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RESEARCH ARTICLE

Walrasian equilibrium theory with and without free-disposal: theorems and counterexamples in an infinite-agent context

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Abstract
This paper provides four theorems on the existence of a free-disposal equilibrium in a Walrasian economy: the first with an arbitrary set of agents with compact consumption sets, the next highlighting the trade-offs involved in the relaxation of the compactness assumption, and the last two with a countable set of agents endowed with a weighting structure. The results generalize theorems in the antecedent literature pioneered by Shafer–Sonnenschein in 1975, and currently in the form taken in He–Yannelis 2016. The paper also provides counterexamples to the existence of non-free-disposal equilibrium in cases of both a countable set of agents and an atomless measure space of agents. One of the examples is related to one Chiaki Hara presented in 2005. The examples are of interest because they satisfy all the hypotheses of Shafer's 1976 result on the existence of a non-free-disposal equilibrium, except for the assumption of a finite set of agents. The work builds on recent work of the authors on abstract economies, and contributes to the ongoing discussion on the modelling of "large" societies.

Keywords Exchange economy · Walrasian equilibrium · Discontinuous preferences · Free-disposal · Arbitrary set · Externalities · Bads

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Contents

1 Introduction

This paper presents

1. a technique, originally developed for Markov Processes and Statistical Decision Theory in Duanmu (2018), for extending theorems on finite models to infinite (including uncountable) models, applied here to free-disposal and non-free-disposal Walrasian equilibrium in exchange economies;
2. a novel model with a continuum of agents in which individual agents are not necessarily negligible, local externalities can be readily formulated, and in which we rely on convexity rather than monotonicity in order to focus our attention on the presence of bads like atmospheric CO_2 emissions;
3. a generalization to a continuum of agents of the state-of-the-art treatment of externalities, price dependencies, discontinuities and nonconvexities in finite and countably infinite economies developed in He and Yannelis (2016);
4. a weakening of the compactness assumptions on consumption sets in He and Yannelis (2016);
5. counterexamples showing that the existence results do not extend to the atomless measure space context without additional assumptions.

In recent work, Anderson et al. (2021), henceforth ADKU, presented a general existence theorem for abstract economies with an arbitrary set of agents, each of whose action sets lie in arbitrarily different locally convex topological vector spaces: that result generalizes benchmark results of Carmona and Podczeck (2016) and He and Yannelis (2016) from the finite or countably infinite agent context to uncountably many agents. That generalization rested on a technique, originally developed in the context of Markov Processes and Statistical Decision Theory by Duanmu (2018). In this paper, we explore the application of the ADKU theorem and Duanmu technique to the existence of Walrasian Equilibrium in exchange economies.

Of the many conceptual differences between abstract economies and exchange economies, we focus on the notion of aggregation. In abstract economies, there is no aggregation of consumptions and endowments, while in exchange economies, aggregation is central to the definitions of allocation and Walrasian equilibrium. In *finite* abstract and exchange economies as well as games, agents act independently, maximizing their preferences taking the environment (which may depend on the choices of others) as given. In exchange economies, the choice is guided by prices; this makes the equilibrium decentralized, in the sense that a social planner is not required to coordinate the actions of different agents. In economies and games with a measure space of agents, by contrast, agents' choices must be measurable, and thus cannot be wholly independent of each other.

ADKU provides a novel model of abstract economies with a continuum of agents in which, because agents' preferences are assumed continuous in the product topology, each agent chooses independently and measurability of the collective choices is not required. In this paper, we adapt that model to exchange economies with externalities that are continuous in the product topology. Our notion of aggregation, arbitrary sums, imposes no measurability requirement.[1] In an atomless exchange economy, individual agents are negligible. By contrast, in our continuum model with arbitrary sums, individual agents need not be negligible. In particular, we can readily formulate a notion of local externalities in which each agent's welfare is affected by the choices of nearby individual agents.

Aumann (1966) showed that the assumption of convexity of preferences can be dispensed with in atomless exchange economies, due to the convexifying effects of aggregation on the choices of individually negligible agents. Since agents need not be negligible in our context, we cannot dispense with the convexity assumption, although we can weaken it somewhat following He and Yannelis (2016). However, Aumann's paper requires the assumption that preferences are monotonic, an assumption that neither we nor (He and Yannelis 2016; Shafer 1976; Shafer and Sonnenschein 1976) require. While the monotonicity assumption is ubiquitous in Walrasian Equilibrium Theory, it is antithetical to the presence of bads, commodities that are harmful to most or all consumers. Hildenbrand (1970) works in a production economy. Instead of monotonicity, he assumes free disposal in production, which guarantees that bads (whether present in the endowments or generated as byproducts of production) can be freely disposed at an equilibrium that purports to exactly equate supply and demand. In an era of rapid climate change, some commodities (notably atmospheric CO_2 emission) fall decidedly into the category of bads and can no longer be brushed aside as peripheral or unimportant. The fact that the convexity assumption can be dispensed with in an atomless continuum of agents is a nice dessert, but not the main course. It is our strongly-held conviction that, in a world facing climate change, assuming convexity

[1] Clearly, there is a tradeoff. Continuity in the product technology imposes constraints on primitives (preferences), while measurability allows weaker constraints on primitives but requires stronger endogenous constraints on outcomes.

is a small price to pay for meaningfully incorporating bads.[2]

If preferences are strongly monotonic, equilibrium prices are strictly positive and the aggregate demand exactly equals the aggregate supply, so we obtain non-free-disposal equilibrium. If one weakens the monotonicity assumption, one must consider free-disposal as well as non-free-disposal equilibrium. In some settings, this is relatively harmless. For example, atmospheric oxygen is essential for life, but the supply is sufficient that everyone[3] can get all they need. While atmospheric CO_2 has doubled since the pre-industrial era, the amount of oxygen used up in that doubling is negligible compared to the supply, and no reasonably foreseeable technology could deplete the supply of oxygen. For that reason, in a model containing goods like food, shelter, …, and oxygen, society is perfectly happy with a free-disposal equilibrium in which the demand for oxygen is strictly less than the supply, and the price of oxygen is zero. However, if we add atmospheric CO_2 to the model, then a free-disposal equilibrium with a zero price for atmospheric CO_2 emissions is a recipe for accelerating climate change. A non-free-disposal equilibrium, or at least a reduced-disposal equilibrium, with a negative price for atmospheric CO_2 emissions, is required to stop or slow climate change. At its heart, free-disposal of bads is a problem of incomplete markets, in that some markets are not required to clear.

He and Yannelis (2016) assume that consumption sets are compact. With a finite number of commodities, there is a standard technique, perfected and consolidated in Debreu (1959), to first establish equilibrium in a truncated economy and then use the continuity of preferences to extend to the case of unbounded consumption sets. However, as He and Yannelis noted in their Remark 6, it is an open problem whether their weakened continuity assumption suffices for this extension. Our Theorems 2 and 3 below work with continuity assumptions considerably weaker than in the antecedent literature due to Debreu, Shafer, Sonnenschein and their followers, but stronger than those assumed by He and Yannelis. These results are new for finite economies, and we further extend them to economies with a continuum of agents aggregated by arbitrary sums. However, whether the continuity assumption of He and Yannelis is sufficient to dispense with the compactness assumption remains open.

We now turn to the central technical driver of this paper, the Duanmu (2018) technique for extending known results for finite models to models of arbitrary cardinality. The Duanmu technique is a novel application of nonstandard analysis. Previous work in nonstandard probability and mathematical economics focused primarily on studying hyperfinite models, uncountable models that nonetheless satisfy all the formal properties of finite models. Scholars used the Loeb measure construction to convert these hyperfinite models into standard measure-theoretic models whose properties could be established, and then derived asymptotic properties for large finite models. Thus,

[2] An anonymous referee kindly pointed out that the 1970s literature on mixed markets associated with the names of Hildenbrand, Mertens, Dreze, Gabszewicz, Schmeidler, Shitovitz and others (Hildenbrand 1974 is a standard reference), makes the convexity assumption only on the atoms. It is important to bear in mind that this literature concerns a setting without aggregated externalities in consumption and it is this that necessitates the convexity assumption on all agents, be they atomic or non-atomic; see for example Assumption A.2 in Khan and Vohra (1985) and their references.

[3] Except individuals who need additional oxygen, from tanks or oxygen concentrators, due to illness or disability. Oxygen concentrators and tanks should be treated as separate commodities from atmospheric oxygen, and carry a positive price.

a hyperfinite random walk becomes a Brownian motion, and this implies Donsker's Theorem. A Stieltjes integral with respect to a hyperfinite random walk becomes an Itô Integral, and this implies Itô's Lemma. Euler–Maruyama schemes on a hyperfinite random walk become solutions of stochastic differential equations (SDEs), and this leads to new standard theorems on strong solutions of SDEs. The nonstandard core of a hyperfinite exchange economy equals the set of nonstandard equilibria, and this leads to new standard theorems on the asymptotic behavior of the core. There are many other examples.

By contrast, Duanmu's technique embeds a standard model of arbitrary cardinality into a hyperfinite model. It has been known since the 1970s that this was possible, but it was not understood that doing so could be very useful. While the previous literature focused on using theorems that were true in the continuum (but false in finite models) to obtain asymptotic properties of large finite models, Duanmu's technique uses theorems that are *true* in finite models to prove new theorems about models of arbitrary cardinality. ADKU applied Duanmu's technique to abstract economies, this paper applies it to exchange economies with arbitrary sums, and we intend in future work to apply it to exchange and production economies with a (not necessarily atomless) measure space of agents.

Section 2 introduces basic notions of nonstandard analysis, and adds a primer on aggregation over an uncountable set. Sections 3, 4 and 5 presents our main results: the first with free disposal, the second without, and both with an arbitrary set of agents, and the third with a countable set of agents endowed with a weighting structures. We observe that Sect. 4 on economies without free-disposal is somewhat negative in tone and spirit, but perhaps the examples it presents are as important as the positive result in Sects. 3 and 5: these examples are inspired by those in Hara (2005, 2006, 2008) and Manelli (1991). Section 6 is devoted to the proofs of the results, and Sect. 7 concludes the paper by returning to the exciting possibilities opened by moving to the next steps in the trajectory of Walrasian theory as laid out in this work: the investigation of economies with continuum of agents and economies with an infinite number of commodities.

