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RESEARCH ARTICLE

Submodular financial markets with frictions

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Abstract

This paper studies arbitrage-free financial markets with bid-ask spreads whose super-hedging prices are submodular. The submodular assumption on the super-hedging price, or the supermodularity usually assumed on utility functions, is the formal expression of perfect complementarity, which dates back to Fisher, Pareto, and Edgeworth, according to Samuelson (J Econ Lit 12:1255–1289, 1974). Our main contribution provides several characterizations of financial markets with frictions that are submodular as a consequence of a more general study of submodular pricing rules. First, a market is submodular if and only if its super-hedging price is a Choquet integral and if and only if its set of risk-neutral probabilities is representable as the core of a submodular non-additive probability that is uniquely defined, called risk-neutral capacity. Second, a market is representable by its risk neutral capacity if and only if it is equivalent to a market, only composed of bid-ask event securities.

Keywords Submodularity · financial markets · Frictions · Bid-ask · Arbitrage · Multi-prior model · Super-hedging price · Super-replication · Risk measure · Pricing rules · Choquet integral · Event securities

JEL Classification D81 · G11 · G12

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1 Introduction

The submodular assumption on the price/cost function, or the supermodularity usually assumed on utility functions, has long been regarded by economists as the formal expression of perfect complementarity. According to Samuelson (1974), the use of supermodularity as a notion of complementarity dates back to Fisher, Pareto, and Edgeworth. Seminal papers on supermodularity and lattice programming in economics are Topkis (1978), Milgrom and Shannon (1994), Vives (1990), Milgrom and Roberts (1990), and a general reference is the book by Topkis (1998).

Financial markets whose bond is frictionless provide a natural rationale for the famous multi-prior model by Gilboa and Schmeidler (1989), extensively used in this paper in its "dual" form. Actually, the Fundamental Theorem of Asset Pricing states that the absence of arbitrage opportunities in the financial market exhibits a risk-free interest rate r, together with a family $\mathcal{P}$ of risk-neutral probabilities, consistent with the bid-ask prices of market securities at date $t = 0$ and the prevalent uncertainty at date $t = 1$. Furthermore, valuing at $t = 0$ a payoff, through the super-hedging price, as the supremum of discounted expectations with respect to the probabilities in the family $\mathcal{P}$, is exactly in the line of the multi-prior model.

Due to the important evidence for the existence of bid-ask, Amihud and Mendelson (1986), the finance literature has developed models that incorporate transactions costs, short sales, and taxes: Luttmer (1996), Prisman (1986), Ross (1987), Bensaid et al. (1992), and general equilibrium models with frictions have also been developed by Cass (2006), Siconolfi (1989), Angeloni and Cornet (2006), Cornet and Gopalan (2010), Cornet and Ranjan (2013), Aouani and Cornet (2009, 2011, 2016, 2017), Markeprand (2008), Martins-da-Rocha and Vailakis (2010), Flam (2020) and Bejan (2020). On the other hand, Jouini and Kallal (1995, 2001) proved the Fundamental Theorem of Asset Pricing for financial markets with bid-ask spreads, extending the theory developed in the frictionless case, first by Ross (1976, 1978) and Cox and Ross (1976), then in a dynamic setting by Harrison and Kreps (1979), Duffie and Huang (1986) and Delbaen and Schachermayer (1994).

The general framework of this paper considers arbitrage-free financial markets M with frictions represented by bid-ask, whose bond is frictionless; the market is said to be submodular whenever its super-hedging price is submodular. Our first contribution (Theorem 2) is two-fold. First, the market M is submodular if and only if its super-hedging price is a Choquet integral with respect to a capacity, which is uniquely defined by M and called the risk-neutral capacity of the market. This allows to have a tractable formula for the super-hedging price, which generalizes the formula expressing the cost of a payoff as a discounted expectation of it, in the frictionless case. Second, the market M is submodular if and only if it is equivalent to a market containing only bid-ask event securities, together with the frictionless bond, and whose risk-neutral capacity is submodular. A bid-ask event security is defined as a security paying either 0 or 1 in the future; here two markets are said to be equivalent if they have the same risk-free interest rate and the same set of risk-neutral probabilities.

Seminal papers introducing and using Choquet integral in economics and decision theory are Schmeidler (1986, 1989) and Gilboa (1987) and a presentation of the theory can be found in Denneberg (1994) and Marinacci and Montrucchio (2004). We will now discuss the related papers that price securities via the use of a Choquet integral. Chateauneuf et al. (1996) show that a pricing rule, which is a Choquet integral with respect to a submodular capacity, allows to explain why, when introduced on the market, puts had a price smaller than the replicated payoff which can be obtained with a corresponding call and security. For markets defined axiomatically by their pricing rules, Cerreia-Vioglio et al. (2015) show that another version of the parity, called Put-Call Parity, characterizes a pricing rule which is a discounted Choquet integral. In a different spirit, Araujo et al. (2012) are mainly concerned with the characterization of the super-replication cost of arbitrage-free frictionless markets on tradable assets, and, in Araujo et al. (2018), they show that efficient complete markets with bid-ask spread are the prevalent case for finite financial markets.

The paper is organized as follows. Section 2 presents the model of a financial market with securities with bid-ask spread in the simplest tractable setting of a two-date stochastic model. The super-hedging price of an arbitrage-free market with bid-ask spread is also presented axiomatically as a pricing rule satisfying positive homogeneity, monotonicity, subaditivity, together with the fact that the bond is frictionless. An equivalent formulation can also be given in terms of coherent risk measures; see Artzner et al. (1999). In essence both formulations are also related to the multi-prior model of Gilboa and Schmeidler (1989).

Section 3 is devoted to the characterization of submodular markets. We recall the definition of submodularity, its equivalence with the so-called "net-cost decreasing property," and we also introduce a weaker form of submodularity, with financial interpretations of these notions. Our main result (Theorem 2) states that the market M is submodular if and only if its super-hedging price is a Choquet integral and if and only if its set of stochastic discount factors is equal to the core of its risk-neutral capacity $\overline{v}_M$, together with the submodularity of $\overline{v}_M$ and if and only if M is equivalent to a market with event securities with bid-ask spread, together with the submodularity of $\overline{v}_M$. In fact, this latter equivalence does not rely on any submodularity assumption (Proposition 5). Finally, we provide some basic examples of submodular markets, namely, perfect complementarity, perfect substituability, and the so-called class of ε-contamination, whose set of risk-neutral probabilities is the ε-contamination of a (given) probability, a basic tool in robustness theory (see Berger 1985; Huber 1981).

The last Sect. 4 provides the proof of our main result (Theorem 2) as a consequence of a more general formulation (Theorem 3) with pricing rules. It appears that such submodular pricing rules always derive from financial markets and are the super-hedging prices of submodular markets with only event securities.

2 Financial markets with securities with bid-ask spread

2.1 General definitions

This paper[1] considers a basic tractable two-date stochastic model, where $t = 0$ (today) is known and $t = 1$ (tomorrow) is uncertain. In the whole paper, Ω is a *finite set*, called

[1] We recall some notations used throughout the paper. Let Ω be a finite set, we let $\mathbb{R}^\Omega$ be the vector space of functions $x : \Omega \to \mathbb{R}$. We say that $x' \geq x$ (resp. $x' > x$, resp. $x' \gg x$) if, for all $\omega \in \Omega$,

the state space, that represents tomorrow's uncertainty by listing the possible states that can prevail at $t = 1$, one and only one of which will be disclosed at $t = 1$.

A *financial market* M, or in short a *market*, is a collection of finitely many *securities with bid-ask spread*, all defined on the same state space Ω, indexed by $j \in \mathbf{J} := \{1, \dots, J\}$, assumed to be finite. Each security j is a contract that promises the payoff $V^j(\omega)$ (resp. $-V^j(\omega)$) at $t = 1$ if state ω prevails, for each unit bought (resp. sold) at $t = 0$ and it is paid $\overline{q}^j$ (resp. $-\underline{q}^j$) at $t = 0$. We adopt the standard sign convention for payoffs that $V^j(\omega)$ is a gain whenever positive and $|V^j(\omega)|$ is a payment whenever $V^j(\omega) < 0$. Similarly, if c is price, or later a cost, then c is a payment if $c > 0$, and $|c|$ is a gain if $c < 0$. For each security j, we denote by $V^j := (V^j(\omega)) \in \mathbb{R}^\Omega$ its vector of payoffs across states (or the random variable $V^j : \Omega \to \mathbb{R}$), simply called the payoff of security j; $\underline{q}^j \in \mathbb{R}$ is called its bid (selling) price, and $\overline{q}^j \in \mathbb{R}$ its ask (buying) price. The security j is summarized by the triple $(V^j, \underline{q}^j, \overline{q}^j)$ and the market by:

$$M = \left(V^j, \underline{q}^j, \overline{q}^j\right)_{j \in \mathbf{J}} \text{ and we let } \underline{q} := (\underline{q}^j)_{j \in \mathbf{J}}, \overline{q} := (\overline{q}^j)_{j \in \mathbf{J}}.$$

Thus, choosing the strategy $(\alpha, \beta) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+$, where $\alpha := (\alpha^j)_{j \in \mathbf{J}}$ (resp. $\beta := (\beta^j)_{j \in \mathbf{J}}$) is the list of quantities bought (resp. sold) of each asset $j \in \mathbf{J}$:

- yields the payoff $\sum_{j \in \mathbf{J}} V^j(\alpha^j - \beta^j) \in \mathbb{R}^\Omega$ across states at $t = 1$,
- in exchange of the payment of $\sum_{j \in \mathbf{J}} \overline{q}^j \alpha^j - \underline{q}^j \beta^j$ at $t = 0$.

