

Weighted temporal utility

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Abstract This paper considers preferences over risky timed outcomes and proposes the weighted temporal utility (WTU) model which separates anticipated subjective evaluations of outcomes from attitudes toward psychological distance induced by risks and delays. Anticipating the subjective evaluation of an outcome requires the decision maker to project himself to the future and to imagine how much he will appreciate the outcome once he receives it. This projection may, but need not, be accurate. We provide a characterization of the WTU model in a static setting and propose a nonparametric method to measure its weighting and utility functions. We also consider a dynamic setting which allows for a varying decision time. The dynamic WTU model can accommodate the standard discounted expected utility model as well as observed deviations from stationarity, time invariance, and time consistency. It therefore enhances our understanding of the drivers of these behavioral phenomena.

Keywords Intertemporal choice · Time preferences · Risk preferences · Stationarity · Time consistency · Time invariance

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JEL Classification D91 · D81**1 Introduction**

Most decisions we take today involve an uncertain outcome at some point in the future. This is not only true for investments and savings, but also for daily decisions about, for instance, what to eat and whether or not to go to the gym. Empirical evidence shows that many decisions are time inconsistent in the sense that the mere passage of time makes people change their plans (Frederick et al. 2002). Such time inconsistencies can cause under-investment and unhealthy lifestyles, which impose large costs on society. A good understanding of the drivers of these time inconsistencies can help to provide solutions to overcome them and to reduce the associated costs.

The literature on intertemporal choice has mostly abstracted from uncertainty in order to focus exclusively on pure time preference (for a survey see Frederick et al. 2002). However, as the future is inherently uncertain, it is also important to examine how pure time preference and risk attitudes interact (Bommier 2006; Takanori and Goto 2009; Epper et al. 2011; Epper and Fehr-Duda 2015a, b; Andreoni and Sprenger 2012, 2015; Cheung 2015; Miao and Zhong 2015). This paper proposes a model of intertemporal choice which contributes to the literature in two ways. First, unlike many existing models of intertemporal choice, our model allows for an interaction between pure time preference and risk attitude. This is in line with empirical evidence on the non-separability of time and risk (Keren and Roelofsma 1995; Abdellaoui et al. 2011; Baucells and Heukamp 2012) and with construal-level theory in psychology (Trope and Liberman 2010) according to which both risk and time induce psychological distance that influences decision making.

The second contribution is that we separate two effects the delay of an outcome can have on its evaluation. A delay requires the decision maker to project himself to the future and to imagine how much he will appreciate the outcome once he receives it. This projection may, but need not, be accurate. For instance, people tend to suffer from projection bias when predicting future utility (Loewenstein et al. 2003). Additionally, the delay of the outcome makes its receipt psychologically more distant, and therefore less salient, than immediate outcomes.

This paper considers preferences over single outcomes to be received with a particular probability at a particular point in time. It, therefore, adds an uncertainty dimension to the framework of Fishburn and Rubinstein (1982). Our weighted temporal utility model evaluates such outcomes by multiplying the time-dependent utility of the outcome with a weight, which depends on the psychological distance induced by the probability and the time at which the outcome is received. Keren and Roelofsma (1995), Abdellaoui et al. (2011), and Baucells and Heukamp (2012) provided empirical evidence for the interaction of the probability and timing of an outcome. To capture this, our model of psychological distance does not require probability and time to be additively separable. Moreover, the magnitude effect, which shows that larger outcomes are discounted at a lower rate than smaller outcomes, suggests that an outcome and its timing may also not be additively separable (Frederick et al. 2002). Thus, in line with the empirical evidence, our model allows for interactions between probabilities

and time on the one hand and between outcomes and time on the other hand, while outcomes and probabilities are assumed to be additively separable for every given point in time. For single outcomes, our model accommodates rank-dependent utility, prospect theory, exponential discounting, and hyperbolic discounting as special cases.

The first part of this paper provides a characterization of the WTU model. In the second part, we consider the WTU model in a dynamic setting. We will show how properties of the weighting and utility function are related to time-inconsistent behavior. In a dynamic setting, one has to distinguish between consumption time, decision time, and temporal distance between consumption and decision time. Time discounting is typically interpreted as depending on consumption and decision time through temporal distance only. In line with this standard approach to time discounting and with the interpretation of psychological distance, we let the weighting function depend on probability and temporal distance only. The utility function, however, can depend on time in various ways. We let utility depend on consumption time only, on temporal distance only, or on decision time only. People who perfectly project themselves to the future and correctly expect their future utilities to change over time will have utilities that depend on consumption time only. People whose projections of the future improve as time passes can be modeled by utility functions that depend on temporal distance as well.

The literature on intertemporal choice has focused almost exclusively on one potential driver of time inconsistencies: non-stationarity. Stationarity holds if a preference between outcomes to be received at different points in time is unaffected by a common additional delay of all outcomes. Deviations from stationarity are often thought to be driven by pure time preference, being the way people weight future points in time, irrespective of the outcomes received at these points in time. Accordingly, hyperbolic discounting models were proposed to accommodate non-stationary behavior (Loewenstein and Prelec 1992; Harvey 1986, 1995; Mazur 1987; Phelps and Pollak 1968). These models can be given a psychological foundation by construal-level theory (Trope and Liberman 2010) and the nonlinear manner in which humans perceive temporal distance (Zauberman et al. 2009). To the extent that these nonlinear perceptions of time are considered irrational, we can view deviations from stationarity caused by pure time preference as irrational.

While deviations from stationarity are a potential cause of time inconsistencies, Halevy (2015) provided empirical evidence that non-stationary behavior neither implies nor is implied by time inconsistency. In his study, only two-thirds of the subjects who exhibit time consistency also exhibit stationarity and half of the subjects whose choices are time inconsistent exhibit stationarity. These findings show that deviations from stationarity are not the sole drivers of time-inconsistent behavior and they cast doubt on the extent to which such deviations are irrational.

In our model, deviations from stationarity are not only caused by pure time preference and nonlinear perception of time (Takahashi et al. 2008), but also by the time dependence of the utility of an outcome. Such time dependence naturally arises whenever the decision maker evaluates an outcome according to the extra utility it generates on top of some baseline consumption (cf. Noor 2009; Gerber and Rohde 2010). If the decision maker expects his baseline consumption to change over time, then the utility of an outcome depends on its timing irrespective of pure time preference. This

dependency, which can induce non-stationarity, can be viewed as foresight of future utility and, therefore, is not irrational as long as it is perfect foresight. We will show that the dependence of the utility of an outcome on its timing and the resulting non-stationarities need not result in time inconsistencies. Depending on whether utility depends on consumption time only, temporal distance only, or decision time only, time consistencies are driven by properties of the weighting function only or by properties of the utility function as well.

As the timing of an outcome influences both its utility and the weight given to the probability that it will be received, measuring the weighting and utility functions of the weighted temporal utility model may seem difficult at first sight. We will show how this can be accomplished in a nonparametric way. This nonparametric approach does not require any assumption about the shape of the utility and weighting functions. In particular, it does not require an assumption of linear utility, which is often used in the literature.

The outline of this paper is as follows. Section 2 introduces the weighted temporal utility (WTU) model and provides an axiomatization. Section 3 discusses the relation between WTU and other models of intertemporal choice. In Sect. 4, we show that the WTU model is consistent with empirical regularities. Section 5 presents a dynamic version of WTU and provides conditions for stationarity, time consistency, and time invariance. Section 6 shows how the weighting and utility function of WTU can be measured. Finally, Sect. 7 concludes.

