Efficiency, equity, and social rationality under uncertainty

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Abstract

In a simple model where agents' monetary payoffs are uncertain, this paper examines the aggregation of subjective expected utility functions which are interpersonally non-comparable. A maximin social welfare criterion is derived from axioms of efficiency, *ex post* equity, and social rationality, combined with the independence of beliefs and risk preferences in riskless situations (Chambers and Echenique in Games Econ Behav 76:582–595, 2012). The criterion compares allocations by the values of the prospects composed of the statewise minimum payoffs evaluated by the certainty equivalents. Because of this construction, the criterion is egalitarian and risk averse.

Keywords Efficiency · Equity · Social rationality · Fair social ordering · Heterogeneous beliefs · Spurious unanimity

JEL Classification D63 · D71 · D81

1 Introduction

Which social welfare criterion should be adopted to evaluate public policies under uncertainty? In the present paper, we address this question by exploring the implications of equity, efficiency, and social rationality. Because these principles are central to the welfare economics of risk and uncertainty, the exploration is important for constructing a reasonable social criterion.

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Table 1 Ex ante Pareto versus statewide dominance & ex post equity

	Prospect <i>U</i>		Prospect <i>V</i>	
	State 1	State 2	State 1	State 2
Ann	100	0	45	45
Bob	0	100	45	45

The path-breaking work is Harsanyi's (1955) aggregation theorem. In the context of risk, this states that if individuals and the social observer are expected utility maximizers, the observer's utility function (social welfare criterion) satisfying the standard *ex ante* Pareto principle¹ is represented by the weighted sum of individual expected utilities. The condition of expected utility maximization by the social observer and the *ex ante* Pareto principle are considered as requirements of social rationality and efficiency, respectively.

Harsanyi's result revealed serious tensions between equity, efficiency, and social rationality.² Most relevantly to this paper, Fleurbaey et al. (2013) show that social welfare criteria satisfying the *ex ante* Pareto principle cannot help the *ex post* worst-off individual without violating a very weak rationality condition, *Statewise Dominance*, which requires that if one allocation is *ex post* preferred to the other in all states of the world, then the former should be *ex ante* preferred to the latter.³ To see this, suppose that there are two individuals, Ann and Bob, and two equiprobable states. Consider two utility prospects *U* and *V* as in Table 1. An equitable social planner should prefer the final distributions in *V* to those in *U* in both states. Then, *Statewise Dominance* demands that *V* should be chosen over *U* to be dynamically consistent with the evaluations of the consequences above. However, the *ex ante* Pareto requires that *U* should be socially preferred to *V* because the expected utilities are unanimously higher in the former.

An important way to obtain reasonable social criteria is to consider weaker efficiency requirements. *Ex ante* Pareto is not necessarily compelling since decision-making under risk and uncertainty is difficult for individuals because of misperceptions of probabilities (Hammond 1981), and it is harsh to require agents to take full responsibility on their *ex post* well-beings because they cannot choose final outcomes (Fleurbaey 2010). For instance, mandatory social security programs are, at least partially, based on this idea.

Under uncertainty, there are yet other reasons to weaken the *ex ante* Pareto principle. First, when agents' beliefs differ, the *ex ante* Pareto principle and the social expected utility condition are incompatible (Mongin 1995). Second, if individuals have different beliefs, unanimous agreement on uncertain allocations may be spurious because of the disagreement of expectations for future outcomes. This is the problem of spurious unanimity pointed out by Mongin (2016). For instance, suppose that the money metric value of a city's environment is \$100 for all citizens. Now, the government is planning

¹ The *ex ante* Pareto principle asserts that if all agents prefer one allocation of prospects to the other, then the observer also prefers the former.

² For a comprehensive survey, see Mongin and Pivato (2016).

³ There is also a conflict between *ex ante* equity and social rationality. See, among others, Diamond (1967) and Mongin and Pivato (2016).

a public project to renovate the city and enhance its attraction. However, there is also a concern that the city may be crowded by tourists after the renovation. There are two states of the world: The city is crowded in state 1, but it is not in state 2. There are two groups of people: For group 1, the crowds of tourists bring business opportunities in state 1 and increase the value of the city to \$120, but the value in state 2 is \$80 because they are not interested in the new environment and have to pay higher taxes to cover the cost. For group 2, the crowds disturb their lives and the value in state 1 is \$80, while they can enjoy the pleasant environment in state 2 and the value is \$120. Group 1 believes that state 1 has probability 0.7, whereas Group 2 believes that state 1 has probability 0.3. Then, both groups agree with the renovation but with different reasons. However, this seems unreasonable because one of the two groups must be wrong.⁴ In the literature, weaker conditions of efficiency under uncertainty have been introduced, but those are typically incompatible with *ex post* equity under *Statewise Dominance* (Miyagishima 2019).

We introduce axioms of *ex post* equity, efficiency, and social rationality that are compelling and mutually compatible, as well as *Invariance of Risk Attitudes and Beliefs for Constant Alternatives* (henceforth referred to as *IRBC*) introduced by Chambers and Echenique (2012). In particular, our efficiency conditions are compatible with *ex post* equity and *Statewise Dominance*, and also avoid the problem of spurious unanimity. It is argued that our axioms have strong implications on inequality and risk aversion. Although our equity condition is fairly weak, we show that our axioms imply a property of *strong priority to the worst-off*, which requires that income should be redistributed to the worst-off agent in each state (Lemma 3). In the main theorem, we characterize a maximin social welfare criterion by the axioms. The criterion assesses each allocation using the prospect composed of the statewise minimum monetary payoffs in the allocation. The value of the prospect is evaluated by the lowest certainty equivalent. This criterion is inequality averse because it focuses on the worst-off agent in each state. It is also risk averse in the sense that it evaluates allocations with the lowest certainty equivalents.

We adopt a simple economic model where agents' future monetary payoffs are uncertain. It is assumed that agents are subjective expected utility maximizers. The assumption of expected utility is stringent because different individuals may follow different principles of decision-making under uncertainty and some of them may be probabilistically unsophisticated. However, our result holds on a broader class of domains and does not depend on the assumption of expected utility. Moreover, it is possible to extend our analysis to various environments such as exchange economies with multiple goods, production economies, and economies with externalities. Therefore, our result could be useful to evaluate public policies in various environments because our axioms are fairly weak and acceptable in many situations. For instance, our social criterion can be used to consider social security reforms that implement second best allocations under incentive constraints.

We consider the aggregation of ordinally measurable and interpersonally non-comparable expected utility functions following the fair social ordering approach

⁴ This example is essentially given by Blume et al. (2018). Gilboa et al. (2004), Brunnermeier et al. (2014), and Gilboa et al. (2014) provide other examples. Other recent contributions to this issue include, among others, Danan et al. (2016), Mongin and Pivato (2015, 2020), Qu (2017), Hayashi and Lombardi (2019).

