Econ Theory (2011) 47:175–188
DOI 10.1007/s00199-010-0530-5

RESEARCH ARTICLE

Axiomatizations of two types of Shapley values for games on union closed systems

René van den Brink · Ilya Katsev · Gerard van der Laan

Received: 16 June 2009 / Accepted: 4 March 2010 / Published online: 8 April 2010

Abstract A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game. A (single-valued) solution for TU-games assigns a payoff distribution to every TU-game. A well-known solution is the Shapley value. In the literature various models of games with restricted cooperation can be found. So, instead of allowing all subsets of the player set N to form, it is assumed that the set of feasible coalitions is a subset of the power set of N. In this paper, we consider such sets of feasible coalitions that are closed under union, i.e. for any two feasible coalitions also their union is feasible. We consider and axiomatize two solutions or rules for these games that generalize the Shapley value: one is obtained as the *conjunctive permission value* using a corresponding *superior graph*, the other is defined as the Shapley value of a modified game similar as the *Myerson value* for games with limited communication.

We thank two referees for their useful comments. This research was partly carried out while the second author was visiting the Tinbergen Institute, VU University Amsterdam, on NWO-Grant 047.017.017 within the framework of Dutch–Russian cooperation. This author was also financially supported by the RFBR Grant 09-06-00155.

R. van den Brink (✉) · G. van der Laan
Department of Econometrics, Tinbergen Institute,
VU University, De Boelelaan 1105, 1081 HV Amsterdam,
The Netherlands
e-mail: jrbrink@feweb.vu.nl

G. van der Laan
e-mail: glaan@feweb.vu.nl

I. Katsev
St. Petersburg Institute for Economics and Mathematics,
Russian Academy of Sciences, Tchaikovsky st. 1,
St. Petersburg 191187, Russia
e-mail: katsev@yandex.ru

Keywords TU-game · Restricted cooperation · Union closed system · Shapley value · Permission value · Superior graph

JEL Classification C71

1 Introduction

A *cooperative game with transferable utility*, or simply a TU-game, is a finite set of players and for any subset (coalition) of players a worth representing the total payoff that the coalition can obtain by cooperating. A (single-valued) solution is a function that assigns to every game a payoff vector which components are the individual payoffs of the players. One of the most applied solutions for cooperative TU-games is the *Shapley value* (Shapley (1953)), which is applied to economic allocation problems in, e.g. Graham et al. (1990), Maniquet (2003), Chun (2006), Tauman and Watanabe (2007), van den Brink et al. (2007), Bergantiños and Lorenzo-Freire (2008), and Ligett et al. (2009).

In its classical interpretation, a TU-game describes a situation in which the players in every coalition S of N can cooperate to form a feasible coalition and earn its worth. In the literature various restrictions on coalition formation are developed.[1] For example, in Myerson (1977) a coalition is feasible if it is connected in a given (communication) graph. In this paper, we consider games in which the collection of feasible coalitions is closed under union, meaning that for any pair of feasible coalitions also their union is feasible. A well-known example of a union closed system is an *antimatroid*.[2] An example of an antimatroid is an acyclic *permission structure* where players need permission from (some of) their superiors in a hierarchical structure when they want cooperate with others. Since the concept of union closed system is more general than the notion of antimatroid, games on union closed systems are more general than the games on antimatroids as considered in Algaba et al. (2003, 2004), and, therefore, also more general than the games with acyclic permission structure, considered in Gilles et al. (1992), van den Brink and Gilles (1996), Gilles and Owen (1994) and van den Brink (1997).

In this paper, we define and axiomatize two solutions for games on union closed systems. The first solution is based on games with a permission structure, the other directly applies the Shapley value to some restricted game. Both solutions generalize the Shapley value in the sense that they are equal to the Shapley value when the union closed system is the power set of player set N. First, we apply a method similar as Myerson (1977) to define a solution for games on union closed systems which generalizes the Shapley value for games on antimatroids as axiomatized in Algaba et al. (2003). To do so, a modified or restricted game is defined. This game is obtained by assigning to any non-feasible coalition the worth of its largest feasible subset.

[1] For a survey we refer to Bilbao (2000).

[2] A collection of feasible coalitions $\mathcal{A} \subseteq 2^N$ is an antimatroid if, besides being union closed and containing $\emptyset$, it satisfies *accessibility* meaning that $S \in \mathcal{A}$ implies that there is a player $i \in S$ such that $S \backslash \{i\} \in \mathcal{A}$, see Dilworth (1940).

By union closedness, this largest feasible subset is unique. Then the *union rule* for games on union closed systems is defined as the Shapley value of this restricted game.

To define the second solution, we define for a union closed system its corresponding *superior graph*. This is the directed graph that is obtained by putting an arc from player i to player j if every feasible coalition containing player j also contains player i. We then consider the game with permission structure induced by this superior graph, and define the *superior rule* as its conjunctive permission value.

This paper is organized as follows. Section 2 is a preliminary section on cooperative TU-games and games with a permission structure. Section 3 introduces the two solutions for games on union closed systems and in Sect. 4 we provide axioms that can be satisfied by solutions for games on union closed systems. In Sect. 5, we give an axiomatization of the superior rule for games on union closed systems, and in Sect. 6 we axiomatize the union rule. Section 7 contains concluding remarks.