2 Mathematical preliminaries

2.1 Basics of nonstandard analysis

For the necessary background to nonstandard analysis we refer the reader to Arkeryd et al. (1996), Cutland et al. (1995) and Loeb and Wolff (2000).[4] We begin this section by a brief introduction to the relevant notation from nonstandard analysis.

We use * to denote the nonstandard extension map taking elements, sets, functions, relations, etc., to their nonstandard counterparts. In particular, $^*\mathbb{R}$ and $^*\mathbb{N}$ denote the nonstandard extensions of the reals and natural numbers, respectively. Given a

[4] See also Duanmu and Roy (2020) and Duanmu et al. (2018) for brief reviews, and to Khan and Sun (1997) for a user-friendly version done for mathematical economists. The content of this subsection is taken from Anderson et al. (2021).

topological space (Y, T) and let $y \in Y$, the monad of ${}^*y$, denoted by $\mu({}^*y)$[5] is defined to be the set $\bigcap\{{}^*U : y \in U \in T\}$. The near-standard points of ${}^*Y$ are points in the monad of some standard points. We use $\mathrm{NS}({}^*Y)$ to denote the subset of near-standard elements in ${}^*Y$.

We let $\mathsf{st} : \mathrm{NS}({}^*Y) \to Y$ denote the standard part map taking near-standard elements to their standard parts. In both cases, the notation elides the underlying space Y and the topology T, because the space and topology will always be clear from context. We conclude this section with the following well-known result of Arkeryd et al. (1996, Theorem 5.1).

Theorem (Arkeryd–Cutland–Henson) *Let X be a topological space. A set $A \subset X$ is compact if and only if for each $y \in {}^*A$, there is an $x \in A$ such that $y \in \mu({}^*x)$. In particular, X is compact if every point of ${}^*X$ is near-standard.*

2.2 Aggregation over uncountable sets

We start by introducing the following definition of arbitrary sum.

Definition 1 Let A be an arbitrary set and f be a non-negative real valued function on A. Then $\sum_{a\in A} f(a)$ is defined to be the supremum of the set of all finite sums $f(a_1)+f(a_2)+\ldots+f(a_k)$ where $k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k$ are distinct elements in A. When f is a vector-valued function, we apply the supremum property coordinate-wise.

Throughout this paper, we use $\sum$ to denote the arbitrary sum. Note that in the context with which we shall use aggregation over an uncountable set in this work, a finite arbitrary sum does *not* necessarily imply an essentially-countable model. For the reader to appreciate this subtlety, we need the definition of a Walrasian equilibrium, and so we return to the point once the definition is in place.

3 Walrasian equilibria with free disposal

In this section, we establish the existence of free-disposal Walrasian equilibrium for exchange economies with possibly infinitely many agents, under similar assumptions of He and Yannelis (2016, Theorem 2).

3.1 Compact consumption sets

Our definition of exchange economy follows from He and Yannelis (2016).

Definition 2 An *exchange economy* is $\mathcal{E}$ is a set of triples $\{(X_i, P_i, e_i) : i \in T\}$, where

1. T is an arbitrary set of agents;
2. $X_i \subset \mathbb{R}^l_{\geq 0}$ is the consumption set of agent i, and $X = \prod_{i\in T} X_i$ with the product topology;

[5] We sometimes write $\mu(y)$ for the monad when the context is clear.

3. $P_i : X \times \Delta' \twoheadrightarrow X_i$ is the preference correspondence of agent i, where $\Delta' = \{p \in \mathbb{R}^l : \|p\| = \sum_{k=1}^{l} |p_k| \leq 1\}$ is the set of all prices;
4. $e_i \in X_i$ is the initial endowment of agent i, where $e = \sum_{i \in T} e_i \neq 0$.

An exchange economy is said to have *finite aggregate endowment* if $\sum_{i \in T} e_i \ll \infty$.

Given a price $p \in \Delta'$, the budget set of agent i is $B_i(p) = \{x_i \in X_i : p \cdot x_i \leq p \cdot e_i\}$. Let $\psi_i(x, p) = B_i(p) \cap P_i(x, p)$ for each $i \in T$, $x \in X$ and $p \in \Delta'$.

Definition 3 Let $\mathcal{E} = \{(X_i, P_i, e_i) : i \in T\}$ be an exchange economy with finite aggregate endowment ($\sum_{i \in T} e_i < \infty$). A *free-disposal Walrasian equilibrium* for the economy $\mathcal{E}$ is $(\bar{x}, \bar{p}) \in X \times \Delta'$ such that

1. $\bar{p} \neq 0$;
2. For each $i \in T$, $\bar{x}_i \in B_i(\bar{p})$ and $\psi_i(\bar{x}, \bar{p}) = \emptyset$;
3. $\sum_{i \in T} \bar{x}_i \leq \sum_{i \in T} e_i$.

We now return to the point that that our assumption of a finite aggregate endowment does *not* necessarily imply a countable model. In the context of a Walrasian economy, while an agent may have zero endowment, this does not guarantee that at an equilibrium (p, x), her consumption need be zero. If the equilibrium price system p is strictly positive, the claim would indeed be true. However, given that our focus is on non-free-disposal, and on commodities that are universally conceived of as being undesirable and as "bads," we are interested in prices in which some components are allowed to be zero or negative. At equilibrium then, an agent with a zero endowment may well end up consuming a non-zero vector! In particular, he or she may choose to consume a positive amount of some bad with a negative price in order to finance the purchase of a positive amount of some good. At any equilibrium, only a countable number of agents will have non-zero consumption, but the countable set may well vary from equilibrium to equilibrium.

There is a further point to be made regarding the equilibrium that we are considering. It is customary to define the free-disposal equilibrium as capturing a situation where aggregate consumption is less than or equal to aggregate endowment, and where the price-system is non-negative. A non-free-disposal equilibrium, then, is where the aggregate consumption *equals* the aggregate endowment, and the price-system is allowed to have negative components. However, this taxonomy misses out an important third case. Consider an economy with two "bad" commodities, one of which can be freely disposed of, and the aggregate consumption of the other is equal to its endowment. Consider, for example, the freely disposed "bad" commodity to possibly be CO_2, while the second may be some composite of the ozone-depleting chlorofluorocarbons (CFCs) and hydrochlorofluorocarbons (HCFCs) that have been phased out under the Montreal Protocol. In the resulting equilibrium, the price of CO_2 is zero and it is freely-disposed, while the price of CFCs and HCFCs is sufficiently negative to deter their production. Thus, our definition of free-disposal equilibrium uses the same price space Δ' that we use for the non-free-disposal case. We show the existence of a free-disposal equilibrium in which the equilibrium price-system can be non-negative if we wish, but there may be other potentially attractive (from a social policy standpoint) equilibria in which some prices are negative.

He and Yannelis (2016) introduce the following "continuous inclusion property".

Definition 4 Let X and Y be convex subsets of $\mathbb{R}^\ell$.[6] A correspondence $F : X \twoheadrightarrow Y$ is said to have the *continuous inclusion property (CIP) at x* if there exists an open set O_x containing x and a nonempty correspondence $G : O_x \twoheadrightarrow Y$ such that $G(z) \subset F(z)$ for every $z \in O_x$ and con(G) has a closed graph, where "con" denotes the convex hull operation applied separately to each $x \in X$.

Our first main result establishes the existence of a free-disposal Walrasian equilibrium for exchange economies with an arbitrary set of agents and with a finite aggregate endowment: it generalizes the following theorem of He and Yannelis (2016, Theorem 2).

Theorem (He–Yannelis (Free-disposal)) *Suppose T is a finite set. Let $\mathcal{E} = \{(X_i, P_i, e_i) : i \in T\}$ be an exchange economy satisfying the following assumptions: for each $i \in T$:*

1. *X_i is a nonempty, compact and convex subset of $\mathbb{R}^l_{\geq 0}$;*
2. *ψ_i has the CIP at each $(x, p) \in X \times \Delta$ with $\psi_i(x, p) \neq \emptyset$ where $\Delta = \{p \in \mathbb{R}^l_{\geq 0} : \sum_{k=1}^{l} p_k = 1\}$;*
3. *$x_i \notin con(\psi_i(x, p))$ for all $(x, p) \in X \times \Delta$.*

Then $\mathcal{E}$ has a free-disposal Walrasian equilibrium.

We now present our result.

Theorem 1 *Let $\mathcal{E} = \{(X_i, P_i, e_i) : i \in T\}$ be an exchange economy with finite aggregate endowment. Suppose, for each $i \in T$:*

1. *X_i is a nonempty, compact and convex subset of $\mathbb{R}^l_{\geq 0}$;*
2. *ψ_i has the CIP at each $(x, p) \in X \times \Delta$ with $\psi_i(x, p) \neq \emptyset$;*
3. *$x_i \notin con(\psi_i(x, p))$ for all $(x, p) \in X \times \Delta$.*

Then $\mathcal{E}$ has a free-disposal Walrasian equilibrium. Moreover, there is at least one equilibrium in which the price vector is an element of Δ.

Next we present two examples of economies that illustrate and supplement the above theorem.[7] The first pertains to an economy with local externalities and infinitely many players, while the second depicts an economy with a continuum of equilibrium allocations such that at any equilibrium, only a countable number of agents will have non-zero consumption, but this countable set varies from equilibrium to equilibrium.

Example 1 The set of agents T is the set of integers $\mathbb{Z}$. For each $t \in T$, let $X_t = [0, 1]$ so each X_t is a non-empty, compact and convex subset of $\mathbb{R}$. The endowment for agent t is $e_t = \frac{1}{2^{|t|}}$. Hence the economy has the finite aggregate endowment property. The utility function of agent t is given by $u_t(x) = 2x_t - \sum_{k \in \mathbb{Z}} \frac{x_{t+k}}{2^{|k|}}$. Each player's consumption affects every other player, but the affects that of player t's consumption

[6] He and Yannelis (2016) prove their result for an abstract economy in the setting of locally convex topological vector spaces: in this paper, we limit ourselves to finite-dimensional Euclidean spaces.

[7] The two examples presented here are motivated by comments of two anonymous referees to whom the authors are grateful.

is greatest on those players closest to t, and diminishes rapidly for players further from t. Note that the set of possible prices is $\Delta = \{1\}$. Notice that in the term for $k = 0$ in the sum, we have $x_{t+0}/2^0 = x_t$ which cancels out half of the first term $2x_t$. Finally, note that there is a unique equilibrium: $x_t = 1/2^{|t|}$ for all $t \in T$ and $p = 1$.

