Dynamic mode decomposition analysis of rotating detonation waves

M. D. Bohon¹ · A. Orchini¹ · R. Bluemner¹ · C. O. Paschereit¹ · E. J. Gutmark²

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Abstract

A rotating detonation combustor (RDC) is a novel approach to achieving pressure gain combustion. Due to the steady propagation of the detonation wave around the perimeter of the annular combustion chamber, the RDC dynamic behavior is well suited to analysis with reduced-order techniques. For flow fields with such coherent aspects, the dynamic mode decomposition (DMD) has been shown to capture well the dominant oscillatory features corresponding to stable limit-cycle or quasi-periodic behavior within its dynamic modes. Details regarding the application of the technique to RDC—such as the number of frames, the effect of subtracting the temporal mean from the processed dataset, the resulting dynamic mode shapes, and the reconstruction of the dynamics from a reduced set of dynamic modes—are analyzed and interpreted in this study. The DMD analysis is applied to two commonly observed operating conditions of rotating detonation combustion, viz., (1) a single spinning wave with weak counter-rotating waves and (2) a clapping operating mode with two counter-propagating waves at equal speed and strength. We show that care must be taken when applying DMD to RDC datasets due to the presence of standing waves (expressed as either counter-propagating azimuthal waves or longitudinal pulsations). Without accounting for these effects, the reduced-order reconstruction fails using the standard DMD approach. However, successful application of the DMD allows for the reconstruction and separation of specific wave modes, from which models of the stabilization and propagation of the primary and counter-rotating waves can be derived.

Keywords Dynamic mode decomposition · Rotating detonation · Pressure gain combustion · Reduced-order dynamics

List of symbols

A	Linear mapping between frames
b	Initial condition coefficients
c	Linear mapping coefficients
D	Eigenvalue matrix of A and S
i	Imaginary unit
r	Residual vector
S	Companion-type mapping matrix
$\mathbf{V}_1^N$	Reconstructed snapshot matrix
$\tilde{\mathbf{V}}_1^N$	Reduced snapshot matrix
v	Vectorized image
$\tilde{\mathbf{v}}$	Reconstruction of vectorized image
x	Eigenvectors of S

y	Eigenvectors of A
z	Mode shape of eigenvector pair
ε	Residual of reconstructed snapshots
φ	Angle of discrete mapping eigenvalues
σ	Growth rate of eigenvalue

Subscripts

<i>i</i>	Index of image
<i>j</i>	Index of eigenvalue

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✉ M. D. Bohon
 mdbohon@gmail.com

¹ Technische Universität Berlin, Müller-Breslau-Str. 8, Berlin, Germany

² Department of Aerospace Engineering, University of Cincinnati, Cincinnati, OH 45220, USA

Image acquisition

f_s	Data acquisition rate, frames per second
IM	Snapshot image
<i>m</i>	Number of rows in image
<i>n</i>	Number of columns in image
<i>N</i>	Number of images
$\text{px}_{m,n}$	Pixel at index (<i>m</i> , <i>n</i>)
<i>t_e</i>	Exposure time (μs)

1 Introduction

Rotating detonation combustors (RDCs) have emerged as one of the most promising concepts to achieve the efficiency gains made possible through pressure gain combustion. In the canonical description of an RDC, a detonation wave continuously propagates around a closed-loop combustion annulus, while fresh reactants are constantly supplied from the head end and product gases are expanded and displaced out of the exhaust end. The simple design, together with the high power density and low-pressure fluctuations in the exhaust gas, makes it an ideal candidate for propulsive and land-based power applications [1].

Apart from the canonical single detonation wave mode, many groups have reported a variety of wave modes, such as the steady operation with counter-rotating waves [2–5] and high-frequency longitudinal pulsed operation [6,7]. A recent study by the authors found that, depending on the injector geometry and reactant mass flow rate, a range of different operating modes could be reliably stabilized [8,9]. These include two counter-rotating waves at equal or different speed: a dominant wave with multiple counter-rotating waves and secondary acoustic or dominant pulsed wave modes. These wave modes lead to complex pressure oscillations in the annulus. In order to interpret and identify the operating mode, these pressure signals were analyzed based on the theoretical speed of sound in the fresh and hot gas, the detonation velocity, and observations from high-speed video [5,9,10].

This study aims to complement the previous studies and to provide a better understanding about the correlation between the pressure signals measured in the combustor annulus and the data contained in the high-speed aft-end video. Toward this goal, the dynamic mode decomposition (DMD) technique will be applied to these videos. The DMD technique was originally developed to extract dynamic information from a sequence of snapshots of a flow field [11]. Since its development, it has been applied in many different fields, including systems with nonlinear dynamics, such as detonation waves [12]. In these nonlinear systems, the DMD is able to extract the dominant dynamic behavior captured in the data sequence and associate this behavior with specific frequencies [11]. A comprehensive review of the technique is provided in [13]. As applied in standard fluid mechanics, the DMD is able to identify stable and unstable modes, as well as their corresponding growth and decay rates. For the steady (i.e., non-varying) operating modes in the RDC, the DMD modes represent stable oscillatory patterns corresponding to stable limit-cycle or quasi-periodic behavior.