(Fleurbaey and Maniquet 2011; Fleurbaey and Zuber 2017). The problem of preference aggregation is important: Social welfare functions are a major tool for the evaluation of public policies such as tax and social security programs and climate policies. When evaluating such policies, we can respect agents' preferences and take into account differences in beliefs and risk attitudes among agents by aggregating their preferences.

The most closely related paper is Fleurbaey and Zuber (2017), who characterized the same social criterion under the assumption that all agents have the same belief.⁵ Our result also holds in their environment. The main differences are as follows. They derived a class of expected equally distributed equivalent criteria based on certainty equivalence by *Statewise Dominance* and *Pareto for Equal or No Risk*, an efficiency condition we also require. Then, they characterized the maximin criterion using the rationality requirement that the observer should also be an expected utility maximizer and share the same belief as the agents. They introduced the domain restriction that there exists an individual who is always the most risk averse, so that the social criterion satisfies the expected utility condition. It was shown that among the class of social criteria above, the maximin criterion is the only criterion satisfying the social expected utility condition under the assumption.

In contrast, we only require *Statewise Dominance*, which is much weaker than the social expected utility condition, without their domain restriction. Instead, we introduce a weak efficiency axiom, *Pareto for Non-reranking Risks*. This is a modified version of Fleurbaey's (2010, p. 660) axiom, and is applied only when all agents have the same expected utility functions. Hence, our result shows that this weak requirement is enough to obtain the criterion in more general situations than Fleurbaey and Zuber (2017, Theorem 2). Although the axiom is used just once in Lemma 3, it is crucial to derive the maximin form, as argued in Sect. 4.

The organization of this paper is as follows. Section 2 presents the model. Section 3 introduces our five axioms. Section 4 gives the main theorem characterizing the social criterion using these axioms. In Sect. 5, we provide some concluding remarks. In the Appendix, we show the proof of the theorem and the independence of the axioms.

2 The model

Let $N = \{1, \dots, n\}$ be the set of agents such that $n \geq 2$. $S = \{s_1, \dots, s_m\}$ is the finite set of states with $m \geq 2$. We denote $f_{is} \in \mathbb{R}_+$ the amount of money agent i receives under state $s \in S$. Agent i 's alternative is denoted by $f_i = (f_{is})_{s \in S} \in \mathbb{R}_+^S$, which is a vector of state-contingent monetary payoffs. Let $A = \mathbb{R}_+^S$ be the set of alternatives. $x = (x_s)_{s \in S} \in A$ is called a *constant alternative* if $x_s = x_{s'}$ for all $s, s' \in S$. Let $\bar{A}$ be the set of constant alternatives. We abuse notation in a standard way by denoting, for each $x \in \bar{A}$, the value of money in every state by x . An allocation is denoted by $f_N = (f_i)_{i \in N}$. $\mathcal{A} = A^N$ is the set of allocations. Let $\bar{\mathcal{A}} = \bar{A}^N$ be the set of *constant allocations*, which are allocations composed of constant alternatives. Let us

⁵ They also obtained another characterization of the same criterion in the model where both agents and the planner have maximin expected utility functions.

also denote $\mathcal{A}^e = \{f_N \in \mathcal{A} \mid f_i = f_j \text{ for all } i, j \in N\}$, which is the set of allocations where all agents have the equal alternatives.

For each $f_i \in A$ and each $s \in S$, let $f_i(s) \in \bar{A}$ be a constant alternative such that $f_{is'}(s) = f_{is''}(s) = f_{is}$ for all $s', s'' \in S$. $f_N(s) \in \bar{A}$ is similarly defined.

We assume that all agents are expected utility maximizers. Given a continuous and increasing utility function $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$, a probability vector $p_i = (p_{is})_{s \in S}$ over S , and an alternative $f_i \in A$, the agent i 's subjective expected utility from f_i is defined by

$$E_{p_i}(u_i \circ f_i) = \sum_{s \in S} p_{is} u_i(f_{is}).$$

$E_{p_i} u_i$ denotes agent i 's subjective expected utility function. Let $\mathcal{U}$ denote the set of such subjective expected utility functions.⁶ Let us denote $\mathcal{D} = \mathcal{U}^N$. A typical profile of subjective expected utility functions is $U = (E_{p_i} u_i)_{i \in N}$. For convenience, we define

$$\mathcal{D}^e = \{U \in \mathcal{D} \mid p_i = p_j \text{ and } u_i = u_j \text{ for all } i, j \in N\}.$$

This is a subset of preference profiles where all agents have the same expected utility functions.

A social quasi-ordering function (SQF) $\mathbf{R}$ is a mapping that determines a reflexive and transitive binary relation over the set of allocations for every profile of subjective expected utility functions. Given $U \in \mathcal{D}$, $\mathbf{R}(U)$ is a *quasi-ordering* over $\mathcal{A}$, that is, it satisfies the following two conditions.

- (1) Reflexivity; for all $f_N \in \mathcal{A}$, $f_N \mathbf{R}(U) f_N$.
- (2) Transitivity; for all $f_N, g_N, h_N \in \mathcal{A}$, $f_N \mathbf{R}(U) g_N$ and $g_N \mathbf{R}(U) h_N$ imply $f_N \mathbf{R}(U) h_N$.

Moreover, if $\mathbf{R}(U)$ satisfies the following condition, we call it an *ordering*.

- (3) Completeness; for all $f_N, g_N \in \mathcal{A}$, $f_N \mathbf{R}(U) g_N$ or $g_N \mathbf{R}(U) f_N$.

We also introduce the following condition.

- (4) Continuity; for all $f_N \in \mathcal{A}$, $\{g_N \in \mathcal{A} \mid g_N \mathbf{R}(U) f_N\}$ and $\{g_N \in \mathcal{A} \mid f_N \mathbf{R}(U) g_N\}$ are closed.

$\mathbf{P}(U)$ and $\mathbf{I}(U)$ denote the strict and indifference parts of $\mathbf{R}(U)$, respectively.

3 Axioms

This section introduces our axioms. The first is *Statewise Dominance*, which is a very weak requirement of rational decision under uncertainty.

Statewise dominance For all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,

- if $f_N(s) \mathbf{R}(U) f'_N(s)$ for all $s \in S$, then $f_N \mathbf{R}(U) f'_N$, and
- if $f_N(s) \mathbf{P}(U) f'_N(s)$ for all $s \in S$, then $f_N \mathbf{P}(U) f'_N$.

⁶ Our result holds even if utility functions are further restricted to be concave.