Example 2 The set of agents T is $[0, \infty)$. For each $t \in T$, let $X_t = [0, 1]^2$ so each X_t is a non-empty, compact and convex subset of $\mathbb{R}^2$. The endowment for agent t is $e_t = (\frac{1}{2^{|t|}}, \frac{1}{2^{|t|}})$ if t is an integer and $e_t = (0, 0)$ otherwise. Hence the economy has the finite aggregate endowment property. For $x \in X = \prod_{t\in T} X_t$, let $x(i)_t$ denote the amount of the i-th commodity for agent t. There are two goods, so we must have the set of prices equaling $\Delta' = \{p : \|p\|_1 \leq 1\}$. The utility function of agent t is given by $u_t(x) = x(1)_t$, and hence is independent of his/her consumption of commodity 2 as well as of the consumption of any other player. If the price of the second good is positive, everyone consumes zero of it, and there is excess supply. If the price of the second good is negative, everyone wants to consume an unbounded amount of it and use the income generated to purchase an unbounded amount of good one, so there is excess demand for both goods. So at equilibrium, the price of the second good must be zero, and the price of the first good must be an arbitrary element $\alpha \in (0, 1]$, since we normalize prices in Δ'. The demand set of agent t at the price $(\alpha, 0)$ is $\{1/2\}^{|t|} \times [0, \infty)$ if t is an integer, $\{0\} \times [0, \infty)$ otherwise. Given any allocation $\{x(2)_t : t \in T\}$ of good 2, let $x(1)_t = e(1)_t$. Then x is an equilibrium allocation of the economy corresponding to the price $(\alpha, 0)$. Note that only a countable agents consume a positive amount; however, for every countable subset $U \subset T$, there is an equilibrium allocation in which the set of agents with nonzero consumption of good 2 is exactly U. Thus, there are an uncountable number of equilibrium allocations.

3.2 Unbounded consumption sets

The existence proofs that are potentially relevant to this paper each come in two steps: the first step for a compactified economy, and the second step showing the irrelevance of the suitably-chosen compactification if it is properly chosen. The approach due to Debreu is set in finite economies and fully exploits the convexity postulate on preferences, whereas that due to Aumann, and Schmeidler and Hildenbrand following him, is set in an atomless measure-theoretic economy, and has therefore no need to assume convexity.[8] Since we are not restricting ourselves to atomless economies, the convexity assumption and Debreu's approach is available to us, and it is interesting that we can exploit it even in the context of economies with an arbitrary set of agents. But there are subtleties here that ought to be noted. First, Debreu's argument makes no use of the continuity postulate in his second step; and second, his postulates on the given (untruncated) economy translate trivially to analogous postulates for the truncated

[8] Note that Debreu uses only one truncation and the notion of "asymptotic cones," while the Aumann–Schmeidler–Hildenbrand approach relies on a sequence of truncations and on versions and generalizations of Fatou's lemma. Aumann and Schmeidler also need to assume monotonicity, which is not required by Debreu or Hildenbrand. However, Hildenbrand (1970), who works in a production economy, assumes free disposal in production, an assumption that guarantees that bads (whether present in the endowment or generated as byproducts of production) can be freely disposed. It should be noted that Florenzano (2003) could be seen as possibly presenting a third approach that synthesizes the two categorized above.

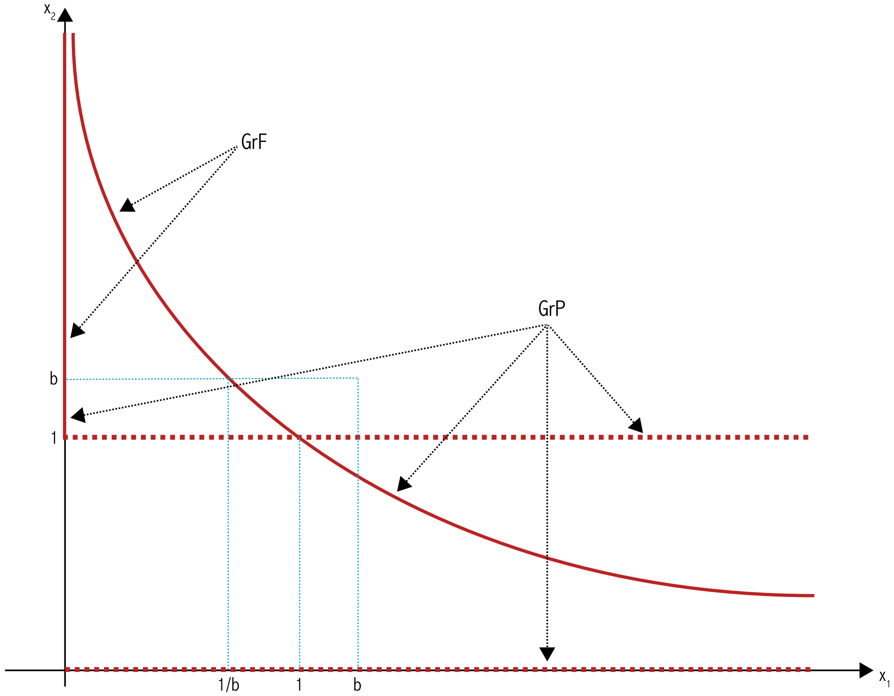

Fig. 1 The failure of the truncation argument under the CIP

one. The truncation technique works well when preferences have open fibers or open graph; see Florenzano (2003) for extended illustration. This is no longer true under the relaxed continuity postulate that we have so far been working with in this paper. This is somewhat of a surprise: to restate it in specific terms, the CIP does not necessarily carry over to the truncated consumption set.[9] We present an example that demonstrates this difficulty: it limits itself to an economy with a single agent and a single commodity (Fig. 1).

Example 3 Let $X = \mathbb{R}_+$ and $P : X \twoheadrightarrow X$ defined as $P(x) = \{1/x\} \cup \{1\}$ if $x > 0$ and rational, $P(x) = \{1/x\} \cup \{0\}$ if x is irrational, and $P(0) = [1, \infty)$. Consider a correspondence $F : X \twoheadrightarrow X$ whose graph is $\{(x, y)|y = 1/x, x > 0\} \cup (\{0\} \times [1, \infty)\})$. Then F has non-empty and convex values, and a closed graph (and also upper hemicontinuous). Since $F(x) \subseteq P(x)$, then P has the CIP.

Let $A = [0, 1]$. Consider any convex and compact set K containing A in its interior. Then $K = [0, b]$, $b > a$. Define the restricted correspondence $P|_K$ on the truncated set K as $P|_K : K \twoheadrightarrow K$ where $P|_K(x) = P(x) \cap K$. Then, $P|_K(0) = [1, b]$ and for all $x \in (0, 1/b)$, $P|_K(x) = \{1\}$ if x is rational and $P|_K(x) = \{0\}$ if x is irrational, and for all $x \geq 1/b$, $P|_K(x) = P(x)$. It is clear that $P|_K$ does not have the CIP at 0.

The example above shows that the preference correspondence in every truncated choice set is not well-behaved and hence proofs based on truncation techniques cannot

[9] See also He and Yannelis (2017) for further discussion on the properties of correspondences with the CIP.

be used. Next, we present a simple result based on the strengthening of the CIP[10] that fulfills the requirement that for each truncation of the economy, there exists a large enough compact, convex consumption set containing it such that the preference correspondence restricted to this covering consumption set has the CIP.

Definition 5 Let X, Y be two subsets of $\mathbb{R}^n$. A correspondence $F : X \twoheadrightarrow Y$ has the *bounded continuous inclusion property* if for each bounded subset A of X, there exists a convex and compact set B containing A such that the restricted correspondence $F|_B : B \twoheadrightarrow Y$ has the CIP at each $x \in B$ such that $F|_B(x) \neq \emptyset$.

Theorem 2 *Let $\mathcal{E} = \{(X_i, P_i, e_i) : i \in T\}$ be an exchange economy with finite aggregate endowment ($\sum_{i \in T} e_i < \infty$). Suppose, for each $i \in T$:*

1. *X_i is a nonempty, closed, bounded below and convex subset of $\mathbb{R}^l$;*
2. *ψ_i has the bounded CIP at each $(x, p) \in X \times \Delta$ with $\psi_i(x, p) \neq \emptyset$;*
3. *$x_i \notin con(\psi_i(x, p))$ for all $(x, p) \in X \times \Delta$;*
4. *for all $(x, p) \in X \times \Delta$, $y_i \in P_i(x, p)$ and all $\lambda \in (0, 1)$, there exists $\delta < \lambda$ such that $\delta y_i + (1 - \delta)x_i \in P_i(x, p)$.*

Then $\mathcal{E}$ has a free-disposal equilibrium.

Remark 1 He–Yannelis write: "We have imposed the compactness condition on the consumption set. It is not clear to us at this stage whether this condition can be dispensed with. [...] Consequently, relaxing the compactness assumption seems to be an open problem."[11] Whereas we partially resolve this open problem in Theorem 2 above, it is important for the reader to appreciate that the example above is not a counterexample to the existence of an equilibrium under the CIP: that remains an open problem.[12]

4 Walrasian equilibria without free disposal

The existence of a non-free-disposal Walrasian equilibrium in an exchange economy with finitely many agents under moderate regularity conditions is well-kown; see for example Shafer (1976, Theorem 2), Bergstrom (1976, Theorem 2) and He and Yannelis (2016, Theorem 4). It is natural to ask whether such an existence result extends to an economy with infinitely many agents. We answer this question in the negative by providing counterexamples. Consider first the following definition, now standard in the literature.

Definition 6 A *(non-free-disposal) Walrasian equilibrium*[13] for the exchange economy $\mathcal{E} = \{(X_i, P_i, e(i)) : i \in T\}$ is $(\bar{x}, \bar{p}) \in X \times \Delta'$ such that

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[10] The treatment of Podczeck and Yannelis referred to in Footnote 1 below does not proceed in this line of development.

[11] Please see He and Yannelis (2016, Remark 6 on page 506) and the accompanying text.

[12] Shortly before the second version of this paper was submitted, we received a note from editor Yannelis with a solution to this problem: Podczeck and Yannelis (2021, June 30).

[13] Note that for his definition of a non-free-disposal equilibrium $(\bar{x}, \bar{p})$, Shafer (1976) requires that for all $i \in T$, $\bar{p} \cdot \bar{x}(i) = \bar{p} \cdot e(i)$ instead of $\bar{x}(i) \in B_i(\bar{p})$. It is trivial that these his definition is equivalent to Definition 6 above.