In this study, the key aspects of the implementation of DMD to specific, relevant aspects of RDC operation will be discussed. Following this, the application of the DMD to example datasets will be explored. A correlation between the

pressure data and the high-speed images of flame luminosity will be shown, and the principle components of the DMD modes will be related to the dynamical operation of the combustor. Two common modes of operation will be explored and compared. The first consists of a single spinning wave with weak difficult-to-detect counter-rotating waves. This will be compared with the second most commonly observed mode, which is a pair of counter-rotating waves moving at the same speed, often called a clapping or slapping wave mode.

2 Overview on dynamic mode decomposition

In this section, a short theoretical introduction to dynamic mode decomposition (DMD) is presented. The theory is adapted from the seminal work of Schmid [11] and is shown for the sake of completeness. DMD aims at identifying the key features of a linear mapping, $\mathbf{A}$, that transforms a given state vector, $\mathbf{v}_i$, measured at time $t = i\Delta t$, into the state vector that is observed at the next time step:

$$\mathbf{v}_{i+1} = \mathbf{A}\mathbf{v}_i. \quad (1)$$

This mapping exists whenever the underlying dynamic does not exhibit transient features, but is displaying steady-state behavior, e.g., exponential growth away from a fixed point or steady oscillatory behavior around a mean flow. The latter is the case for the analysis presented in this study. The core idea of DMD is that, having at hand a sufficiently large number of snapshots N , these will eventually become linearly dependent, so that (at least) the N th state, $\mathbf{v}_N$, can be expressed as a linear combination of the previous states $\mathbf{v}_i$, for $1 \leq i < N$. This enables us to approximate $\mathbf{A}$ via its dominant eigenfrequencies/eigenvectors.

In our application of DMD to RDC, we will process snapshots of the rotating detonation process captured by a camera at a frame rate of f_s . Each image $\mathbf{IM}_i$ is composed of $m \times n$ pixels

$$\mathbf{IM}_i \equiv \begin{bmatrix} \text{px}_{1,1}^{(i)} & \text{px}_{1,2}^{(i)} & \cdots & \text{px}_{1,n}^{(i)} \\ \text{px}_{2,1}^{(i)} & \text{px}_{2,2}^{(i)} & \cdots & \text{px}_{2,n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ \text{px}_{m,1}^{(i)} & \text{px}_{m,2}^{(i)} & \cdots & \text{px}_{m,n}^{(i)} \end{bmatrix}. \quad (2)$$

For the datasets considered in this manuscript, the number of pixels per image is larger than the number of recorded snapshots, $mn > N$. We then construct the state vector $\mathbf{v}_i$, of length mn , by stacking the columns of $\mathbf{IM}_i$:

$$\mathbf{v}_i \equiv [\text{px}_{1,1}^{(i)}, \text{px}_{2,1}^{(i)}, \cdots, \text{px}_{m,1}^{(i)}, \text{px}_{1,2}^{(i)}, \cdots, \text{px}_{m,n}^{(i)}]^T. \quad (3)$$

Lastly, the state vectors are collected together into a so-called snapshot matrix $\mathbf{V}_1^N$, with dimension $mn \times N$, whose i th column identifies the i th state vector:

$$\mathbf{V}_1^N \equiv \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}. \quad (4)$$

Note that if the mapping $\mathbf{A}$ were known, it would be possible to completely reconstruct the entire snapshot matrix (4) from the first state vector only, by iteratively applying the mapping (1):

$$\mathbf{V}_1^N \equiv \left\{ \mathbf{v}_1, \mathbf{A}\mathbf{v}_1, \mathbf{A}^2\mathbf{v}_1, \dots, \mathbf{A}^{N-1}\mathbf{v}_1 \right\}. \quad (5)$$

2.1 General theory

In order to find an approximation for the eigendecomposition of $\mathbf{A}$, the number of snapshots must be increased until the vectors $\mathbf{v}_i$ become linearly dependent. Then, the last frame, $\mathbf{v}_N$, can be expressed as a linear combination of the first $N-1$ vectors

$$\begin{aligned} \mathbf{v}_N &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_{N-1}\mathbf{v}_{N-1} + \mathbf{r} = \dots \\ &= \mathbf{V}_1^{N-1}\mathbf{c} + \mathbf{r}, \end{aligned} \quad (6)$$

where $\mathbf{r}$ is a residual vector, which accounts also for experimental noise and/or stochastic components in the data, and c_i are coefficients to be identified. By applying the mapping (1) to frames $i = 1 \dots N-1$, frames $i = 2 \dots N$ are obtained

$$\begin{aligned} \mathbf{A}\mathbf{V}_1^{N-1} &= \mathbf{V}_2^N = \{\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_N\} \dots \\ &= \left\{ \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{V}_1^{N-1}\mathbf{c} + \mathbf{r} \right\} \dots \\ &\approx \left\{ \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{V}_1^{N-1}\mathbf{c} \right\}. \end{aligned} \quad (7)$$

The approximation in the last step indicates that we have neglected the error contained in the residual vector $\mathbf{r}$, assumed to be small, when substituting $\mathbf{v}_n$ with its linear approximation (6). From (7), expressing the last line in matrix form, the following relations hold

$$\mathbf{A}\mathbf{V}_1^{N-1} = \mathbf{V}_2^N \approx \mathbf{V}_1^{N-1}\mathbf{S}, \quad (8)$$

where we have introduced the matrix $\mathbf{S}$, which is of companion type [11]. From (8), we can calculate the companion matrix $\mathbf{S}$ by means of a QR decomposition of the matrix $\mathbf{V}_1^{N-1}$:

$$\mathbf{V}_2^N = \mathbf{V}_1^{N-1}\mathbf{S} = \mathbf{Q}\mathbf{R}\mathbf{S} \quad (9a)$$

$$\mathbf{S} = (\mathbf{Q}\mathbf{R})^{-1}\mathbf{V}_2^N = \mathbf{R}^{-1}\mathbf{Q}^{-1}\mathbf{V}_2^N = \mathbf{R}^{-1}\mathbf{Q}^H\mathbf{V}_2^N. \quad (9b)$$