2 The model

This paper considers preferences \succsim over *risky timed outcomes* (x, p, t) which give *outcome* $x \in \mathbb{R}_+ = [0, \infty)$ with *probability* $p \in [0, 1]$ at *time* $t \in \mathbb{R}_+$ and zero otherwise. Like [Baucells and Heukamp \(2012\)](#), we therefore add an uncertainty dimension to the framework of [Fishburn and Rubinstein \(1982\)](#). Strict preference \succ and indifference \sim are defined as usual. We make the following basic assumptions on \succsim :

Assumption 1

- A1 (Continuity) \succsim is a continuous weak order.
 A2 (Zero Equivalence) For all outcomes x, y , for all probabilities p, q , and for all times $t, s \in \mathbb{R}_+$,

$$(x, p, t) \sim (y, q, s) \text{ whenever } px = qy = 0.$$

A1 is a standard assumption that is sufficient for \succsim to be representable by a utility function. A2 requires that the decision maker is indifferent between any two risky dated outcomes that are both equivalent to receiving zero for sure, either because the outcome is zero itself or because a positive outcome is received with probability zero.

Weighted temporal utility (WTU) holds if \succsim can be represented by

$$V(x, p, t) = w(p, t)v(x, t),$$

where w is a *weighting function* and v is a *utility function*. Under WTU, a decision maker evaluates a risky timed outcome (x, p, t) by first determining the utility $v(x, t)$

that outcome x will yield at time t , irrespective of the probability that it will be received, and then discounting this utility by a weight $w(p, t)$, which can be viewed as a time-dependent probability-weighting function.¹

WTU captures two ways in which the time at which a risky outcome is received can influence its evaluation. First of all, the utility derived from outcome x may depend on time t . A special case is the one where utility $v(x, t)$ equals the additional utility u outcome x gives on top of baseline consumption b_t at time t , i.e., where $v(x, t) = u(b_t + x) - u(b_t)$. A decision maker who expects to be wealthier in the future ($b_t > b_0$) will then expect $x > 0$ to generate less utility in the future than now if u is strictly concave. Ambrus et al. (2015) provided evidence that future income expectations indeed can influence choices over delayed rewards.

Second, as the utility is generated in the future and only with a probability p , it can be viewed as a psychologically distant utility. The weighting function $w(p, t)$ transforms the two components, p and t , of this psychological distance into a discount which is applied to the instantaneous utility $v(x, t)$. In psychology, construal-level theory (Trope and Liberman 2010) has been proposed as a theory which shows how psychological distance resulting from a.o. risk and time influences decision making. Prelec and Loewenstein (1991) showed that there are many parallels between the impacts of risk and time on decision making. Moreover, Halevy (2008) argues that any delay in the receipt of an outcome involves a risk due to the hazard of mortality. All this supports the idea that risk and time can be summarized into one variable: psychological distance. Our model puts construal-level theory into a (mathematical) weighting function. The weighting function w can be thought of as a function that first combines probability and delay into psychological distance and then gives a weight to this distance. Keren and Roelofsma (1995), Abdellaoui et al. (2011), and Baucells and Heukamp (2012) provided empirical evidence for the non-separability of probability and time. Hence, we do not assume that $w(p, t)$ can be written as $w(p, t) = f(p)g(t)$ for some functions f and g . Yet, for single outcomes, rank-dependent utility, prospect theory, exponential discounting, and hyperbolic discounting are special cases of WTU.

In the remainder of this section, we will provide a characterization of WTU. The following axioms will be shown to be necessary and sufficient for WTU to hold.

Axiom 1 (*Monotonicity*) \succsim satisfies the following conditions:

M1 (Monotonicity in Probabilities) For all outcomes x with $x > 0$, for every time t , and for all probabilities p, q with $p > q$,

$$(x, p, t) \succ (x, q, t).$$

M2 (Monotonicity in Outcomes) For all outcomes x, y , with $x > y$, for every time t , and for all probabilities p with $p > 0$,

$$(x, p, t) \succ (y, p, t).$$

¹ One way to extend the model to a setting with multiple possible outcomes would be to first compute the rank-dependent utilities of the lotteries to be received at all points in time, using the time-dependent weighting and utility functions, and then to sum over these rank-dependent utilities (in the spirit of the separable case in the online appendix of Epper and Fehr-Duda 2015a).

Axiom 2 (*Present Solvability*) For all outcomes x and every time t , there exists an outcome x_0 such that²

$$(x_0, 1, 0) \sim (x, 1, t).$$

Axiom 3 (*Hexagon Condition at Time t*) For all outcomes $x, y, z > 0$ and all probabilities $p, q, l > 0$, if

$$(y, p, t) \sim (x, q, t) \text{ and } (y, q, t) \sim (x, l, t),$$

then

$$(z, p, t) \sim (y, q, t) \text{ implies } (z, q, t) \sim (y, l, t).$$

The hexagon condition has been introduced by [Debreu \(1960\)](#) to characterize additively separable utility functions if there are only two essential factors. It can be interpreted as follows. Assume that the trade-off between p and q equals the trade-off between q and l in the sense that they both offset the trade-off between y and x at time t . If the trade-off between p and q also offsets the trade-off between z and y at time t , then the hexagon condition implies that the trade-off between q and l offsets the trade-off between z and y at time t as well. Thus, the hexagon condition allows us to conclude that, at time t , the trade-off between p and q equals the trade-off between q and l , irrespective of the outcomes. [Wakker \(1989\)](#) showed that the hexagon condition is weaker than the often used Thomsen condition ([Thomsen 1927](#)). [Karni and Safra \(1998\)](#) provided conditions which imply the equivalence of the Thomsen and hexagon conditions.

For WTU to represent preferences, we need more than only additive separability of x and p at every single point in time. Such a separability condition would imply a WTU representation at every given point in time. Yet, we also need a condition which allows us to use WTU to compare outcomes that are received at *different* points in time. Consider, for instance, the evaluation function $V(x, p, t) = (\ln(x) + \ln(p))^{1/(1+t)}$. This function satisfies additive separability of x and p at every point in time t , but is not a WTU representation. The following condition provides the missing link between the different points in time.

Axiom 4 (*Probability-Independent Time–Outcome Trade-off*) For all outcomes $x, y, x_0, y_0 > 0$, for all probabilities p, p_0 , and every time t , if

$$(x, 1, t) \sim (x_0, 1, 0) \text{ and } (y, 1, t) \sim (y_0, 1, 0),$$

then

$$(x, p, t) \sim (x_0, p_0, 0) \text{ implies } (y, p, t) \sim (y_0, p_0, 0).$$

² Observe that present solvability follows from continuity and monotonicity in outcomes if the decision maker is weakly impatient, i.e., if $(x, 1, 0) \succsim (x, 1, t)$ for all $x \in \mathbb{R}_+$ and for all $t \in \mathbb{R}_+$.

Probability-independent time–outcome trade-off can be interpreted as follows. Assume that the trade-off between x for sure and x_0 for sure equals the trade-off between y for sure and y_0 for sure in the sense that they both offset the trade-off between time t and time 0. Assume that the trade-off between x with probability p and x_0 with probability p_0 also offsets the trade-off between time t and time 0. Then, probability-independent time–outcome trade-off implies that the trade-off between y with probability p and y_0 with probability p_0 offsets the trade-off between time t and time 0 as well.

The following theorem imposes additive separability of x and p at time $t = 0$ through the hexagon condition. Probability-independent time–outcome trade-off then implies additive separability of x and p at every time $t \geq 0$ and ensures that we can use WTU to compare outcomes that are received at different points in time. The proof is in the Appendix.³

Theorem 2.1 *Under present solvability and monotonicity, the following statements are equivalent:*

- (i) *Probability-independent time–outcome trade-off and the hexagon condition at time 0 hold.*
- (ii) *Preferences \succsim can be represented by*

$$V(x, p, t) = w(p, t)v(x, t)$$

with $w(0, t) = v(0, t) = 0$ for all t . Moreover, for all x, p, t we have $w(p, t) \geq 0$ and $v(x, t) \geq 0$ with w increasing in p and v increasing in x .