This axiom states that if, for all states, the constant allocations generated from the outcomes of f_N are better (resp. weakly better) than the one generated from f'_N , f_N is socially preferred to (resp. at least as good as) f'_N . Intuitively, it requires to choose the prospect with better outcome in every state.

Next, we introduce two efficiency axioms. The first efficiency condition is introduced by Fleurbaey and Zuber (2017).

Pareto for Equal or No Risk For all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}^e \cup \bar{\mathcal{A}}$,
 if $E_{p_i}(u_i \circ f_i) \geq E_{p_i}(u_i \circ f'_i)$ for all $i \in N$, then $f_N \mathbf{R}(U) f'_N$, and
 if $E_{p_i}(u_i \circ f_i) > E_{p_i}(u_i \circ f'_i)$ for all $i \in N$, then $f_N \mathbf{P}(U) f'_N$.

When comparing uncertain allocations where all agents have equal alternatives and are therefore under the egalitarian condition, it is compelling to judge that the unanimously preferred allocation should be more desirable (Fleurbaey 2010). If allocations are constant, unanimous improvements are also socially desirable. *Pareto for Equal or No Risk* combines these ideas, but is still much weaker than the *ex ante* Pareto principle. This axiom also avoids spurious unanimity: Consider again the example of city renovation in the Introduction. Here, suppose that both groups share the same benefit and cost in each state after the renovation, and the uncertainty allocation f_N is such that $f_1 = f_2 = (200, 90)$. Remember that agents are risk neutral, the initial allocation is such that $x_1 = x_2 = (100, 100)$, and the beliefs are $p_1 = (0.7, 0.3)$ and $p_2 = (0.3, 0.7)$. Then, all agents unanimously improve for the same reason that the benefit is sufficiently large in state 1 compared with the cost in state 2. In this case, although the beliefs are heterogeneous, the unanimous improvement is not spurious because the values of the consequences in f_N are the same among agents.

Pareto for Equal or No Risk is compelling, but can be applied only when allocations are either constant or equal. The next Pareto axiom extends the application to the allocations where agents' relative positions are the same in every state. This set of allocations is introduced by Fleurbaey (2010). Formally, define the set of non-reranking allocations as

$$\mathcal{A}^{nr} = \{f_N \in \mathcal{A} \mid (f_{is} - f_{js})(f_{is'} - f_{js'}) \geq 0 \text{ for all } i, j \in N \text{ and all } s, s' \in S\}.$$

Note that $\mathcal{A}^e \cup \bar{\mathcal{A}} \subset \mathcal{A}^{nr}$.

Pareto for Non-reranking Risks For all $U \in \mathcal{D}^e$, all $f_N \in \mathcal{A}^{nr} \setminus \bar{\mathcal{A}}$ and all $x_N \in \bar{\mathcal{A}}$,
 if $E_{p_i}(u_i \circ f_i) > u_i(x_i)$ for all $i \in N$, then $f_N \mathbf{P}(U) x_N$.

This axiom states that if all agents with the same preference are willing to take non-reranking risks, then the risk-taking decisions should be socially supported. By the condition that all agents have the same preference, there is a consensus in the sense of Sprumont (2012) that everyone has a better prospect. This Pareto condition also avoids the problem of spurious unanimity caused by different beliefs among agents, because the agents are supposed to have the same belief. To see this, consider again the example of city renovation. Here, suppose that the initial allocation is such that $y_1 = (100, 100)$ and $y_2 = (120, 120)$, and that they have $g_1 = (120, 90)$ and $g_2 = (150, 100)$ after the renovation. Note that $g_N \in \mathcal{A}^{nr}$. Suppose also that $p_1 = p_2 = (0.5, 0.5)$. Then,

we can see that agents unanimously improve for the same reason that the benefit is larger than the cost under the common belief.

We will show that the above Pareto conditions and *Statewise Dominance* are compatible with the following *ex post* equity condition.

Ex Post Transfer among Equals For all $U \in \mathcal{D}^e$ and all $x_N, x'_N \in \bar{\mathcal{A}}$, if there exist j, k such that $x_i = x'_i$ for all $i \neq j, k$, then for all $t > 0$,

$$[x_j = x'_j - t > x_k = x'_k + t] \Rightarrow x_N \mathbf{R}(U) x'_N.$$

This axiom requires that if there is an *ex post* inequality between two agents with the same preference, it should be socially accepted to reduce the inequality by transfers. The restriction to individuals with the same preference is meaningful in terms of equal treatment of equals.

The next invariance axiom is essentially based on that introduced by Chambers and Echenique (2012).

Invariance to Risk Attitudes and Beliefs for Constant Alternatives (IRBC) For all $U = (E_{p_i} u_i)_{i \in N}$, $U' = (E_{p'_i} u'_i)_{i \in N} \in \mathcal{D}$ and all $x_N, x'_N \in \bar{\mathcal{A}}$, $x_N \mathbf{R}(U) x'_N$ if and only if $x_N \mathbf{R}(U') x'_N$.

This axiom claims that social judgments over constant allocations should be invariant of risk preferences and beliefs. The idea is that as long as riskless outcomes are compared, agents' risk preferences are irrelevant for the comparisons.

4 The social criterion and its characterization

In this section, we derive the social welfare criterion from the five axioms in Sect. 3. For convenience, we introduce several notations. Given $f_i \in A$ and $E_{p_i} u_i \in \mathcal{U}$, let

$$C(f_i, E_{p_i} u_i) = \inf\{c \in \mathbb{R}_+ | u_i(c) \geq E_{p_i}(u_i \circ f_i)\},$$

which is the certainty equivalent of f_i with respect to $E_{p_i} u_i$. Moreover, for each $f_N \in \mathcal{A}$, we denote $m(f_N) = (\min_{i \in N} f_{is})_{s \in S} \in A$, which is the prospect composed of the statewise minimum monetary payoffs in f_N .

Then, we obtain the following result.

Theorem *A continuous SQF $\mathbf{R}$ satisfies the five axioms in Sect. 3 if and only if, for all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,*

$$f_N \mathbf{R}(U) f'_N \iff \min_{i \in N} C(m(f_N), E_{p_i} u_i) \geq \min_{i \in N} C(m(f'_N), E_{p_i} u_i).$$

The proof is given in the Appendix. We give the following example to show how to calculate $\min_{i \in N} C(m(f_N), E_{p_i} u_i)$. Let $N = \{1, 2\}$ and $S = \{s, s'\}$. Suppose that $u_1(x) = \sqrt{x}$, $u_2(x) = x$, $p_1 = (0.6, 0.4)$ and $p_2 = (0.4, 0.6)$. Consider f_N such that

Fig. 1 The construction of $\min_{i \in N} C(m(f_N), E_{p_i} u_i)$. Because $f_{is} < f_{js}$ and $f_{js'} < f_{is'}$, we have $m(f_N) = (f_{is}, f_{js'})$

$f_1 = (900, 100)$ and $f_2 = (400, 500)$. Then, we obtain $m(f_N) = (400, 100)$. In this case,

$$E_{p_1}(u_1 \circ m(f_N)) = 0.4\sqrt{400} + 0.6\sqrt{100} = 14 = u_1(196),$$

and hence $C(m(f_N), E_{p_1} u_1) = 196$. Moreover,

$$E_{p_1}(u_2 \circ m(f_N)) = 0.6 \times 400 + 0.4 \times 100 = 280 = u_2(280),$$

and hence $C(m(f_N), E_{p_2} u_2) = 280$. Therefore, we have $\min_{i \in N} C(m(f_N), E_{p_i} u_i) = 196$.