2 Preliminaries

2.1 TU-games

A situation in which a finite set of players can obtain certain payoffs by cooperating can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N = \{1, \dots, n\}$ is a finite set of n players and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. For any coalition $E \subseteq N$, $v(E)$ is the worth of coalition E, i.e., the members of coalition E can obtain a total payoff of $v(E)$ by agreeing to cooperate. Since we take the player set N to be fixed, we denote the game (N, v) just by its characteristic function v. We denote the collection of all characteristic functions on N by $\mathcal{G}^N$ and $n = |N|$ denotes the cardinality of N. A game $v \in \mathcal{G}^N$ is *monotone* if $v(E) \leq v(F)$ for all $E \subseteq F \subseteq N$. We denote by $\mathcal{G}^N_M$ the class of all monotone TU-games on N.

A payoff vector for a game is a vector $x \in \mathbb{R}^n$ assigning a payoff x_i to every $i \in N$. In the sequel, for $E \subseteq N$ we denote $x(E) = \sum_{i \in E} x_i$. A (single-valued) solution f is a function that assigns to any $v \in \mathcal{G}^N$ a unique payoff vector. The most well-known (single-valued) solution is the Shapley value given by

$$Sh_i(v) = \sum_{\{E \subseteq N | i \in E\}} \frac{(|N| - |E|)!(|E| - 1)!}{|N|!}(v(E) - v(E \backslash \{i\})) \quad \text{for all } i \in N.$$

For each non-empty $T \subseteq N$, the *unanimity* game u_T is given by $u_T(E) = 1$ if $T \subseteq E$, and $u_T(E) = 0$ otherwise. It is well known that the unanimity games form a basis for $\mathcal{G}^N$. For every $v \in \mathcal{G}^N$ it holds that $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T$, where $\Delta_v(T) = \sum_{E \subseteq T} (-1)^{|T| - |E|} v(E)$ are the *Harsanyi dividends*, see Harsanyi (1959).

2.2 Cooperative games with a permission structure

A game with a permission structure on N describes a situation where some players in a TU-game need permission from other players before they are allowed to cooperate within a coalition. Formally, a permission structure can be described by a directed graph on N. A directed graph or *digraph* is a pair (N, D) where $N = \{1, \ldots, n\}$ is a finite set of nodes (representing the players) and $D \subseteq N \times N$ is a binary relation on N. In the sequel we simply refer to D for a digraph (N, D) and we denote the collection of all digraphs on N by $\mathcal{D}^N$. For $i \in N$ the nodes in $S_D(i) := \{j \in N \mid (i, j) \in D\}$ are called the *successors* of i, and the nodes in $P_D(i) := \{j \in N \mid (j, i) \in D\}$ are called the *predecessors* of i. By $\widehat{S}_D(i)$ we denote the set of successors of i in the *transitive closure* of D, i.e., $j \in \widehat{S}_D(i)$ if and only if there exists a sequence of players $(h_1, \ldots, h_t)$ such that $h_1 = i$, $h_{k+1} \in S_D(h_k)$ for all $1 \leq k \leq t-1$, and $h_t = j$. Further, for $T \subseteq N$, we denote $\widehat{S}_D(T) = \cup_{i \in T} \widehat{S}_D(i)$. We call digraph D *acyclic* if $i \notin \widehat{S}_D(i)$ for all $i \in N$. Note that acyclicity of a digraph D implies that D is irreflexive, i.e., $(i, i) \notin D$ for all $i \in N$, and that D has at least one node that does not have a predecessor.

A tuple (v, D) with $v \in \mathcal{G}^N$ a TU-game and $D \in \mathcal{D}^N$ a digraph on N is called a *game with a permission structure*. In this paper, we follow the *conjunctive approach* as introduced in Gilles et al. (1992) and van den Brink and Gilles (1996) in which it is assumed that a player needs permission from all its predecessors in order to cooperate with other players.[3] Therefore, a coalition is feasible if and only if for any player in the coalition all its predecessors are also in the coalition. So, for permission structure D the set of *conjunctive feasible coalitions* is given by

$$\Phi_D^c = \{E \subseteq N \mid P_D(i) \subseteq E \quad \text{for all } i \in E\}.$$

For any $E \subseteq N$, let $\overline{\sigma}_D^c(E) = \bigcup_{\{F \in \Phi_D^c \mid F \subseteq E\}} E = E \setminus \widehat{S}_D(N \setminus E)$ be the largest conjunctive feasible subset of E in the collection Φ_D^c.[4] Then, the induced *restricted* game of the pair (v, D) under the conjunctive approach is the game $\overline{r}_{v,D}^c : 2^N \to \mathbb{R}$, given by

$$\overline{r}_{v,D}^c(E) = v(\overline{\sigma}_D^c(E)) \quad \text{for all } E \subseteq N,$$

i.e., the restricted game $\overline{r}_{v,D}^c$ assigns to each coalition $E \subseteq N$ the worth of its largest conjunctive feasible subset of E. Then the *conjunctive permission value* φ^c is the solution that assigns to every game with a permission structure the Shapley value of the restricted game, thus

$$\varphi^c(v, D) = Sh(\overline{r}_{v,D}^c).$$

3 Solutions for games on union closed systems

We consider tuples (v, Ω), where v is a TU-game on player set N and $\Omega \subseteq 2^N$ is a collection of subsets of N. We call such a tuple a *game with limited cooperation*. In

[3] As an alternative, Gilles and Owen (1994) and van den Brink (1997) consider the disjunctive approach, where it is assumed that a player needs permission to cooperate of at least one of its predecessors (if it has any).