Furthermore, $V'(x, p, t) = w'(p, t)v'(x, t)$ also represents \succsim if and only if there exist $\alpha(t) > 0$, $\beta > 0$, and $\gamma > 0$ such that

$$w'(p, t) = \alpha(t) (w(p, t))^\gamma \text{ and}$$

$$v'(x, t) = \frac{\beta}{\alpha(t)} (v(x, t))^\gamma$$

for all (x, p, t) .

3 Relation to other models of intertemporal choice

3.1 Discounted expected utility

A special case of WTU is discounted expected utility where utility is derived from adding an outcome to baseline consumption at the time when the outcome is received.

³ Recently, Gilboa et al. (2016) characterized the evaluation function $f(x, y, z) = u(x, y) + v(y, z)$ for $(x, y, z) \in X \times Y \times Z$ with X, Y , and Z convex subsets of Euclidian spaces. Their key condition is cross-consistency, which cannot directly be related to our probability-independent time–outcome trade-off and the hexagon condition at time 0. In our setting, the latter conditions have a more intuitive interpretation than cross-consistency, as time $t = 0$ has a very natural interpretation.

If b_t is baseline consumption at time t and u is the decision maker's utility function, then the discounted expected utility of a risky timed outcome (x, p, t) is

$$V(x, p, t) = p\delta(t)(u(b_t + x) - u(b_t)), \quad (1)$$

where δ is the time discount function. The utility generated by receiving outcome x at time t , therefore, is the extra utility outcome x generates on top of the utility derived from baseline consumption at time t . This model is consistent with the one proposed by Noor (2009). Gerber and Rohde (2015) derived testable hypotheses on the discount function when baseline consumption is unobserved. Baseline consumption can be interpreted as any status quo to which additional outcomes are added. Gerber and Rohde (2010) considered the case where baseline consumption can be stochastic rather than deterministic.

3.2 Probability and time trade-off model

The WTU model separates attitudes toward psychological distance and attitudes toward outcomes, where the former are captured by the weighting function $w(p, t)$ and the latter by the utility function $v(x, t)$. Baucells and Heukamp (2012) had introduced an alternative probability and time trade-off (PTT) model, where the utility of a risky timed outcome is given by $V(x, p, t) = w(pe^{-r_x t})v(x)$ with w increasing and $r_x > 0$ decreasing in x . Thus, like WTU, PTT allows the trade-off between probability p and time t to depend on the outcome x , but unlike WTU, it does not assume x and p to be additively separable given t .

For further comparison, we consider the characterizing axioms of WTU and PTT. Both models satisfy standard continuity and monotonicity conditions in outcomes and probabilities, but they behave differently with respect to the following axioms. We will show that WTU is more general than PTT in the sense that it may but need not satisfy monotonicity in time and probability–time trade-off. Yet, it is less general than PTT in the sense that PTT may but need not satisfy probability-independent time–outcome trade-off. Finally, we will show that WTU can also account for subendurance, the condition that ensures r_x to be decreasing in x for PTT.

3.2.1 Monotonicity in t

Under PTT, utility is always decreasing in time t , while it can be both decreasing and increasing in t under WTU. This is due to the fact that under WTU, time does not only affect psychological distance as in PTT, but it also affects the instantaneous utility of the outcome. Hence, even though an increase in delay t decreases $w(p, t)$ through an increase in psychological distance, it may increase overall utility $V(x, p, t)$ if $v(x, t)$ is increasing in t to a sufficiently strong extent.⁴

⁴ For example, consider the discounted expected utility model, where $V(x, p, t) = p\delta(t)(u(b_t + x) - u(b_t))$, and assume that $u'' < 0$ and b_t is decreasing in t . Then, $v(x, t) = u(b_t + x) - u(b_t)$ is increasing in t .

3.2.2 Probability–time trade-off

One of the characterizing axioms of PTT, probability–time trade-off, requires that for all $(x, p, t), (x, q, s)$ with $x > 0, p > 0$ and $q > 0$, and for all $\Delta > 0$ and $\theta \in (0, 1)$, if $(x, p, t + \Delta) \sim (x, p\theta, t)$, then $(x, q, s + \Delta) \sim (x, q\theta, s)$. Thus, if adding a delay Δ is equivalent to multiplying probabilities by a factor θ for some probability and point in time, then this equivalence holds for all probabilities and points in time. One could say that in this sense there is a fixed probability–time trade-off for each outcome x . In general, WTU need not satisfy this condition.⁵

3.2.3 Probability-independent time–outcome trade-off

One of the characterizing axioms of WTU is probability-independent time–outcome trade-off. It ensures separability of outcomes and probabilities for $t > 0$. PTT need not satisfy this separability and therefore need not satisfy this axiom.⁶ Note, however, that PTT satisfies additive separability at $t = 0$ which also implies that it satisfies the hexagon condition at $t = 0$.

3.2.4 Subendurance

Baucells and Heukamp (2012) provided empirical evidence that the willingness to wait in exchange for a higher probability to receive a reward increases in the size of the reward. They call this behavioral pattern *subendurance*. Formally, it holds if for all (x, p, t) with $px > 0$, for all $\theta \in (0, 1)$, for all $\Delta > 0$ and for all $0 < y < x$ we have that $(x, p, t + \Delta) \sim (x, p\theta, t)$ implies $(y, p, t + \Delta) \preceq (y, p\theta, t)$. *Isoendurance* holds if the implied weak preference \preceq is always an indifference \sim . Baucells and Heukamp (2012) showed that subendurance can be rationalized by a weighting function w which depends on the outcome x via a decreasing probability discount rate r_x . In our model, the dependence on outcomes of the trade-off between probability and time is captured by the utility function v and the following proposition follows immediately.

Proposition 3.1 *Under WTU, subendurance (isoendurance) is equivalent to*

$$v(x, t)/v(x, t + \Delta)$$

being weakly decreasing (constant) in x for all $x > 0$, all t , and all $\Delta > 0$.

In particular, under WTU isoendurance holds if $v(x, t)$ is independent of time.

3.2.5 Probability–time exchange

To conclude the comparison with the PTT model, we consider the special case of WTU where the weighting function is given by $w(pe^{-rt})$ for some constant $r > 0$.

⁵ It does so in special cases, e.g., $V(x, p, t) = p\delta^t v(x)$.

⁶ It is satisfied by PTT in special cases, e.g., if $w(p) = p$ for all p .

Hence, unlike in PTT, the probability discount rate r does not depend on x . As we have argued above, under PTT r is assumed to be decreasing in x to be able to capture subendurance. By contrast, in our WTU model subendurance is driven by the utility function v . Probability–time exchange, as defined next, generates the weighting function $w(p, t) = w(pe^{-rt})$. It requires a Δ increase in temporal distance to be equivalent to a $e^{-r\Delta}$ discount of probability.

Axiom 5 (*Probability–Time Exchange*) There exists an r such that for all outcomes x, y , every time t , every $\Delta \geq 0$, and every probability p we have that

$$(x, pe^{-r\Delta}, t) \sim (y, p, t + \Delta) \text{ implies } (x, qe^{-r\Delta}, t) \sim (y, q, t + \Delta)$$

for all probabilities q . Moreover, for every risky timed outcome (x, p, t) there is an x_0 such that

$$(x_0, pe^{-rt}, 0) \sim (x, p, t).$$

Theorem 3.2 *Under monotonicity the following statements are equivalent:*

- (i) *Probability–time exchange and the hexagon condition at time 0 hold.*
- (ii) *Preferences \succsim can be represented by*

$$V(x, p, t) = w(pe^{-rt})v(x, t)$$

with $w(0) = v(0, t) = 0$ for all t . Moreover, for all x, p, t , we have $w(p) \geq 0$ and $v(x, t) \geq 0$ with w increasing and v increasing in x . Furthermore, for every x and t we can find an x_0 such that $v(x_0, 0) = v(x, t)$.