We recall that $\mathbf{R}$ is upper triangular, and thus, the calculation of its inverse (generally intended as a pseudo-inverse for non-square matrices in this study) is cheaper than that of $\mathbf{V}_1^{N-1}$,

and that $\mathbf{Q}$ is unitary, so that its inverse equals its Hermitian conjugate, $\mathbf{Q}^H = \mathbf{Q}^{-1}$. Using (8), we can now relate $\mathbf{S}$ to $\mathbf{A}$:

$$\mathbf{A} = \mathbf{V}_1^{N-1}\mathbf{S}(\mathbf{V}_1^{N-1})^{-1}. \quad (10)$$

Assuming that $\mathbf{V}_1^{N-1}$ is full rank ($N-1$), this amounts to a similarity transformation between $\mathbf{S}$ and $\mathbf{A}$. As a consequence, the eigenvalues of the $N \times N$ matrix $\mathbf{S}$ correspond to (some of) the eigenvalues of the $mn \times mn$ matrix $\mathbf{A}$. To show this, one calculates the eigendecomposition of $\mathbf{S}$

$$\mathbf{S} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1}, \quad (11)$$

where the elements d_j of the diagonal matrix $\mathbf{D}$ and the column vectors $\mathbf{x}_j$ of $\mathbf{X}$ are the eigenvalues and eigenvectors of $\mathbf{S}$, respectively. Then, it follows from (8) that

$$\mathbf{A}\mathbf{V}_1^{N-1} = \mathbf{V}_1^{N-1}\mathbf{X}\mathbf{D}\mathbf{X}^{-1} \quad (12a)$$

$$\mathbf{A}\mathbf{V}_1^{N-1}\mathbf{X} = \mathbf{V}_1^{N-1}\mathbf{X}\mathbf{D} \quad (12b)$$

$$\mathbf{A}\mathbf{Y} = \mathbf{Y}\mathbf{D}, \quad (12c)$$

where $\mathbf{Y} \equiv \mathbf{V}_1^{N-1}\mathbf{X}$. Equation (12c) is a partial (low rank) eigendecomposition of $\mathbf{A}$, yielding $N-1$ eigenvalues d_j (the eigenvalues of $\mathbf{S}$) and eigenvectors $\mathbf{y}_j$, the columns of $\mathbf{Y}$, which will be called the dynamic modes. These can be sorted by relevance, so that the dynamics can be well approximated using only a small subset of dynamic modes.

We conclude by noting that if the objective of the DMD is the construction of a reduced-order model, an alternative and more robust approach to the one presented above involves the use of the singular value decomposition (SVD) of the matrix $\mathbf{V}_1^{N-1}$ [11]. This has the advantage of naturally identifying coherent structures in $\mathbf{V}_1^{N-1}$ and sorting them in descending relevance order. For the purpose of this manuscript, however, the QR decomposition method is sufficient, and we avoid discussing the further mathematical details involved with the SVD method.

2.2 Standing waves

When operating the RDC, several types of oscillation patterns can be observed as discussed above. These include a single spinning detonation wave traveling in the clockwise or counter-clockwise direction, the coexistence of a primary (P) detonation wave together with one or multiple secondary (S) waves traveling at lower speeds and in opposite direction than the P wave, and the coexistence of two counter-rotating P waves that give rise to a so-called clapping oscillation pattern. In the latter case, and generally whenever two waves spinning in opposite directions are observed, a decomposition into coherent structures of the observed dynamics will contain standing components. As was noted in

[13], the DMD method discussed above cannot describe the dynamics of standing waves. This is because any oscillatory behavior that needs to be described by means of (12c) must contain a pair of complex-conjugate eigenvalues d . However, a standing wave is detected by the DMD as a single real-valued structure: Lacking a complex-conjugate doppelgänger, its associated eigenvalue must be real, so that the DMD detected structure can grow, decay, or remain constant in amplitude, but cannot oscillate. If a standing-like mode dominates the dynamics, as, for example, in the clapping oscillation pattern, the reconstruction of the dynamics using a subset of the dynamic modes will fail. All dynamic modes must be retained to reproduce the dynamics, and it is impossible to clearly identify dominant dynamical features.

A simple remedy to capture standing structures is the subtraction of the temporally averaged mean field from the snapshot matrix $\mathbf{V}_1^N$. As was noted in [14], subtracting the temporal mean—calculated over the whole set of N considered snapshots—from the snapshot matrix has the unexpected result of pinning the eigenvalues d_j of the companion matrix $\mathbf{S}$ to the roots of unity, independently from its content

$$d_j = \exp\left(2\pi i \frac{j-1}{N-1}\right), \quad j = 1, \dots, N-1. \quad (13)$$

In this sense, the DMD with mean flow subtraction is analogous to a discrete Fourier transform. This has pros and cons. On the positive side, pinning down the eigenfrequencies makes it possible for the method outlined in Sect. 2 to detect standing waves, as the frequency content is prescribed a priori and standing-like components are collected into structures at the closest frequency. On the negative side, all identified modes are by construction neutrally stable, and a large number of snapshots are required in order to have a sufficiently high frequency resolution as per (13). For the purposes of this study, the latter are, however, minor concerns as (i) we are processing data at steady-state conditions, for which neutrally stable eigenvalues are expected, and (ii) a sufficiently high number of snapshots are available. In Sect. 4, we present results for the DMD analysis both with and without mean subtraction to clarify our arguments further.