[4] This largest feasible subset is unique since Φ_D^c is closed under union, see Definition 1.

such a game the collection of subsets Ω restricts the cooperation possibilities of the players in N. A set $S \subseteq N$ of players can attain its value $v(S)$ if $S \in \Omega$. When $S \notin \Omega$ then not all players are able to cooperate within S, so that $v(S)$ can not be realised. We say that a coalition $S \in 2^N$ is *feasible* if $S \in \Omega$. In this paper, we only consider sets of feasible coalitions that are closed under union.

Definition 1 A collection $\Omega \subseteq 2^N$ is a union closed system of coalitions if

1. $\emptyset, N \in \Omega$,
2. If $S,\ T \in \Omega$, then $S \cup T \in \Omega$.

Notice that $\Omega = \{\emptyset, N\}$ is the smallest union closed system and that $\Omega = 2^N$ is the largest one. Also notice that every antimatroid is a union closed system by definition. Also the collection of conjunctive feasible coalitions of a permission structure is union closed (see Gilles et al. 1992) and this collection is an antimatroid when the permission structure is acyclic (see Algaba et al. 2004).

We assume that the 'grand coalition' N is feasible for notational convenience. The results in this paper can be modified to hold without this assumption if in the axioms we distinguish between players that belong to at least one feasible coalition and those that do not belong to any feasible coalition. Note that by condition 2 in Definition 1 the 'grand coalition' is feasible if every player belongs to at least one feasible coalition. So, instead of assuming that $N \in \Omega$ we could do with the weaker *normality* assumption stating that every player belongs to at least one feasible coalition. In the sequel we denote the collection of all union closed systems in 2^N by $\mathcal{C}^N$.

A solution for games on union closed systems is a function f that assigns a payoff distribution $f(v, \Omega) \in \mathbb{R}^n$ to every $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$. In the following, we introduce two solutions.

For a tuple (v, Ω), let $\sigma_\Omega \colon 2^N \to \Omega$ be given by

$$\sigma_\Omega(S) = \bigcup_{\{E \in \Omega | E \subseteq S\}} E,$$

i.e., $\sigma_\Omega(S)$ is the largest feasible subset of S in the system Ω. By union closedness this largest feasible subset is unique. Then the *restricted* game $r_{v,\Omega} \in \mathcal{G}^N$ of (v, Ω) is defined by

$$r_{v,\Omega}(S) = v(\sigma_\Omega(S)),$$

and thus assigns to each coalition $S \subseteq N$ the worth of its largest feasible subset. Notice that when v is monotone, it holds that for every $\Omega \in \mathcal{C}^N$ also the restricted game $r_{v,\Omega}$ is monotone, since $S \subseteq T$ implies that $\sigma_\Omega(S) \subseteq \sigma_\Omega(T)$. Now, the first solution is the *union rule*, which is defined similar as the Myerson rule for games with limited communication in Myerson (1977) and the Shapley value for games on antimatroids in Algaba et al. (2003). The union rule, denoted by U, is given by

$$U_i(v, \Omega) = Sh_i(r_{v,\Omega}) \quad \text{for all } i \in N,$$

i.e., the union rule assigns to every (v, Ω) the Shapley value of the restriced game $r_{v,\Omega}$.

The second solution applies the conjunctive permission value to a digraph associated with the union closed system, called the *superior graph*. For a union closed system $\Omega \in \mathcal{C}^N$, the associated superior graph is the graph that assigns an arc from a player i to a player j if and only if every feasible coalition containing player j also contains player i. So, the arcs can be seen as some kind of dominance relation in the sense that a player is a subordinate of another player if it 'needs' the other player to be in a feasible coalition. For two players $i, j \in N, i \neq j$, player i is a *superior* of player j in $\Omega \in \mathcal{C}^N$, if $i \in S$ for every $S \in \Omega$ with $j \in S$. In that case we call player j a *subordinate* of player i.

Definition 2 For $\Omega \in \mathcal{C}^N$, the *superior graph* of Ω is the directed graph $D^\Omega \in \mathcal{D}^N$ with

$$D^\Omega = \{(i, j) \in N \times N \mid i \text{ is superior of } j \text{ in } \Omega\}.$$

Notice that i is a subordinate (superior) of j in $\Omega \in \mathcal{C}^N$ if and only if i is a successor (predecessor) of j in D^Ω. The next corollary is straightforward.

Corollary 1 *Let $\Omega \in \mathcal{C}^N$. If i is a superior of j in Ω, and k is a superior of i in Ω, then k is a superior of j in Ω.*

Having constructed the superior graph D^Ω of a union closed system Ω, we consider now the set of feasible coalitions of the permission structure D^Ω according to the conjunctive approach, and we denote this collection of coalitions by $\Sigma^\Omega = \Phi^c_{D^\Omega}$.

Proposition 1 *For $\Omega \in \mathcal{C}^N$ it holds that $\Omega \subseteq \Sigma^\Omega$.*

Proof Let $S \in \Omega$. By definition of superior it holds that S includes all superiors of i for every $i \in S$. On the other hand it holds that $(j, i) \in D^\Omega$ if and only if j is superior of i, $i \in S$. It follows that S is feasible for the permission structure D^Ω according to the conjunctive approach. Hence $\Omega \subseteq \Sigma^\Omega$. □

The *superior rule*, denoted by *SUP*, is the solution for games on union closed systems given by

$$SUP_i(v, \Omega) = \varphi_i^c(v, D^\Omega) = Sh_i(\overline{r}^c_{v,D^\Omega}) \quad \text{for all } i \in N,$$

i.e., the superior rule assigns to every (v, Ω) the conjunctive permission value of the game v with the induced permission structure D^Ω. The following example shows that the union and superior rule are different.