The market M is said to be *arbitrage-free* if it is both *present arbitrage-free (PAF)* and *future arbitrage-free (FAF)*, that is, for all $(\alpha, \beta) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+$

$$\sum_{j \in \mathbf{J}} V^j(\alpha^j - \beta^j) \geq 0 \implies \sum_{j \in \mathbf{J}} \overline{q}^j \alpha^j - \underline{q}^j \beta^j \geq 0 \,[PAF],$$
$$\sum_{j \in \mathbf{J}} V^j(\alpha^j - \beta^j) > 0 \implies \sum_{j \in \mathbf{J}} \overline{q}^j \alpha^j - \underline{q}^j \beta^j > 0 \,[FAF].$$

Footnote 1 continued
$x'(\omega) \geq x(\omega)$ (resp. $x' \geq x$ and $x' \neq x$, resp. for all $\omega \in \Omega$, $x'(\omega) > x(\omega)$); moreover, $x \leq x'$ means that $x' \geq x$ and similarly for the other two relations. The lattice operations $\wedge$ and $\vee$ in $\mathbb{R}^\Omega$ are defined by $(x \wedge x')(\omega) := \min\{x(\omega), x'(\omega)\}$, $(x \vee x')(\omega) := \max\{x(\omega), x'(\omega)\}$ for all $\omega \in \Omega$. Then $\mathbb{R}^\Omega_+ := \{x \in \mathbb{R}^\Omega : x \geq 0\}$) denotes the set of non-negative functions and $\mathbb{R}^\Omega_{++} := \{x \in \mathbb{R}^\Omega : x \gg 0\}$. For $A \subseteq \Omega$, we denote by A^c the complement set of A and $\mathbf{1}_A$ the indicator (or characteristic) function of A, i.e., $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$, and $\mathbf{1}_A(\omega) = 0$ otherwise, and, by convention, $\mathbf{1}_\omega = \mathbf{1}_{\{\omega\}}$ for all ω, and $\mathbf{1}_\emptyset = 0$. When $\Omega = \{1, \dots, n\}$, we can identify $\mathbb{R}^\Omega$ with $\mathbb{R}^n$, thus a function $x : \Omega \to \mathbb{R}$ can also be viewed as the n-tuple $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. The previously defined order $\geq$ is then identified with the coordinate-wise order of $\mathbb{R}^n$, i.e., $x' = (x'_1, \dots, x'_n) \geq x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ means $x'_i \geq x_i$ for every $i = 1, \dots, n$. With the previous identification, for $A \subseteq \{1, \dots, n\}$, $\mathbf{1}_A$ will now be the vector in $\mathbb{R}^n$ such that $x_i = 1$ if $i \in A$ and $x_i = 0$ otherwise. Thus we denote by $\mathbf{1}_i := \mathbf{1}_{\{i\}}$ (resp. $\mathbf{1}_\Omega$) the vector with all coordinates equal to zero, but the i-th equal to 1 (resp. with all coordinates equal to 1) so that $x = (x_1, \dots, x_n) = x_1\mathbf{1}_1 + \dots + x_n\mathbf{1}_n$. Without any risk of confusion, we will use indifferently the same notation μ to represent the function $\mu : \Omega \to \mathbb{R}$, the vector in $\mathbb{R}^\Omega$, the associated linear function $x \to x \cdot \mu$, or the associated set-function $A \to \mu(A) := \mathbf{1}_A \cdot \mu = \sum_{\omega \in A} \mu(\omega)$ for all $A \subseteq \Omega$.

Given the market M, the payoff $x \in \mathbb{R}^\Omega$ is said to be *super-replicable* if

$$\exists (\alpha, \beta) \in \mathbb{R}_+^J \times \mathbb{R}_+^J, \sum_{j \in \mathbf{J}} V^j (\alpha^j - \beta^j) \geq x,$$

and the *super-hedging price* $c_+(x)$ of $x \in \mathbb{R}^\Omega$ is then defined by:

$$c_+(x) := \inf \left\{ \sum_{j \in \mathbf{J}} \overline{q}^j \alpha^j - \underline{q}^j \beta^j : (\alpha, \beta) \in \mathbb{R}_+^J \times \mathbb{R}_+^J, \sum_{j \in \mathbf{J}} V^j (\alpha^j - \beta^j) \geq x \right\}.$$

We notice that $c_+(x) \in [-\infty, +\infty]$, with the convention that $c_+(x) = +\infty$ if x is not super-replicable. We will see hereafter that $c_+(x)$ will not take the value $-\infty$ whenever the market is present arbitrage-free (see Theorem 1).

The payoff $x \in \mathbb{R}^\Omega$ is said to be *frictionless* if $-c_+(-x) = c_+(x)$.[2] Similarly, the j-th security of M is said to be frictionless if its payoff V^j is frictionless, and the market M is said to be frictionless if all its securities are frictionless. If the market M is arbitrage-free, we recall that $\underline{q}^j \leq -c_+(-V^j) \leq c_+(V^j) \leq \overline{q}^j$ for all $j \in \mathbf{J}$; thus, in particular the inequality $\underline{q}^j \leq \overline{q}^j$ always holds for all j. Moreover, if $\underline{q}^j = \overline{q}^j$, then the j-th security is frictionless. Finally, for the (frictionless) market $M = \left(V^j, q^j, q^j\right)_{j \in \mathbf{J}}$, the present (resp. future) arbitrage-free notion coincides with the standard one, i.e., for all $\theta \in \mathbb{R}^J$, $\sum_{j \in \mathbf{J}} V^j \theta^j \geq 0$ implies $\sum_{j \in \mathbf{J}} q^j \theta^j \geq 0$ (resp. $\sum_{j \in \mathbf{J}} V^j \theta^j > 0$ implies $\sum_{j \in \mathbf{J}} q^j \theta^j > 0$).

2.2 Stochastic discount factors and risk-neutral probabilities

Given the market $M = \left(V^j, \underline{q}^j, \overline{q}^j\right)_{j \in \mathbf{J}}$ we let:

$$\mathcal{M}_+ = \left\{ \mu \in \mathbb{R}_+^\Omega : \underline{q}^j \leq V^j \cdot \mu \leq \overline{q}^j \quad \forall j \in \mathbf{J} \right\} \text{ (resp. } \mathcal{M}_{++} = \mathcal{M}_+ \cap \mathbb{R}_{++}^\Omega),$$

be the set of nonnegative (resp. strictly positive) *stochastic discount factors* μ for which the discounted payoff $V^j \cdot \mu$ of each security j belongs to the price spread $[\underline{q}^j, \overline{q}^j]$ (hence is equal to the asset price $q^j := \underline{q}^j$ whenever $\underline{q}^j = \overline{q}^j$).

We now recall the *Fundamental Theorem of Asset Pricing* [the equivalence $(i) \iff (ii)$ below] that characterizes an arbitrage-free (resp. present arbitrage-free) market by the existence of a strictly positive (resp. nonnegative) stochastic discount factor $\mu \in \mathcal{M}_{++}$ (resp. $\mu \in \mathcal{M}_+$) and the *Duality Theorem of Asset Pricing* [the equivalence $(ii) \iff (iii)$ below]. For the first part, see Ross (1976, 1978) and Cox and Ross (1976) in the frictionless case, and in a dynamic setting Harrison and Kreps (1979), Duffie and Huang (1986) and Delbaen and Schachermayer (1994). The duality part is a direct consequence of the first part and the Strong Duality Theorem of Linear Programming. The following result synthesizes the two parts in a unique statement

[2] We point out (see Proposition 1 hereafter) that the function c_+ is *sublinear.* if the market is arbitrage-free. Hence, $-c_+(-x) \leq c_+(x)$ for every payoff $x \in \mathbb{R}^\Omega$.

that also consider both arbitrage-free and present arbitrage-free markets, the latter being used extensively throughout the paper. The proof of the theorem is standard.

Theorem 1 (Fundamental Theorem of Asset Pricing) *Consider the market* $M = \left(V^j, \underline{q}^j, \overline{q}^j\right)_{j\in\mathbf{J}}$. *Then the following three assertions are equivalent:*

(i) M is arbitrage-free (resp. present arbitrage-free);
(ii) $\mathcal{M}_{++} \neq \emptyset$ (resp. $\mathcal{M}_{+} \neq \emptyset$);
(iii) $\mathcal{M}_{++} \neq \emptyset$ and $\sup_{\mu\in\mathcal{M}_{++}} x \cdot \mu = c_+(x)$ for all $x \in \mathbb{R}^\Omega$,

(resp. $\mathcal{M}_{+} \neq \emptyset$ and $\sup_{\mu\in\mathcal{M}_{+}} x \cdot \mu = c_+(x)$ for all $x \in \mathbb{R}^\Omega$).

It is worth noticing that Theorem 1 provides a natural rationale for the famous multi-prior model of Gilboa and Schmeidler (1989), extensively used in this paper in its "dual" form, i.e., with a "sup" since we are dealing with cost function, instead of an "inf" when dealing with utility functions.

Consider an arbitrage-free market M whose bond $\mathbf{1}_\Omega$ is frictionless, then the stochastic discount factors $\mu \in \mathcal{M}_{++}$ can be uniformly normalized as *risk-neutral probabilities*, in a standard way, as follows.

Remark 1 (Risk-neutral Probabilities) Consider an arbitrage-free market $M = \left(V^j, \underline{q}^j, \overline{q}^j\right)_{j\in\mathbf{J}}$ whose bond $\mathbf{1}_\Omega$ is frictionless. By Theorem 1 the present discounted value of the bond is independent of the choice of μ in $\mathcal{M}_{++}$ since $\mathbf{1}_\Omega \cdot \mu = c_+(\mathbf{1}_\Omega)$ for all $\mu \in \mathcal{M}_{++}$.[3] Thus $c_+(\mathbf{1}_\Omega) > 0$ is interpreted as a *risk-free discount factor* and r can be (uniquely) defined by:

$$c_+(\mathbf{1}_\Omega) = \frac{1}{1+r} \text{ and } r \text{ is interpreted as the risk-free interest rate.}$$

Finally, we can (uniformly) normalize each stochastic discount factor $\mu \in \mathcal{M}_{++}$ as a probability, called *risk-neutral probability*, as follows[4]:

$$(1+r)\mathcal{M}_{++} = \left\{P \in \text{Proba}_{++}(\Omega) : \underline{q}^j \leq \frac{1}{1+r} E_P(V^j) \leq \overline{q}^j \ \forall j \in \mathbf{J}\right\} \neq \emptyset.$$

Thus, for every risk-neutral probability P, the discounted expected payoff of every security j belongs to the price spread $[\underline{q}^j, \overline{q}^j]$ (hence is equal to its asset price whenever $\underline{q}^j = \overline{q}^j$). □

We end this section with the definition of equivalent markets. The markets $M = \left(V^j, \underline{q}^j, \overline{q}^j\right)_{j\in\mathbf{J}}$, and $M' = (W^j, \underline{r}^j, \overline{r}^j)_{j\in\mathbf{J}'}$ are said to be **equivalent**, denoted $M \sim M'$, if they have the same set of nonnegative stochastic discount factors, that is, $\mathcal{M}_+ = \mathcal{M}'_+$. Then two markets are equivalent if and only if they have the same super-hedging price (from Theorem 1 and Proposition 1).

[3] Indeed, from Theorem 1, we have $-c_+(-\mathbf{1}_\Omega) = \inf_{\mu\in\mathcal{M}_{++}} \mathbf{1}_\Omega \cdot \mu \leq \sup_{\mu\in\mathcal{M}_{++}} \mathbf{1}_\Omega \cdot \mu = c_+(\mathbf{1}_\Omega)$. But the inequality is an equality since $\mathbf{1}_\Omega$ is frictionless.