We provide some remarks on our result. The social criterion evaluates each allocation f_N by $m(f_N)$ based on the minimum value of certainty equivalents among individuals. Figure 1 illustrates the construction of $\min_{i \in N} C(m(f_N), E_{p_i} u_i)$ for an allocation f_N . This criterion is inequality averse because it focuses on the worst-off agent in each state, and is also risk averse in the sense that it evaluates allocations using the lowest certainty equivalents.

This criterion respects agents' preferences, and is based particularly on the most risk averse agents' preferences. It is also dynamically consistent by *Statewise Dominance*. Because of this property, the criterion first evaluates consequential distributions in each state, and then aggregates the evaluations to $m(f_N)$. In general, however, it does not satisfy the social expected utility condition because the most risk averse agents may vary among different allocations. The criterion satisfies the expected utility condition when all agents share the same belief and there exists an agent who is always the most risk averse (Fleurbaey and Zuber 2017). This is because, in this case, $m(f_N)$ is evaluated for all f_N by the same agent's expected utility function, and thus the criterion is also an expected utility function. This stronger rationality condition is obtained from the characterization.

It is worth emphasizing that *Pareto for Non-reranking Risks* is indispensable for our result, as shown in the Appendix. The following R^* violates *Pareto for Non-reranking*

Risks but satisfies the other four axioms. For all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,

$$f_N \mathbf{R}^*(U) f'_N \iff \sum_{i \in N} \frac{C(\sum_{i \in N} f_i/n, E_{p_i} u_i)}{n} \geq \sum_{i \in N} \frac{C(\sum_{i \in N} f'_i/n, E_{p_i} u_i)}{n}.$$

Note that this is not a maximin ordering. Thus, *Pareto for Non-reranking Risks* is crucial to derive our maximin criterion despite it is an efficiency requirement.

5 Concluding remarks

In this paper, we analyzed the implications of equity, efficiency and social rationality under uncertainty. We obtained the social criterion that is sensitive to the statewise worst-off individuals. Compared with Fleurbaey and Zuber (2017) who characterized the same criterion using the social expected utility condition with the domain restriction that there exists an agent who is always the most risk averse, we instead used *Pareto for Non-reranking Risks* and *Statewise Dominance* without the domain condition. Moreover, we analyzed the case where agents' beliefs as well as utility functions may be heterogeneous.

To simplify our analysis, we considered the model where each agent's *ex post* well-being is measured in monetary terms and thus single dimensional. The analysis can be extended to the case where agents have preferences over multidimensional outcomes, following the approach developed by Fleurbaey and Zuber (2017). In that case, a criterion for interpersonal comparison is adopted to evaluate well-being *ex post* using the fair social ordering approach (Fleurbaey and Maniquet 2011).

We mention some open questions. First, the maximin social ordering may be criticized as extremely egalitarian. It remains for future research to obtain a class of social orderings that are more modestly egalitarian.⁷ Second, we focused on the domain of subjective expected utility functions. Extensions to Hurwicz criterion and Bewley criterion are worth exploring. Third, it is interesting to derive optimal public policies such as social insurance under incentive and budget constraints using the maximin social ordering.

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⁷ In the environment of interpersonally comparable utility, Fleurbaey and Zuber (2013) derived such a class of social orderings under risk.

6 Appendix: proofs

In this appendix, we provide the proof of the main theorem and show the independence of the axioms. The proof proceeds via several lemmas, which are interesting in their own right.

The first lemma states that *Ex Post Transfer among Equals*, *Statewise Dominance*, and *IRBC* together imply the following strong equity axiom.

Transfer For all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$, if there exist j, k such that $f_i = f'_i$ for all $i \neq j, k$, then for all $\Delta \in \mathbb{R}_{++}^S$,

$$[f_j = f'_j - \Delta \gg f_k = f'_k + \Delta] \Rightarrow f_N \mathbf{R}(U) f'_N.$$

This axiom states that for two agents, if one has more income in every state than the other, a transfer in each state to reduce the inequality should be acceptable.

Lemma 1 *Statewise Dominance, Ex Post Transfer among Equals, and IRBC together imply Transfer.*

Proof Let $f_N, f'_N \in \mathcal{A}$ be such that $f_i = f'_i$ for all $i \neq j, k$, and

$$f_j = f'_j - \Delta \gg f_k = f'_k + \Delta, \quad \Delta \in \mathbb{R}_{++}^S.$$

Consider $f_N(s), f'_N(s) \in \bar{\mathcal{A}}$ for each $s \in S$. Let U' be such that $E_{p'_i} u'_i = E_{p_j} u'_j$ for all $i, j \in N$. By assumption and *Ex Post Transfer among Equals*, we have $f_N(s) \mathbf{R}(U') f'_N(s)$ for all $s \in S$. It follows from *IRBC* that $f_N(s) \mathbf{R}(U) f'_N(s)$ for all $s \in S$. Then, $f_N \mathbf{R}(U) f'_N$ follows from *Statewise Dominance*. $\square$

Lemma 1 has an important normative implication. From the fundamental incompatibility of equity and efficiency shown by Fleurbaey and Trannoy (2003), we can see that there is no SQF satisfying both *Transfer* and the *ex ante* Pareto principle. Thus, we have to give up the *ex ante* Pareto if *IRBC* and *Statewise Dominance* are required in addition to the weak equity condition, *Ex Post Transfer among Equals*.

The next lemma is an implication of combining *Pareto for Equal or No Risk* and *Statewise Dominance*. Note that the Pareto condition is applied only for constant allocations in this lemma.

Lemma 2 *If $\mathbf{R}$ satisfies Pareto for Equal or No Risk and Statewise Dominance, then for all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,*

- (1) $f_N \gg f'_N \implies f_N \mathbf{P}(U) f'_N$,
- (2) $f_N \geq f'_N \implies f_N \mathbf{R}(U) f'_N$.