4 WTU in a static setting

In the following, we will show that the WTU model can account for the empirical regularities that have been documented in the literature. This section considers a static setting where the decision maker makes decisions at one point in time and only the timing of the receipt of outcomes varies. The next section considers a dynamic setting where the decision time varies as well. In this dynamic setting, we will show that the WTU model can explain phenomena that cannot be explained by existing models like PTT.

For single outcomes, as considered in this paper, rank-dependent utility and prospect theory are special cases of WTU. Hyperbolic discounting is a special case of WTU as well. Consider, for instance, $V(x, p, t) = w(p)\delta(t)v(x)$. By restricting attention to $t = 0$, we have rank-dependent utility and prospect theory. By restricting attention to $p = 1$, we have hyperbolic discounting if δ is defined appropriately. WTU therefore accommodates the common ratio effect and the common difference effect (Prelec and Loewenstein 1991).

The magnitude effect, which says that larger outcomes are discounted less than smaller outcomes, can be accommodated by WTU through a utility function $v(x, t)$ which is concave in x .

Concerning the interaction between risk and time it has been frequently observed that decision makers are less risk averse for later points in time, i.e., the *risk premium*

decreases with t (Abdellaoui et al. 2011). WTU can accommodate this finding through the weighting function, which allows for interactions between probability and time, and also through the utility function, which allows for interactions between outcomes and time. Abdellaoui et al. (2011) attributed the effect to the weighting function: they found utility to be time independent and the probability-weighting function to be time dependent. Ambrus et al. (2015), however, provided evidence that utility can be time dependent. If, for instance, decision makers integrate future outcomes with future baseline consumption, expected changes in baseline consumption can make utility time dependent. The utility function in this case would be $v(x, t) = u(b_t + x) - u(b_t)$, with b_t denoting baseline consumption at time t . Ambrus et al. (2015) indeed found that future income expectations (b_t) can influence choices over delayed outcomes. Given that there is no unambiguous evidence in favor of time-independent utility, it seems reasonable to allow the utility of an outcome to depend on time. In a dynamic setting, we will see that this time dependence allows WTU to explain empirical phenomena in a practically more appealing manner than other existing models.

Finally, the following example shows that WTU is compatible with all the empirical findings that Baucells and Heukamp (2012) use to support their model. Note that the WTU function specified below also satisfies the empirical regularities discussed before.

Example 4.1 Let time be denoted in weeks. Consider a decision maker with WTU function $V(x, p, t) = w(p, t)v(x, t)$, where $w(p, t) = e^{-(-\ln(p)+0.023t)^{0.65}}$ for all p, t , $v(x, t) = \sqrt{100 + \frac{5}{4}t + x} - \sqrt{100 + \frac{5}{4}t}$ for all x and all $t < 4$, and $v(x, t) = \sqrt{105 + x} - \sqrt{105}$ for all x and $t \geq 4$. Thus, the decision maker adds x to his baseline consumption, which equals $100 + \frac{5}{4}t$ for $t < 4$ and 105 for $t \geq 4$. The weighted temporal utilities of the prospects in Baucells and Heukamp (2012, Table 1) are summarized in Table 1. These utility levels yield modal choices as reported in Baucells and Heukamp (2012). Thus, WTU can account for their empirical findings.

5 WTU in a dynamic setting

Choices between risky timed outcomes involve three important moments: *consumption time*—the time at which the outcome is received, *decision time*—the time at which the decision is made, and *resolution time*—the time at which uncertainty is resolved. Throughout this paper, we assume that resolution time coincides with consumption time. So far we have considered a static setting where the decision time was fixed and only the consumption time could vary. This section will consider a dynamic setting where the decision time can vary as well.

We assume that for every decision time τ the decision maker has a preference relation \succsim^τ over risky timed outcomes to be received from time τ onwards. By $\{\succsim^\tau\}_\tau$ we denote the set of preferences for all decision times τ . Strict preference $>^\tau$ and indifference \sim^τ are defined as usual. Halevy (2015) considered three notions of consistency: stationarity, time invariance, and time consistency. The definitions are as follows.

Table 1 Prospects and utility values for Example 4.1

	Prospect A	Prospect B	$V(A)$	$V(B)$
1.	(€9, 100%, now)	(€12, 80%, now)	0.4403	0.3998
2.	(€9, 10%, now)	(€12, 8%, now)	0.0789	0.0939
3.	(€9, 100%, 3 months)	(€12, 80%, 3 months)	0.2789	0.3014
4.	(fl.100, 100%, now)	(fl.110, 100%, 4 weeks)	2.0603	1.7820
5.	(fl.100, 100%, 26 weeks)	(fl.110, 100%, 30 weeks)	0.9867	1.0041
6.	(fl.100, 50%, now)	(fl.110, 50%, 4 weeks)	0.9369	0.9373
7.	(€100, 100%, 1 month)	(€100, 90%, now)	3.2930	3.2858
8.	(€5, 100%, 1 month)	(€5, 90%, now)	0.1951	0.1959

$V(A)$ and $V(B)$ denote the weighted temporal utilities of prospect A and B , respectively. Rows 4–6 have outcomes denoted in Dutch Guilders. We transformed them into Euro by using the conversion rate at the introduction of the Euro: fl.100 is approximately €45.45 and fl.110 is approximately €50. We set 1 month equal to 4 weeks and 3 months equal to 12 weeks

Definition 5.1 Preferences $\{\succsim^\tau\}_\tau$ are stationary if for every x, y, p, q, τ, s, t , with $0 \leq \tau \leq s, t$, and for every $\Delta \geq 0$,

$$(x, p, t) \sim^\tau (y, q, s) \iff (x, p, t + \Delta) \sim^\tau (y, q, s + \Delta).$$

Stationarity means that preferences remain unchanged if the decision time remains unchanged and all consumption times are delayed by a common time interval.

Definition 5.2 Preferences $\{\succsim^\tau\}_\tau$ are time invariant if for every x, y, p, q, τ, s, t with $0 \leq \tau \leq s, t$, and for every $\Delta \geq 0$,

$$(x, p, t) \sim^\tau (y, q, s) \iff (x, p, t + \Delta) \sim^{\tau+\Delta} (y, q, s + \Delta).$$

Time invariance means that preferences remain unchanged if the temporal distance between consumption and decision time remain unchanged.

Definition 5.3 Preferences $\{\succsim^\tau\}_\tau$ are time consistent if for every $x, y, p, q, \tau, \tau', s, t$, with $0 \leq \tau, \tau' \leq s, t$,

$$(x, p, t) \sim^\tau (y, q, s) \iff (x, p, t) \sim^{\tau'} (y, q, s).$$

Time consistency means that preferences remain unchanged if consumption time remains unchanged.

Any two of the three properties (stationarity, time invariance, and time consistency) imply the third, and hence, either none or at least two of the properties must be violated (Halevy 2015). Thus, there are five possible preference types as summarized in Table 2.

The literature on intertemporal choice has focused almost exclusively on deviations from stationarity and much less on other potential causes of time inconsistencies. Yet, Halevy (2015) provided experimental evidence for all five preference types in Table 2. In particular, in his experiment only two-thirds of the subjects who exhibit time

Table 2 The five possible preference types

Type	Stationary	Time invariant	Time consistent
I	✓	✓	✓
II	–	✓	–
III	✓	–	–
IV	–	–	✓
V	–	–	–

“✓” (“–”) means that preferences have (do not have) the corresponding property

consistency also exhibit stationarity and half of the subjects whose choices are time inconsistent exhibit stationarity. This shows that non-stationary behavior, e.g., due to decreasing impatience, is not equivalent to time inconsistency. We will show how the WTU model can efficiently accommodate all preference types of Table 2.