[4] Where $\text{Proba}_{++}(\Omega) := \{P \in \mathbb{R}^\Omega_{++} : \sum_{\omega\in\Omega} P(\omega) = 1\}$ is the set of strictly positive probabilities on Ω and $E_P(X) := X \cdot P$ is the expected payoff of $X \in \mathbb{R}^\Omega$.

Finally this paper will mainly consider markets *whose bond is frictionless,* without always distinguishing them from markets *with the frictionless bond*, since when $M = \big(V^j, \underline{q}^j, \overline{q}^j\big)_{j\in\mathbf{J}}$ is present arbitrage-free, the following assertions are equivalent:

(i) M is a market whose bond is frictionless, i.e., $c_+(\mathbf{1}_\Omega) = -c_+(-\mathbf{1}_\Omega)$;
(ii) $M \sim M' := ((\mathbf{1}_\Omega, q^0, q^0), (V^j, \underline{q}^j, \overline{q}^j)_{j\in\mathbf{J}}\big)$ for some price q^0.

2.3 Pricing rules

We now deduce from Theorem 1 that the super-hedging price of a present arbitrage-free market is a pricing rule in the following sense.

Definition 1 (Pricing Rule) We call pricing rule every real-valued function $f : \mathbb{R}^\Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$ satisfying the following properties:

$[Finitenesss]$ f is finite-valued, i.e., $f : \mathbb{R}^\Omega \to \mathbb{R}$;
$[Positive\ Homogeneity]$ $f(tx) = tf(x)$ for all $x \in \mathbb{R}^\Omega$, all $t \geq 0$;
$[Subadditivity]$ $f(x + x') \leq f(x) + f(x')$ for all x, x';
$[Monotonicity]$ $f(x) \leq f(x')$ for all $x \leq x'$;
$[Frictionless Bond]$ $-f(-\mathbf{1}_\Omega) = f(\mathbf{1}_\Omega)$.

Moreover, f is said to be arbitrage-free if $f(x) > 0$ for all $x > 0$.

Note that $f : \mathbb{R}^\Omega \to \mathbb{R}$ is a pricing rule if and only if the associated function $\rho : \mathbb{R}^\Omega \to \mathbb{R}$ defined by $\rho(x) = f(-x)$ is a *coherent risk measure*; see Artzner et al. (1999). For the study of *pricing rules* we refer to Jouini (2000), Jouini and Kallal (2001), Castagnoli et al. (2002), Araujo et al. (2012, 2018) and Chateauneuf and Cornet (2018).

The following proposition summarizes the main properties of pricing rules that will be used in this paper.

Proposition 1 (Pricing Rule) *(a) Consider the present arbitrage-free market:*

$$M := \big((\mathbf{1}_\Omega, q^0, q^0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big).$$

Then its super-hedging price c_+ is a pricing rule and $\partial c_+(0) = \mathcal{M}_+$.
(b) *Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be a pricing rule, then it satisfies the following properties:*

$[Constant\ Additivity]$ $f(x + t\mathbf{1}_\Omega) = f(x) + tf(\mathbf{1}_\Omega)$ *for all x, all $t \in \mathbb{R}$;*
$[Nonnegative\ Spread]$ $-f(-x) \leq f(x)$ *for all x;*
$[Subdifferential]$ $\partial f(0) = \{\mu \in \mathbb{R}^\Omega : x \cdot \mu \leq f(x)\ \ \forall x \in \mathbb{R}^\Omega\} \neq \emptyset$[5] *and*

$f(x) = \sup\{x \cdot \mu \ : \ \mu \in \partial f(0)\}$ *for all $x \in \mathbb{R}^\Omega$.*

The proof of the equality $\partial c_+(0) = \mathcal{M}_+$ is given in Sect. 4.2 and the remaining part is left to the reader.

[5] The subdifferential is defined by $\partial f(0) := \{\mu \in \mathbb{R}^\Omega : x \cdot \mu + f(0) \leq f(x)\ \ \forall x \in \mathbb{R}^\Omega\}$ (see Rockafellar 1970) and the equality holds since $f(0) = 0$.

2.4 Risk-neutral capacities

To every pricing rule and every present arbitrage-free market, whose bond $\mathbf{1}_\Omega$ is frictionless, we associate the notions of upper and lower risk-neutral capacities which are defined as follows.

Definition 2 (Risk-neutral Capacities) If $v : 2^\Omega \to \mathbb{R}$ is a set function satisfying $v(\emptyset) = 0$ we let:[6]

$$\overline{\text{core}}(v) := \left\{\mu \in \mathbb{R}^\Omega : \forall A \subseteq \Omega, \mu(A) \le v(A) \text{ and } \mu(\Omega) = v(\Omega)\right\},$$
$$\underline{\text{core}}(v) := \left\{\mu \in \mathbb{R}^\Omega : \forall A \subseteq \Omega, \mu(A) \ge v(A) \text{ and } \mu(\Omega) = v(\Omega)\right\}.$$

A capacity, is a set function $v : 2^\Omega \to \mathbb{R}$ such that $v(\emptyset) = 0$, and v is monotone, i.e., $v(A) \le v(B)$ for all $A \subseteq B \subseteq \Omega$.

If f is a pricing rule, the upper and lower risk-neutral capacities $\overline{v}_f : 2^\Omega \to \mathbb{R}$ and $\underline{v}_f : 2^\Omega \to \mathbb{R}$ are defined by:

$$\overline{v}_f(A) := f(\mathbf{1}_A) \text{ and } \underline{v}_f(A) := -f(-\mathbf{1}_A) \text{ for all } A \subseteq \Omega.$$

Similarly, if M is a present-arbitrage-free market whose bond is frictionless, the upper and lower risk-neutral capacities $\overline{v}_M$ and $\underline{v}_M$ are defined by

$$\overline{v}_M(A) := c_+(\mathbf{1}_A) \text{ and } \underline{v}_M(A) := -c_+(-\mathbf{1}_A) \text{ for all } A \subseteq \Omega.$$

We refer to Chateauneuf and Cornet (2022) for a study of the risk-neutral capacities, also called risk-neutral non-additive probabilities (under the assumption that $f(\mathbf{1}_\Omega) = 1$). The following proposition summarizes the basic properties that will be used in this paper.

Proposition 2 (Risk-neutral Capacities) *Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be a pricing rule. Then, the following assertions hold:*

- $\overline{v}_f(A) = \sup\{\mu(A) : \mu \in \partial f(0)\}$ *for all* $A \subseteq \Omega$;
- $\overline{v}_f$ *and* $\underline{v}_f$ *are capacities which are mutually conjugate, in the sense that:*
$$\overline{v}_f(A) + \underline{v}_f(A^c) = \overline{v}_f(\Omega) = \underline{v}_f(\Omega) \textit{ for all } A \subseteq \Omega;$$
- $\overline{\text{core}}(\overline{v}_f) = \underline{\text{core}}(\underline{v}_f) \subseteq \mathbb{R}^\Omega_+$;
- $\partial f(0) \subseteq \overline{\text{core}}(\overline{v}_f)$.

  Let M be a present-arbitrage-free market with frictionless bond, then
- $\overline{v}_M(A) = \sup\{\mu(A) : \mu \in \mathcal{M}_+\}$ *for all* $A \subseteq \Omega$;
- $\overline{v}_M$ *and* $\underline{v}_M$ *are capacities which are mutually conjugate;*
- $\overline{\text{core}}(\overline{v}_M) = \underline{\text{core}}(\underline{v}_M) \subseteq \mathbb{R}^\Omega_+$;
- $\mathcal{M}_+ \subseteq \overline{\text{core}}(\overline{v}_M)$.

[6] The two notions of core, sometimes called anti-core and core, only differ by the sense of the inequalities in their definitions. We have adopted the notations $\overline{\text{core}}(v)$ and $\underline{\text{core}}(v)$ to keep the parallel with the notations of the upper and lower risk-neutral capacities. Hereafter we will use the same term of **core** in the two different contexts.

The proof of the proposition is left to the reader.

3 Submodular markets

3.1 Submodularity and decreasing differences

A pricing rule $f : \mathbb{R}^{\Omega} \to \mathbb{R}$ (resp. a market M) is said to be *submodular* if f (resp. $f := c_+$, the super-hedging price of M) is finite-valued and submodular:

$$[\textit{Submodularity}]\, f(x \vee y) + f(x \wedge y) \leq f(x) + f(y) \text{ for all } x, y \text{ in } \mathbb{R}^{\Omega}.$$

We recall that the function f is submodular if and only if it satisfies:

$$[\textit{Decreasing Differences}] \text{ for all } x \in \mathbb{R}^{\Omega}, \text{ all } \alpha, \beta \text{ in } \mathbb{R}_+, \text{ for all } \omega, \omega' \text{ in } \Omega :$$

$$f(x + \alpha \mathbf{1}_{\omega} + \beta \mathbf{1}_{\omega'}) - f(x + \beta \mathbf{1}_{\omega'}) \leq f(x + \alpha \mathbf{1}_{\omega}) - f(x) \text{ if } \omega \neq \omega' \text{ in } \Omega,$$

and if and only if it satisfies:

[Weak Cost Complementarity] for all $x' \in \mathbb{R}^{\Omega}$, α, β in $\mathbb{R}^{\Omega}_+$[7]:

$$f(x' + \alpha + \beta) - f(x' + \beta) \leq f(x' + \alpha) - f(x') \text{ if } \alpha \wedge \beta = 0.$$

We refer to Topkis (1998) for a proof of the equivalence between the three above properties. The assumptions of decreasing differences and of weak cost complementarity can be interpreted for financial markets as follows.

Remark 2 (Financial Interpretation) Let x be a payoff, and interpret state ω as "my house is burnt tomorrow" and state ω' as "my car is stolen tomorrow". Buying (today) an insurance to cover state ω (resp. ω') means buying $\alpha \geq 0$ (resp. $\beta \geq 0$) units of the Arrow security on state ω (resp. ω'). Thus the assumption of decreasing differences is saying that:

Net cost for insuring the house (once the car is insured)

$$\begin{aligned} &:= c_+(x + \alpha \mathbf{1}_{\omega} + \beta \mathbf{1}_{\omega'}) - c_+(x + \beta \mathbf{1}_{\omega'}) \\ &\leq c_+(x + \alpha \mathbf{1}_{\omega}) - c_+(x) := \text{ Net cost for insuring the house.} \end{aligned}$$

A similar interpretation can be given for weak cost complementarity, replacing Arrow securities by bundles of insurance α, β in $\mathbb{R}^{\Omega}_+$ on disjoint sets of states, i.e., such that $\alpha \wedge \beta = 0$. □

The next proposition shows that strengthening the weak cost complementarity by removing the assumption $\alpha \wedge \beta = 0$, leads to linear pricing rules, thus the assumption becomes too strong to study markets with frictions.