3 Experimental setup and methodology

The RDC under investigation uses a radially inward injector design, where air is injected through a narrow slot of variable height at the bottom of the annulus, and fuel (hydrogen) is injected through a large number of discretely spaced holes. A crosscut view of the RDC and imaging setup is shown in Fig. 1. Fuel and air mix in a jet-in-cross flow configuration. The RDC was ignited with a pre-detonator tube

Fig. 1 Cross-sectional view of the RDC and high-speed imaging setup; $D = 90$ mm, $L = 120$ mm, $\Delta = 7.6$ mm, and $g = 1.6$ mm

from the outer annulus wall near the injector head. Operation is computer-controlled and monitored by a 500-kHz data acquisition system. Run times were in the order of 300 ms to prevent sensor damage due to the high temperatures in the combustion zone. Dynamic pressure sensors (PCB112A05) were mounted in a recessed cavity in the annulus outer wall close to the combustion zone to measure the passage of the detonation wave, as well as any other pressure oscillations in the RDC annulus. The cavity has been designed to exhibit a Helmholtz resonance at a frequency much higher than that of the combustor operation [15,16].

A high-speed camera (Photron SA-Z) was used to image the natural flame luminosity from the aft end of the RDC via a visible wavelength mirror (see Fig. 1). For all the datasets considered in this study, images were recorded at a rate of $f_s = 87,500$ frames per second, with an exposure time of $8.75 \mu\text{s}$ and an inter-frame time of $11.4 \mu\text{s}$. The high-speed images allow for an assessment of the number, direction, and location of the waves in the annulus. Sample images for an operating condition dominated by a spinning wave are shown in Fig. 2. For many operating points, however, the dynamics is more intricate, and the separation of the observed dynamics into individual waves can be difficult without the aid of a reduction-order tool, such as DMD. DMD is performed on a sequence of up to 2500 frames of high-speed video, starting at a run time of 200 ms. At a camera frame rate of 87,500 fps, this results in a captured time interval of $\Delta t = 28.57$ ms. Based on typically observed wave speeds in RDCs of 50–80% of Chapman–Jouguet (CJ) speed, between 100 and 170 full laps of a wave are captured. In order to exclude any modes occurring outside of the annulus, the views of the exhaust plume around the combustor and the center body are masked prior to the analysis. The annulus location is automatically detected in each frame by an image processing

Fig. 2 Sample images of aft-end high-speed video in the RDC. Operating conditions: mass flow rate of air is 200 g/s, stoichiometric equivalence ratio. The dynamics is dominated by a spinning wave. Gray circles indicate the inner and outer walls of the combustion annulus. The luminosity outside these circles is set to zero

algorithm, as described in [8]. This post-processing centers the combustor within the frame and eliminates the tracking of the image due to flexure of the mirror in the exhaust flow.

4 Results

In this section, we present results on the application of the DMD method described in Sect. 2 to the high-frequency images collected when operating the RDC at various conditions as discussed in Sect. 3. We first discuss the details of the application of the DMD to the specific RDC problem, examining the influence of number of frames with and without mean subtraction, the correlation between the natural luminosity and high-speed pressure measurements, and the form of the resultant modes. We then proceed in analyzing in detail the dynamic modes identified in RDCs at two operating conditions, first by continuing the analysis of a single spinning wave with weak counter-rotating waves and then by considering the often observed clapping mode.

4.1 Application of DMD to RDC images

Using the approach described in Sect. 2, we conduct a DMD analysis of the aft-end high-speed videos of RDC operation. Several works utilizing results from aft-end video in this combustor have been presented in [8,9], to which we redirect the interested reader for experimental implementation details.

The test case considered is shown in Fig. 2. This test case was conducted at an air mass flow rate of 200 g/s and $\phi = 1$. A dataset of 1000 frames at a frame rate f_s of 87,500 frames per second was acquired with each image spanning 329 by

329 pixels. A snapshot matrix $\mathbf{V}_1^N$ of size $108,241 \times 1000$ is created by vectorizing each frame. In order to limit the analysis to the relevant domain within the combustion annulus, the areas inside the smaller gray circle and outside the larger gray circle were masked out to zeros prior to processing.

When the DMD process is applied, the resultant eigenvalue matrix $\mathbf{D}$ has $N - 1$ entries along the diagonal, which we shall refer to individually as d_j . Because the snapshot images are real valued, all d_j are either real valued or form complex-conjugate pairs. These eigenvalues are shown in Fig. 3 in the upper half of the complex plane only; the lower half of the complex plane is symmetric and is therefore omitted in the figure. In Fig. 3a, we plot the real and imaginary parts of d_j for $N = 100$ frames without mean subtraction; in Fig. 3b for 1000 frames without mean subtraction; and in Fig. 3c for 1000 frames with mean subtraction. In all figures, the markers indicating the eigenvalues d_j are scaled in size and color by the norm of their corresponding eigenvectors (shown in Fig. 4), which can be considered as a metric of the importance of the individual dynamic modes.