Example 1 Consider the tuple (v, Ω) on $N = \{1, 2, 3, 4\}$ with game $v = u_{\{3\}}$ and union closed system $\Omega = \{\emptyset, \{1, 2\}, \{1, 3\}, \{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$. The restricted game $r_{v,\Omega}$ is given by $r_{v,\Omega} = u_{\{1,3\}} + u_{\{3,4\}} - u_{\{1,3,4\}}$, and thus the union rule gives payoff vector $U(v, \Omega) = Sh_i(r_{v,\Omega}) = (\frac{1}{6}, 0, \frac{2}{3}, \frac{1}{6})$.

On the other hand, the superior graph is given by $D^\Omega = \{(1, 2), (3, 4)\}$, and the corresponding restricted game under the permission structure is given by $\overline{r}^c_{v,D^\Omega} = u_{\{3\}} = v$. It follows that $SUP(v, \Omega) = Sh_i(\overline{r}^c_{v,D^\Omega}) = (0, 0, 1, 0)$.

4 Axioms

In this section, we state several axioms that can be satisfied by solutions for games on union closed systems. The first five axioms are generalizations of axioms used to axiomatize the conjunctive permission value in van den Brink and Gilles (1996) and the Shapley value for games on poset antimatroids in Algaba et al. (2003). First, efficiency states that the total sum of payoffs equals the worth of the 'grand' coalition.

Axiom 1 (*Efficiency*) For every game $v \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have $\sum_{i \in N} f_i(v, \Omega) = v(N)$.

Additivity is a straightforward generalization of the well-known additivity axiom for TU-games.

Axiom 2 (*Additivity*) For every pair of cooperative TU-games $v, w \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have $f(v + w, \Omega) = f(v, \Omega) + f(w, \Omega)$.

Next, we introduce a generalization of the inessential player property stating that a null player in v whose subordinates in Ω are all null players in v, earns a zero payoff. We say that player $i \in N$ is *inessential* in (v, Ω) if $v(E \cup \{j\}) = v(E)$ for all $j \in \{i\} \cup S_{D^\Omega}(i)$ and $E \subseteq N \backslash \{j\}$. For $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$, we denote by $I(v, \Omega)$ the set of all inessential players in (v, Ω).

Axiom 3 (*Inessential player property*) For every game $v \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have that $f_i(v, \Omega) = 0$ for all $i \in I(v, \Omega)$.

The next axiom generalizes the necessary player property (which holds for monotone TU-games) in a straightforward way, stating that a necessary player in a monotone game earns at least as much as any other player, irrespective of the coalitions in the union closed system. A player $i \in N$ is *necessary* in game v if $v(E) = 0$ for all $E \subseteq N \backslash \{i\}$.

Axiom 4 (*Necessary player property*) For every monotone game $v \in \mathcal{G}_M^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have $f_i(v, \Omega) \geq f_j(v, \Omega)$ for all $j \in N$, when $i \in N$ is a necessary player in v.

Finally, structural monotonicity is generalized using the superior graph, stating that whenever player i is a superior of player j in the union closed system and the game is monotone, then player i earns at least as much as player j.

Axiom 5 (*Structural monotonicity*) For every monotone game $v \in \mathcal{G}_M^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have $f_i(v, \Omega) \geq f_j(v, \Omega)$ if $i \in N$ and $j \in S_{D^\Omega}(i)$.

In the next section, we show that the superior rule is characterized by these five axioms. The union rule satisfies these axioms except the inessential player property.[5]

[5] Consider, for instance, Example 1. Since $D^\Omega = \{(1, 2), (3, 4)\}$, the set of inessential players is given by $I(v, \Omega) = \{1, 2, 4\}$. However, $U(v, \Omega) = (\frac{1}{6}, 0, \frac{2}{3}, \frac{1}{6})$ and thus the inessential player property is not satisfied.

However, the union rule satisfies the weaker axiom requiring zero payoffs for inessential players only in games where the worth of any coalition equals the worth of its largest feasible subset.[6]

Axiom 6 (*Inessential player property for union closed games*) For every game $v \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$ such that $v(E) = v(\sigma_\Omega(E))$ for all $E \subseteq N$, it holds that $f_i(v, \Omega) = 0$ for every $i \in I(v, \Omega)$.

Of course, also the superior rule satisfies this weaker axiom. Finally, we introduce another axiom that is satisfied by the union rule but not by the superior rule. It states that the payoffs only depend on the worths of feasible coalitions.

Axiom 7 (*Independence of irrelevant coalitions*) *For every pair of cooperative TU-games $v, w \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have $f(v, \Omega) = f(w, \Omega)$ whenever $v(E) = w(E)$ for all $E \in \Omega$.*

To show that this axiom is not satisfied by the superior rule, consider again Example 1 and let game w be given by $w = r_{v,\Omega}$. Obviously $w(E) = v(E)$ for all $E \in \Omega$. Since the superior graph is given by $D^\Omega = \{(1, 2), (3, 4)\}$, we have that $\overline{r}^c_{v,D^\Omega} = u_{\{3\}} = v$ and $\overline{r}^c_{w,D^\Omega} = u_{\{1,3\}} + u_{\{3,4\}} - u_{\{1,3,4\}} = w$. Then $SUP(v, \Omega) = (0, 0, 1, 0)$ and $SUP(w, \Omega) = (\frac{1}{6}, 0, \frac{2}{3}, \frac{1}{6})$, so the axiom is not satisfied. In Sect. 6 we show that the union rule is characterized by the latter two axioms together with the first four axioms.