[7] This is Definition (2.6.1) by Topkis (1998) p. 53, but for a function defined on $\mathbb{R}^{\Omega}_+$.

Remark 3 (Cost Complementarity Topkis 1998) The pricing rule $f : \mathbb{R}^\Omega \to \mathbb{R}$ is said to satisfy cost complementarity if:

$$\text{for all } x' \in \mathbb{R}^\Omega, \alpha, \beta \text{ in } \mathbb{R}^\Omega_+ : f(x' + \alpha + \beta) - f(x' + \beta) \leq f(x' + \alpha) - f(x').$$

First introduced in the study of cost games (see Sharkey and Telser 1978; Moulin 1992), the properties of such functions (or its opposite called ultra-modular) have been studied by Marinacci and Montrucchio (2005), and by Müller and Scarsini (2012) (and called infra-modular).

Proposition 3 *Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be positively homogeneous, constant additive, (e.g., if f is a pricing rule) and satisfy cost complementarity, then f is linear.*

Proof

- [Subadditivity] Let $x \in \mathbb{R}^\Omega$ and $y \in \mathbb{R}^\Omega$, we can always write $x = \alpha + t\mathbf{1}_\Omega$, $y = \beta + \tau\mathbf{1}_\Omega$, for some $\alpha \geq 0$, $\beta \geq 0$, $t \in \mathbb{R}$, $\tau \in \mathbb{R}$. Thus

$$\begin{aligned} f(x+y) &= f((t+\tau)\mathbf{1}_\Omega + \alpha + \beta) \\ &\leq f((t+\tau)\mathbf{1}_\Omega + \alpha) + f((t+\tau)\mathbf{1}_\Omega + \beta) - f((t+\tau)\mathbf{1}_\Omega) \\ &= f(t\mathbf{1}_\Omega + \alpha) + f(\tau\mathbf{1}_\Omega + \beta)[\text{from Constant Additivity}] \\ &= f(x) + f(y). \end{aligned}$$

- [$f(-\alpha) = -f(\alpha)$ for all $\alpha \geq 0$] First, taking $x' = -\alpha$ and $\beta = 2\alpha$ in the cost complementarity inequality, we get $f(2\alpha) + f(-\alpha) \leq f(\alpha) + f(0) = f(\alpha)$[since $f(0) = 0$] Hence $f(-\alpha) \leq f(\alpha) - f(2\alpha) = -f(\alpha)$ from positive homogeneity. Second, since f is subadditive, we have $0 = f(\alpha - \alpha) \leq f(\alpha) + f(-\alpha)$.
- [Super-additivity] Let $x \in \mathbb{R}^\Omega$ and $y \in \mathbb{R}^\Omega$, we can always write $x = \alpha + t\mathbf{1}_\Omega$, $y = \beta + \tau\mathbf{1}_\Omega$, for some $\alpha \geq 0$, $\beta \geq 0$, $t \in \mathbb{R}$, $\tau \in \mathbb{R}$. Thus

$$\begin{aligned} & f(x+y-y) - f(x+y) \\ & \leq f(-y) \quad [\text{from subadditivity}] \\ & = f(-\beta - \tau\mathbf{1}_\Omega) = f(-\beta) - \tau f(\mathbf{1}_\Omega) \quad [\text{from constant additivity}] \\ & = -f(\beta) - \tau f(\mathbf{1}_\Omega) \quad [\text{from } \beta \geq 0 \text{ and the homogeneity property}] \\ & = -f(\beta + \tau\mathbf{1}_\Omega) = -f(y) \quad [\text{from constant additivity}] \end{aligned}$$

  Consequently, $f(x) + f(y) \leq f(x+y)$.
- [Homogeneity : $f(tx) = tf(x)$ for all $x \in \mathbb{R}^\Omega, t \in \mathbb{R}$] From positive homogeneity, it suffices to prove that $f(-x) = -f(x)$. Indeed, since f is additive, for all $x \in \mathbb{R}^\Omega$, one has $0 = f(x-x) = f(x) + f(-x)$. □

3.2 A weak form of submodularity

This section introduces for the pricing rule $f : \mathbb{R}^{\Omega} \to \mathbb{R}$ (hence also for markets when applied to $f = c_+$) the following weaker form of submodularity:

$$[0 - \mathit{Submodularity}]\, f(x \vee 0) + f(x \wedge 0) \leq f(x), \text{ for all } x \in \mathbb{R}^{\Omega},$$

and we notice that, for a pricing rule f (since f is sub-additive), it is equivalent to the following Choquet Decomposition:

$$[\mathit{Choquet\ Decomposition}]\, f(x) = f(x \vee 0) + f(x \wedge 0) \text{ for all } x \in \mathbb{R}^{\Omega}.$$

We end this section with several remarks on the two last assumptions, which will be shown to be equivalent to submodularity for markets (Theorem 2) and for pricing rules (Theorem 3). The Choquet Decomposition Property can be interpreted for financial markets as follows.

Remark 4 (Financial Interpretation) Splitting a payoff x in two buying and selling parts in the following way $x = [x]_+ - [x]_-$, the cost $f(x)$ is equal to the difference between the payment $f([x]_+) \geq 0$ for the purchase $[x]_+$ and the gain $-f(-[x]_-) \geq 0$ from the sale of $[x]_-$.

Moreover, the previous splitting strategy leads to the smallest cost in the following sense. Among all strategies of splitting x in two buying and selling parts, i.e., $x = a - b$ with $a \geq 0$, $b \geq 0$, then the aggregate cost $f(a) + f(-b)$ is the smallest one for $a = [x]_+$ and $b = [x]_-$. Formally:

$$f([x]_+) + f(-[x]_-) = f(x) = \inf \left\{ f(a) + f(-b) : x = a - b, a \geq 0, b \geq 0 \right\}.$$

The above assertion follows from the subadditivity of the pricing rule f. □

Remark 5 (Comonotonic Additivity) Since $[x]_+$ and $-[x]_-$ are comonotonic, the Choquet Decomposition Property follows from the Comonotonic Additivity Property (see Remark 6), a basic property of the Choquet integral. Hence the Choquet Decomposition Property is a necessary condition for the Choquet representation of the super-hedging price of a market (or of a pricing rule) and we will see that it is in fact necessary and sufficient for the Choquet representation of markets (Theorem 2) and of pricing rules (Theorem 3). □

3.3 Characterization of submodular markets

We now state the main theorem of our paper that provides several characterization properties of the submodularity of the market M, namely (ii) the 0-submodularity or the Choquet Decomposition Property of its super-hedging price c_+, (iii) the Choquet representation of its super-hedging price c_+, (iv) the representation of its set of nonnegative stochastic discount factors $\mathcal{M}_+$ as the core of its risk-neutral capacity $\overline{v}_M$, together with the submodularity of $\overline{v}_M$, and finally (v) the equivalence of the market M with a market with only event securities whose upper risk-neutral capacity is submodular.

An *event security* is a security whose payoff is $\mathbf{1}_A$ for some event $A \subseteq \Omega$. The Choquet integral (Choquet 1954) of $x \in \mathbb{R}^\Omega$ with respect to the capacity v, is denoted $\int_\Omega^C x\, dv$, and we refer to Denneberg (1994), and Marinacci and Montrucchio (2004) for standard references.

We can now state the main result of the paper.

Theorem 2 *Consider the present arbitrage-free market:*

$$M := \big((\mathbf{1}_\Omega, q^0, q^0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big).$$

Then the following assertions are equivalent:

(i) the market M is submodular, i.e., c_+ is submodular;
(ii) $c_+(x \vee 0) + c_+(x \vee 0) \leq c_+(x)$ for all $x \in \mathbb{R}^\Omega$;
(iii) c_+ is a Choquet integral, i.e., $c_+(x) = \int_\Omega^C x\, d\overline{v}_M$ for all $x \in \mathbb{R}^\Omega$;
(iv) $\mathcal{M}_+ = \overline{\mathrm{core}}(\overline{v}_M)$ and $\overline{v}_M$ is submodular[8]*;*
(v) M is equivalent to a present arbitrage-free market M'

with event securities, i.e., $M \sim M' := \big((\mathbf{1}_\Omega, r^0, r^0), (\mathbf{1}_{A^j}, \underline{r}^j, \overline{r}^j)_{j \in \mathbf{J}'}\big)$, for some events $A^j \subseteq \Omega$ and prices $\underline{r}^j, \overline{r}^j$ ($j \in \mathbf{J}'$ finite), and $\overline{v}_M$ is submodular, or equivalently $\overline{v}_{M'}$ is submodular.

The proof of the theorem is given in Sect. 4, without invoking comonotonic additivity. Interestingly, the submodularity of the market proves to be equivalent to the comonotonic additivity of its super-hedging price.

Remark 6 (Comonotonic Additivity) The following assertions are equivalent

(i) the market M is submodular;
(i') the super-hedging price c_+ of M is comonotonic additive, i.e.,

$c_+(x + y) = c_+(x) + c_+(y)$ whenever x, y in $\mathbb{R}^\Omega$ are comonotonic, i.e., $(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \geq 0$ for all ω, ω' in Ω.

Indeed, from Theorem 2 the submodularity of the market is equivalent to the fact that its super-hedging price c_+ is a Choquet integral. Thus, by Schmeidler (1986), it is equivalent to the comonotonic additivity of c_+. □

The following remark reformulates Assertions (iii) and (iv) of Theorem 2 with the lower risk-neutral capacity $\underline{v}_M$ of the market M (see Definition 2).

Remark 7 (Lower risk-neutral capacity) Let M be present arbitrage-free with frictionless bond, then M is submodular if and only if (iii') or (iv') holds:

(iii') $-c_+(-x) = \int_\Omega^C x\, d\underline{v}_M$ for all $x \in \mathbb{R}^\Omega$;
(iv') $\mathcal{M}_+ = \underline{\mathrm{core}}(\underline{v}_M)$ and $\underline{v}_M$ is *super-modular*.

[8] The set function $v : 2^\Omega \to \mathbb{R}$ satisfying $v(\emptyset) = 0$ is submodular, also called concave by Shapley (1971), if $v(A_1 \cup A_2) + v(A_1 \cap A_2) \leq v(A_1) + v(A_2)$ for all $A_1 \subseteq \Omega$, $A_2 \subseteq \Omega$.