In general, we can let any model account for violations of time invariance and time consistency by assuming that its functional forms and parameters depend on decision time. For WTU in its most general form, we then have that at decision time τ preferences \succsim_τ can be represented by

$$V_\tau(x, p, t) = w_\tau(p, t)v_\tau(x, t),$$

where $t \geq \tau$ is consumption time. Similarly, the PTT model of [Baucells and Heukamp \(2012\)](#) in its most general form is $V_\tau(x, p, t) = w_\tau(pe^{-\lambda x, \tau t})v_\tau(x)$. We say that outcome x to be received with probability p at consumption time $t \geq \tau$ is at temporal distance $t - \tau$ viewed from decision time τ .

These models in their general forms may be too general for use in applications. Their generality, for instance, implies that the weighting and utility functions measured at one decision time cannot be used to predict choices at other decision times. Predictions of dynamic decision making then require a separate measurement of these weighting and utility functions at every possible decision time. In applications, less general versions of the models may be preferred. Ideally, weighting and utility functions measured at one decision time, i.e., in a static setting, carry over to other decision times, so that no additional measurements are needed to predict choices in a dynamic setting. This is possible if these functions depend on decision time through temporal distance only, i.e., if these functions depend on time through consumption time and temporal distance only. This section will show that the WTU model can be specified in such an ideal way and still accommodate violations of stationarity, time invariance, and time consistency in various combinations, which, for instance, discounted expected utility and PTT cannot.

In a static setting with decision time $\tau = 0$, we have that consumption time t coincides with temporal distance. The distinction between consumption time and temporal distance therefore does not matter in a static setting. In a dynamic setting, this distinction becomes important, though. Thus, it becomes important to specify whether time t in the static setting is consumption time or temporal distance.

Let us first consider discounted expected utility and the PTT model and assume that we would like the functions measured in a static setting to carry over to the

dynamic setting, i.e., we want the functions to depend on time through consumption time and/or temporal distance only. If time is considered to be temporal distance, as it is commonly interpreted, then time invariance follows automatically. Alternatively, if time is considered to be consumption time, time consistency follows automatically. Thus, for discounted expected utility and PTT, carrying over the functions measured in a static setting to a dynamic setting implies that either time invariance or time consistency cannot be violated. For WTU, time enters both the weighting function and the utility function. In a dynamic setting, we therefore need to specify for each of these functions whether time is consumption time or temporal distance. We will show that by letting the weighting function depend on temporal distance only, as is common, and letting the utility function depend on consumption time only, we can accommodate violations of both time invariance and time consistency.

Discounting in intertemporal choice is typically thought to be driven only by temporal distance between consumption and decision time. We therefore let the weighting function be a function which only depends on psychological distance to the outcome. Thus, it combines distance resulting from risk and temporal distance: $w_\tau(p, t) = w(p, t - \tau)$. We therefore assume that the weighting function is independent of decision time: once we have measured it in a static setting, it carries over to the dynamic setting without requiring additional measurements.

The literature usually assumes that utility is independent of time, so that $v_\tau(x, t) = v(x)$, while our WTU allows for time-dependent utility. In its most general form, utility depends both on consumption and decision time: $v_\tau(x, t)$. It then also automatically depends on temporal distance. Yet, working with such a specification can be inconvenient in practice, as it does not allow utility measured in a static setting to carry over to a dynamic setting: It requires a separate measurement of the utility function at each decision time. Utility measured in a static setting would carry over to a dynamic setting if it would depend on consumption time only ($v_\tau(x, t) = v(x, t)$) or on temporal distance only ($v_\tau(x, t) = v(x, t - \tau)$).

The psychological processes driving the dependence of utility on time, determine whether utility depends on consumption time, decision time, and temporal distance, or only on a subset of these. The WTU model, with the well-known discounted utility model as a special case, captures two psychological steps that can be taken to evaluate future outcomes. When evaluating a future outcome, the DM first projects himself to the future and determines how much utility the outcome will generate in the future. Then, the DM determines how much this future utility is worth to him now. One can think of the first step to relate to the utility function, and the second one to the discount function.

In the most rational scenario, the DM would perfectly project himself to the future and would correctly predict his future utility. In this case, utility depends only on consumption time and not on decision time or temporal distance. As discussed by [Loewenstein et al. \(2003\)](#), however, we suffer from projection bias when predicting future utility. When the DM imperfectly projects himself to the future, utility will also depend on temporal distance or decision time. Suppose that utility depends only on temporal distance. If $v(x, t - \tau)$ is continuous, we then have that predictions of future utility become more accurate as temporal distance is reduced: $v(x, t - \tau)$ approaches $v(x, 0)$, the utility of receiving x immediately. If, instead, utility depends on decision

Table 3 Relations of stationarity, time invariance, and time consistency with properties of w and v

	Stationarity	Time invariance	Time consistency
$w(p, t - \tau)v(x)$	w	✓	w
$w(p, t - \tau)v(x, t)$	w and v	v	w
$w(p, t - \tau)v(x, t - \tau)$	w and v	✓	w and v
$w(p, t - \tau)v_\tau(x)$	w	v	w and v

This table shows, for various specifications of the utility function v , on which functions (w or v) the various intertemporal consistency concepts impose restrictions. “✓” means that the property is always satisfied, no matter the exact form of w or v

time with $v_\tau(x, t) = v_\tau(x)$, predictions need not become more accurate as t moves closer to τ . Moreover, we then have a different utility function at each decision time so that utility measured in a static setting no longer carries over to a dynamic setting.

These different versions of WTU have different implications in terms of stationarity, time invariance, and time consistency, as we will see next. We will show to what extent stationarity, time invariance, and time consistency are driven by properties of the weighting function only, the utility function only, or the interaction between both functions. Table 3 summarizes our findings. For instance, it shows that if utility is independent of decision time, consumption time, and temporal distance, stationarity and time consistency are driven by properties of the weighting function only, and stationarity is equivalent to time consistency. If, however, utility depends on consumption time only, stationarity depends on properties of the interaction between the weighting and utility function, while time consistency depends on properties of the weighting function only. Moreover, time invariance is now driven by properties of the utility function. Thus, stationarity and time consistency are no longer equivalent.

The typical assumption in the literature is that utility does not depend on consumption time, decision time, or temporal distance. In this case preferences are time invariant and deviations from stationarity imply deviations from time consistency and vice versa. Moreover, stationarity and time consistency are determined by the weighting function, as the next theorem summarizes.

Theorem 5.4 *Assume that time- τ preferences \succsim^τ are represented by the utility function*

$$V_\tau(x, p, t) = w(p, t - \tau)v(x).$$

Then, preferences are time invariant. Moreover, preferences are stationary if and only if they are time consistent. Preferences are stationary and time consistent if and only if

$$\frac{w(p, t - \tau)}{w(q, s - \tau)} = \frac{w(p, t - \tau + \Delta)}{w(q, s - \tau + \Delta)} \tag{2}$$

for all p, q, s, t, τ, Δ with $q > 0$.

A special and well-known weighting function which satisfies Eq. (2) is $w(p, t) = w(p)\delta^t$ with $\delta \in (0, 1)$.

If utility depends on consumption time only, time consistency is still solely determined by the weighting function. Yet, time invariance is no longer guaranteed and is determined by properties of the utility function only. Stationarity is determined by properties of the interaction between the weighting and utility functions.