1. $\bar{p} \neq 0$;
2. For each $i \in T$, $\bar{x}(i) \in B_i(\bar{p})$ and $\psi_i(\bar{x}, \bar{p}) = \emptyset$;
3. $\sum_{i\in T} \bar{x}(i) = \sum_{i\in T} e(i)$.

He and Yannelis (2016) establish the following result.[14]

Theorem (He–Yannelis (Non-free-disposal)) *Let T be a finite set of agents. Let $\mathcal{E}$ be an exchange economy satisfying the following assumptions: for each $i \in T$,*

1. *X_i is a nonempty compact convex subset of $\mathbb{R}^l_{\geq 0}$;*
2. *ψ_i has the CIP at each $(x, p) \in X \times \Delta'$ with $\psi_i(x, p) \neq \emptyset$;*
3. *$x_i \notin con(\psi_i(x, p))$ for all $(x, p) \in X \times \Delta'$;*
4. *For each $p \in \Delta'$ and each x in the set of feasible allocations $\mathcal{A} = \left\{x' \in X : \sum_{t\in T} x'(t) = \sum_{t\in T} e(t)\right\}$, there exists $t \in T$ such that $P_t(x, p) \neq \emptyset$.*

Then $\mathcal{E}$ has a (non-free-disposal) Walrasian equilibrium.

The following example shows that the above theorem does not hold without additional assumptions when T is infinite.

Example 4 Let $T = \{0, 1, 2, \ldots\}$ be the set of agents. For each $t \in T$, let $X_t = [0, 3] \times [0, 3]$ so each X_t is a non-empty, compact and convex subset of $\mathbb{R}^2_{\geq 0}$. So there are two items in this economy, we should refer them as the first item and the second item. The endowment for agent t is $e(t) = \left(\frac{1}{2^t}, \frac{1}{2^t}\right)$. Hence the economy has the finite aggregate endowment property. For $x \in X = \prod_{t\in T} X_t$, let $x(i)$ denote the i-th coordinate of x. The utility function of agent 0 is given by $u_0(x(0)) = (x(0))_1 - (x(0))_2$, where $(x(0))_1$ and $(x(0))_2$ denote the first and second coordinates of $x(0)$, respectively. For $t > 0$, the utility function of agent t is given by $u_t(x(t)) = (x(t))_1 - \frac{(x(t))_2}{t}$.

The preference correspondence of agent $t \in T$ generated by its utility function is defined as $P_t : X \times \Delta' \to X_t$, where $P_t(x, p) = \{y(t) \in X_t : u_t(y(t)) > u_t(x(t))\}$. Hence preference correspondences do not depend on prices or the actions of the other agents. Note that, for each agent t, the preference correspondence P_t, induced by a continuous utility function, has open graph. Moreover, for each agent t, the endowment $e(t)$ is in the interior of the consumption set X_t, and hence the budget correspondence B_t is both upper and lower hemicontinuous. Hence, for each agent $t \in T$, the correspondence $\psi_t = B_t \cap P_t$ satisfies the CIP (by the argument in the proof of Corollary 1 in He and Yannelis 2016). In fact, it is straightforward to verify that all assumptions of Theorem [He–Yannelis (Non-free-disposal)], except for finiteness of T, are satisfied. We now show that there is no (non-free-disposal) Walrasian equilibrium. Let the price vector be (p_1, p_2).

1. If $p_1 \leq 0$, then every agent demands 3 units of the first commodity. So there is excess demand for the first commodity, hence such a price system cannot be an equilibrium price.

[14] It maybe worth pointing out to the reader that the theorem below can be proved in two alternative ways, one involving the modified budget set of Bergstrom–Shafer, and the other with the budget set as conventionally defined; see the last paragraph on page 507 of He and Yannelis (2016, Theorem 4). Please note that in the statement of the theorem, unlike Conditions 1,2 and 3, Condition 4 has no reference to index i.

2. Suppose $p_1 > 0$. Without loss of generality, we can set $p_1 = 1$ via normalization. Note that

   – If $p_2 \geq 0$, then every agents demands 0 unit of the second commodity. So there is excess supply for the second commodity, hence such a price system cannot be an equilibrium price.
   – If $p_2 < 0$, then for every agent $t > \frac{1}{|p_2|}$, the "bad" commodity, commodity 2, becomes less costly in terms of utility, hence agent t consumes 3 units of the second commodity in order to consume highest possible amount of the first commodity. Then agent t' s excess demand for commodity 2 is positive, and as t increases his excess demand of commodity 2 increases. Even though every agent $t < \frac{1}{|p_2|}$ sells their endowment of the second commodity and consumes only commodity 1, the aggregate demand of the second commodity is infinite. As there is excess demand for the second commodity, such a price system cannot be an equilibrium price.

   Therefore, there is no non-free disposal equilibrium.

Our second example, closely related to the example in Hara (2005), shows that the second theorem of He–Yannelis for a non-free disposal setting reported above does not hold, without additional assumptions in an economy with a measure space of agents.[15]

Example 5 Let $T = (0, 1)$, endowed with the Lebesgue measure. There are two items in this economy, we should refer them as the first item and the second item. For each $t \in T$, let $e(t) = (t, t)$ and $X_t = [0, \frac{1}{t}] \times [0, t + \frac{1}{t^2}]$. So each X_t is a non-empty, compact and convex subset of $\mathbb{R}^2_{\geq 0}$. Note that we also have $\int e(t)\mathrm{d}t = (\frac{1}{2}, \frac{1}{2})$. The utility function for each agent t is $u_t(x) = (x(t))_1 - t(x(t))_2$. The preference correspondence of each agent is generated by its utility function. As in Example 4, it can be verified that all assumptions of Theorem [He–Yannelis (Non-free-disposal)], except for finiteness of T, are satisfied.

We now show that there is no (non-free-disposal) Walrasian equilibrium. Let the price vector be (p_1, p_2).

1. If $p_1 \leq 0$, then agent t would demand $(\frac{1}{t}, 0)$. The aggregate demand for the first item is infinite hence there is no equilibrium.
2. If $p_1 > 0$, without loss of generality, we can set $p_1 = 1$ via normalization. Note that

   – If $p_2 \geq 0$, then every agent demands 0 unit of the second item hence there is excess supply of the second item. So there is no equilibrium.
   – Suppose $p_2 < 0$, consider those agents t with $t < \frac{1}{|p_2|}$. Then agent t's demand is $(\frac{1}{t}, \frac{(p_2+1)t-\frac{1}{t}}{p_2})$. The aggregate demand for the first item is no less than $\int_0^{\frac{1}{|p_2|}} \frac{1}{t}\mathrm{d}t = \infty$. Hence there is no equilibrium.

[15] We plan to address this question in forthcoming work.

5 Weighted economies with a countable set of agents

For exchange economies with an infinite agent space, it is common to impose a measure-theoretical structure on the agent space. In this section, we establish the existence of free and non-free-disposal equilibrium for such exchange economies with a countable agent space endowed with a finite measure. Precisely, we consider the following type of weighted exchange economies:

Definition 7 A *weighted exchange economy* $\mathcal{E}$ is a quadruple $\{(X_i, P_i, e(i), \nu) : i \in T\}$, where:

1. T is a countable set;
2. ν is a probability measure on T;
3. $X_i \subset \mathbb{R}^l_{\geq 0}$ is the consumption set of agent i, and $X = \prod_{i \in T} X_i$ with the product topology;
4. $P_i : X \times \Delta' \twoheadrightarrow X_i$ is the preference correspondence of agent i;
5. $e(i) \in X_i$ is the initial endowment of agent i, where $0 < \int_{i \in T} e(i)\nu(\mathrm{d}i) < \infty$;

We now give the definition of free and non-free-disposal equilibrium for weighted exchange economies.

Definition 8 Let $\mathcal{E} = \{(X_i, P_i, e(i), \nu) : i \in T\}$ be a weighted exchange economy defined in Definition 7. A *free-disposal Walrasian equilibrium* is $(\bar{x}, \bar{p}) \in X \times \Delta'$ such that the following conditions are satisfied:

1. $\bar{p} \neq 0$;
2. $\bar{x}(i) \in B_i(\bar{p})$ and $\psi_i(\bar{x}, \bar{p}) = \emptyset$ for almost all $i \in T$;
3. $\int \bar{x}(i)\nu(\mathrm{d}i) \leq \int e(i)\nu(\mathrm{d}i)$.

A *non-free-disposal Walrasian equilibrium* is $(\bar{y}, \bar{q}) \in X \times \Delta'$ is a free-disposal Walrasian such that $\int \bar{y}(i)\nu(\mathrm{d}i) = \int e(i)\nu(\mathrm{d}i)$.

Our first major result of this section is the following:

Theorem 3 *Let $\mathcal{E} = \{(X_i, P_i, e(i), \nu) : i \in T\}$ be a weighted exchange economy satisfying the following assumptions: for each $i \in T$:*

1. *X_i is a nonempty, compact and convex subset of $\mathbb{R}^l_{\geq 0}$;*
2. *$\nu(\{i\}) > 0$;*
3. *ψ_i has the CIP at each $(x, p) \in X \times \Delta$ with $\psi_i(x, p) \neq \emptyset$;*
4. *$x(i) \notin con(\psi_i(x, p))$ for all $(x, p) \in X \times \Delta$.*

Then $\mathcal{E}$ has a free-disposal Walrasian equilibrium. Moreover, there is at least one free-disposal equilibrium in which the equilibrium price is an element of Δ. Suppose, in addition,

1. *T is finite;*
2. *For each $p \in \Delta'$ and each x in the feasible allocations $\mathcal{A} = \{y \in X : \sum_{t \in T} y(t)\mu(\{t\}) = \sum_{t \in T} e(t)\mu(\{t\})\}$, there exists $t \in T$ such that $P_t(x, p) \neq \emptyset$.*

Then $\mathcal{E}$ has a non-free-disposal Walrasian equilibrium.

Our next result extends Theorem 3 to weighted economies in which some agents may have weight zero.