Proof We only prove (1). Let $f_N, f'_N \in \mathcal{A}$ be such that $f_N \gg f'_N$. Since $f_i(s) \gg f'_i(s)$ for all $i \in N$ and all $s \in S$, *Pareto for Equal or No Risk* implies $f_N(s) \mathbf{P}(U) f'_N(s)$ for all $s \in S$. The desired conclusion follows from *Statewise Dominance*. (2) can be similarly shown. $\square$

The next lemma establishes an infinite *ex post* inequality aversion, which is captured by the following axiom.

Strong Priority to the Worst-off For all $U \in \mathcal{D}$ and all $x_N, x'_N \in \bar{\mathcal{A}}$, if there exist $j, k \in N$ such that $x_k = \min_{i \in N} x_i$, $x'_k = \min_{i \in N} x'_i$, and $x_i = x'_i$ for all $i \neq j, k$, then

$$[x'_j > x_j > x_k > x'_k] \implies x_N \mathbf{P}(U) x'_N.$$

Lemma 3 *The five axioms in Sect. 3 together imply Strong Priority to the Worst-off.*

Proof The result is obvious if $x_k - x'_k \geq x'_j - x_j$. Thus, we assume that $x_k - x'_k < x'_j - x_j$.

First, let $y_N \in \bar{\mathcal{A}}$ be such that $0 \leq x'_k < y_k < x_k$, and given $\bar{x} > \max_{i \in N} x'_i$, $y_h = \bar{x}$ for all $h \neq k$. *Pareto for Equal or No Risk* implies $y_N \mathbf{P}(U) x'_N$. We show that $x_N \mathbf{R}(U) y_N$ below. Then, we can obtain the desired result by transitivity.

Since x_N and y_N are constant allocations, we can invoke *IRBC* to arbitrarily modify the subjective expected utility functions. For the purpose of constructing an appropriate expected utility function, we introduce parameters $a, b, \alpha, \beta, k, \epsilon, \pi, \bar{f} \in \mathbb{R}_{++}$ such that,

$$a(x_k - 2\beta - y_k) + y_j + \alpha - (x_k - 2\beta) = \pi(a(x_k - 2\beta - y_k) + \bar{f} + (n + k + 1)\epsilon - (x_k - 2\beta)), \quad (1)$$

$$a(x_k - 2\beta - y_k) + \beta = \pi(a(x_k - 2\beta - y_k) + \bar{f} + (n + k)\epsilon - (x_k - 2\beta)) - (1 - \pi)ba\epsilon, \quad (2)$$

$$a(x_k - 2\beta - y_k) = \pi(a(x_k - 2\beta - y_k) + \bar{f} + n\epsilon - (x_k - 2\beta)) - (1 - \pi)ba(k + 1)\epsilon, \quad (3)$$

$$0 = \pi(a(x_k - 2\beta - y_k) + \bar{f} - (x_k - 2\beta)) - (1 - \pi)ba(n + k + 1)\epsilon, \quad (4)$$

$$\pi \in \left(\frac{n + 1}{2n}, \frac{n + 2}{2n} \right), \quad (5)$$

where π is sufficiently close to $(n + 1)/2n$. We also assume that α, β , and ϵ are close to 0. a, b, k and $\bar{f}$ will be specified later.

Define $E_{p_0} u_0$ as

$$u_0(x) = \begin{cases} ba(x - y_k) & \text{for } x < y_k, \\ a(x - y_k) & \text{for } x \in [y_k, x_k - 2\beta], \\ a(x_k - 2\beta - y_k) + x - (x_k - 2\beta) & \text{for } x > x_k - 2\beta, \end{cases}$$

and $p_{0s} = \pi$ and $p_{0s'} = (1 - \pi)/(m - 1)$ for all $s' \neq s$.⁸ $a > 1$ and $b > 1$ as shown later, and thus u_0 is concave. Let $U \in \mathcal{D}$ denote the preference profile where all agents have $E_{p_0} u_0$.

⁸ Remember that m is the cardinality of S .

Define $f_N, g_N, h_N \in \mathcal{A}^{nr} \setminus \bar{\mathcal{A}}$ such that

$$\begin{aligned} f_{ks} &= \bar{f} + \epsilon, \quad f_{ks'} = y_k - (n+k)\epsilon \text{ for all } s' \neq s, \\ f_{js} &= \bar{f} + (n+k+1)\epsilon, \quad f_{js'} = y_k \text{ for all } s' \neq s \text{ and all } j \neq k, \\ g_{ks} &= \bar{f} + n\epsilon, \quad g_{ks'} = y_k - (k+1)\epsilon \text{ for all } s' \neq s, \\ g_{js} &= \bar{f} + (n+k)\epsilon, \quad g_{js'} = y_k - \epsilon \text{ for all } s' \neq s \text{ and all } j \neq k, \\ h_{is} &= \bar{f} + (n+k+\delta)\epsilon, \quad h_{is'} = y_k - \epsilon \text{ for all } s' \neq s \text{ and all } i \in N, \end{aligned}$$

where $\delta > 0$ is small enough that $u_0(x_k) > E_{p_0}(u_0 \circ h_i)$. Then, from (1)–(4), we can see that

$$\begin{aligned} u_0(y_j + \alpha) &= a(x_k - 2\beta - y_k) + y_j + \alpha - (x_k - 2\beta) \\ &= \pi(a(x_k - 2\beta - y_k) + \bar{f} + (n+k+1)\epsilon - (x_k - 2\beta)) \\ &\quad + (1-\pi)ba0, \\ &= \pi(a(x_k - 2\beta - y_k) + \bar{f} + (n+k+1)\epsilon - (x_k - 2\beta)) \\ &\quad + (1-\pi)ba(y_k - y_k) \\ &= E_{p_0}(u_0 \circ f_j), \\ u_0(x_k - \beta) &= a(x_k - 2\beta - y_k) + x_k - \beta - (x_k - 2\beta) = a(x_k - 2\beta - y_k) + \beta \\ &= \pi(a(x_k - 2\beta - y_k) + \bar{f} + (n+k)\epsilon - (x_k - 2\beta)) - (1-\pi)ba\epsilon \\ &= \pi(a(x_k - 2\beta - y_k) + \bar{f} + (n+k)\epsilon + (x_k - 2\beta)) \\ &\quad + (1-\pi)ba(y_k - \epsilon - y_k) \\ &= E_{p_0}(u_0 \circ g_j), \\ u_0(x_k - 2\beta) &= a(x_k - 2\beta - y_k) + x_k - 2\beta - (x_k - 2\beta) = a(x_k - 2\beta - y_k) \\ &= \pi(a(x_k - 2\beta - y_k) + \bar{f} + (n+k)\epsilon - (x_k - 2\beta)) \\ &\quad - (1-\pi)ba(k+1)\epsilon \\ &= \pi(a(x_k - 2\beta - y_k) + \bar{f} + n\epsilon - (x_k - 2\beta)) \\ &\quad + (1-\pi)ba(y_k - (k+1)\epsilon - y_k)\epsilon \\ &= E_{p_0}(u_0 \circ g_k), \\ u_0(y_k) &= 0 \\ &= \pi(a(x_k - 2\beta - y_k) + \bar{f} - (x_k - 2\beta)) - (1-\pi)ba(n+k+2)\epsilon \\ &= \pi(a(x_k - 2\beta - y_k) + \bar{f} + (x_k - 2\beta)) \\ &\quad + (1-\pi)ba(y_k - (n+k+1)\epsilon - y_k)\epsilon \\ &< \pi(a(x_k - 2\beta - y_k) + \bar{f} + \epsilon + (x_k - 2\beta)) \\ &\quad + (1-\pi)ba(y_k - (n+k)\epsilon - y_k)\epsilon \\ &= E_{p_0}(u_0 \circ f_k). \end{aligned}$$