5 Axiomatization of the superior rule

The following theorem characterizes the superior rule for games on union closed systems.

Theorem 1 *A solution f for cooperative games on union closed systems is equal to the superior rule SUP if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property and structural monotonicity.*

Proof First, the superior rule satisfies the five axioms. By efficiency of the conjunctive permission value (i.e., $\sum_{i \in N} \varphi^c_i(v, D) = v(N)$ for every $v \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$) we have that $\sum_{i \in N} SUP_i(v, \Omega) = \sum_{i \in N} \varphi^c_i(v, D^\Omega) = v(N)$, showing that the superior rule satisfies efficiency. Additivity, the inessential player property, the necessary player property and structural monotonicity follow from the corresponding axioms of the conjunctive permission value for games with a permission structure, see van den Brink and Gilles (1996, Theorem 3.6).

The proof of uniqueness follows similar steps as the uniqueness proof for the conjunctive permission value in van den Brink and Gilles (1996, Theorem 3.6). Suppose that solution f satisfies the five axioms. Let v_0 be the *null game* given by $v_0(E) = 0$ for all $E \subseteq N$. The inessential player property then implies that $f_i(v_0, \Omega) = 0$ for all $i \in N$.

[6] Note that the union rule satisfies the stronger property requiring zero payoffs for all null players in games v such that $v(E) = v(\sigma_\Omega(E))$ for all $E \subseteq N$.

Now, consider a union closed system Ω and game $w_T = c_T u_T$, for some $c_T > 0$ and $\emptyset \neq T \subseteq N$. Note that $w_T \in \mathcal{G}_M^N$. We distinguish the following three cases with respect to $i \in N$:

1. If $i \in T$ then the necessary player property implies that there exists a $c^* \in \mathbb{R}$ such that $f_i(w_T, \Omega) = c^*$ for all $i \in T$, and $f_i(w_T, \Omega) \leq c^*$ for all $i \in N \backslash T$.
2. If $i \in N \backslash T$ and $T \cap (\{i\} \cup S_{D^\Omega}(i)) \neq \emptyset$ then structural monotonicity implies that $f_i(w_T, \Omega) \geq f_j(w_T, \Omega)$ for every $j \in T \cap (\{i\} \cup S_{D^\Omega}(i))$, and thus with case 1 that $f_i(w_T, \Omega) = c^*$.
3. If $i \in N \backslash T$ and $T \cap (\{i\} \cup S_{D^\Omega}(i)) = \emptyset$ then the inessential player property implies that $f_i(w_T, \Omega) = 0$.

From 1 and 2 follows that $f_i(w_T, \Omega) = c^*$ for $i \in T \cup P_{D^\Omega}(T)$. Efficiency and 3 then imply that $\sum_{i \in N} f_i(w_T, \Omega) = |T \cup P_{D^\Omega}(T)| c^* = c_T$, implying that c^*, and thus $f(w_T, \Omega)$, is uniquely determined.

Next, consider (w_T, Ω) with $w_T = c_T u_T$ for some $c_T < 0$ (and thus we cannot apply the necessary player property and structural monotonicity since w_T is not monotone). Since $-w_T = -c_T u_T$ with $-c_T > 0$, and $v_0 = w_T + (-w_T)$, it follows from additivity of f that $f(w_T, \Omega) = f(v_0, \Omega) - f(-w_T, \Omega) = -f(-w_T, \Omega)$ is uniquely determined because $-w_T$ is monotone.

Finally, since every characteristic function $v \in \mathcal{G}^N$ can be written as a linear combination of unanimity games $v = \sum_{T \subseteq N} \Delta_v(T) u_T$ (with $\Delta_v(T)$ the Harsanyi dividend of coalition T), additivity uniquely determines $f(v, \Omega) = \sum_{T \subseteq N} f(\Delta_v(T) u_T, \Omega)$ for any $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$. □

We conclude this section by showing that the five axioms stated in Theorem 1 are logically independent.

1. The solution that assigns to every game on union closed system simply the Shapley value of game v, i.e., $f(v, \Omega) = \mathit{Sh}(v)$, satisfies efficiency, additivity, the inessential player property and the necessary player property. It does not satisfy structural monotonicity.
2. For $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$, let $\overline{v} \in \mathcal{G}^N$ be given by $\overline{v}(E) = v(\bigcup_{i \in E} \{i\} \cup S_{D^\Omega}(i))$ for all $E \subseteq N$. The solution $f(v, \Omega) = \mathit{Sh}(\overline{v})$ satisfies efficiency, additivity, the inessential player property and structural monotonicity. It does not satisfy the necessary player property.
3. The equal division solution given by $f_i(v, \Omega) = \frac{v(N)}{|N|}$ for all $i \in N$, satisfies efficiency, additivity, the necessary player property and structural monotonicity. It does not satisfy the inessential player property.
4. The equal division over essential players, given by

$$f_i(v, \Omega) = \begin{cases} \frac{v(N)}{|N \backslash I(v, \Omega)|} & \text{if } i \in N \backslash I(v, \Omega) \\ 0 & \text{if } i \in I(v, \Omega), \end{cases}$$

satisfies efficiency, the inessential player property, the necessary player property and structural monotonicity. It does not satisfy additivity.