If we represent the eigenvalues in polar form as

$$d_j = |d_j|e^{i\varphi_j}, \text{ where } \varphi_j \equiv \arctan \frac{\text{Im}[d_j]}{\text{Re}[d_j]}, \quad (14)$$

we can extract the frequency and growth rate describing the evolution of each mode. In fact, the angle φ is linked to the frequency of the dynamic mode by

$$f_j = \frac{1}{\Delta t} \frac{\varphi_j}{2\pi} = f_s \frac{\varphi_j}{2\pi}, \quad (15)$$

and the magnitude $|d_j|$ is related to the growth rate of the dynamic mode by

$$\sigma_j = \frac{\ln |d_j|}{\Delta t} = f_s \ln |d_j|. \quad (16)$$

Figure 3 shows what was discussed in Sect. 2.2, i.e., that mean subtraction impacts the resulting distribution of modes. In particular, without mean subtraction the most significant mode is the real-valued eigenvalue $d_j = 1$ (corresponding to a frequency of 0 Hz). In Fig. 3a, most of the dominant modes sit on or near the unit circle, while a number of modes are distributed around the interior, indicating that these modes will be damped over time ($\sigma < 0$). However, increasing the number of frames to 1000 in Fig. 3b shows that these modes converge to the unit circle and are not damped but neutrally stable ($\sigma = 0$). This is consistent with the stable, periodic oscillatory state of the combustor dynamics at this operating point. Also note that the dominant frequencies identified by the DMD for $N = 100$ (Fig. 3a) and $N = 1000$ (Fig. 3b) vary only by a few hertz. This hints to the fact that the DMD is capable of providing good estimates and converge toward

Fig. 3 Complex-valued eigenvalues d_j found by the DMD method. Marker size and color are scaled by the norm of the corresponding eigenvector (the darker/larger the marker, the larger the dynamic mode relevance to the dynamics). Numbers indicate frequency of the eigenvalue in hertz

Fig. 4 Norm of the dynamic modes y plotted against their corresponding frequency. Comparison with FFT of PCB pressure trace measured at the base of the combustor. The DMD spectrum has been shifted up for clarity. Mean subtracted, $N = 1000$

the correct dominant frequencies using only a few snapshots, $\mathcal{O}(10^2)$. On the other hand, subtracting the mean before performing the DMD analysis results in a prescribed, uniform distribution of eigenvalues at zero growth rate, with

$$f_j = f_s \frac{j-1}{N-1} \quad \text{and} \quad \sigma_j = 0, \quad (17)$$

obtained by substituting (13) into (15) and (16), respectively. This is visible in Fig. 3c. By doing so, the energy of the standing modes (at this operating condition, this is due to the counter-propagating waves) becomes associated with the eigenvalues at the appropriate frequency and enforces zero growth rate for all eigenvalues. Because the presence of standing wave components is unavoidable for the description of non-trivial operating modes, the mean-subtracted approach will prove more successful for the remainder of the text. This will be shown in further detail in Sect. 4.3.

Figure 4 shows the amplitude of the norm of the dynamic modes, normalized by $\max_j \|y_j\|$, as a function of the mode's frequency. Along with it, the figure shows the Fourier spectrum of a PCB pressure sensor installed in the combustor annulus. As one might reasonably expect, it is evident that the scaling of the dynamic modes as identified by the DMD method applied to luminosity data strongly correlates with that of the FFT of the pressure sensor.

The highest peak in both spectra, at 4988 Hz, corresponds to the primary wave frequency (P). Contributions at higher harmonics are also observed ($n \times P$), with content up to $8 \times P$. A characteristic feature in this operating range is the presence of one or more counter-rotating waves. In the case presented, a triplet is observed—i.e., a set of three acoustic or weak shock waves spinning in the opposite direction of the primary wave—that corresponds to the peak at 10,413 Hz (labeled $S3$). Generally, these operating modes have been identified by deductive reasoning based on the relationship between observed wave propagation speeds (using either video imaging or pressure traces). For example, the process for analyzing the spectra in Fig. 4 is as follows. The largest peak at 4987 Hz is easily identifiable as the primary wave. Additionally, the harmonics of this primary wave are easily identifiable as integer multiples of this primary wave frequency. The remaining peaks require deeper reasoning. In this example, the peak at 10,412 Hz is faster than the predicted CJ (maximum expected) velocity of a detonation wave. Therefore, there is likely more than one wave. A pair of waves would leave the resultant wave speed faster than the primary wave, which is unlikely. Therefore, it is reasonable to expect this peak to correspond to a triplet of waves. The fact that this speed corresponds to the speed of sound in the products further supports this conclusion. However, the direction of propagation is undetermined and is simply reasoned to be counter-propagating to the primary wave. This approach is

generally adopted and is described in a number of works in the literature [5,9,10]. It is possible (although somewhat difficult) to see these counter-rotating waves in the high-speed video snapshots in Fig. 2 (e.g., frame $i = 13$), by a periodic increase in the luminosity of the primary wave when intersected by them. Lastly, several features of the spectrum can be seen to correspond to the interactions between the primary and secondary waves. These interactions occur at combinations of the primary and secondary frequencies, e.g., $S3 - P$ and $S3 + P$.