Theorem 4 *Let $\mathcal{E} = \{(X_i, P_i, e_i, \nu) : i \in T\}$ be a weighted exchange economy satisfying the following assumptions: for each $i \in T$:*

1. *X_i is a nonempty, compact and convex subset of $\mathbb{R}^l_{\geq 0}$;*
2. *ψ_i has the CIP at each $(x, p) \in X \times \Delta$ with $\psi_i(x, p) \neq \emptyset$;*
3. *$x_i \notin con(\psi_i(x, p))$ for all $(x, p) \in X \times \Delta$.*

Then $\mathcal{E}$ has a free-disposal Walrasian equilibrium. Suppose, in addition,

1. *T is finite;*
2. *For each $p \in \Delta'$ and each x in the feasible allocations $\mathcal{A} = \{y \in X : \sum_{t \in T} y(t)\mu(\{t\}) = \sum_{t \in T} e(t)\mu(\{t\})\}$, there exists $t \in T$ with $\mu(\{t\}) > 0$ such that $P_t(x, p) \neq \emptyset$.*

Then $\mathcal{E}$ has a non-free-disposal Walrasian equilibrium.

Remark 2 It is not difficult to show that every finite (countably) weighted economy is isomorphic to a finite (countably) unweighted economy with finite aggregate endowment. Therefore, in Theorems 3 and 4, the existence of a free-disposal equilibrium follows from Theorem 1, and the existence of a non-free-disposal equilibrium follows from He and Yannelis (2016, Theorem 4). Hence, relaxation of the compactness of the consumption sets follows from Theorems 2 and 3 above. However, the direct measure-theoretic proofs that we provide are of independent interest.

Even though this paper is exclusively tailored to Walrasian theory, the importance of the transitivity assumption for such a theory leads us to present a simple example of a normal form game.[16] The point is that the following game has no Nash equilibrium in the canonical setting of a large non-anonymous game when all of the assumptions of standard results are satisfied except that of transitivity.[17]

Example 6 The set of agents T is $[0, 1]$ with Lebesgue measure. For each $t \in T$, let $X_t = \{1, 2, 3\}$. The agents have the same preference relation, and each agent cares only about her own action. For each $t \in T$ and $x, x', x'', x''' \in X = \prod_{i \in T} X_i$, $(1, x_{-t}) \succ (2, x'_{-t}) \succ (3, x''_{-t}) \succ (1, x'''_{-t})$, and $(1, x_{-t}) \sim (1, x'_{-t})$, $(2, x_{-t}) \sim (2, x'_{-t})$, $(3, x_{-t}) \sim (3, x'_{-t})$. However, it is easy to see that this game has no Nash equilibrium. Notice that the preference relation of each agent is continuous, complete and non-transitive, and also it depends trivially on the average of the society's action. Hence, all assumptions of Theorem 2 of Schmeidler (1973) are satisfied except transitivity.

[16] We leave it to the reader to execute the routine transformation of this example for a game to into one for a Walrasian economy.

[17] The canonical result on a large non-anonymous game that we have in mind is of course Theorem 2 of Schmeidler (1973) and its generalizations till 2000 comprehensively surveyed in Khan and Sun (2002).

6 Proof of the Theorems

Proof of Theorem 1 We give a proof by using nonstandard analysis. We start by setting up a few notations. For each $t \in T$, pick an element ϵ_t from X_t. We fix the collection $\{\epsilon_t : t \in T\}$ of points for the rest of the section. It is not important which ϵ_t we pick for each $t \in T$, any point suffices. Note that, by the transfer principle, we have a fixed collection $\{^*\epsilon_t : t \in {}^*T\}$ such that $^*\epsilon_t \in {}^*X_t$ for all $t \in {}^*T$. For every $B \subset T$ and every $x \in \prod_{t\in B} X_t$, define the extension $E(x)$ of x to be the point in $\prod_{t\in T} X_t$ such that $E(x)_t = x_t$ for $t \in B$ and $E(x)_t = \epsilon_t$ for $t \notin B$.

Definition 9 Let $F_i : X \twoheadrightarrow X_i$ be a correspondence and let B be a subset of T. The restriction of F_i to B is a correspondence $F_i^B : \prod_{t\in B} X_t \twoheadrightarrow X_i$ such that $F_i^B(y) = F_i(E(y))$.

We show that the restriction of a correspondence preserves upper hemicontinuity.

Lemma 1 *Suppose F_i is a upper hemicontinuous correspondence from X to X_i. Then, for every $B \subset T$, F_i^B is also upper hemicontinuous.*

Proof Fix $B \subset T$ and a point $x \in \prod_{t\in B} X_t$. Let V be an open set containing $F_i^B(x) = F_i(E(x))$. As F_i is upper hemicontinuous, there exists a basic open set U containing $E(x)$ such that $F_i(a) \subset V$ for every $a \in U$. Then $\pi_B(U)$ is an open set containing x. Note that, for every $y \in \pi_B(U)$, $E(y)$ is an element of U. Thus, for every $y \in \pi_B(U)$, we have $F_i^B(y) = F_i(E(y)) \subset V$. Hence, F_i^B is upper hemicontinuous. □

Lemma 2 *Suppose F_i is a correspondence from X to X_i such that $x_i \notin F_i(x)$ for every $x \in X$. Then, for every $B \subset T$ with $i \in B$ and every $y \in \prod_{t\in B} X_t$, $y_i \notin F_i^B(y)$.*

Proof Fix $B \subset T$ with $i \in B$ and pick $y \in \prod_{t\in B} X_t$. Then it is easy to see that $y_i = E(y)_i$. Thus, we have $y_i = E(y)_i \notin F_i(E(y)) = F_i^B(y)$. □

We now show that, if a correspondence has closed graph, then so is its restriction.

Lemma 3 *Suppose F_i is a correspondence from X to X_i that has a closed graph. Then, for every $B \subset T$, F_i^B also has a closed graph.*

Proof Let (x, z) be a point in $\prod_{t\in B} X_t \times X_i$ such that $z \notin F_i^B(x)$. This means that $z \notin F_i(E(x))$. As F_i has a closed graph, there is an open set U containing $(E(x), z)$ such that U is disjoint from the set $\{(a, b) \in X \times X_i | b \in F_i(a)\}$. Let V be the projection of U to the set $\prod_{t\in B} X_t \times X_i$. Note that V is an open set containing (x, z). For every point $(c, d) \in V$, we have $d \notin F_i(E(c)) = F_i^B(c)$. Thus, we conclude that F_i^B has a closed graph. □

We now show that the restriction of a correspondence preserves the CIP.

Lemma 4 *Let F_i be a correspondence from X to X_i that has CIP at every $x \in X$ such that $F_i(x) \neq \emptyset$. Then, for every $B \subset T$, F_i^B has CIP at every $y \in \prod_{t\in B} X_t$ such that $F_i^B(y) \neq \emptyset$.*

Proof Fix $B \subset T$ and pick $y \in \prod_{t \in B} X_t$ with $F_i^B(y) \neq \emptyset$. Then we have $E(y) \in \prod_{t \in T} X_t$ and $F_i(E(y)) = F_i^B(y) \neq \emptyset$. Thus, there exists an open set O around $E(y)$ and a non-empty correspondence $G : O \twoheadrightarrow X_i$ such that $G(z) \subset F_i(z)$ for every $z \in O$ and $\text{con}(G)$ has a closed graph. Without loss of generality, we assume that O is a basic open set. Then $\pi_B(O)$ is an open set containing y, and for every $a \in \pi_B(O)$, $E(a)$ is an element in O. Let $G^B : \pi_B(O) \twoheadrightarrow X_i$ be $G^B(a) = G(E(a))$. Then, G^B is clearly non-empty and, for every $a \in \pi_B(O)$, we have

$$G^B(a) = G(E(a)) \subset F_i(E(a)) = F_i^B(a).$$

We now show that $\text{con}(G^B)$ has a closed graph. For $a \in \pi_B(O)$, we have

$$\text{con}(G^B)(a) = \text{con}(G^B(a)) = \text{con}(G(E(a))) = \text{con}(G)(E(a)) = \text{con}(G)^B(a)$$

By Lemma 3, as $\text{con}(G)$ has a closed graph, we conclude that $\text{con}(G^B)$ has a closed graph. Thus, F_i^B has CIP at every $y \in \prod_{t \in B} X_t$ such that $F_i^B(y) \neq \emptyset$. □

By saturation, let $S_T \subset {}^*T$ be a hyperfinite set that contains T as a subset. For every $\theta \in S_T$, define ${}^*\psi_\theta^{S_T} : \prod_{t \in S_T} {}^*X_t \times {}^*\Delta \twoheadrightarrow {}^*X_\theta$ to be

$${}^*\psi_\theta^{S_T}(x, p) = {}^*\psi_\theta({}^*E(x), p) = {}^*B_\theta(p) \cap {}^*P_\theta({}^*E(x), p).$$

We consider the hyperfinite exchange economy $\{({}^*X_\theta, {}^*P_\theta^{S_T}, {}^*e_\theta) : \theta \in S_T\}$. Then, for $\theta \in S_T$, we have

1. ${}^*X_\theta$ is a nonempty, *compact and *convex subset of ${}^*\mathbb{R}^l_{\geq 0}$;
2. By the transfer of Lemma 4, ${}^*\psi_\theta^{S_T}$ has the *CIP at each $(x, p) \in \prod_{t \in S_T} {}^*X_t \times {}^*\Delta$ with ${}^*\psi_\theta^{S_T}(x, p) \neq \emptyset$;
3. By the transfer of Lemma 2, we have $x_\theta \notin {}^*H({}^*\psi_\theta^{S_T}(x, p))$ for all $(x, p) \in \prod_{t \in S_T} {}^*X_t \times {}^*\Delta$.

By the transfer of Theorem [He–Yannelis (Free-disposal)], there exists a *free-disposal Walrasian equilibrium for the hyperfinite exchange economy $\{({}^*X_\theta, {}^*P_\theta^{S_T}, {}^*e_\theta) : \theta \in S_T\}$. That is, there exists $(\bar{x}, \bar{p}) \in \prod_{t \in S_T} {}^*X_t \times {}^*\Delta$ such that

1. For each $\theta \in S_T$, $\bar{x}_\theta \in {}^*B_\theta(\bar{p})$ and ${}^*\psi_\theta^{S_T}(\bar{x}, \bar{p}) = \emptyset$;
2. $\sum_{t \in S_T} \bar{x}_t \leq \sum_{t \in S_T} {}^*e_t$.

Let $\bar{y} \in X$ be such that $\bar{y}_i = \mathsf{st}(\bar{x}_i)$ for every $i \in T$ and let $\bar{q} = \mathsf{st}(\bar{p})$. Note that $\bar{q} \in \Delta$. We now show that $(\bar{y}, \bar{q})$ is a free-disposal Walrasian equilibrium for $\mathcal{E}$.