Figure 2 illustrates the construction of $E_{p_0}u_0$ and the allocations in the two-person case. The indifference curves are depicted by the lines sloping downward to the rights.

 Fig. 2 $E_{p_0}u_0$ and the allocations

Here, we specify a , k , and $\bar{f}$. From (4), we have

$$b = \frac{\pi(a(x_k - 2\beta - y_k)) + \bar{f} - (x_k - 2\beta)}{(1 - \pi)a(n + k + 1)\epsilon}.$$

From this and (1)–(2), we obtain

$$y_j + \alpha - x_k + \beta = \pi \left(\epsilon + \frac{\pi(a(x_k - 2\beta - y_k)) + \bar{f} - (x_k - 2\beta)}{n + k + 1} \right). \quad (6)$$

Similarly, by (2)–(3) and (3)–(4), we obtain respectively

$$\beta = \pi k \left(\epsilon + \frac{\pi(a(x_k - 2\beta - y_k)) + \bar{f} - (x_k - 2\beta)}{n + k + 1} \right), \quad (7)$$

$$a(x_k - 2\beta - y_k) = \pi n \left(\epsilon + \frac{\pi(a(x_k - 2\beta - y_k)) + \bar{f} - (x_k - 2\beta)}{n + k + 1} \right). \quad (8)$$

Solving (6)–(8) with respect to a , we have

$$a = \frac{(n + k + 1)(y_j + \alpha - x_k + \beta - \pi\epsilon) - \pi(\bar{f} - x_k + 2\beta)}{\pi(x_k - 2\beta - y_k)} \quad (9)$$

$$= \frac{\frac{(n+k+1)}{k}(\beta - k\pi\epsilon) - \pi(\bar{f} - x_k + 2\beta)}{\pi(x_k - 2\beta - y_k)} \quad (10)$$

$$= \frac{\pi n((n + k + 1)\epsilon + \bar{f} - x_k + 2\beta)}{(n(1 - \pi) + k + 1)(x_k - 2\beta - y_k)}. \quad (11)$$

Considering the denominators of (9)–(11), we set

$$k = n(2\pi - 1) - 1, \quad (12)$$

which implies $\pi = n(1 - \pi) + k + 1$. Note that, by (5), we obtain $k \in (0, 1)$. Moreover, k is close to 0 because π is close to $(n + 1)/2n$ as we assumed.

From (9) and (10), we have

$$(n + k + 1)(y_j + \alpha - x_k + \beta - \pi\epsilon) = \frac{(n + k + 1)}{k}(\beta - k\pi\epsilon),$$

which implies

$$\beta = \frac{k}{1 - k}(y_j + \alpha - x_k). \quad (13)$$

Because $\beta > 0$, k must be in $(0, 1)$. This can be assured by (5), as mentioned in the last paragraph.

From (9), (11), and (13), we obtain

$$\bar{f} = \left(n + (n - 2) \frac{k}{1 - k} \right) (y_j + \alpha - x_k) - (n + k + 1)\epsilon + x_k.$$

Then, from this and (13), we have

$$a = \frac{n(1 + \frac{k}{1 - k})(y_j + \alpha - x_k)}{x_k - y_k - 2\frac{k}{1 - k}(y_j + \alpha - x_k)}.$$

Remember that π is sufficiently close to $(n + 1)/2n$. Then, k is close to 0, and a is close to

$$\frac{n(y_j + \alpha - x_k)}{x_k - y_k},$$

and hence $a > 1$ because $x_k - y_k < y_j - x_k$. Similarly, we can check that $b > 1$ when ϵ is sufficiently small.

Now, we show the result. *Pareto for Non-reranking Risks* implies $f_N \mathbf{P}(U) y_N$. By repeated applications of *Transfer* (Lemma 1), we have $g_N \mathbf{R}(U) f_N$. Lemma 2 implies $h_N \mathbf{R}(U) g_N$. Moreover, $x_N \mathbf{P}(U) h_N$ follows from *Pareto for Equal or No Risk*. By transitivity, we obtain $x_N \mathbf{P}(U) y_N$, as sought. $\square$

Here, we discuss differences between the present paper and Fleurbaey and Zuber (2017, Lemma A.1). The latter used a form of additive representation result obtained by the social expected utility condition. They constructed two expected utility functions for different agents because they only use *Pareto for Equal or No Risk*. In contrast, we constructed an expected utility function which is common for all agents since we

use *Pareto for No-reranking Risks* that can be applied only when all agents have the same expected utility function.

The next lemma is useful to prove our main theorem.

Lemma 4 Suppose that $\mathbf{R}$ satisfies the five axioms in Sect. 3. Then, for all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$, $m(f_N) \gg m(f'_N)$ implies $f_N \mathbf{P}(U) f'_N$.

Proof Suppose $m(f_N) \gg m(f'_N)$. Consider $f_N(s), f'_N(s) \in \bar{\mathcal{A}}$ for each $s \in S$. Note that by assumption,

$$\min_{i \in N} f_i(s) > \min_{i \in N} f'_i(s) \text{ for every } s \in S.$$

For our purpose, it is sufficient to show $f_N(s) \mathbf{P}(U) f'_N(s)$ for every $s \in S$, which implies $f_N \mathbf{P}(U) f'_N$ by *Statewise Dominance*, as sought.

The rest of the proof is divided into two cases.

Case 1. Suppose $f'_j(s) = \min_{i \in N} f'_i(s)$ for all $j \in N$. Then,

$$f_j(s) \geq \min_{i \in N} f_i(s) > f'_j(s) \text{ for all } j \in N,$$

and hence $f_N(s) \mathbf{P}(U) f'_N(s)$ by *Pareto for Equal or No Risk*.