Figure 5 shows the mode shapes z of the P , $2 \times P$, $S3$, and $S3 - P$ modes. These mode shapes are determined by the combination of the instantaneous state of the dynamic mode y_j over one period with its complex conjugate

$$z_j \equiv y_j e^{2\pi i f_j t} + \bar{y}_j e^{-2\pi i f_j t}, \quad (18)$$

where t spans $[0, 1/f_j]$ and complex conjugation is denoted by overbar. Each row in Fig. 5 illustrates the progression of a mode over one period, returning to its initial state at 2π . We can see that the P mode has azimuthal order one and is rotating counter-clockwise (consistent with the video shown in Fig. 2). The second harmonic ($2 \times P$) has azimuthal order two and is also rotating counter-clockwise. This pattern repeats for all of the higher harmonics of P . On the other hand, the $S3$ mode has azimuthal order three and rotates clockwise. This is consistent with the identification of this mode as a triplet of waves as discussed above. However, reaching this identification does not require the deductive reasoning applied previously. Lastly, the $S3 - P$ mode has azimuthal order four and rotates clockwise.

4.2 Analysis of single spinning wave modes

The strength of the DMD approach becomes finally clear when we use a reduced subset of dynamic modes to reconstruct the original snapshot matrix. The first step in the reconstruction is to determine the initial scaling of the eigenvectors in $\mathbf{Y}$. This can be accomplished solving for a coefficient vector $\mathbf{b}$ of initial conditions that, when multiplied by $\mathbf{Y}$, would recreate a particular snapshot. In particular, it is typical to use as a reference snapshot the first one [13], $\mathbf{v}_1$, so that

$$\mathbf{v}_1 = \sum_{k=1}^N b_k \mathbf{y}_k e^{(\sigma_k + 2\pi i f_k)t_1} = \sum_{k=1}^N b_k \mathbf{y}_k = \mathbf{Y} \mathbf{b}. \quad (19)$$

In the latter, we have assumed without loss of generality that $t_1 = 0$. The coefficient vector $\mathbf{b}$ can then be calculated as $\mathbf{b} = \mathbf{Y}^{-1} \mathbf{v}_1$. From here, a reduced-order reconstruction of

the original dataset can be computed as

$$\tilde{\mathbf{v}}_i = \sum_{k=1}^{N_{\text{red}}} b_k \mathbf{y}_k e^{(\sigma_k + 2\pi i f_k)t_i}, \quad (20)$$

where the sum over k is taken on a set of chosen modes (complex-conjugate pairs). The vectors $\tilde{\mathbf{v}}_i$ approximate the state vector $\mathbf{v}_i$ at time $t_i \equiv (i - 1)\Delta t$ and can be collected into an approximation snapshot matrix $\tilde{\mathbf{V}}_1^N$ similarly to (4). If k spans then entire spectrum, then the resulting reconstruction equals the initial snapshot matrix. In the case at hand, we select a subset of 28 dynamic mode pairs from the 499 available pairs. The selected modes are chosen based on their energy content as per Fig. 4. They correspond to the primary mode and its harmonics, the secondary modes, and the modes resulting from the interaction of the primary and secondary modes. Since the peaks of the first few primary modes are somewhat broad, the dynamic modes allocated in the neighboring bins are also included. Also, the first three modes at low frequency are included, as they have some spectral content, albeit at quite low frequency. The resulting reconstruction from this reduced dynamic mode set is shown in the second row of Fig. 6. For clarity, the mean has not been included in the reconstruction to enable better comparison with the original, mean-subtracted image sequence (top row). In both the original and reconstructed sequences, the mean is identical. As a quantification of the quality of the reconstruction, the residual between the original snapshot matrix and the reduced-order reconstruction can be calculated as

$$\varepsilon = \frac{\|\mathbf{V}_1^N - \tilde{\mathbf{V}}_1^N\|}{\|\mathbf{V}_1^N\|}. \quad (21)$$

By maintaining these 28 mode pairs, the resulting residual is $\varepsilon = 0.134$, indicating that the reconstruction captures 86.6% of the original image set.

The good quality of this reduced-order reconstruction can be seen by comparison with the original sequence of frames in the top row of Fig. 6. The major features of the detonation wave propagation are preserved. The primary wave can be seen to rotate counter-clockwise. Important features such as the periodic increase in the intensity of the wave, due to the interaction between the primary and secondary waves, can be observed in the reconstruction in the first, fourth, and last snapshots.

In the third and fourth rows of Fig. 6, we separate the contributions of the primary and secondary waves. In the third row, the reconstruction was conducted using only the primary wave dynamic mode, its harmonics, and the shoulders as described earlier, totaling 12 mode pairs, labeled as ΣP . Given the spinning nature of the dynamics of the RDC operation considered here, the primary modes alone approximate the dynamics very well. In the fourth row, the

Fig. 5 Mode shapes z_j for four major dynamical features. The columns correspond to the progression through one period, with $\theta \equiv f_j t$, while each row corresponds to a different mode. Mean subtracted, $N = 1000$

reconstruction shows only the contributions of the secondary waves and their interactions with the primary, totaling 11 mode pairs labeled $\Sigma S + B$. This reconstruction explicitly shows how the secondary waves interact with the primary. In the original images, a region of high local intensity can be seen in frame number 12. This location coincides with the high-intensity region in the secondary wave reconstruction. Moving forward in time, the secondary wave continues to rotate clockwise through frames 13 and 14 while simultaneously decaying in strength. By frame 15, the primary wave has progressed counter-clockwise toward the intersection with the second wave of the counter-rotating triplet. Through frames 15 and 16, the process of forming a local region of high intensity followed by a gradual decay repeats for the second wave in the triplet. Meanwhile, the first wave of the triplet has decayed to a relatively weak, but constant, intensity. Finally, as the process marches further forward through frames 17 and 18, the primary wave meets the third member of the counter-rotating triplet, again repeating the process. Conducting such a decomposition, i.e., distinguishing the primary and secondary components, it is possible to better investigate and quantify the interaction of these wave in order to better understand their stabilizing mechanisms.