Theorem 5.5 *Assume that time- τ preferences \succsim^τ are represented by the utility function*

$$V_\tau(x, p, t) = w(p, t - \tau)v(x, t).$$

Then,

(i) *stationarity holds if and only if*

$$\frac{w(p, t - \tau)v(x, t)}{w(q, s - \tau)v(y, s)} = \frac{w(p, t - \tau + \Delta)v(x, t + \Delta)}{w(q, s - \tau + \Delta)v(y, s + \Delta)} \tag{3}$$

for all $x, y, p, q, s, t, \tau, \Delta$ with $y > 0$ and $q > 0$,

(ii) *time invariance holds if and only if*

$$\frac{v(x, t)}{v(y, s)} = \frac{v(x, t + \Delta)}{v(y, s + \Delta)} \tag{4}$$

for all x, y, s, t, Δ with $y > 0$,

(iii) *time consistency holds if and only if*

$$\frac{w(p, t - \tau)}{w(q, s - \tau)} = \frac{w(p, t - \tau')}{w(q, s - \tau')} \tag{5}$$

for all p, q, s, t, τ, τ' with $q > 0$.

Under the assumption of Theorem 5.5, the weighting function $w(p, t) = w(p)\delta^t$ with $\delta \in (0, 1)$ induces time consistency (Eq. 5), but neither time invariance nor stationarity. Thus, when the weighting function depends on temporal distance only and the utility function on consumption time only, these functions measured in a static setting carry over to the dynamic setting, and we can have violations of stationarity, time invariance, and time consistency. This specification of the WTU model could therefore be particularly useful to model dynamic decision making.

If utility depends on temporal distance only, time consistency is no longer determined by only the weighting function. Yet, time invariance is then guaranteed and stationarity coincides with time consistency.

Theorem 5.6 *Assume that time- τ preferences \succsim^τ are represented by the utility function*

$$V_\tau(x, p, t) = w(p, t - \tau)v(x, t - \tau).$$

Then, time invariance holds. Moreover, preferences are stationary if and only if they are time consistent. Preferences are stationary and time consistent if and only if

$$\frac{w(p, t - \tau)v(x, t - \tau)}{w(q, s - \tau)v(y, s - \tau)} = \frac{w(p, t - \tau + \Delta)v(x, t - \tau + \Delta)}{w(q, s - \tau + \Delta)v(y, s - \tau + \Delta)} \tag{6}$$

for all $x, y, p, q, s, t, \tau, \Delta$ with $y > 0$ and $q > 0$.

If utility depends on decision time only, time consistency is also determined by the interaction between the weighting and utility function. Time invariance is determined by the utility function only and stationarity by the weighting function only.

Theorem 5.7 Assume that time- τ preferences \succsim^τ are represented by the utility function

$$V_\tau(x, p, t) = w(p, t - \tau)v_\tau(x).$$

Then,

(i) stationarity holds if and only if

$$\frac{w(p, t - \tau)}{w(q, s - \tau)} = \frac{w(p, t - \tau + \Delta)}{w(q, s - \tau + \Delta)} \tag{7}$$

for all p, q, s, t, τ, Δ with $q > 0$,

(ii) time invariance holds if and only if

$$\frac{v_\tau(x)}{v_\tau(y)} = \frac{v_{\tau+\Delta}(x)}{v_{\tau+\Delta}(y)} \tag{8}$$

for all x, y, τ, Δ with $y > 0$,

(iii) time consistency holds if and only if

$$\frac{w(p, t - \tau)v_\tau(x)}{w(q, s - \tau)v_\tau(y)} = \frac{w(p, t - \tau')v_{\tau'}(x)}{w(q, s - \tau')v_{\tau'}(y)} \tag{9}$$

for all p, q, s, t, τ, τ' with $y > 0$ and $q > 0$.

In this case, the weighting function $w(p, t) = w(p)\delta^t$ with $\delta \in (0, 1)$ implies stationarity, but neither time consistency nor time invariance.

This section showed that we have to measure both w and v in order to get a complete picture of how a decision maker’s preferences respond to changes in decision or consumption time. This measurement is the topic of the next section.

6 Parameter-free elicitation of $V(x, p, t)$

This section presents a parameter-free method for eliciting the weighting function $w(p, t)$ and the utility function $v(x, t)$ of WTU for a given continuous preference

relation \succsim over risky timed outcomes which satisfies impatience and monotonicity. The method is closely related to the trade-off method (Wakker and Deneffe 1996; Abdellaoui 2000). We start with an elicitation of $w(p, 0)$.

6.1 Elicitation of $w(p, 0)$

Fix an arbitrary outcome $x > 0$, an arbitrary probability p_0 with $0 < p_0 < 1$, and a parameter κ with $0 < \kappa < 1$. Without loss of generality we can normalize w so that

$$w(p_0, 0) = \kappa \text{ and } w(1, 0) = 1.$$

Elicit y_1 such that

$$(x, p_0, 0) \sim (y_1, 1, 0) \tag{10}$$

and p_1 such that

$$(x, p_1, 0) \sim (y_1, p_0, 0). \tag{11}$$

By monotonicity and continuity, y_1 and p_1 exist, are unique, and satisfy $y_1 < x$ and $p_1 < p_0$. Indifference (10) is equivalent to

$$\kappa v(x, 0) = v(y_1, 0) \tag{12}$$

and (11) is equivalent to

$$w(p_1, 0)v(x, 0) = \kappa v(y_1, 0). \tag{13}$$

From (12) and (13), it follows that

$$w(p_1, 0) = \kappa^2.$$

We can continue like this and elicit y_i and p_i for $i = 2, 3, \dots$, such that

$$(x, p_{i-1}, 0) \sim (y_i, 1, 0) \tag{14}$$

and

$$(x, p_i, 0) \sim (y_i, p_{i-1}, 0). \tag{15}$$

It follows that

$$w(p_i, 0) = \kappa^{2^i} \text{ for all } i, \tag{16}$$

which can be shown as follows. For $i = 1$, we already verified that (16) holds. Now suppose that $w(p_{i-1}, 0) = \kappa^{2^{i-1}}$. From indifference (14), we have

$$w(p_{i-1}, 0)v(x, 0) = v(y_i, 0)$$

From indifference (15), we have

$$w(p_i, 0)v(x, 0) = w(p_{i-1}, 0)v(y_i, 0).$$

It follows that

$$w(p_i, 0) = w(p_{i-1}, 0)^2 = (\kappa^{2^{i-1}})^2 = \kappa^{2^i}.$$

By choosing the starting point p_0 arbitrarily close to 1, we can make the grid on which we determine the weighting function $w(p, 0)$ arbitrarily fine.

6.2 Elicitation of $v(x, 0)$

Given $w(p, 0)$ with $w(1, 0) = 1$, it is straightforward to elicit $v(x, 0)$. Fix an arbitrary outcome $x > 0$. Without loss of generality, we can normalize v so that

$$v(x, 0) = 1.$$

Then, for any outcome y with $y < x$ elicit p such that

$$(y, 1, 0) \sim (x, p, 0).$$

By monotonicity and continuity, p exists and is unique. Then, we have $v(y, 0) = w(p, 0)$. Similarly, for any outcome y with $y > x$ elicit q such that

$$(x, 1, 0) \sim (y, q, 0).$$

It follows that $v(y, 0) = \frac{1}{w(q, 0)}$.