The proof of the remark relies on the standard result that $\underline{\text{core}}(\underline{v}_M) = \overline{\text{core}}(\overline{v}_M)$ since $\overline{v}_M$ and $\underline{v}_M$ are mutually conjugate by Proposition 2. □

The two following remarks give examples of markets M that are not submodular since either $\mathcal{M}_+ \neq \overline{\text{core}}(\overline{v}_M)$ even if $\overline{v}_M$ is submodular (Remark 8) or $\overline{v}_M$ is not submodular even if the core condition holds (Remark 9).

Remark 8 ($\overline{v}_M$ submodular and $\mathcal{M}_+ \neq \overline{\text{core}}(\overline{v}_M)$) Consider $\Omega = \{1, 2, 3\}$ and the market $M := \big((\mathbf{1}_\Omega, 1, 1), (V, 0, .5)\big)$, where $V := (1, -1, 0)$. Then $\overline{v}_M$ is submodular and $(3/4, 0, 1/4) \in \overline{\text{core}}(\overline{v}_M) \setminus \mathcal{M}_+$. [9] □

Remark 9 ($\overline{v}_M$ not submodular and $\mathcal{M}_+ = \overline{\text{core}}(\overline{v}_M)$) Let $\overline{v} : 2^\Omega \to \mathbb{R}$ be a capacity that is exact but not submodular. We denote $\underline{v}$ its conjugate, and we let $M := (\mathbf{1}_A, \underline{v}(A), \overline{v}(A))_{A \in 2^\Omega}$ be the market with all event securities. Then the bond is frictionless since $\underline{v}(\Omega) = \overline{v}(\Omega)$ and M is present-arbitrage-free since $\mathcal{M}_+ = \overline{\text{core}}(\overline{v}) \neq \emptyset$ (for $\overline{v}$ is exact). Moreover, $\overline{v} = \overline{v}_M$, the upper risk-neutral capacity of M, since $\overline{v}$ is exact, hence from Theorem 1 for all $A \subseteq \Omega$ one has $\overline{v}(A) = \sup_{\mu \in \overline{\text{core}}(\overline{v})} \mathbf{1}_A \cdot \mu = \sup_{\mu \in \mathcal{M}_+} \mathbf{1}_A \cdot \mu = c_+(\mathbf{1}_A) = \overline{v}_M(A)$. □

We end this section with a remark exhibiting a submodular market with securities which are not event securities; in other words the equivalence ($\sim$) in Assertions (v) of Theorem 2 cannot be replaced by an equality.

Example 1 Let $\Omega = \{1, 2\}$ and consider the two equivalent markets: $M = \big(([\begin{smallmatrix}1\\1\end{smallmatrix}], 1, 1), ([\begin{smallmatrix}2\\1\end{smallmatrix}], 1.4, 1.5)\big) \sim M' = \big(([\begin{smallmatrix}1\\1\end{smallmatrix}], 1, 1), ([\begin{smallmatrix}1\\0\end{smallmatrix}], .4, .5), ([\begin{smallmatrix}0\\1\end{smallmatrix}], 0, 1)\big)$.
Then the second security of M is not an event security but M is submodular since $M \sim M'$ which is submodular by Proposition 4 hereafter. □

3.4 A class of submodular markets: epsilon-contamination

This section provides basic and important examples of submodular pricing rules, together with the markets from which they derive. We will consider markets with perfect complementarity, perfect substituability, and markets whose set of risk-neutral probabilities are the ε-contamination of a (given) probability, a basic tool in robustness theory (see Berger 1985; Huber 1981). These three examples belong to the following general class of markets/pricing rules that we now define. We let $q^0 \in \mathbb{R}_+$, $\underline{q} \in \mathbb{R}_+^\Omega$ and we define:

- the market M with frictionless bond and all bid-ask Arrow securities:

  $$M := \big((\mathbf{1}_\Omega, q^0, q^0), (\mathbf{1}_\omega, \underline{q}(\omega), \overline{q}(\omega))_{\omega \in \Omega}\big) \text{ where } \overline{q}(\omega) = q^0 - \underline{q}(\omega^c),$$

  and we denote by $\mathcal{M}_+$ its set of stochastic discount factors;

[9] Indeed, first $\mathcal{M}_+ := \{\mu \in \mathbb{R}^3_+ : \mu_1 + \mu_2 + \mu_3 = 1, 0 \le \mu_1 - \mu_2 \le .5\}$. Then one checks that $\overline{v}_M$ is submodular and $(3/4, 0, 1/4) \in \overline{\text{core}}(\overline{v}_M) \setminus \mathcal{M}_+$. Assume that M is submodular, then for $x = (3/4, 0, 1/4)$, $\int_\Omega^C x d\overline{v}_M = 5/8$ and $c_+(x) = 4/8$, a contradiction with Theorem 2.

- the function $f : \mathbb{R}^\Omega \to \mathbb{R}$ by:

$$f(x):=x \cdot \underline{q} + \alpha \max_{\omega\in\Omega} x(\omega) \text{ where } \alpha := q^0 - \mathbf{1}_\Omega \cdot \underline{q};$$

- the set function $\overline{v} : 2^\Omega \to \mathbb{R}$ by:

$$\overline{v}(A) = \begin{cases} q^0 - \sum_{\omega\in A^c} \underline{q}(\omega) & \text{if } A \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition shows the market M is present arbitrage-free and submodular, f is its super-hedging price, and $\overline{v}$ is its risk-neutral capacity. Moreover the different characterizations properties of submodularity listed in Theorem 2 are also satisfied by M.

Proposition 4 *Assume that $q^0 \in \mathbb{R}_+$, $\underline{q} \in \mathbb{R}^\Omega_+$, and $\alpha := q^0 - \mathbf{1}_\Omega \cdot \underline{q} \geq 0$. Then the following assertions hold:*

- $\mathcal{M}_+ = \underline{q} + \alpha\Delta \neq \emptyset$, *where Δ is the simplex of $\mathbb{R}^\Omega$;*
- $[Arbitrage - free]$ *the market M is present arbitrage-free;*
- $[f = c_+]$ *f is the super-hedging price of the market M;*
- $[Submodularity]$ *f is a submodular pricing rule, thus M is submodular;*
- $[Risk - neutral\ Capacity]$ $\overline{v} = \overline{v}_M = \overline{v}_f$;
- $[Choquet\ Integral]$ $f(x) = \int_\Omega^C x(\omega)\, d\overline{v}(\omega)$ *for all $x \in \mathbb{R}^\Omega$;*
- $[Core\ Representation]$ $\mathcal{M}_+ = \overline{\text{core}}(\overline{v}_M)$ *and $\overline{v}_M$ is submodular.*

The proof of Proposition 4 is given at the end of the section. We first notice that the previous framework covers three important subclasses of submodular markets. In each case the market M^i ($i = 1, \ldots, 3$) is present arbitrage-free and submodular, and we denote by c^i_+ its super-hedging price, $\overline{v}^i$ its risk-neutral capacity, and $\mathcal{M}^i_+$ its set of nonnegative stochastic discount factors.

Example 2 (Perfect Complementarity) Let $q^0 > 0$ (and $\underline{q} := 0$), then

- $M^1 = (\mathbf{1}_\Omega, q^0, q^0)$;
- $c^1_+ : \mathbb{R}^\Omega \to \mathbb{R}$ is defined by $c^1_+(x) := q^0 \max_{\omega\in\Omega} x(\omega)$;[10]
- $\overline{v}^1 : 2^\Omega \to \mathbb{R}$ is defined by: $\overline{v}^1(A) = q^0$ if $A \neq \emptyset$ and $\overline{v}^1(\emptyset) = 0$;
- $\mathcal{M}^1_+ = q^0\Delta$. □

Example 3 (Perfect Substituability) Let $q = \underline{q} \in \mathbb{R}^\Omega_+$ and $q^0 := \sum_{\omega\in\Omega} q(\omega)$,

- $M^2 = \left(\mathbf{1}_\omega, q(\omega), q(\omega)\right)_{\omega\in\Omega}$;
- $c^2_+ : \mathbb{R}^\Omega \to \mathbb{R}$ is defined by $c^2_+(x) := x \cdot q$;
- $\overline{v}^2 : 2^\Omega \to \mathbb{R}$ is defined by: $\overline{v}^2(A) = \sum_{\omega\in A} q(\omega)$;
- $\mathcal{M}^2_+ = \{q\}$. □

[10] The definition of perfect complementarity for cost functions is “dual" from the one given for utility functions, where the “max” is replaced by a “min” (see Topkis 1998).

Example 4 (ε-contamination) Given a probability P on Ω and $\varepsilon \in [0, 1]$;

- $M^3 = \left((\mathbf{1}_\Omega, 1, 1), (\mathbf{1}_\omega, \underline{q}(\omega), \overline{q}(\omega))_{\omega\in\Omega}\right)$
  with $\underline{q}(\omega) := (1-\varepsilon)P(\omega)$ and $\overline{q}(\omega) := P(\omega) + \varepsilon P(\omega^c)$ for all $\omega \in \Omega$;
- $c^3_+ : \mathbb{R}^\Omega \to \mathbb{R}$ is defined by $c^3_+(x) := (1-\varepsilon)x \cdot P + \varepsilon \max_{\omega\in\Omega} x(\omega)$;
- $\overline{v}^3 : 2^\Omega \to \mathbb{R}$ is defined by $\overline{v}^3(A) = P(A) + \varepsilon P(A^c)$ if $A \neq \emptyset$ and $\overline{v}^3(\emptyset) = 0$;
- $\mathcal{M}^3_+ = (1-\varepsilon)P + \varepsilon\Delta$. [$\varepsilon$-contamination] □

We now give the proof of Proposition 4.

Proof of Proposition 4 • [$\mathcal{M}_+ = \underline{q} + \alpha\Delta$] We have:
$\mathcal{M}_+ := \left\{\mu \in \mathbb{R}^\Omega_+ \ : \ \mathbf{1}_\Omega \cdot \mu = q^0, \ \underline{q} \leq \mu \leq \overline{q}\right\}$ with $q^0 := \mathbf{1}_\Omega \cdot \underline{q} + \alpha$.
⋆ [$\mathcal{M}_+ \subseteq \underline{q} + \alpha\Delta$] Let $\mu \in \mathcal{M}_+$. Suppose first that $\alpha = 0$, then $\mu = \underline{q}$ since $\underline{q} \leq \mu$ and $\mathbf{1}_\Omega \cdot \underline{q} = \mathbf{1}_\Omega \cdot \mu = q^0$. Thus, $\mu \in \underline{q} + 0\Delta$.