From this process, it seems that these counter-rotating waves are generally stabilized by the periodic interaction with the primary wave. After collision, they appear to decay

in strength while moving away from the primary wave. This observation, however, must be tempered by the knowledge that the underlying data are based on the luminosity in the flame. As the secondary wave propagates—first through the products of the primary wave and then into the refilling region—the decay in intensity may not coincide with a decay in strength, but rather be affected by the gas composition.

4.3 Analysis of clapping wave modes

In this section, a second, commonly observed form of RDC operation, known as the “clapping” or “slapping” mode will be examined. Whereas the example in Sects. 4.1 and 4.2 was predominantly characterized by a primary wave spinning in a constant direction with a very weak counter-rotating component, the major feature of the clapping mode is a pair of equal strength waves moving in opposite directions.

Following the approach shown in Fig. 4 in Sect. 4.1 (using $N = 1000$ snapshots and subtracting the mean from the data), Fig. 7 presents the frequency spectrum for the norm of the dynamic modes y overlaid on the spectrum of a pressure sensor installed in the combustor wall. Again, there is good agreement between the two spectra. The dominant peak occurs at 4025 Hz, which is significantly slower than the previous spinning wave case, and is consistent with observations of wave speeds in these modes. The harmonics of the pri-

Fig. 6 Comparison of original, mean-subtracted snapshots (top row) with various reconstructions for $N = 1000$. The second row shows a reconstruction using a reduced set of 28 energetic pairs. The third row shows a reconstruction obtained using only the primary modes (P). The

bottom row shows a reconstruction obtained using only the secondary waves (S) and the interaction modes (B). *Note:* The color scale of the fourth row is different from that of the first three

Fig. 7 Norm of the dynamic modes y plotted against their corresponding frequency for the clapping operating mode. Comparison with FFT of PCB pressure trace measured at the base of the combustor. The DMD spectrum has been shifted up for clarity. Mean subtracted, $N = 1000$

major peak are also observable, up to $10 \times P$. The other features observable in the spectra are small peaks at 2013 and 5950 Hz.

Figure 8 shows the mode shapes z_j for these features plotted over the course of one period. These modes have been obtained from the mean subtracted data. Similar structures were also identified for the data including the mean (not shown) with the inclusion of a dominant zero frequency

mode. In the first row, freezing the first instant, the primary feature may appear similar to the spinning wave case in Fig. 5. However, as the period evolves, the mode does not spin in the same way. In fact, the primary mode is a standing wave mode which results in a symmetry line crossing through the intersecting points (i.e., the position of the maximum does not vary over the course of the period). As the period progresses, the dominant standing wave seesaws in phase. Contrary to the spinning case, the evolution of the primary mode shape alone is not very representative of the actual dynamics of the luminosity (see Fig. 9). This indicates that the harmonics of the primary wave play a key role in the reconstruction of the clapping oscillation. The $2 \times P$ mode shape has a rather complicated structure. Here again, rather than spinning, the mode shape evolves along the same symmetry line. However, contrary to the shape of the P mode for which a nodal line (a line at which the mode shape has 0 intensity at every time) perpendicular to the symmetry line could be identified, no nodal line can be observed for the $2 \times P$ mode. This suggests that the $2 \times P$ dynamical mode consists of both standing and spinning components, which is representative of the clapping oscillation pattern. Each of the higher harmonic modes contains additional information necessary to reconstruct the non-sinusoidal (steep-fronted) wave. Consequently, physical

Fig. 8 Mode shapes z_j for several major features of the clapping operating mode. The columns correspond to the progression through one period, while each row corresponds to a different mode. Mean subtracted, $N = 1000$

interpretations of single higher-harmonic modes are difficult since they primarily contain the higher-frequency content of the fundamental mode.

The third row illustrates another important feature in this operating mode, one that is also quite likely present in many other combustors. This mode corresponds to a longitudinal pulsation within the combustor. Here, a longitudinally propagating wave is moving along the axial length of the combustor. This results in a nearly uniform and in-phase variation in the mode intensity throughout the entire annulus over the course of the period. As discussed in Sect. 2.2, the DMD without mean subtraction cannot handle a standing wave nor a longitudinal component. By conducting the DMD on a mean subtracted dataset, these modes in Fig. 8 can be captured by allowing the energy in the mode to be assigned to a nonzero frequency (i.e., a complex valued) eigenvalue. Without mean subtraction, all of this structure would be collapsed into a single real-valued eigenvalue at zero frequency, and this analysis would not be possible.

Lastly, the fourth row demonstrates how the nonlinear effects couple the dynamics of the P and L modes. We can conclude this because the frequency of $P+L$ mode is the sum of the individual modes, and the mode shape is the product of the two individual mode shapes.

As in Sect. 4.2, a reconstruction of the original image sequence using a reduced subset of modes can be conducted.