6.3 Elicitation of $w(p, t)$ and $v(x, t)$ for $t > 0$

In order to elicit $w(p, t)$ and $v(x, t)$ for $t > 0$, we use the method in the proof of Theorem 2.1. For every $x > 0$ elicit $x_0(x, t)$ such that

$$(x, 1, t) \sim (x_0(x, t), 1, 0)$$

and define

$$v(x, t) = v(x_0(x, t), 0). \tag{17}$$

Fix $x > 0$. For every $p > 0$ elicit $p_0(p, t)$ such that

$$(x, p, t) \sim (x_0(x, t), p_0(p, t), 0)$$

and define

$$w(p, t) = w(p_0(p, t), 0). \tag{18}$$

7 Conclusion

This paper introduced the weighted temporal utility (WTU) model, which separates attitudes to temporal distances and projections of future utilities. The model evaluates a risky timed outcome by the product of time-dependent utility generated by this outcome and a time-dependent probability weight. A special case of WTU arises when the decision maker evaluates an outcome at a specific point in time by the extra utility it generates on top of the utility derived from baseline consumption. If baseline consumption is expected to change over time, then the utility generated by an outcome is indeed time-dependent.

The first part of the paper provided a characterization of WTU and the second part considered WTU in a dynamic setting. We considered projections of future utility that depend on consumption time only, on decision time only, or on the temporal distance between these times only. We showed how these specifications of utility lead to stationarity, time invariance, and time consistency to depend on the weighting function, on the utility function, or on the interaction between these two functions. The WTU model therefore enhances our understanding of the drivers of time-inconsistent behavior.

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Appendix

Lemma 1 *Under monotonicity, the following statements are equivalent:*

- (i) *The hexagon condition at time 0 holds.*
- (ii) *Preferences at time 0 can be represented by*

$$V(x, p, 0) = w_0(p)u_0(x).$$

with $w_0(0) = u_0(0) = 0$ and w_0 and u_0 increasing.

Furthermore, $V'(x, p, 0) = w'_0(p)u'_0(x)$ also represents \succsim if and only if there exist $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ such that

$$\begin{aligned} w'_0(p) &= \alpha (w_0(p))^\gamma \text{ and} \\ u'_0(x) &= \beta (u_0(x))^\gamma \end{aligned}$$

for all (x, p) .

Proof of Lemma 1: The fact that (ii) implies (i) can easily be shown and also follows directly from Theorem III.4.1 in [Wakker \(1989\)](#).

Now assume that (i) holds. Then, Theorem III.4.1 in Wakker (1989) shows that we have a representation

$$V(x, p, 0) = w_0(p)u_0(x)$$

for all positive outcomes and positive probabilities. Moreover, $w_0(p) > 0$ for all $p > 0$ and $u_0(x) > 0$ for all $x > 0$. Since \succcurlyeq satisfies monotonicity and $w_0(p) > 0$ for all $p > 0$ and $u_0(x) > 0$ for all $x > 0$ it is straightforward to show that w_0 and u_0 are increasing. From Theorem III.4.1 in Wakker (1989), it also follows that $V'(x, p, 0) = w'_0(p)u'_0(x)$ also represents \succcurlyeq for positive outcomes and probabilities if and only if there are $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ such that

$$\begin{aligned} w'_0(p) &= \alpha (w_0(p))^\gamma \text{ and} \\ u'_0(x) &= \beta (u_0(x))^\gamma. \end{aligned}$$

We will first show that $w_0(p)$ goes to zero as p goes to zero. Since $w_0(p)$ is increasing and bounded below by 0, it follows that there exists $W \geq 0$ with $\lim_{p \rightarrow 0} w_0(p) = W$. Suppose by way of contradiction that $W > 0$. Now consider any two outcomes $y > x > 0$ and a small probability $\varepsilon > 0$. Then,

$$(y, \varepsilon, 0) \succ (x, \varepsilon, 0) \succ (0, \varepsilon, 0) \sim (y, 0, 0)$$

By continuity, there must be a probability $\kappa > 0$ with

$$(x, \varepsilon, 0) \sim (y, \kappa, 0),$$

which implies

$$w_0(\varepsilon)u_0(x) = w_0(\kappa)u_0(y).$$

Note that $\kappa < \varepsilon$. Yet, when ε and κ are small enough, we have that $\frac{w_0(\varepsilon)}{w_0(\kappa)} \approx \frac{W}{W} = 1$, which contradicts the fact that $\frac{u_0(y)}{u_0(x)} > 1$. Hence, $W = 0$, i.e., $w_0(p)$ goes to zero as p goes to zero. A similar argument shows that $u_0(x)$ goes to zero as x goes to zero.

Define $u_0(0) = 0$ and $w_0(0) = 0$. Consider $(x, p, 0) \succcurlyeq (y, q, 0)$. If x, y, p , and q are all positive, then we have that

$$w_0(p)u_0(x) \geq w_0(q)u_0(y).$$

If $x = 0$ or $p = 0$, then by monotonicity we must have $y = 0$ or $q = 0$, which implies that $w_0(p)u_0(x) \geq w_0(q)u_0(y)$. If $y = 0$ or $q = 0$ and $x > 0$ and $p > 0$, then $w_0(p)u_0(x) \geq w_0(q)u_0(y)$ follows as well. This shows that preferences at time 0 can be represented by $V(x, p, 0) = w_0(p)u_0(x)$. The uniqueness properties of w_0 and u_0 now follow immediately from the uniqueness result of Theorem III.4.1 in Wakker (1989). □

Proof of Theorem 2.1 We first prove that (i) implies (ii). Assume that probability-independent time–outcome trade-off and the hexagon condition at time 0 hold. By Lemma 1, preferences at time 0 can be represented by $V(x, p, 0) = w_0(p)u_0(x)$ with w_0 and u_0 nonnegative and increasing, and $w_0(0) = u_0(0) = 0$. For every outcome $x \geq 0$ and time t , define the outcome $x_0(x, t)$ by $(x, 1, t) \sim (x_0(x, t), 1, 0)$. By present solvability and monotonicity, $x_0(x, t)$ is always well defined. For every $x > 0$ and every p, t , define the probability $p_0(x, p, t)$ by $(x, p, t) \sim (x_0(x, t), p_0(x, p, t), 0)$. By monotonicity and continuity, $p_0(x, p, t)$ is always defined for $x > 0$. Probability-independent time–outcome trade-off implies that $p_0(x, p, t) = p_0(y, p, t)$ for all $x, y > 0$. Thus, we define $p_0(p, t) = p_0(x, p, t)$.

Then, we have for $x, y > 0$

$$\begin{aligned} &(x, p, t) \succcurlyeq (y, q, s) \\ \iff &(x_0(x, t), p_0(p, t), 0) \succcurlyeq (x_0(y, s), p_0(q, s), 0) \\ \iff &w_0(p_0(p, t))u_0(x_0(x, t)) \geq w_0(p_0(q, s))u_0(x_0(y, s)). \end{aligned}$$

Now we can define

$$v(x, t) = u_0(x_0(x, t))$$

for all x, t with $x > 0$ and set $v(0, t) = 0$. Further, we can define

$$w(p, t) = w_0(p_0(p, t))$$

for all p, t . It follows that for strictly positive outcomes $V(x, p, t) = w(p, t)v(x, t)$ represents \succcurlyeq .

If $x = 0$ or $y = 0$, then it is straightforward to verify that $(x, p, t) \succcurlyeq (y, q, s)$ if and only if $w(p, t)v(x, t) \geq w(q, s)v(y, s)$. Thus, $V(x, p, t) = w(p, t)v(x, t)$ represents \succcurlyeq . Moreover, by definition, $w(p, t)$ and $v(x, t)$ are nonnegative and increasing in p and x , respectively, and $w(0, t) = v(0, t) = 0$ for all t .