Case 2. Suppose $f'_j(s) > \min_{i \in N} f'_i(s)$ for some $j \in N$. Let $x_N \in \bar{\mathcal{A}}$ be such that

$$x_j = \frac{\min_{i \in N} f_i(s) + \min_{i \in N} f'_i(s)}{2} \text{ for all } j \in N,$$

Then, it is straightforward to show that repeated applications of Lemma 3, *Pareto for Equal or No Risk*, and transitivity together imply $x_N \mathbf{P}(U) f'_N(s)$. $f_N(s) \mathbf{P}(U) x_N$ follows from *Pareto for Equal or No Risk*. We obtain $f_N(s) \mathbf{P}(U) f'_N(s)$ by transitivity. $\square$

While *Ex Post Transfer among Equals* is quite weak, if it is combined with the other four axioms in Sect. 3, the social criterion should be sensitive to the statewise worst-off individuals. The statewise worst-offs are crucial for our main theorem.

Now, we prove the main theorem.

Proof of Theorem First, we discuss the “if” part and show that the criterion satisfies all the axioms.

On *Statewise Dominance*, let f_N and g_N be such that

$$\min_{i \in N} C(m(f_N(s)), E_{p_i} u_i) > \min_{i \in N} C(m(g_N(s)), E_{p_i} u_i) \text{ for all } s \in S.$$

This means $\min_{i \in N} f_{is} > \min_{i \in N} g_{is}$ for all $s \in S$. Then, $m(f_N) \gg m(g_N)$, and hence $\min_{i \in N} C(m(f_N), E_{p_i} u_i) > \min_{i \in N} C(m(g_N), E_{p_i} u_i)$ by monotonicity of expected utility. The case of weak relation is similarly proved.

Fig. 3 Illustration of the proof

On *Pareto for Equal or No Risk*, let $f_N, g_N \in \mathcal{A}^e \cup \bar{\mathcal{A}}$ be such that $E_{p_i}(u_i \circ f_i) > E_{p_i}(u_i \circ g_i)$ for all $i \in N$. If $f_N, g_N \in \mathcal{A}^e$ or $f_N, g_N \in \bar{\mathcal{A}}$, the criterion obviously satisfies the axiom. If $f_N \in \mathcal{A}^e$ and $g_N \in \bar{\mathcal{A}}$, then $f_i = m(f_N)$ for all $i \in N$, and $\min_{i \in N} g_{is} = \min_{i \in N} C(m(g_N), E_{p_i} u_i)$ for all $s \in S$. Therefore, $\min_{i \in N} C(m(f_N), E_{p_i} u_i) > \min_{i \in N} C(m(g_N), E_{p_i} u_i)$. The case of $g_N \in \mathcal{A}^e$ and $f_N \in \bar{\mathcal{A}}$ is similarly proved.

On *Pareto for No-reranking Risks*, let $U \in \mathcal{D}^e$ and $f_N \in \mathcal{A}^{nr}$ and $x_N \in \bar{\mathcal{A}}$ be such that $E_{p_i}(u_i \circ f_i) > u_i(x_i)$ for all $i \in N$. Then, $m(f_N) = \min_{i \in N} (f_i)_{s \in S}$ and $\min_{i \in N} x_i = \min_{i \in N} C(m(x_N), E_{p_i} u_i)$. Since all agents have the same expected utility function, we have $\min_{i \in N} C(m(f_N), E_{p_i} u_i) > \min_{i \in N} C(m(x_N), E_{p_i} u_i)$.

On *Ex Post Transfer among Equals*, let $x_N, y_N \in \bar{\mathcal{A}}$ be such that, for some $i, j \in N$, $x_i = y_i + t < y_j - t = x_j$ and $x_k = y_k$ for all $k \neq i, j$. Note that $x_i = C(x_i, E_{p_i} u_i)$ and $y_i = C(y_i, E_{p_i} u_i)$ for all $i \in N$, $\min_{i \in N} x_i = \min_{i \in N} C(m(x_N), E_{p_i} u_i)$ and $\min_{i \in N} y_i = \min_{i \in N} C(m(x_N), E_{p_i} u_i)$. Hence, we have $\min_{i \in N} C(m(f_N), E_{p_i} u_i) \geq \min_{i \in N} C(m(x_N), E_{p_i} u_i)$.

On *IRBC*, let $x_N, y_N \in \bar{\mathcal{A}}$ be such that $\min_{i \in N} C(m(f_N), E_{p_i} u_i) \geq \min_{i \in N} C(m(x_N), E_{p_i} u_i)$. Since $x_i = C(x_i, E_{p_i} u_i)$ and $y_i = C(y_i, E_{p_i} u_i)$ for all $i \in N$ and all $U \in \mathcal{D}$. Thus, $\min_{i \in N} C(m(f_N), E_{p_i} u_i) \geq \min_{i \in N} C(m(x_N), E_{p_i} u_i)$ for all $U \in \mathcal{D}$.

Next, we prove the “only-if” part. First, we show that

$$\min_{i \in N} C(m(f_N), E_{p_i} u_i) > \min_{i \in N} C(m(f'_N), E_{p_i} u_i) \implies f_N \mathbf{P}(U) f'_N.$$

Figure 3 illustrates the proof.

Without loss of generality, let agent 1 be such that

$$C(m(f'_N), E_{p_1} u_1) = \min_{i \in N} C(m(f'_N), E_{p_i} u_i).$$

Consider $\epsilon = (\epsilon, \dots, \epsilon) \in \mathbb{R}_{++}^S$ and $g_N, g'_N \in \mathcal{A}$ such that

$$\begin{aligned} g_i &= m(g_N) = m(f_N) \text{ for all } i \in N, \\ g'_i &= m(g'_N) = m(f'_N) + \epsilon \text{ for all } i \in N, \\ \min_{j \in N} C(m(g_N), E_{p_j} u_j) &> \min_{j \in N} C(m(g'_N), E_{p_j} u_j) + 3\epsilon. \end{aligned}$$

We have $f_N \mathbf{R}(U) g_N$ by Lemma 2, and $g'_N \mathbf{P}(U) f'_N$ by Lemma 4. In the following, we show $g_N \mathbf{P}(U) g'_N$. Then, by transitivity, we have the desired result.

