ. **32**, 1–19 (1999)
Webb, C.S., Zank, H.: Accounting for optimism and pessimism in expected utility. J. Math. Econ. **47**, 706–717 (2011)
Werner, J.: A simple axiomatization of risk-averse expected utility. Econom. Letters **88**, 73–77 (2005)
Werner, K.M., Zank, H.: A revealed reference point for prospect theory. Econ. Theory **67**, 731–773 (2019)
Yaari, M.: The dual theory of choice under risk. Econometrica **55**, 95–116 (1987)
Zank, H.: Consistent probability attitudes. Econ. Theory **44**(2), 167–185 (2010)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.