From Lemma 1, we know that w_0 and u_0 can be substituted by w'_0 and u'_0 to represent preferences at time $t = 0$ if and only if there exist $\alpha(0) > 0$, $\beta(0) > 0$, and $\gamma(0) > 0$ such that $w'_0(p) = \alpha(0)(w_0(p))^{\gamma(0)}$ and $u'_0(x) = \beta(0)(u_0(x))^{\gamma(0)}$. Similarly, preferences at time t can be represented by $w'(p, t)v'(x, t)$ if and only if $w'(p, t) = \alpha(t)(w(p, t))^{\gamma(t)}$ and $v'(x, t) = \beta(t)(v(x, t))^{\gamma(t)}$ for some $\alpha(t) > 0$, $\beta(t) > 0$, and $\gamma(t) > 0$.

Let $x \geq 0$ and $t > 0$. Then, by present solvability there exists an x_0 such that $(x_0, 1, 0) \sim (x, 1, t)$. Hence,

$$\begin{aligned} &w_0(1)u_0(x_0) = w(1, t)v(x, t) \\ \text{and } &\alpha(0)\beta(0)[w_0(1)u_0(x_0)]^{\gamma(0)} = \alpha(t)\beta(t)[w(1, t)v(x, t)]^{\gamma(t)} \end{aligned}$$

which implies that

$$\frac{\alpha(0)\beta(0)}{\alpha(t)\beta(t)} = [w(1, t)v(x, t)]^{\gamma(t)-\gamma(0)}. \tag{19}$$

If $\gamma(t) \neq \gamma(0)$, the right-hand side of (19) depends on x , while the left-hand side does not which leads to an immediate contradiction. Hence, (19) implies that $\gamma(t) = \gamma(0)$ for all t and $\alpha(0)\beta(0) = \alpha(t)\beta(t)$ for all t . This proves the uniqueness properties of w and v .

Now we need to prove that (ii) implies (i). Assume that preferences \succsim can be represented by

$$V(x, p, t) = w(p, t)v(x, t),$$

where $w(p, t)$ and $v(x, t)$ are nonnegative and increasing in p and x , respectively, and $w(0, t) = v(0, t) = 0$ for all t . The hexagon condition at time 0 follows from Lemma 1. Assume that $x, y > 0$ and $(x, 1, t) \sim (x_0, 1, 0)$, $(x, p, t) \sim (x_0, p_0, 0)$, and $(y, 1, t) \sim (y_0, 1, 0)$. Then,

$$\frac{v(x, t)}{v(x_0, 0)} = \frac{v(y, t)}{v(y_0, 0)}.$$

If $p > 0$, then

$$\frac{v(x, t)}{v(x_0, 0)} = \frac{w(p_0, 0)}{w(p, t)}$$

and it follows that

$$\frac{v(y, t)}{v(y_0, 0)} = \frac{w(p_0, 0)}{w(p, t)}$$

Thus, $(y, p, t) \sim (y_0, p_0, 0)$ if $p > 0$. If $p = 0$, then $(x, p, t) \sim (x_0, p_0, 0)$ implies that $p_0 = 0$, and hence, $(y, p, t) \sim (y_0, p_0, 0)$ holds in this case as well. \square

Proof of Theorem 3.2 We first prove that (i) implies (ii). Preferences at time 0 can be represented by $V(x, p, 0) = w(p)u_0(x)$ with w and u_0 nonnegative and increasing and $w(0) = u_0(0) = 0$. Let r be the rate as defined in the definition of probability–time exchange. For every (x, p, t) , find $x_0(x, p, t)$ which satisfies

$$(x_0(x, p, t), pe^{-rt}, 0) \sim (x, p, t).$$

This $x_0(x, p, t)$ exists by probability–time exchange. Define

$$v(x, p, t) = u_0(x_0(x, p, t)).$$

By probability–time exchange, $x_0(x, p, t)$ and, therefore, $v(x, p, t)$ are independent of p . We can therefore define $v(x, t) = v(x, p, t)$. It follows that

$$\begin{aligned} &(x, p, t) \succcurlyeq (y, q, s) \\ \iff &(x_0(x, t), pe^{-rt}, 0) \succcurlyeq (x_0(y, s), qe^{-rs}, 0) \\ \iff &w(pe^{-rt})u_0(x_0(x, t)) \geq w(qe^{-rs})u_0(x_0(y, s)) \\ \iff &w(pe^{-rt})v(x, t) \geq w(qe^{-rs})v(y, s). \end{aligned}$$

We now prove that (ii) implies (i). Consider (x, p, t) . Then,

$$V(x, p, t) = w(pe^{-rt})v(x, t)$$

and Lemma 1 implies that the hexagon condition at time 0 holds. Moreover, (ii) implies that there exists an x_0 such that $v(x_0, 0) = v(x, t)$. Then, we have

$$(x_0, pe^{-rt}, 0) \sim (x, p, t).$$

Now suppose that for y and Δ , we have

$$(x, pe^{-r\Delta}, t) \sim (y, p, t + \Delta).$$

It follows that

$$w(pe^{-r\Delta}e^{-rt})v(x, t) = w(pe^{-r(t+\Delta)})v(y, t + \Delta).$$

Thus, $v(x, t) = v(y, t + \Delta)$. It follows that

$$(x, qe^{-r\Delta}, t) \sim (y, q, t + \Delta)$$

for all $q \leq 1$. Hence, probability–time exchange holds which proves the theorem. \square

Proof of Theorems 5.4, 5.5, 5.6, and 5.7 These theorems all assume

$$V_\tau(x, p, t) = w(p, t - \tau)v_\tau(x, t).$$

Stationarity requires that

$$\begin{aligned} w(p, t - \tau)v_\tau(x, t) &= w(q, s - \tau)v_\tau(y, s) \\ \iff & \\ w(p, t + \Delta - \tau)v_\tau(x, t + \Delta) &= w(q, s + \Delta - \tau)v_\tau(y, s + \Delta) \end{aligned}$$

for all $x, y, p, q, s, t, \tau, \Delta$. This is equivalent to requiring

$$\frac{w(p, t - \tau)v_\tau(x, t)}{w(q, s - \tau)v_\tau(y, s)} = \frac{w(p, t + \Delta - \tau)v_\tau(x, t + \Delta)}{w(q, s + \Delta - \tau)v_\tau(y, s + \Delta)}$$

for all $x, y, p, q, s, t, \tau, \Delta$ with $y > 0$ and $q > 0$.

Time invariance requires that

$$\begin{aligned} w(p, t - \tau)v_\tau(x, t) &= w(q, s - \tau)v_\tau(y, s) \\ &\iff \\ w(p, t - \tau)v_{\tau+\Delta}(x, t + \Delta) &= w(q, s - \tau)v_{\tau+\Delta}(y, s + \Delta) \end{aligned}$$

for all $x, y, p, q, s, t, \tau, \Delta$. This is equivalent to requiring

$$\frac{v_\tau(x, t)}{v_\tau(y, s)} = \frac{v_{\tau+\Delta}(x, t + \Delta)}{v_{\tau+\Delta}(y, s + \Delta)}$$

for all $x, y, p, q, s, t, \tau, \Delta$ with $y > 0$ and $q > 0$.

Time consistency requires that

$$\begin{aligned} w(p, t - \tau)v_\tau(x, t) &= w(q, s - \tau)v_\tau(y, s) \\ &\iff \\ w(p, t - \tau')v_{\tau'}(x, t) &= w(q, s - \tau')v_{\tau'}(y, s) \end{aligned}$$

for all $x, y, p, q, s, t, \tau, \tau'$. This is equivalent to requiring

$$\frac{w(p, t - \tau)v_\tau(x, t)}{w(q, s - \tau)v_\tau(y, s)} = \frac{w(p, t - \tau')v_{\tau'}(x, t)}{w(q, s - \tau')v_{\tau'}(y, s)}$$

for all $x, y, p, q, s, t, \tau, \tau'$ with $y > 0$ and $q > 0$.

The results of the theorems now follow easily. □

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