Consider $x_N, y_N \in \bar{\mathcal{A}}$ such that

$$\begin{aligned} x_1 &= \min_{j \in N} C(m(g'_N), E_{p_j} u_j) + \epsilon, \quad u_i(x_i) > E_{p_i}(u_i \circ g'_i), \quad x_i > x_1 + 3\epsilon \text{ for all } i \neq 1, \\ y_1 &= x_1 + 2\epsilon, \quad y_i = x_1 + 3\epsilon \text{ for all } i \neq 1. \end{aligned}$$

By *Pareto for Equal or No Risk*, we obtain $x_N \mathbf{P}(U) g'_N$. It follows from repeated applications of *Strong Priority to the Worst-off* (Lemma 3) that $y_N \mathbf{P}(U) x_N$. Note that $m(g_N) = g_i$ and $E_{p_i}(u_i \circ g_i) > y_i$ for all $i \in N$. Hence, *Pareto for Equal or No Risk* implies $g_N \mathbf{P}(U) y_N$. By transitivity, we obtain $g_N \mathbf{P}(U) g'_N$ as sought.

The remaining is to show that

$$\min_{i \in N} C(m(f_N), E_{p_i} u_i) = \min_{i \in N} C(m(f'_N), E_{p_i} u_i) \implies f_N \mathbf{I}(U) f'_N.$$

However, this can be proved by a usual argument using continuity, and therefore we can safely omit it. $\square$

Next, we give the following social orderings to the independence of the axioms. We only show the examples of violating *Pareto for No-reranking Risks* and *Statewise Dominance*.

Dropping Pareto for Equal or No Risk: Consider $\mathbf{R}^1$ such that, for all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,

- (i) if $U \in \mathcal{D}^e$, then $f_N \mathbf{R}^1(U) f'_N \iff \min_{i \in N} C(m(f_N), E_{p_i} u_i) \geq \min_{i \in N} C(m(f'_N), E_{p_i} u_i)$;
- (ii) if $U \notin \mathcal{D}^e$, then $f_N \mathbf{R}^1(U) f'_N \iff \min_{i \in N} \min_{s \in S} f_{is} \geq \min_{i \in N} \min_{s \in S} f'_{is}$.

Dropping Pareto for No-reranking Risks: Given $f_N \in \mathcal{A}$, let $a(f_N) = (\sum_{i \in N} f_{is} / n)_{s \in S}$. Consider $\mathbf{R}^2$ such that, for all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,

$$f_N \mathbf{R}^2(U) f'_N \iff \sum_{i \in N} \frac{C(a(f_N), E_{p_i} u_i)}{n} \geq \sum_{i \in N} \frac{C(a(f'_N), E_{p_i} u_i)}{n}.$$

It is straightforward to check that $\mathbf{R}^2$ satisfies all of the axioms except *Pareto for No-reranking Risks*. To see that $\mathbf{R}^2$ violates *Pareto for No-reranking Risks*, consider

the following two-agent and two-state example: Let $N = \{1, 2\}$, $S = \{s_1, s_2\}$,

$$u_1(x) = u_2(x) = \begin{cases} 2x & \text{for } x \in [0, 10], \\ 20 + (x - 10) = 10 + x & \text{for } x > 10, \end{cases}$$

and $p_{1s} = p_{2s} = 0.5$ for each $s \in S$. Define $f_N \in \mathcal{A}^{nr}$ and $x_N \in \bar{\mathcal{A}}$ as $f_1 = (10, 0)$, $f_2 = (70, 1)$, $x_1 = (4, 4)$, and $x_2 = (30, 30)$. Then, we can see that

$$\begin{aligned} E_{p_1}(u_1 \circ f_1) &= 0.5 \times 20 + 0.5 \times 0 = 10 > 8 = u(x_1), \\ E_{p_1}(u_2 \circ f_2) &= 0.5 \times 80 + 0.5 \times 2 = 41 > 40 = u(x_2). \end{aligned}$$

However, because $a(f_N) = (f_1 + f_2)/2 = (40, 0.5)$ and $a(x_N) = (x_1 + x_2)/2 = (17, 17)$, we have

$$\begin{aligned} E_{p_1}(u_1 \circ a(f_N)) &= E_{p_2}(u_2 \circ a(f_N)) = 0.5 \times 50 + 0.5 \times 1 = 25.5 < 27 \\ &= u_1(17) = u_2(17). \end{aligned}$$

Hence, we obtain

$$\sum_{i \in N} \frac{C(a(f_N), E_{p_i} u_i)}{2} < \sum_{i \in N} \frac{C(a(x_N), E_{p_i} u_i)}{2}.$$

This shows the violation of *Pareto for No-reranking Risks*.

Dropping Statewise Dominance: Consider $\mathbf{R}^3$ such that, for all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$, $f_N \mathbf{R}^3(U) f'_N \iff \min_{i \in N} C(f_i, E_{p_i} u_i) \geq \min_{i \in N} C(f'_i, E_{p_i} u_i)$. To see that this rule violates *Statewise Dominance*, let $N = \{1, 2\}$, $S = \{s_1, s_2\}$, $u_1(x) = u_2(x) = x$, and $p_{1s_1} = 0.2$, $p_{2s_1} = 0.8$. Define f_N and x_N such that $f_1 = (0, 5)$, $f_2 = (5, 0)$, $x_1 = x_2 = (3, 3)$. Then, because $f_N(s_1) = ((0, 0), (5, 5))$ and $f_N(s_2) = ((5, 5), (0, 0))$,

$$\min_{i \in N} C(f_i(s_1), E_{p_i} u_i) = \min_{i \in N} C(f_i(s_2), E_{p_i} u_i) = 0 < 3 = \min_{i \in N} C(x_i, E_{p_i} u_i),$$

but

$$\min_{i \in N} C(f_i, E_{p_i} u_i) = 4 > \min_{i \in N} C(x_i, E_{p_i} u_i) = 3.$$

This means that $\mathbf{R}^3$ violates *Statewise Dominance*.

Dropping Ex Post Transfer among Equals: For each $f_N \in \mathcal{A}$, let $M(f_N) = (\max_{i \in N} f_{is})_{s \in S}$. Consider $\mathbf{R}^4$ such that, for all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,

$$f_N \mathbf{R}^4(U) f'_N \iff \max_{i \in N} C(M(f_N), E_{p_i} u_i) \geq \max_{i \in N} C(M(f'_N), E_{p_i} u_i).$$

Dropping IRBC: Let $\hat{C}(f_i, E_{p_i}u_i) = \inf\{c \in \mathbb{R}_+ | E_{p_i}u_i(2c, c, \dots, c) \geq E_{p_i}u_i(f_i)\}$ for each $f_i \in A$ and $E_{p_i}u_i$. Consider R^5 such that, for all $U \in \mathcal{D}$, for all $f_N, f'_N \in \mathcal{A}$,

$$f_N R^5(U) f'_N \iff \min_{i \in N} \hat{C}(m(f_N), E_{p_i}u_i) \geq \min_{i \in N} \hat{C}(m(f'_N), E_{p_i}u_i).$$

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