5. The zero solution given by $f_i(v, \Omega) = 0$ for all $i \in N$ satisfies additivity, the inessential player property, the necessary player property and structural monotonicity. It does not satisfy efficiency.

6 Axiomatization of the union rule

As mentioned in Sect. 4, all axioms that are used to characterize the superior rule in Theorem 1 are also satisfied by the union rule, except the inessential player property. Instead, it satisfies the weaker inessential player property for union closed games and independence of irrelevant coalitions. Replacing in Theorem 1, the inessential player property by the weaker inessential player property for union closed games, and adding independence of irrelevant coalitions, characterizes the union rule. In that case we can do without structural monotonicity. To prove this characterization, we use the following lemma. For $\Omega \in \mathcal{C}^N$ and $T \subseteq N$, we define $\Omega_T = \{H \in \Omega \mid T \subseteq H\}$ as the set of feasible coalitions containing coalition T.

Lemma 1 *For every $\Omega \in \mathcal{C}^N$, $T \subseteq N$ and $c \in \mathbb{R}$, there exist numbers $\delta_H \in \mathbb{R}$, $H \in \Omega_T$, such that $r_{cu_T,\Omega} = \sum_{H \in \Omega_T} \delta_H u_H$.*

Proof Consider $\Omega \in \mathcal{C}^N$, $T \subseteq N$ and $c \in \mathbb{R}$. If $T \in \Omega$ then $T \in \Omega_T$, and we have $\delta_T = c$ and $\delta_H = 0$ for all $H \in \Omega_T \backslash \{T\}$. If $T \notin \Omega$, then define

$$\mathcal{T}^1 = \{H \in \Omega \mid T \subset H \text{ and there is no } Z \in \Omega \text{ such that } T \subset Z \subset H\}$$

and, recursively, for $k \geq 2$

$$\mathcal{T}^k = \left\{H \in \Omega \mid T \subset H \text{ and } Z \in \cup_{p=1}^{k-1} \mathcal{T}^p \text{ for all } Z \in \Omega \text{ with } T \subset Z \subset H\right\}.$$

Since N is finite there exists an $M < \infty$ such that $\mathcal{T}^k \neq \emptyset$ for all $k \in \{1, \ldots M\}$, $\mathcal{T}^{M+1} = \emptyset$ and $\bigcup_{k=1}^M \mathcal{T}^k = \Omega_T$. Since by definition $\mathcal{T}^k \cap \mathcal{T}^l = \emptyset$ for all $k, l \in \mathbb{N}$, we have that $\mathcal{T}^1, \ldots, \mathcal{T}^M$ is a partition of the set $\{H \in \Omega \mid T \subset H\}$ of feasible coalitions containing non-feasible coalition T. (Note that this set equals Ω_T since $T \notin \Omega$.) Then $\delta_H = c$ for all $H \in \mathcal{T}^1$ and, recursively for $k = 2, \ldots, M$, the numbers δ_H, $H \in \mathcal{T}^k$, are determined by

$$\delta_H + \sum_{\{Z \subset H \mid Z \in \bigcup_{l=1}^{k-1} \mathcal{T}^l\}} \delta_Z = c.$$

$\square$

Theorem 2 *A solution f for cooperative games on union closed systems is equal to the union rule U if and only if it satisfies efficiency, additivity, the inessential player property for union closed games, the necessary player property and independence of irrelevant coalitions.*

Proof We first prove that U satisfies the five axioms. Let $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$.

1. By efficiency of the Shapley value and $\sigma_\Omega(N) = N$, we have $\sum_{i \in N} U_i(v, \Omega) = \sum_{i \in N} Sh_i(r_{v,\Omega}) = v(N)$, showing that U satisfies efficiency.
2. Additivity of the Shapley value and the fact that $r_{v,\Omega}(S) + r_{w,\Omega}(S) = v(\sigma_\Omega(S)) + w(\sigma_\Omega(S)) = (v + w)(\sigma_\Omega(S)) = r_{v+w,\Omega}(S)$ for all $S \subseteq N$, imply for $i \in N$ that $U_i(v, \Omega) + U_i(w, \Omega) = Sh_i(r_{v,\Omega}) + Sh_i(r_{w,D^\Omega}) = Sh_i(r_{v+w,\Omega}) = U_i(v + w, \Omega)$, showing that U satisfies additivity.
3. U satisfying the inessential player property for union closed games follows directly from the null player property of the Shapley value.
4. Let v be a monotone game on N. Since $S \subseteq T$ implies that $\sigma_\Omega(S) \subseteq \sigma_\Omega(T)$, by monotonicity of v we have that $r_{v,\Omega}$ is a monotone game on N. The necessary player property then follows from the necessary player property of the Shapley value.
5. If $v(S) = w(S)$ for all $S \in \Omega$, then $r_{v,\Omega} = r_{w,\Omega}$, showing that the union rule U satisfies independence of irrelevant coalitions.

To prove that the five axioms characterize a unique solution, let $\Omega \in \mathcal{C}^N$ and consider $v = cu_T$ for some $c \in \mathbb{R}$ and $\emptyset \neq T \subseteq N$. We distinguish two cases.