Suppose now that $\alpha > 0$ and let $\delta := (\mu - \underline{q})/\alpha \geq 0$. Then $\mathbf{1}_\Omega \cdot \delta = \mathbf{1}_\Omega \cdot (\mu - \underline{q})/\alpha = [q^0 - (q^0 - \alpha)]/\alpha = 1$. Thus $\mu = \underline{q} + \alpha\delta \in \underline{q} + \alpha\Delta$.
⋆ [$\underline{q} + \alpha\Delta \subseteq \mathcal{M}_+$] Let $\mu = \underline{q} + \alpha\delta$ for some $\delta \in \Delta$. First, $\mu \geq 0$ since $\underline{q} \geq 0$, $\alpha \geq 0$, and $\delta \geq 0$. Second, $\mathbf{1}_\Omega \cdot \mu = \mathbf{1}_\Omega \cdot \underline{q} + \alpha = q^0$. Third, $\mu - \underline{q} = \alpha\delta \geq 0$, and finally, for all ω, one has $\overline{q}(\omega) - \mu(\omega) = q^0 - \underline{q}(\omega^c) - \mu(\omega) = \mu(\omega^c) - \underline{q}(\omega^c) \geq 0$. Hence $\mu \in \mathcal{M}_+$. □

- [M is present arbitrage-free] The market M is present arbitrage-free since $\mathcal{M}_+ = \underline{q} + \alpha\Delta \neq \emptyset$ [from above] by Theorem 1. □
- [$f = c_+$] From the Fundamental Theorem of Asset Pricing (Theorem 1), using the fact that $\mathcal{M}_+ = \underline{q} + \alpha\Delta$, we get

$$\begin{aligned} c_+(x) &= \sup\{x \cdot \mu : \mu \in \mathcal{M}_+\} = \sup\{x \cdot \mu : \mu \in \underline{q} + \alpha\Delta\} \\ &= x \cdot \underline{q} + \alpha \sup_{\delta\in\Delta} x \cdot \delta = \underline{q} \cdot x + \alpha \max_{\omega\in\Omega} x(\omega) = f(x). \end{aligned}$$

- [f is a submodular pricing rule] The function f is a submodular pricing rule as the sum of two submodular pricing rules. That is, $f = f_1 + f_2$ with $f_1(x) := x \cdot \underline{q}$, $f_2(x) := \alpha \max_{\omega\in\Omega} x(\omega)$ and both functions f_1 and f_2 are clearly submodular pricing rules. Hence f is also a submodular pricing rule. □
- [$\overline{v} = \overline{v}_f = \overline{v}_M$] One easily sees that for all $A \subseteq \Omega$, $\overline{v}_f(A) := f(\mathbf{1}_A) = \overline{v}(A)$. Since $f = c_+$ we have also $\overline{v}_M(A) := c_+(\mathbf{1}_A) = f(\mathbf{1}_A) = \overline{v}(A)$. □
- [Choquet Integral] Since we proved that M is present arbitrage-free, $f = c_+$ is submodular, and $\overline{v}_M = \overline{v}$, from Theorem 2 we deduce that

$$f(x) = c_+(x) = \int_\Omega^C x(\omega)\, d\overline{v}(\omega) \text{ for all } x \in \mathbb{R}^\Omega.$$

The Choquet property can also be proved directly noticing that $\overline{v} = \underline{q} + \alpha\,\delta^*_\Omega$ where $\delta^*_\Omega(A) = 1$ if $A \neq \emptyset$ and $\delta^*_\Omega(\emptyset) = 0$. Then, for all $x \in \mathbb{R}^\Omega$

$$\int_\Omega^C x(\omega)\, d\overline{v}(\omega) = \int_\Omega^C x(\omega)\, d(\underline{q} + \alpha\,\delta^*_\Omega)(\omega)$$

$$= \int_{\Omega}^{C} x(\omega)\, d\underline{q}(\omega) + \alpha \int_{\Omega}^{C} x(\omega)\, d\delta^{*}_{\Omega}(\omega)$$
$$= x \cdot \underline{q} + \alpha \max_{\omega\in\Omega} x(\omega).$$

- $[\mathcal{M}_+ = \overline{\text{core}}(\overline{v}_M)$ and $\overline{v}_M$ is submodular$]$ It follows from Theorem 2 since we have checked that all the assumptions of Theorem 2 are satisfied. □

3.5 Markets represented by their risk-neutral capacities

This section provides a proof of the equivalence $[(iv) \iff (v)]$ between the last two assertions of Theorem 2, without any assumption of submodularity, together with some other equivalent formulations of Assertion (iv) and (v).

A present arbitrage-free market M whose bond is frictionless is said to satisfy the Core Property if:

$$\mathcal{M}_+ = \overline{\text{core}}(v) \cap \mathbb{R}^{\Omega}_+ \text{ for some set function } v : 2^{\Omega} \to \mathbb{R},\, v(\emptyset) = 0.$$

This property is weaker than Condition (iv), $\mathcal{M}_+ = \overline{\text{core}}(\overline{v}_M)$,[11] and the following result (Proposition 5) will show that it is in fact equivalent to it. We also refer to Proposition 6 for a formulation of this result with pricing rules.

Proposition 5 *Consider the present arbitrage-free market:*
$M := \big((\mathbf{1}_\Omega, q^0, q^0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big).$
Then the following assertions are equivalent:
$(\tilde{iv})$ $\mathcal{M}_+ = \overline{\text{core}}(\overline{v}_M)$;
$(\tilde{iv}')$ $\mathcal{M}_+ = \overline{\text{core}}(v) \cap \mathbb{R}^{\Omega}_+$ *for some set function* $v : 2^{\Omega} \to \mathbb{R}$, $v(\emptyset) = 0$;
$(\tilde{v}')$ *M is equivalent to some present arbitrage-free market M_E with all event securities, i.e.,* $M \sim M_E := \big((\mathbf{1}_\Omega, v_\Omega, v_\Omega), (\mathbf{1}_A, 0, v_A)_{A\subsetneq\Omega}\big)$, *for some* $v_A \in \mathbb{R}$, $v_\emptyset = 0$;
$(\tilde{v})$ *M is equivalent to some present arbitrage-free market M' with event securities, i.e.,* $M \sim M' := \big((\mathbf{1}_\Omega, r^0, r^0), (\mathbf{1}_{A^j}, \underline{r}^j, \overline{r}^j)_{j\in\mathbf{J}'}\big)$ *for some events $A^j \subseteq \Omega$ and prices $\underline{r}^j, \overline{r}^j$ ($j \in \mathbf{J}'$ finite).*

We prepare the proof of the proposition with a lemma.

Lemma 1 *(a) Let $v : 2^{\Omega} \to \mathbb{R}$ be a set function such that $v(\emptyset) = 0$, then*
$\overline{\text{core}}(v) \cap \mathbb{R}^{\Omega}_+ = \mathcal{M}_+(M_E)$,
where $M_E := \big((\mathbf{1}_\Omega, v(\Omega), v(\Omega)), (\mathbf{1}_A, 0, v(A))_{A\subsetneq\Omega}\big)$.
(b) Consider the present arbitrage-free market
$M' := \big((\mathbf{1}_\Omega, r^0, r^0), (\mathbf{1}_{A^j}, \underline{r}^j, \overline{r}^j)_{\mathbf{J}'}\big)$ *with* $A^j \subseteq \Omega$ $(j \in \mathbf{J}')$,
and let $\overline{v}_{M'}$ be its risk-neutral capacity and $\mathcal{M}'_+$ be its set of stochastic discount factors. Then $\mathcal{M}'_+ = \overline{\text{core}}(\overline{v}_{M'})$.

[11] Notice that $\overline{\text{core}}(\overline{v}_M) \subseteq \mathbb{R}^{\Omega}_+$ since $\overline{v}_M$ is monotone from the monotonicity of c_+.

Proof of Lemma 1 *Part (a)* We have:

$$\begin{aligned}\overline{\text{core}}(v)\cap\mathbb{R}_+^{\Omega} &= \{\mu\in\mathbb{R}_+^{\Omega} : \mu(\Omega)=v(\Omega) \text{ and } \mu(A)\le v(A)\ \forall A\neq\Omega\}\\ &= \{\mu\in\mathbb{R}_+^{\Omega} : \mathbf{1}_{\Omega}\cdot\mu=v(\Omega) \text{ and } 0\le\mathbf{1}_A\cdot\mu\le v(A)\ \forall A\neq\Omega\}\\ &= \mathcal{M}_+(M_E).\end{aligned}$$

Part (b) In the proof we will use several times the equality:

$\sup\{\mu'(A) : \mu'\in\mathcal{M}'_+\}=\overline{v}_{M'}(A)$ for all $A\subseteq\Omega$ [by Proposition 2].

⋆ [$\mathcal{M}'_+\subseteq\overline{\text{core}}(\overline{v}_{M'})$] Let $\mu\in\mathcal{M}'_+$. First we have:

$\mu(\Omega)=r^0=\sup\{\mu'(\Omega) : \mu'\in\mathcal{M}'_+\}=\overline{v}_{M'}(\Omega)$.

Moreover, for all $A\subseteq\Omega$, $\mu(A)\le\sup\{\mu'(A) : \mu'\in\mathcal{M}'_+\}=\overline{v}_{M'}(A)$. Thus $\mu\in\overline{\text{core}}(\overline{v}_{M'})$. □

⋆ [$\overline{\text{core}}(\overline{v}_{M'})\subseteq\mathcal{M}'_+$] Indeed, let $\mu\in\overline{\text{core}}(\overline{v}_{M'})$, that is, $\mu(\Omega)=\overline{v}_{M'}(\Omega)$ and $\mu(A)\le\overline{v}_{M'}(A)$ for all $A\subseteq\Omega$.

First, we have $\mu(\Omega)=\overline{v}_{M'}(A)=\sup\{\mu'(A) : \mu'\in\mathcal{M}'_+\}=r^0$.

Second, taking $A:=A_j$ ($j\in\mathbf{J}'$), we get

$\mu(A_j)\le\overline{v}_{M'}(A_j)=\sup\{\mu'(A_j) : \mu'\in\mathcal{M}'_+\}\le\overline{r}^j$.