Claim 1 For each $i \in T$, $\bar{y}_i \in B_i(\bar{q})$ and $\psi_i(\bar{y}, \bar{q}) = \emptyset$.

Proof Note that $\bar{x}_i \in {}^*B_i(\bar{p}) = \{z \in {}^*X_i : \bar{p} \cdot z \leq \bar{p} \cdot e_i\}$. As $\bar{q} \cdot \bar{y}_i \approx \bar{p} \cdot \bar{x}_i$ and $\bar{p} \cdot e_i \approx \bar{q} \cdot e_i$, we conclude that $\bar{y}_i \in B_i(\bar{q})$.

Suppose $\psi_i(\bar{y}, \bar{q})$ is not empty for some $i \in T$. By the CIP, there exists a basic open set O containing $(\bar{y}, \bar{q})$ and a nonempty correspondence $F : O \twoheadrightarrow X_i$ such that

$F(z) \subset \psi_i(z)$ for every $z \in O$. Note that $({}^*E(\bar{x}), \bar{p}) \in {}^*O$. Thus, by the transfer principle, we know that

$$ {}^*\psi_i^{S_T}(\bar{x}, \bar{p}) = {}^*\psi_i({}^*E(\bar{x}), \bar{p}) \neq \emptyset. $$

As $i \in S_T$, this is a contradiction. Thus, we have $\psi_i(\bar{y}, \bar{q}) = \emptyset$ for every $i \in T$. □

Claim 2 $\sum_{i \in T} \bar{y}_i \leq \sum_{i \in T} e_i$.

Proof By the transfer principle, we have $\sum_{i \in T} e_i = \sum_{t \in {}^*T} {}^*e_t$. Thus, we have $\sum_{t \in S_T} {}^*e_t \leq \sum_{i \in T} e_i$. As $\sum_{i \in T} e_i$ is finite, so is $\sum_{t \in S_T} {}^*e_t$. For every finite set $B \subset T$, we have

$$ \sum_{i \in B} \bar{y}_i \approx \sum_{i \in B} \bar{x}_i \leq \sum_{t \in S_T} \bar{x}_t. $$

Thus, we have

$$ \sum_{i \in T} \bar{y}_i \lessapprox \sum_{t \in S_T} \bar{x}_t < \sum_{t \in S_T} {}^*e_t \leq \sum_{i \in T} e_i. $$

As both $\sum_{i \in T} \bar{y}_i$ and $\sum_{i \in T} e_i$ are standard, we have $\sum_{i \in T} \bar{y}_i \leq \sum_{i \in T} e_i$. □

By Claims 1 and 3, $(\bar{y}, \bar{q})$ is a free-disposal Walrasian equilibrium for the exchange economy $\mathcal{E}$. □

We now turn to the proof of Theorem 2 in two steps: first, a result for a finite economy, and second, in keeping with a methodological innovation articulated in this paper, uplifting the finite economy result to that with an arbitrary number of agents using the methods of non-standard analysis.

Lemma 5 *Let $\mathcal{E} = \{(X_i, P_i, e_i) : i \in T\}$ be an exchange economy where T is finite. Let $X = \prod_{i \in T} X_i$. Suppose, for each $i \in T$:*

1. *X_i is a nonempty, closed, bounded below and convex subset of $\mathbb{R}^l$;*
2. *$\psi_i(x, p) = P_i(x, p) \cap B_i(p)$ has the bounded CIP;*
3. *$x_i \notin con(\psi_i(x, p))$ for all $(x, p) \in X \times \Delta$;*
4. *for all $(x, p) \in X \times \Delta$, $y_i \in P_i(x, p)$ and all $\lambda \in (0, 1)$, there exists $\delta < \lambda$ such that $\delta y_i + (1 - \delta)x_i \in P_i(x, p)$.*

Then $\mathcal{E}$ has a free-disposal equilibrium.

We then define the set of attainable consumption set for each consumer.

Definition 10 An *attainable state* of an economy $\mathcal{E} = \{(X_i, P_i, e_i) : i \in T\}$ is $x \in X$ such that $\sum_{i \in T}(x_i - e_i) \leqq 0$. The *attainable consumption set of consumer i* is

$$ \hat{X}_i = \{x_i \in X_i \mid \exists x_{-i} \in X_{-i} \text{ such that } (x_i, x_{-i}) \text{ is an attainable state}\}. $$

Let $\hat{X} = \prod_{i \in T} X_i$.

Proof of Lemma 5 *Step 1: Equilibrium in the Truncated Economy.* Note that since e_i is finite and X_i is bounded below for all i, $\hat{X}_i$ is a bounded set. Let K be compact and convex set in $\mathbb{R}^\ell$ containing in its interior $\hat{X}_i$ for all i. Define

$$\widehat{X}_i = X_i \cap K.$$

Define an economy $\widehat{\mathcal{E}} = \{(\widehat{X}_i, \widehat{P}_i, e_i) : i \in T\}$ where $\widehat{X} = \prod_{i \in T} \widehat{X}_i$ and $\widehat{P}_i : \widehat{X} \times \Delta \twoheadrightarrow \widehat{X}_i$ with $\widehat{P}_i(x, p) = P_i(x, p) \cap \widehat{X}_i$. Define $\widehat{\psi} : \widehat{X} \times \Delta \twoheadrightarrow \widehat{X}_i$ as $\widehat{\psi}_i(x, p) = \psi_i(x, p) \cap \widehat{X}_i$. By assumption (2), $\widehat{\psi}_i$ has the CIP. Note that $x_i \notin \text{con}(\widehat{\psi}_i(x, p))$ for all $(x, p) \in \widehat{X} \times \Delta$.

By Theorem (He–Yannelis), there exists a free disposal equilibrium (x^*, p^*) for the economy $\widehat{\mathcal{E}}$. Hence, $p^* \in \Delta$, $\sum_{i \in T}(x_i^* - e_i) \leqq 0$, and for all $i \in T$, $p^* \cdot x_i^* \leqq p^* \cdot e_i$ and $y_i \in \widehat{P}_i(x^*, p^*)$ implies $p^* \cdot y_i > p^* \cdot e_i$.

Step 2: Equilibrium in the Original Economy. We next show that (x^*, p^*) is a free disposal equilibrium for the economy $\mathcal{E}$. Establishing the following is sufficient:

$(*)$ *For every* $i \in T$, $x_i \in P_i(x^*, p^*)$ *implies* $p^* \cdot x_i > p^* \cdot e_i$.

It is clear from step 1 that $y_i \in P_i(x^*, p^*) \cap X_i \cap K$ implies $p^* \cdot y_i > p^* \cdot e_i$. Assume towards a contradiction that there exists $x_i \in X_i$ such that $x_i \in P_i(x^*, p^*)$ and $p^* \cdot x_i \leqq p^* \cdot e_i$. Since x_i^* is in the interior of K, one can find on the straight-line segment $[x_i^*, x_i']$ a point x_i'' different from x_i^* but sufficiently close to x_i^* to be in K. By assumption (4), there exists $\hat{x}_i \in [x_i^*, x_i'']$ such that $\hat{x}_i \in P_i(x^*, p^*)$. Note that $\hat{x}_i \in K$, hence in $\widehat{X}_i$. Since $p^* \cdot x_i \leqq p^* \cdot e_i$ and $p^* \cdot x_i^* \leqq p^* \cdot e_i$, $p^* \cdot \hat{x}_i \leqq p^* \cdot e_i$. This yields a contradiction. □

We now extend Lemma 5 to the setting with an arbitrary agent space. We first establish that a few key properties are preserved by restriction of correspondences.

Lemma 6 *Let B be a subset of T with $i \in B$. Under the conditions of Theorem 2, the correspondence $\psi_i^B : \prod_{t \in B} X_t \times \Delta \twoheadrightarrow X_i$ has the bounded CIP.*

Proof Pick a bounded set $K \subset \prod_{t \in B} X_t \times \Delta$. Without loss of generality, we assume that $K = K_1 \times K_2$, where K_1 is a bounded subset of $\prod_{t \in B} X_t$ and K_2 is a subset of Δ. Let $K_1' = \{E(x) | x \in K_1\}$. Clearly, K_1' is a bounded subset of X, hence $K_1' \times K_2$ is a bounded subset of $X \times \Delta$. As ψ_i satisfies the bounded continuous property, there is a compact and convex set V such that $\psi_i|_V : V \twoheadrightarrow X_i$ has the CIP. The projection V' of V into $\prod_{t \in B} X_t \times \Delta$ is a compact and convex set. By Lemma 3, $\psi_i^B|_V'$ has the CIP, completing the proof. □

Lemma 7 *Let B be a subset of T with $i \in B$. Under the conditions of Theorem 2, for all $(x, p) \in \prod_{t \in B} X_t \times \Delta$, all $a_i \in P_i^B(x, p)$ and all $\lambda \in (0, 1)$, there exists $\delta < \lambda$ such that $\delta a_i + (1 - \delta)x_i \in P_i^B(x, p)$.*

Proof Pick $(x, p) \in \prod_{t \in B} X_t \times \Delta$, $a_i \in P_i^B(x, p)$ and $\lambda \in (0, 1)$. Then there exists $\delta < \lambda$ such that $\delta a_i + (1 - \delta)E(x)_i \in P_i(E(x), p)$. As $i \in B$ and $P_i^B(x, p) = P_i(E(x), p)$, we conclude that $\delta a_i + (1 - \delta)E(x)_i = \delta a_i + (1 - \delta)x_i \in P_i^B(x, p)$, hence obtaining the desired result. □

We are now at the place to prove Theorem 2.

Proof of Theorem 2 By saturation, let $S_T \subset {}^*T$ be a hyperfinite set that contains T as a subset. We consider the hyperfinite exchange economy $\{({}^*X_\theta, {}^*P_\theta^{S_T}, {}^*e_\theta) : \theta \in S_T\}$. Then, for $\theta \in S_T$, we have

1. ${}^*X_\theta$ is a nonempty, *closed, *bounded below and *convex subset of ${}^*\mathbb{R}^l_{\geq 0}$;
2. By the transfer of Lemma 6, ${}^*\psi_\theta^{S_T} : \prod_{t \in S_T} {}^*X_t \times {}^*\Delta \twoheadrightarrow {}^*X_\theta$ has the *bounded CIP;
3. By the transfer of Lemma 2, we have $x_\theta \notin {}^*\mathrm{con}({}^*\psi_\theta^{S_T}(x, p))$ for all $(x, p) \in \prod_{t \in S_T} {}^*X_t \times {}^*\Delta$;
4. For all $(x, p) \in \prod_{t \in S_T} {}^*X_t \times {}^*\Delta$, all $a_\theta \in {}^*P_\theta^{S_T}(x, p)$ and all $\lambda \in {}^*(0, 1)$, by the transfer of Lemma 7, there exists $\delta < \lambda$ such that $\delta a_\theta + (1 - \delta)x_\theta \in {}^*P_\theta^{S_T}(x, p)$.