1. Let $T \in \Omega$, i.e., T is feasible. Then $r_{cu_T,\Omega} = cu_T$. From the necessary player property it follows that there exists a $c^* \in \mathbb{R}$ such that $f_i(cu_T, \Omega) = c^*$ for all $i \in T$. Since $i \in N \backslash T$ is a null player in cu_T, and $cu_T(E) = cu_T(\sigma_\Omega(E))$ for all $E \subseteq N$ (since $T \in \Omega$), the inessential player property for union closed games implies that $f_i(cu_T, \Omega) = 0$ for all $i \in N \backslash T$. Then efficiency implies that $c^* = f_i(cu_T, \Omega) = \frac{c}{|T|}$ for all $i \in T$, and thus $f(cu_T, \Omega)$ is determined.
2. Suppose that $T \notin \Omega$, i.e., T is not feasible. Recall that $\Omega_T = \{H \in \Omega \mid T \subseteq H\}$ is the collection of feasible subsets of N that contain T. (Note that $T \notin \Omega_T$ since $T \notin \Omega$.) By Lemma 1 there exist numbers δ_H, $H \in \Omega_T$, such that $r_{cu_T,\Omega} = \sum_{H \in \Omega_T} \delta_H u_H$. Since $cu_T(E) = r_{cu_T,\Omega}(E)$ for all $E \in \Omega$, by independence of irrelevant coalitions it then follows that $f(cu_T, \Omega) = f(r_{cu_T,\Omega}, \Omega) = f(\sum_{H \in \Omega_T} \delta_H u_H, \Omega)$. By additivity we then have that
$$f(cu_T, \Omega) = f\left(\sum_{H \in \Omega_T} \delta_H u_H, \Omega\right) = \sum_{H \in \Omega_T} f(\delta_H u_H, \Omega).$$
Since all $H \in \Omega_T$ are feasible in Ω, we know from case 1 that $f(\delta_H u_H, \Omega)$ is uniquely determined for every $H \in \Omega_T$. Thus, $f(cu_T, \Omega) = \sum_{H \in \Omega_T} f(\delta_H u_H, \Omega)$ is uniquely determined.

Finally, it follows that additivity uniquely determines $f(v, \Omega) = \sum_{T \subseteq N} f(\Delta_v(T) u_T, \Omega)$ for every $v \in \mathcal{G}^N$. $\square$

Also the axioms of Theorem 2 are logically independent as shown by the following alternative solutions.

1. The superior rule satisfies efficiency, additivity, the inessential player property for union closed games and the necessary player property. It does not satisfy independence of irrelevant coalitions.
2. The solution that assigns to every game on union closed system the weighted Shapley of the restricted game $r_{v,\Omega}$ for some exogenous weight system $\omega \in \mathbb{R}^n$ with $\omega_i \neq \omega_j$ for some $i, j \in N$, satisfies efficiency, additivity, the inessential player property for union closed games and independence of irrelevant coalitions. It does not satisfy the necessary player property.
3. The equal division solution given by $f_i(v, \Omega) = \frac{v(N)}{|N|}$ for all $i \in N$, satisfies efficiency, additivity, the necessary player property and independence of irrelevant coalitions. It does not satisfy the inessential player property for union closed games.
4. The equal division over non-null players, given by

$$f_i(v, \Omega) = \begin{cases} \frac{v(N)}{|N \backslash Null(v,\Omega)|} & \text{if } i \in N \backslash \text{Null}(v, \Omega) \\ 0 & \text{if } i \in \text{Null}(v, \Omega), \end{cases}$$

where $\text{Null}(v, \Omega)$ denotes the set of null players in the restricted game $r_{v,\Omega}$, satisfies efficiency, the inessential player property for union closed games, the necessary player property and independence of irrelevant coalitions. It does not satisfy additivity.
5. The zero solution given by $f_i(v, \Omega) = 0$ for all $i \in N$ satisfies additivity, the inessential player property for union closed games, the necessary player property and independence of irrelevant coalitions. It does not satisfy efficiency.

7 Concluding remarks

In this paper, we introduced two generalizations of the Shapley value to games on union closed systems. The superior rule applies the conjunctive permission value to an associated game with permission structure, while the union rule is obtained as the Shapley value of the restricted game. Both rules satisfy efficiency, additivity, the inessential player property for union closed games, the necessary player property and structural monotonicity. We obtain an axiomatization of the superior rule by strengthening the inessential player property for union closed games to the stronger inessential player property. This stronger property is not satisfied by the union rule. We obtain a characterization of the union rule by adding independence of irrelevant coalitions. In that case we can do without structural monotonicity.

As mentioned in Sect. 3, both the superior and the union rule can also be defined and axiomatized without assuming in Definition 1 that the 'grand coalition' N is feasible. By condition 2 in that definition, the players that do not belong to the largest feasible subset of N do not belong to any feasible coalition. Referring to these players as *irrelevant* players, we can define the superior rule and the union rule by applying these two rules to the game and union closed system restricted to $R(\Omega) = \{i \in N \mid$ there is an $S \in \Omega$ with $i \in S\}$, and assigning zero payoff to all irrelevant players. The two rules can be axiomatized for such situations by requiring the axioms for the relevant players, and by adding the *irrelevant player property* stating that irrelevant players get zero payoff.