Third, taking $A:=A_j^c$ ($j\in\mathbf{J}'$) and using the fact that $\overline{v}_{M'}(\Omega)=\mu'(\Omega)$ for all $\mu'\in\mathcal{M}'_+$ (since $\mathbf{1}_{\Omega}$ is frictionless) we get $\mu(A_j)\ge\underline{r}^j$ since:

$$\begin{aligned}\overline{v}_{M'}(\Omega)-\mu(A_j)=\mu(\Omega)-\mu(A_j)=\mu(A_j^c)&\le\overline{v}_{M'}(A_j^c)\\ &=\sup\{(\mathbf{1}_{\Omega}-\mathbf{1}_{A_j})\cdot\mu' : \mu'\in\mathcal{M}'_+\}\\ &=\overline{v}_{M'}(\Omega)-\inf\{\mathbf{1}_{A_j}\cdot\mu' : \mu'\in\mathcal{M}'_+\}\\ &\le\overline{v}_{M'}(\Omega)-\underline{r}^j.\end{aligned}$$

To show that $\mu\in\mathcal{M}'_+$, it only remains to prove that $\mu\ge 0$. Indeed, let $\omega\in\Omega$ and take $A:=\{\omega\}^c$. Using the fact that $\overline{v}_{M'}(\Omega)=\mu'(\Omega)$ for all $\mu'\in\mathcal{M}'_+$ (since $\mathbf{1}_{\Omega}$ is frictionless) we get:

$$\begin{aligned}\overline{v}_{M'}(\Omega)-\mu(\{\omega\})=\mu(\Omega)-\mu(\{\omega\})=\mu(\{\omega\}^c)&\le\overline{v}_{M'}(\{\omega\}^c)\\ &=\sup\{(\mathbf{1}_{\Omega}-\mathbf{1}_{\omega})\cdot\mu' : \mu'\in\mathcal{M}'_+\}\\ &=\overline{v}_{M'}(\Omega)-\inf\{\mathbf{1}_{\omega}\cdot\mu' : \mu'\in\mathcal{M}'_+\}.\end{aligned}$$

Hence, $\mu(\{\omega\})\ge\inf\{\mathbf{1}_{\omega}\cdot\mu' : \mu'\in\mathcal{M}'_+\}\ge 0$ since $\mathcal{M}'_+\subseteq\mathbb{R}_+^{\Omega}$. Thus $\mu\ge 0$. □

Proof of Proposition 5

- [$(\imath\tilde{v})\implies(\imath\tilde{v}')$] Assertion $(\imath\tilde{v}')$ holds with $v:=\overline{v}_M$ since $\overline{\text{core}}(\overline{v}_M)\subseteq\mathbb{R}_+^{\Omega}$ for $\overline{v}_M$ is monotone. □
- [$(\imath\tilde{v}')\implies(\tilde{v}')$] From $(\imath\tilde{v}')$ one has $\mathcal{M}_+=\overline{\text{core}}(v)\cap\mathbb{R}_+^{\Omega}$ and by Lemma 1.a, one has $\overline{\text{core}}(v)\cap\mathbb{R}_+^{\Omega}=\mathcal{M}_+(M_E)$. Thus, $\mathcal{M}_+=\mathcal{M}_+(M_E)$, i.e., $M\sim M_E$. □
- [$(\tilde{v}')\implies(\tilde{v})$] Immediate.
- [$(\tilde{v})\implies(\imath\tilde{v})$] From $(\tilde{v})$, the two markets M and M' have the same set of stochastic discount factors, that is, $\mathcal{M}_+=\mathcal{M}'_+$. Hence, from Proposition 2, their risk-neutral capacities are also equal, that is, $\overline{v}_M=\overline{v}_{M'}$. But, by Lemma 1.$(b)$ we have $\mathcal{M}'_+=\overline{\text{core}}(\overline{v}_{M'})$. Thus, $\mathcal{M}_+=\overline{\text{core}}(\overline{v}_M)$. □

4 Proof of the characterization theorem

4.1 Characterization of submodular pricing rules

We prepare the second part of the proof of Theorem 2, namely the equivalence
$[(i) \iff (ii) \iff (iii) \iff (iv)]$
with the following result on pricing rules, which is also of interest for itself.

Theorem 3 *Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be a pricing rule. Then the following assertions are equivalent:*

(1) *f is submodular;*
(2) *$f(x \vee 0) + f(x \wedge 0) \leq f(x)$ for all $x \in \mathbb{R}^\Omega$;*
(3) *f is a Choquet integral, i.e., $f(x) = \int_\Omega^C x \, d\overline{v}_f$ for all $x \in \mathbb{R}^\Omega$;*
(4) *$\partial f(0) = \overline{\text{core}}(\overline{v}_f)$ and $\overline{v}_f$ is submodular;*
(4′) *$\partial f(0) = \overline{\text{core}}(v) \cap \mathbb{R}_+^\Omega$ for some set function $v : 2^\Omega \to \mathbb{R}$, $v(\emptyset) = 0$ and $\overline{v}_f$ is submodular;*
(5) *f is the super-hedging price of some present arbitrage-free market:*
$M' := \big((\mathbf{1}_\Omega, r^0, r^0), (\mathbf{1}_{A^j}, \underline{r}^j, \overline{r}^j)_{j \in \mathbf{J}'}\big)$,
for some events $A^j \subseteq \Omega$ and prices $\underline{r}^j, \overline{r}^j$ ($j \in \mathbf{J}'$ finite)
and $\overline{v}_f$ is submodular;

Proof of Theorem 3 • [(1) $\implies$ (2)] Immediate. □

- [(2) $\implies$ (3)] We first recall the definition of the Choquet integral. Let $x \in \mathbb{R}^\Omega$ with its values ranked in decreasing order $x_1 > x_2 > \cdots > x_k > \cdots > x_K$, let $A_k := \{\omega \in \Omega \ : \ x(\omega) = x_k\}$, then:

$$x = \sum_{k=1}^{K} y_i \text{ with } y_k := (x_k - x_{k+1}) \mathbf{1}_{A_1 \cup \cdots \cup A_k} (k \leq K-1), \; y_K := x_K \mathbf{1}_\Omega,$$

and the Choquet integral is defined as follows:

$$\int_\Omega^C x \, d\overline{v}_f := \sum_{k=1}^{K-1} (x_k - x_{k+1}) \overline{v}_f (A_1 \cup \cdots \cup A_k) + x_K \overline{v}_f(\Omega).$$

Then the equality $f(x) = \int_\Omega^C x \, d\overline{v}_f$ is a consequence of the following claims.

Claim 4.1 *$f(y_k) := (x_k - x_{k+1}) \overline{v}_f (A_1 \cup \cdots \cup A_k)$ ($k \leq K-1$), and $f(y_K) = x_K \overline{v}_f(\Omega)$.*

Proof First, for $k \leq K-1$, since f is positively homogenous, one has:

$$\begin{aligned} f(y_k) &= f\big((x_k - x_{k+1}) \mathbf{1}_{A_1 \cup \cdots \cup A_k}\big) \\ &= (x_k - x_{k+1}) f(\mathbf{1}_{A_1 \cup \cdots \cup A_k}) \text{ (since } x_k - x_{k+1} \geq 0) \\ &= (x_k - x_{k+1}) \overline{v}_f (A_1 \cup \cdots \cup A_k). \end{aligned}$$

Second, for $k = K$, since $f(-\mathbf{1}_\Omega) = -f(\mathbf{1}_\Omega)$ and f is positively homogenous, one gets

$$f(y_K) = f(x_K \mathbf{1}_\Omega) = x_K f(\mathbf{1}_\Omega) = x_K \overline{v}_f(\Omega). \qquad \square$$

Claim 4.2 $f(x) = \sum_{k=1}^{K} f(y_k)$.

Proof Since $x = \sum_{k=1}^{K} y_k$, we only need to prove that:

$$f(y_1 + \cdots + y_k) = f(y_1 + \cdots + y_{k-1}) + f(y_k) \text{ for all } k = 1, \ldots, K.$$

First, for $k = K$, we have

$$\begin{aligned} f(y_1 + \cdots + y_K) &= f(y_1 + \cdots + y_{K-1} + x_K \mathbf{1}_\Omega) \\ &= f(y_1 + \cdots + y_{K-1}) + x_K f(\mathbf{1}_\Omega) [\text{ from the constant additivity of f}] \\ &= f(y_1 + \cdots + y_{K-1}) + f(y_K) \text{ [since } f(y_K) = x_K f(\mathbf{1}_\Omega)]. \end{aligned}$$

Second, for $k \le K - 1$, since f is subadditive, we have:

$$f(y_1 + \cdots + y_k) \le f(y_1 + \cdots + y_{k-1}) + f(y_k).$$

We now prove the converse inequality. We let:

$$a := y_1 + \cdots + y_k \text{ and } b := (x_k - x_{k+1})\mathbf{1}_\Omega.$$

Recalling that $x_1 > \cdots > x_K$ one gets:

$$\begin{aligned} y_1 + \cdots + y_k &= (x_1 - x_2)\mathbf{1}_{A_1} + \cdots + (x_k - x_{k+1})\mathbf{1}_{A_1 \cup \cdots \cup A_k}, \\ &= (x_1 - x_{k+1})\mathbf{1}_{A_1} + \cdots + (x_k - x_{k+1})\mathbf{1}_{A_k} \\ a \vee b :&= (y_1 + \cdots + y_k) \vee (x_k - x_{k+1})\mathbf{1}_\Omega \\ &= ((x_1 - x_{k+1})\mathbf{1}_{A_1} + \cdots + (x_k - x_{k+1})\mathbf{1}_{A_k}) \vee (x_k - x_{k+1})\mathbf{1}_\Omega \\ &= (x_1 - x_{k+1})\mathbf{1}_{A_1} + \cdots + (x_{k-1} - x_{k+1})\mathbf{1}_{A_{k-1}} \\ &\quad + (x_k - x_{k+1})\mathbf{1}_{[A_1 \cup \cdots \cup A_{k-1}]^c} \\ &= (x_1 - x_k)\mathbf{1}_{A_1} + \cdots + (x_{k-1} - x_k)\mathbf{1}_{A_{k-1}} + (x_k - x_{k+1})\mathbf{1}_\Omega \\ &= y_1 + \cdots + y_{k-1} + (x_k - x_{k+1})\mathbf{1}_\Omega. \\ a \wedge b :&= (y_1 + \cdots + y_k) \wedge (x_k - x_{k+1})\mathbf{1}_\Omega \\ &= ((x_1 - x_{k+1})\mathbf{1}_{A_1} + \cdots + (x_k - x_{k+1})\mathbf{1}_{A_k}) \wedge (x_k - x_{k+1})\mathbf{1}_\Omega \\ &= (x_k - x_{k+1})\mathbf{1}_{A_1} + \cdots + (x_k - x_{k+1})\mathbf{1}_{A_k} = (x_k - x_{k+1})\mathbf{1}_{A_1 \cup \cdots \cup A_k} \\ &= y_k, \end{aligned}$$