By the transfer of Lemma 5, the hyperfinite exchange economy $\{({}^*X_\theta, {}^*P_\theta^{S_T}, {}^*e_\theta) : \theta \in S_T\}$ has a free-disposal equilibrium $(\bar{x}, \bar{p}) \in \prod_{t \in S_T} {}^*X_t \times {}^*\Delta$. That is:

1. $\sum_{\theta \in S_T} \bar{x}_\theta \leq \sum_{\theta \in S_T} {}^*e_\theta$;
2. For all $\theta \in S_T$, $\bar{p} \cdot \bar{x}_\theta \leq \bar{p} \cdot {}^*e_\theta$ and ${}^*\psi_\theta^{S_T}(\bar{x}, \bar{p}) = \emptyset$.

As $\sum_{i \in T} e_i < \infty$, by the transfer principle, we conclude that $\sum_{\theta \in S_T} {}^*e_\theta < \infty$. Thus, for all $\theta \in S_T$, $\bar{x}_\theta$ is near-standard. Let $\bar{y} \in X$ be such that $\bar{y}_t = \mathsf{st}(\bar{x}_t)$ for every $t \in T$ and let $\bar{q} = \mathsf{st}(\bar{p})$. Note that $\bar{q} \in \Delta$. We now show that $(\bar{y}, \bar{q})$ is a free disposal Walrasian equilibrium for $\mathcal{E}$. For every $i \in T$, we have $\bar{q} \cdot \bar{y}_i \approx \bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot {}^*e_i \approx \bar{q} \cdot e_i$.

Claim 3 $\sum_{i \in T} \bar{y}_i \leq \sum_{i \in T} e_i$.

Proof By the transfer principle, we have $\sum_{i \in T} e_i = \sum_{t \in {}^*T} {}^*e_t$. Thus, we have $\sum_{t \in S_T} {}^*e_t \leq \sum_{i \in T} e_i$. As $\sum_{i \in T} e_i$ is finite, so is $\sum_{t \in S_T} {}^*e_t$. For every finite set $B \subset T$, we have

$$\sum_{i \in B} \bar{y}_i \approx \sum_{i \in B} \bar{x}_i \leq \sum_{t \in S_T} \bar{x}_t.$$

Thus, we have

$$\sum_{i \in T} \bar{y}_i \lessapprox \sum_{t \in S_T} \bar{x}_t < \sum_{t \in S_T} {}^*e_t \leq \sum_{i \in T} e_i.$$

As both $\sum_{i \in T} \bar{y}_i$ and $\sum_{i \in T} e_i$ are standard, we have $\sum_{i \in T} \bar{y}_i \leq \sum_{i \in T} e_i$. □

Claim 4 For all $i \in T$, $\psi_i(\bar{y}, \bar{q}) = \emptyset$.

Proof Suppose $\psi_i(\bar{y}, \bar{q})$ is not empty for some $i \in T$. Pick some bounded set $K \subset X \times \Delta$ that contains $(\bar{y}, \bar{q})$ as an interior point. By the bounded CIP, there exists a basic open set $O \subset K$ containing $(\bar{y}, \bar{q})$ and a nonempty correspondence $F : O \twoheadrightarrow X_i$ such that $F(z) \subset \psi_i(z)$ for every $z \in O$. Note that $({}^*E(\bar{x}), \bar{p}) \in {}^*O$. Thus, by the transfer principle, we know that

$${}^*\psi_i^{S_T}(\bar{x}, \bar{p}) = {}^*\psi_i({}^*E(\bar{x}), \bar{p}) \neq \emptyset.$$

As $i \in S_T$, this is a contradiction. Thus, we have $\psi_i(\bar{y}, \bar{q}) = \emptyset$ for every $i \in T$. □

Combining Claims 3 and 4, we see that $(\bar{y}, \bar{q})$ is a free-disposal equilibrium. □
We now turn to the proof of Theorem 3

Remark 3 Before proving Theorem 3, we set up a few notations. For $p \in \mathbb{R}$ and $A \subset \mathbb{R}^l$, let pA denote the set $\{px : x \in A\}$. For a correspondence $F : X \twoheadrightarrow \mathbb{R}^l$ and $p \neq 0$, let $pF : pX \twoheadrightarrow \mathbb{R}^l$ be the correspondence such that $(pF)(x) = pF(\frac{1}{p}x)$.

Remark 4 Since we are assuming $\nu(\{t\}) > 0$ for all $t \in T$, if $(\bar{x}, \bar{p})$ is a free-disposal Walrasian equilibrium, then $\bar{x}(t) \in B_t(\bar{p})$ and $\psi_t(\bar{x}, \bar{p}) = \emptyset$ for all $t \in T$.

Proof of Theorem 3 Let $\nu_t = \nu(\{t\})$ for every $t \in T$. Note that $\nu_t > 0$ for every $t \in T$. We consider the scaled exchange economy $\mathcal{E}' = \{(X'_t, P'_t, e'_t) : t \in T\}$, where $X'_t = \nu_t X_t$, $P'_t = \nu_t P_t$ and $e'_t = \nu_t e(t)$. Let $X' = \prod_{t \in T} X'_t$. To be precise, $P'_t : X' \times \Delta \twoheadrightarrow X'_t$ is a correspondence such that $P'_t(x, p) = \nu_t P_t(y, p)$, where $y_i = \frac{1}{\nu_i} x_i$ for all $i \in T$. The price set for $\mathcal{E}'$ is still $\Delta = \{p \in \mathbb{R}^l_{\geq 0} : \sum_{k=1}^{l} p_k = 1\}$. Given $p \in \Delta$, the budget set of agent t is $B'_t(p) = \{x_t \in X'_t : p \cdot x_t \leq p \cdot e'_t\}$. Note that $B'_t(p) = \nu_t B_t(p)$. Let $\psi'_t(x, p) = B'_t(p) \cap P'_t(x, p)$ for each $t \in T$, $x \in X'$ and $p \in \Delta$. Finally, if T is finite, let $\mathcal{A}' = \{b \in X' : \sum_{t \in T} b(t) = \sum_{t \in T} e'_t\}$.

Claim 5 ψ'_t has the CIP at each $(x, p) \in X' \times \Delta$ with $\psi'_t(x, p) \neq \emptyset$.

Proof Pick $(x, p) \in X' \times \Delta$ such that $\psi'_t(x, p) \neq \emptyset$. Let $y \in X$ be a point such that $y_t = \frac{1}{\nu_t} x_t$. By construction, we have $\psi'_t(x, p) = \nu_t \psi_t(y, p)$, hence $\psi_t(y, p) \neq \emptyset$. Then there exists an open set $U = V \times O$ ($y \in V$ and $p \in O$) and a nonempty correspondence $F : U \twoheadrightarrow X_t$ such that

1. $F(a, b) \subset \psi_t(a, b)$ for all $(a, b) \in U$;
2. $\mathrm{con}(F)$ has a closed graph.

Without loss of generality, we can assume that $V = \prod_{t \in T} V_t$, where V_t is an open set containing y_t for all $t \in T$. Let $V' = \prod_{t \in T} \nu_t V_t$ and $U' = V' \times O$. Then U' is an open set containing (x, p). Define $F' : U' \twoheadrightarrow X'_t$ to be a correspondence such that, for every $(a, b) \in U'$, $F'(a, b) = \nu_t F(z, b)$, where $z_t = \frac{1}{\nu_t} a_t$ for every $t \in T$. Clearly F' is nonempty and $F'(a, b) \subset \psi'_t(a, b)$ for all $(a, b) \in U'$. It remains to show that $\mathrm{con}(F')$ has a closed graph.

To show that $\mathrm{con}(F')$ has a closed graph, it is sufficient to show that, for every $(a, b) \in U'$, we have $\mathrm{con}(F')(a, b) = \nu_t \mathrm{con}(F)(z, b)$, where $z_t = \frac{1}{\nu_t} a_t$ for every $t \in T$. Clearly, we have $F'(a, b) \subset \nu_t \mathrm{con}(F)(z, b)$. As $\nu_t \mathrm{con}(F)(z, b)$ is also convex, we conclude that $\mathrm{con}(F')(a, b) \subset \nu_t \mathrm{con}(F)(z, b)$. Let C be any convex set that contains $F'(a, b)$ as a subset. Then $\frac{1}{\nu_i} C$ is a convex set containing $F(z, b)$ as a subset, hence containing $\mathrm{con}(F)(z, b)$ as a subset. Thus, we can conclude that $\nu_t \mathrm{con}(F)(z, b) \subset C$, which implies that $\nu_t \mathrm{con}(F)(z, b) \subset \mathrm{con}(F')(a, b)$. □

Claim 6 For each $t \in T$, we have $x_t \notin \mathrm{con}(\psi'_t(x, p))$ for all $(x, p) \in X' \times \Delta$.

Proof Pick $(x, p) \in X' \times \Delta$. Let $y \in X$ be a point such that $y_t = \frac{1}{\nu_t} x_t$. Then we have $y_t \notin \mathrm{con}(\psi_t(y, p))$. By the same proof as in Claim 5, we have $\mathrm{con}(\psi'_t(x, p)) = \nu_t \mathrm{con}(\psi_t(y, p))$. Hence we conclude that $x_t \notin \mathrm{con}(\psi'_t(x, p))$. □

Claim 7 For each $p \in \Delta'$ and a in the set of feasible allocations $\mathcal{A}' = \{b \in X' : \sum_{t \in T} b(t) = \sum_{t \in T} e'_t\}$, there exists $t \in T$ such that $P'_t(a, p) \neq \emptyset$.