The axioms discussed in this paper all are applied to a fixed union closed system Ω. Myerson (1980) characterized the Myerson rule for conference structures using fairness. A straightforward modification of the fairness axiom in Myerson (1977, 1980) states that deleting a feasible coalition $S \in \Omega$, such that $\Omega \backslash \{S\}$ is still union closed, changes the payoffs of players in S by the same amount. The superior rule does not satisfy this fairness, but the union rule does. However, the union rule is not the only solution for games on union closed systems that satisfies (component) efficiency, fairness and the irrelevant player property.[7] Axiomatizations using axioms that require us to allow to change the set of feasible coalitions, such as fairness, will be studied in future research.

Another point we like to mention is that the notion of conference structure does not put any condition on the collection of feasible sets, i.e., a conference structure is an arbitrary collection of subsets of N. However, by Myerson (1980)'s definition of connectedness, every single player is connected and thus earns its own worth in the restricted game. So, even if a singleton does not belong to the conference structure, a single player earns its worth in the restricted game. This differs from our approach, in which $r_{v,\Omega}(\{i\}) = v(\{i\})$ if $\{i\} \in \Omega$, and $r_{v,\Omega}(\{i\}) = 0$ otherwise. Alternatively, in line with Myerson (1980) we could always take $r_{v,\Omega}(\{i\}) = v(\{i\})$ irrespective of whether $\{i\}$ is feasible or not. Because of Myerson's definition of connectedness and thus the restricted game, it does not matter whether a conference structure does or does not contain $\{i\}$ for any $i \in N$. Consequently, an arbitrary conference structure $\mathcal{F}$ yields the same Myerson rule payoffs as the conference structure $\mathcal{F} \cup \{\{i\} \mid i \in N\}$, obtained from $\mathcal{F}$ by adding all singleton coalitions.[8] This does not hold for the class of union closed systems. When adding all singleton coalitions to a union closed system Ω, the resulting collection of coalitions $\Omega \cup \{\{i\} \mid i \in N\}$ is not a union closed system anymore, since by condition 2 of Definition 1, the unique union closed system containing all singletons is the collection $\Omega = 2^N$.

References

Algaba, E., Bilbao, J.M., van den Brink, R., Jiménez-Losada, A.: Axiomatizations of the Shapley value for cooperative games on antimatroids. Math Methods Oper Res **57**, 49–65 (2003)

Algaba, E., Bilbao, J.M., van den Brink, R., Jiménez-Losada, A.: Cooperative games on antimatroids. Discret Math **282**, 1–15 (2004)

[7] Another solution that satisfies these axioms on the class of games on union closed systems is the *modified union rule* where we take two disjoint coalitions of equal cardinality and in case both are feasible we subtract a fixed amount, say 1, from all players in one coalition and give it to all players in the other coalition.

[8] Considering the subclass of conference structures where all singletons are feasible, the proof that there is a unique solution satisfying component efficiency and fairness is similar to Myerson's proof for the full class of conference structures.

Bergantiños, G., Lorenzo-Freire, S.: A characterization of optimistic weighted Shapley rules in minimum cost spanning tree problems. Econ Theory **35**, 523–538 (2008)

Bilbao, J.M.: Cooperative Games on Combinatorial Structures. Dordrecht: Kluwer (2000)

Chun, Y.: No-envy in queueing problems. Econ Theory **29**, 151–162 (2006)

Dilworth, R.P.: Lattices with unique irreducible decompositions. Ann Math **41**, 771–777 (1940)

Gilles, R.P., Owen, G.: Cooperative Games and Disjunctive Permission Structures. Blacksburg: Department of Economics, Virginia Polytechnic Institute and State University (1994)

Gilles, R.P., Owen, G., van den Brink, R.: Games with permission structures: the conjunctive approach. Int J Game Theory **20**, 277–293 (1992)

Graham, D.A., Marshall, R.C., Richard, J.F.: Differential payments within a bidder coalition and the Shapley value. Am Econ Rev **80**, 493–510 (1990)

Harsanyi, J.C: A bargaining model for cooperative n-person games. In: Tucker, A.W., Luce, R.D. (eds.) Contributions to the Theory of Games IV, pp. 325–355.Princeton: Princeton University Press (1959)

Ligett, T.M., Lippman, S.A., Rumelt, R.P.: The asymptotic Shapley value for a simple market game. Econ Theory **40**, 333–338 (2009)

Maniquet, F.: A characterization of the Shapley value in queueing problems. J Econ Theory **109**, 90–103 (2003)

Myerson, R.B.: Graphs and cooperation in games. Math Oper Res **2**, 225–229 (1977)

Myerson, R.B.: Conference structures and fair allocation rules. Int J Game Theory **9**, 169–182 (1980)

Shapley, L.S: A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (eds.) Annals of Mathematics Studies, vol. 28 (Contributions to the Theory of Games, vol. 2), pp. 307–317. Princeton: Princeton University Press (1953)

Tauman, Y., Watanabe, N.: The Shapley value of a patent licensing game: the assymptotic equivalence to non-cooperative results. Econ Theory **30**, 135–149 (2007)

van den Brink, R.: An axiomatization of the disjunctive permission value for games with a permission structure. Int J Game Theory **26**, 27–43 (1997)

van den Brink, R., Gilles, R.P.: Axiomatizations of the conjunctive permission value for games with permission structures. Games Econ Behav **12**, 113–126 (1996)

van den Brink, R., van der Laan, G., Vasil'ev, V.: Component efficient solutions in line-graph games with applications. Econ Theory **33**, 349–364 (2007)