Consequently, since f is constant additive and $b := (x_k - x_{k+1})\mathbf{1}_\Omega$, we get:

$$\begin{aligned} 0 &\le f(a - b) - f([a - b] \vee 0) - f([a - b] \wedge 0) \;\; \text{[from (2)]} \\ &= f(a) + f(b) - f(b + [a - b] \vee 0) - f(b + [a - b] \wedge 0) \end{aligned}$$

$$
\begin{aligned}
&= f(a) + f(b) - f(a \vee b) - f(a \wedge b) \\
&:= f(y_1 + \cdots + y_k) + f\big((x_k - x_{k+1})\mathbf{1}_\Omega\big) \\
&\quad - f\big(y_1 + \cdots + y_{k-1} + (x_k - x_{k+1})\mathbf{1}_\Omega) - f(y_k) \text{ [from above]} \\
&:= f(y_1 + \cdots + y_k) + f\big((x_k - x_{k+1})\mathbf{1}_\Omega\big) \\
&\quad - f(y_1 + \cdots + y_{k-1}) - f((x_k - x_{k+1})\mathbf{1}_\Omega\big) - f(y_k) \\
&= f(y_1 + \cdots + y_k) - f(y_1 + \cdots + y_{k-1}) - f(y_k).
\end{aligned}
$$

This ends the proof of the claim and the proof of the implication. □

• [(3) $\Longrightarrow$ (4)] ⋆ [$\overline{v}_f$ is submodular] Since f is a Choquet integral and f is subadditive (as part of the properties of a pricing rule) then f is submodular (see, for example Marinacci and Montrucchio 2004). Finally, since f is submodular, one easily deduces that $\overline{v}_f$ is also submodular. □
⋆ [$\overline{\text{core}}(\overline{v}_f) \subseteq \partial f(0)$] From Schmeidler (1986), from (3) and the submodularity of $\overline{v}_f$, we deduce that:

$$
f(x) = \sup\big\{x \cdot \mu \ : \ \mu \in \overline{\text{core}}(\overline{v}_f)\big\} \text{ for all } x \in \mathbb{R}^\Omega. \tag{*}
$$

We now prove the inclusion $\overline{\text{core}}(\overline{v}_f) \subseteq \partial f(0)$. Let $\mu \in \overline{\text{core}}(\overline{v}_f)$, from ($*$) we deduce that $x \cdot \mu \le f(x)$ for all $x \in \mathbb{R}^\Omega$, hence $\mu \in \partial f(0)$ since $f(0) = 0$.
⋆ [$\partial f(0) \subseteq \overline{\text{core}}(\overline{v}_f)$] Let $\mu \in \partial f(0)$, then for all $x \in \mathbb{R}^\Omega$, $x \cdot \mu \le f(x)$ since $f(0) = 0$. Thus for $A \subseteq \Omega$, taking $x := \mathbf{1}_A$ we deduce that $\mu(A) = \mathbf{1}_A \cdot \mu \le f(\mathbf{1}_A) := \overline{v}_f(A)$. Taking successively $x := \mathbf{1}_\Omega$ and $x := -\mathbf{1}_\Omega$ we deduce that $\mu(\Omega) = \mathbf{1}_\Omega \cdot \mu \le f(\mathbf{1}_\Omega) =: \overline{v}_f(\Omega)$ and $-\mu(\Omega) = (-\mathbf{1}_\Omega) \cdot \mu \le f(-\mathbf{1}_\Omega) = -f(\mathbf{1}_\Omega) =: -\overline{v}_f(\Omega)$ since $\mathbf{1}_\Omega$ is frictionless as part of the definition of f pricing rule; thus $\mu(\Omega) = \overline{v}_f(\Omega)$. We have thus proved that $\mu \in \overline{\text{core}}(\overline{v}_f)$. □
• [(4) $\Longrightarrow$ (1)] For all $x \in \mathbb{R}^\Omega$, one has:

$$
\begin{aligned}
f(x) &= \sup\big\{x \cdot \mu \ : \ \mu \in \partial f(0)\big\} \ \text{ [from Proposition1]} \\
&= \sup\big\{x \cdot \mu \ : \ \mu \in \overline{\text{core}}(\overline{v}_f)\big\} \ \text{ [since } \partial f(0) = \overline{\text{core}}(\overline{v}_f) \text{ by (4)]} \\
&= \int_\Omega^C x \, d\overline{v}_f \ \text{ [by } \textit{Schmeidler} \text{ (1986) since } \overline{v}_f \text{ is submodular]}.
\end{aligned}
$$

Thus, the function f is a Choquet integral with respect to a submodular capacity. Hence, f is also submodular (see, for example Marinacci and Montrucchio 2004).
• [(4) $\Longleftrightarrow$ (4′) $\Longleftrightarrow$ (5)] The proof follows from the following Proposition 6. □
The following Proposition 6 provides a formulation of Proposition 5 for pricing rules, which is also of interest for itself. Note that, as in Proposition 5, no submodularity assumption is made in the following proposition.

Proposition 6 *Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be a pricing rule. The following assertions are equivalent:*

($\tilde{4}$) $\partial f(0) = \overline{\text{core}}(\overline{v}_f)$;
($\tilde{4}'$) $\partial f(0) = \overline{\text{core}}(v) \cap \mathbb{R}^\Omega_+$ *for some set function* $v : 2^\Omega \to \mathbb{R}$, $v(\emptyset) = 0$;

($\tilde{5}'$) *f is the super-hedging price of some present arbitrage-free market M_E with all event securities:*
$M_E := \big((\mathbf{1}_\Omega, v_\Omega, v_\Omega), (\mathbf{1}_A, 0, v_A))_{A \subsetneq \Omega}\big)$ *for some $v_A \in \mathbb{R}$, $v_\emptyset = 0$;*

($\tilde{5}$) *f is the super-hedging price of some present arbitrage-free market:*

$M' := \big((\mathbf{1}_\Omega, r^0, r^0), (\mathbf{1}_{A^j}, \underline{r}^j, \overline{r}^j)_{j \in \mathbf{J}'}\big),$
for some events $A^j \subseteq \Omega$ and prices $\underline{r}^j, \overline{r}^j$ ($j \in \mathbf{J}'$ finite).

Proof of Proposition 6 • [($\tilde{4}$) $\implies$ ($\tilde{4}'$)] Assertion ($\tilde{4}'$) holds with $v := \overline{v}_f$ since $\overline{\text{core}}(v) \subseteq \mathbb{R}_+^\Omega$ for $v := \overline{v}_f$ is monotone. □

- [($\tilde{4}'$) $\implies$ ($\tilde{5}'$)] From ($\tilde{4}'$) one has $\partial f(0) = \overline{\text{core}}(v) \cap \mathbb{R}_+^\Omega$ and by Lemma 1.a, we have $\overline{\text{core}}(v) \cap \mathbb{R}_+^\Omega = \mathcal{M}_+(M_E)$. Thus, $\partial f(0) = \mathcal{M}_+(M_E)$. Hence by Proposition 1, $f(x) = \sup_{\partial f(0)} x \cdot \mu = \sup_{\mathcal{M}_+(M_E)} x \cdot \mu$ for all x, that is, f is the super-hedging price of the market M_E by Theorem 1. □
- [($\tilde{5}'$) $\implies$ ($\tilde{5}$)] Immediate.
- [($\tilde{5}$) $\implies$ ($\tilde{4}$)] From ($\tilde{5}$), f is the super-hedging price of the market M', thus $\overline{v}_f = \overline{v}_{M'}$. Moreover, $\partial f(0) = \mathcal{M}'_+$ by Proposition 1 (see also Lemma 2 hereafter). But, by Lemma 1.(*b*) we have $\mathcal{M}'_+ = \overline{\text{core}}(\overline{v}_{M'})$. Thus, $\partial f(0) = \overline{\text{core}}(\overline{v}_f)$. □

4.2 Proof of Theorem 2

We now give the proof of Theorem 2 as a consequence of Theorem 3. We prepare the proof with a lemma that provides a direct proof of the assertion that $\partial c_+(0) = \mathcal{M}_+ \neq \emptyset$ (see Proposition 1).

Lemma 2 *Consider the present arbitrage-free market:*

$$M := \big((V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big).$$

Then $\partial c_+(0) = \mathcal{M}_+ \neq \emptyset$.

Proof of Lemma 2 $\star$ [$\mathcal{M}_+ \subseteq \partial c_+(0)$] From the Fundamental Theorem of Asset Pricing (Theorem 1), since the market is present arbitrage-free we have:

$$c_+(x) = \sup \left\{x \cdot \mu' \ : \ \mu' \in \mathcal{M}_+\right\} \text{ for all } x \in \mathbb{R}^\Omega.$$

Let $\mu \in \mathcal{M}_+$. From above, $x \cdot \mu \le c_+(x)$ for all $x \in \mathbb{R}^\Omega$, hence $\mu \in \partial c_+(0)$ (since $c_+(0) = 0$). □

$\star$ [$\partial c_+(0) \subseteq \mathcal{M}_+$] Let $\mu \in \partial c_+(0)$, then for all $x \in \mathbb{R}^\Omega$, $x \cdot \mu \le c_+(x)$ (since $c_+(0) = 0$). We first prove that $\mu \ge 0$. Indeed, for all $\omega \in \Omega$, taking $x := -\mathbf{1}_\omega$ we get $-\mu(\omega) = (-\mathbf{1}_\omega) \cdot \mu \le c_+(-\mathbf{1}_\omega) \le c_+(0) = 0$ since c_+ is monotone and $c_+(0) = 0$ (Proposition 1); thus $\mu \ge 0$.

Now taking successively $x := V^j$ and $x := -V^j$ we deduce that $V^j \cdot \mu \le c_+(V^j) \le \overline{q}^j$ and $(-V^j) \cdot \mu \le c_+(-V^j) \le -\underline{q}^j$ from the definition of the super-hedging price c_+ (taking respectively $(\alpha, \beta) := (\mathbf{1}_j, 0)$ and $(\alpha, \beta) := (0, \mathbf{1}_j)$). Consequently, $\underline{q}^j \le V^j \cdot \mu \le \overline{q}^j$ for all $j \in \mathbf{J}$. Thus $\mu \in \mathcal{M}_+$. □

Proof of Theorem 2 Since the market M is present arbitrage-free, with frictionless bond $\mathbf{1}_\Omega$, its super-hedging price c_+ is a pricing rule (Proposition 1). From Theorem 3, taking $f := c_+$, we notice that $\overline{v}_f = \overline{v}_M$ and $\partial f(0) = \partial c_+(0) = \mathcal{M}_+$ by Lemma 2. Moreover, if $f := c_+$, the super-hedging price of M, is equal to c'_+, the super-hedging price of the market M', then M and M' have the same set of stochastic discount factors, since $\mathcal{M}_+ = \partial c_+(0) = \partial c'_+(0) = \mathcal{M}'_+$ by Lemma 2. Thus $M \sim M'$. Consequently, the proof of Theorem 2 follows directly from Theorem 3. □

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