Proof Pick $p \in \Delta'$ and $a \in \mathcal{A}'$. Let $x \in X$ be such that $x_i = \frac{1}{\nu_i} a_i$. We can conclude that $x \in \mathcal{A} = \{y \in X : \sum_{t \in T} y(t)\mu(\{t\}) = \sum_{t \in T} e(t)\mu(\{t\})\}$. Thus, there exists $t \in T$ such that $P_t(x, p) \neq \emptyset$. Note that $P'_t(a, p) = \nu_t P_t(x, p) \neq \emptyset$, completing the proof. □

We first establish the existence of a free-disposal Walrasian equilibrium. It is straightforward to verify that each X'_t is a nonempty, convex and compact subset of $\mathbb{R}^l_{\geq 0}$. By Claims 5 and 6, and Theorem 1, there exists a free-disposal Walrasian equilibrium $(\bar{x}, \bar{p}) \in X' \times \Delta$ for the scaled economy $\mathcal{E}'$. That is, we have

1. for each $t \in T$, $\bar{x}_t \in B'_t(\bar{p})$ and $\psi'_t(\bar{x}, \bar{p}) = \emptyset$;
2. $\sum_{t \in T} \bar{x}_t \leq \sum_{t \in T} e'_t$.

As $\sum_{i \in T} e'_i = \sum_{i \in T} \nu(\{i\}) e(i) = \int e(i)\nu(\mathrm{d}i) < \infty$, both $\sum_{i \in T} \bar{x}_i$ and $\sum_{i \in T} e'_i$ are well-defined countable sums. Let $\bar{y} \in X$ be such that $\bar{y}(t) = \frac{1}{\nu_t} \bar{x}_t$.

Claim 8 $(\bar{y}, \bar{p})$ is a free-disposal Walrasian equilibrium for $\mathcal{E}$.

Proof As $B'_i(\bar{p}) = \nu_i B_i(\bar{p})$, we conclude that $\bar{y}(i) \in B_i(\bar{p})$. Similarly, we have $\psi'_i(\bar{x}, \bar{p}) = \nu_i \psi(\bar{y}, \bar{p})$. Hence we conclude that $\psi_i(\bar{y}, \bar{p}) = \emptyset$. Finally, we know that

$$\int \bar{y}(i)\nu(\mathrm{d}i) = \sum_{i \in T} \bar{x}_i \leq \sum_{i \in T} e'_i = \int e(i)\nu(\mathrm{d}i),$$

completing the proof. □

Thus, by Claim 8, $\mathcal{E}$ has a free-disposal Walrasian equilibrium.

We now establish the existence of non-free-disposal Walrasian equilibrium, under two additional assumptions:

1. T is finite;
2. For each $p \in \Delta'$ and x in the set of feasible allocations $\mathcal{A} = \{y \in X : \sum_{t \in T} y(t)\mu(\{t\}) = \sum_{t \in T} e(t)\mu(\{t\})\}$, there exists $t \in T$ such that $P_t(x, p) \neq \emptyset$.

By Claims 5, 6 and 7, and Theorem [He–Yannelis (Non-free-disposal)], there exists a non-free-disposal Walrasian equilibrium $(\bar{a}, \bar{q}) \in X' \times \Delta'$ for the scaled economy $\mathcal{E}'$. That is, we have

1. $\|\bar{q}\| \neq 0$;
2. For each $t \in T$, $\bar{a}_t \in B'_t(\bar{q})$ and $\psi'_t(\bar{a}, \bar{q}) = \emptyset$;
3. $\sum_{t \in T} \bar{a}_t = \sum_{t \in T} e'(t)$.

Let $\bar{b} \in X$ be such that $\bar{b}_t = \frac{1}{\nu_t} \bar{a}_t$.

Claim 9 $(\bar{b}, \bar{q})$ is a non-free-disposal Walrasian equilibrium for $\mathcal{E}$.

Proof As $B'_i(\bar{q}) = \nu_i B_i(\bar{q})$, we conclude that $\bar{b}(i) \in B_i(\bar{q})$. Similarly, we have $\psi'_i(\bar{a}, \bar{q}) = \nu_i \psi(\bar{b}, \bar{q})$. Hence we conclude that $\psi_i(\bar{b}, \bar{q}) = \emptyset$. Finally, we know that

$$\sum_{t \in T} \bar{b}_t \nu_t = \sum_{t \in T} \bar{a}_t = \sum_{t \in T} e'_t = \sum_{t \in T} e_t \nu_t,$$

completing the proof. □

Thus, by Claim 9, $\mathcal{E}$ has a non-free-disposal Walrasian equilibrium. □

Finally, we provide a proof of Theorem 4.

Proof of Theorem 4 Let $T' = \{i \in T : \nu(\{i\}) > 0\}$. Note that $\nu(T') = 1$. Pick $\epsilon_t \in X_t$ for every $t \notin T'$. For $y \in \prod_{t \in T'} X_t = X'$, let $E(y)$ be the point in $X = \prod_{t \in T} X_t$ such that

1. $E(y)_t = y_t$ for all $t \in T'$;
2. $E(y)_t = e(t)$ for all $t \notin T'$.

For each $\theta \in T'$, define $\psi_\theta^{T'} : X' \times \Delta \twoheadrightarrow X_\theta$ to be

$$\psi_\theta^{T'}(x, p) = \psi_\theta(E(x), p) = B_\theta(p) \cap P_\theta(E(x), p).$$

Let $\mathcal{A}' = \{y \in X' : \sum_{t \in T'} y_t \nu_t = \sum_{t \in T'} e_t \nu_t\}$. The sub-economy $\mathcal{E}' = \{(X_\theta, P_\theta^{T'}, e_\theta, \nu) : \theta \in T'\}$ has the following properties:

Lemma 8 *$\psi_\theta^{T'}$ has the CIP at each $(x, p) \in X' \times \Delta$ such that $\psi_\theta^{T'}(x, p) \neq \emptyset$.*

Lemma 9 *$x_\theta \notin con(\psi_\theta^{T'}(x, p))$ for all $(x, p) \in X' \times \Delta$.*

Lemma 10 *Suppose T is finite. For each $p \in \Delta'$ and $x \in \mathcal{A}'$, there exists $\theta \in T'$ such that $P_\theta^{T'}(x, p) \neq \emptyset$.*

Proof Pick some $p \in \Delta'$ and $x \in \mathcal{A}'$. Then we have $E(x) \in \mathcal{A}$. Thus, there exists $\theta \in T'$ such that $P_\theta(E(x), p) \neq \emptyset$. So we can conclude that $P_\theta^{T'}(x, p) \neq \emptyset$. □

Consider the sub-economy $\mathcal{E}'$, for each $\theta \in T'$:

1. X_θ is a nonempty, compact and convex subset of $\mathbb{R}^l_{\geq 0}$;
2. $\nu(\{\theta\}) > 0$;
3. By Lemma 8, $\psi_\theta^{T'}$ has the CIP at each $(x, p) \in X' \times \Delta$ such that $\psi_\theta^{T'}(x, p) \neq \emptyset$;
4. By Lemma 9, we have $x_\theta \notin \text{con}(\psi_\theta^{T'}(x, p))$ for all $(x, p) \in X' \times \Delta$.

Under additional assumptions, for each $\theta \in T'$, the sub-economy $\mathcal{E}'$ has the following properties:

1. T' is finite;
2. By Lemma 10, for each $p \in \Delta'$ and $x \in \mathcal{A}'$, there exists $\theta \in T'$ such that $P_\theta^{T'}(x, p) \neq \emptyset$.

By Theorem 3, the sub-economy $\mathcal{E}'$ has a free-disposal equilibrium $(\bar{x}, \bar{p})$. Under additional assumptions, $\mathcal{E}'$ also has a non-free-disposal equilibrium $(\bar{a}, \bar{q})$.

Claim 10 $(E(\bar{x}), \bar{p})$ is a free-disposal equilibrium of $\mathcal{E}$.

Proof For each $i \in T'$, we have $E(\bar{x})_i = \bar{x}_i \in B_i(\bar{p})$. For each $i \in T'$, we also have:

$$\psi_i(E(\bar{x}), \bar{p}) = \psi_i^{T'}(\bar{x}, \bar{p}) = \emptyset.$$

As $\nu(T') = 1$, we conclude that $E(\bar{x})_i \in B_i(\bar{p})$ and $\psi_i(E(\bar{x}), \bar{p}) = \emptyset$ for almost all $i \in T$. Note that we have:

$$\sum_{i \in T} \nu(\{i\}) E(\bar{x})_i = \sum_{i \in T'} \nu(\{i\}) \bar{x}_i \leq \sum_{i \in T'} \nu(\{i\}) e_i = \sum_{i \in T} \nu(\{i\}) e_i,$$

completing the proof. □

Claim 11 $(E(\bar{a}), \bar{q})$ is a non-free-disposal equilibrium of $\mathcal{E}$.

Proof By the same argument as in Claim 10, we conclude that $E(\bar{a})_i \in B_i(\bar{q})$ and $\psi_i(E(\bar{a}), \bar{q}) = \emptyset$ for almost all $i \in T$. Finally, we have

$$\sum_{i \in T} E(\bar{a})_i \nu_i = \sum_{i \in T'} \bar{a}_i \nu_i = \sum_{i \in T'} e_i \nu_i = \sum_{i \in T} e_i \nu_i, \tag{1}$$

completing the proof. □

By Claim 10, $(E(\bar{x}), \bar{p})$ is a free-disposal Walrasian equilibrium for $\mathcal{E}$. By Claim 11, $(E(\bar{a}), \bar{q})$ is a non-free-disposal Walrasian equilibrium of $\mathcal{E}$. □

7 Concluding remarks

This paper has had many strands which the reader has to put together to obtain a coherent and unified view: to be sure, the principal result is a theorem on the existence of Walrasian equilibrium with an arbitrary set of agents, externalities in consumption and price-dependent preferences that are convex and continuous precisely where those conditions are needed, but not necessarily everywhere, and with or without free disposal of commodities. The issue is how this theorem is framed by the two counterexamples pertaining to economies without free disposal and with or without measure-theoretic structures. And to fully appreciate how far we can go in a measure-theoretic setting, we have a result that we could only prove with the infinite number of agents severely restricted to a countable set. The fact that the setting allows weighted markets is hardly a consolation. In any case, what we should like to emphasize in this concluding section is our determination to take the vision of Walrasian theory articulated here as a launching pad for a deeper investigation into the relaxation of the free-disposal postulate with consumption externalities in the canonical rendering of an uncountable number agents; namely with an atomless measure space and/or a hyperfinite set of agents. This is a topical task that is surely urgent for our times.

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