This is shown in Fig. 9 for the same dataset, but with mean subtraction (Fig. 9a) and while retaining the mean (Fig. 9b). Whereas as shown in Fig. 6 the mode pairs were purposefully selected, the mode selection here was based on the importance of the sorted peaks as shown in Fig. 7. For the case with mean subtraction, the top 5, 25, and 60 dynamic mode pairs (10, 50, and 120 total modes, respectively) were selected and the image sequence reconstructed. Their reconstruction residuals, from (21), are, respectively, $\varepsilon_{10} = 0.389$, $\varepsilon_{50} = 0.225$, and $\varepsilon_{120} = 0.174$. Naturally, the greater the number of mode pairs included, the more accurate is the reconstruction. We note that, despite the large error in the intensity, the main spinning and frequency features of the original luminosity are already well preserved using only five mode pairs. These are very promising results—taking steps toward the description of the highly non-harmonic dynamics of pressure gain combustion processes with a reduced set of fundamental modes.

On the other hand, retaining the mean in the snapshot matrix results in redistributing the content of the standing waves components throughout all of the dynamic modes. Consequently, a reconstruction based on a reduced set will inevitably omit a significant portion of the energy of the original image sequence. We can see this clearly in Fig. 9b. Comparing the original image set in the top row with the reconstruction in the second row using all the 999 of the

(a) With mean subtraction: each row compares a different set of reconstructions. Residuals are $\varepsilon_{120} = 0.174$, $\varepsilon_{50} = 0.225$, and $\varepsilon_{10} = 0.389$.

(b) Without mean subtraction: each row compares a different set of reconstructions. Residuals are $\varepsilon_{all} = 0.006$, $\varepsilon_{501} = 0.101$, and $\varepsilon_{121} = 0.326$.

Fig. 9 Comparison of original snapshots with several reconstructions. **a** The top row shows the original image sequence with mean subtraction. The subsequent rows show a reconstruction using a reduced set of 60, 25, and 5 dynamic mode pairs, respectively. **b** The top row shows

the original image sequence without mean subtraction. The subsequent rows show a reconstruction using all of the dynamic modes and two reduced sets of 501 and 121 dynamic modes, respectively; $N = 1000$

dynamic modes (498 complex-conjugate pairs and one real-valued zero-frequency mode) shows a perfect reconstruction. This demonstrates that the method has been applied correctly and that an exact reconstruction is possible using all modes. However, results from the reduced-order reconstruction rapidly deteriorate when considering smaller subsets of DMD modes. A reduced set of 501 dynamic modes (250 pairs and one zero-frequency mode) maintains most of the major features, but already shows a significant amount of noise. At 121 modes (60 pairs and one zero-frequency mode), even the major features are difficult to identify; the noise is much greater and the reconstruction even shows negative luminosity intensity, which is non-physical (recall that the mean is not subtracted here). The latter reconstruction has the same order as the one shown in the second row of Fig. 9a. It is evident that, at the same order, the analysis with mean is significantly worse.

The discussion above emphasizes even further that the reconstruction failure is due to the presence of standing components, which is always present in the clapping scenario and in many RDC operating conditions. This is an important consideration when studying RDC derived datasets using standard DMD. If counter-propagating or longitudinal waves are present in the dynamics, coherent structures will contain standing components. Their dynamical content cannot then be captured when retaining the mean, and reconstruction of the dynamics from a reduced set of modes is not possible. Since a “clean” RDC dataset without counter-propagating waves is somewhat trivial, these considerations with respect to standing waves will likely affect a majority of relevant experiments.

5 Conclusions and outlook

DMD is a powerful method that can be applied to a variety of problems, with the objective of reducing the dynamics of a system to a small set of coherent structures that capture the dominant features of the flow. In this work, we applied DMD to the analysis of aft-end high-speed video of the natural luminosity of the RDC. We explored various aspects regarding the application of DMD to RDC problems. First, the distribution of eigenvalues as a function of the number of frames in the snapshot matrix—as well as the impact of mean subtraction—was explored. It was shown that, for a steady quasi-periodic oscillatory operation, the dynamic modes require a relatively large number of frames before fully converging to the zero growth rate unit circle. This effect was partly attributed to the limitations of the DMD in describing standing waves. By incorporating mean subtraction, the resulting prescription of the eigenvalue locations resolved this issue at the expense of always requiring a greater number of frames in order to increase frequency resolution.

In analyzing the resulting outputs, it was shown that the norm of the dynamic modes matches well with the FFT of a pressure sensor installed in the wall. This observation aids the interpretation of the resulting dynamic modes from luminosity snapshots. The nature of the pressure spectrum peaks was inferred through a deductive process that leaves some uncertainties. On the other hand, the dynamic mode shapes unequivocally determine the nature of the spectrum peaks.

Two commonly observed operating modes were analyzed by reconstructing the original image sequence from a reduced set of dynamic modes. The first case was a single rotating wave with a triplet of counter-rotating waves. By selecting a subset of 28 pairs of modes, the original sequence could be well reconstructed. This allowed for the separation of the contributions of the primary wave from the counter-rotating waves and for the quantification of the interaction between the two sets of waves (e.g., the change in intensity or wave speed during and after intersection). The second case examined a clapping mode that was heavily dependent on a number of standing waves. The original image sequence could again be reliably reconstructed using only few dynamic modes when mean contributions were ignored.

From these results, it appears that DMD can be a powerful tool for analyzing RDC operation. However, due to the general presence of standing components within the RDC azimuthal dynamics, care must be taken when applying the method. In future work, we will investigate alternative data analysis methods that allow for a more accurate DMD analysis while retaining the mean flow.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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