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ORIGINAL ARTICLE

Improved estimates for the linear Molodensky problem

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Abstract

The paper deals with the linearized Molodensky problem, when data are supposed to be square integrable on the telluroid S, proving that a solution exists, is unique and is stable in a space of harmonic functions with square integrable gradient on S. A similar theorem has already been proved by Sansò and Venuti (J Geod 82:909–916, 2008). Yet the result basically requires that S should have an inclination of less than 60° with respect to the vertical, or better to the radial direction. This constraint could result in a severe regularization for the telluroid specially in mountainous areas. The paper revises the result in an effort to improve the above estimates, essentially showing that the inclination of S could go up to 75°. At the same time, the proof is made precise mathematically and hopefully more readable in the geodetic community.

Keywords Geodetic boundary value problem · Regularity of the telluroid · Spaces of harmonic functions

1 Introduction

The geodetic boundary value problem (GBVP), or Molodensky problem (MP) Eremeev et al. (1962), is one of the theoretical pillars of physical geodesy in that it studies under what conditions we can provide on the Earth surface gravity field data such that it can be determined, in a stable way, in the space outside the masses.

Since the GBVP is basically a mathematical continuous model to which we can arrive only after a long process of approximations as well as taking ideally a limit on the frame of real data (Sansò 1995), which is always discrete and finite, it has been only natural that it received different formulations in different epochs of evolution of the available data and geodetic theory, depending on progress both in mathematical analysis and in its numerical implementation. Here we mention only the spherical formulation of Stokes (Stokes 1849), with its corollary of Pizzetti and Somigliana for the normal field (Pizzetti 1894; Somigliana 1929), the Helmert solution for accounting of the effects of topographic masses (Helmert 1884), the work of Molodensky and others, representing the first step into a modern linearized analysis (Eremeev et al. 1962), followed by the advances of Heiskanen and Moritz (1967), Moritz (1980) and Krarup (2006), who first gave a completely rigorous formulation of the GBVP in linear form. Then the basic turning point of the analysis by Hörmander (1976) who first has attacked the GBVP in its natural form of a free boundary, oblique derivative BVP for the Laplace operator and, as such, a nonlinear problem of potential theory, immediately followed by the analysis exploiting the so-called *gravity space approach* (Sansò 1977) that provides more favorable results, requiring a lower degree of regularity of the data. For a more general analysis of the oblique derivative problem, one can consult (Yanushauskas 1989).

One important point to underline here is that it is due to L. Hörmander the intuition that introducing as data lower degree harmonic coefficients of the potential would have simplified the analysis of the linear and hence of the nonlinear problem. On the other hand, the introduction of these harmonic coefficients as data is quite justified today that a number of global models derived from satellite only observations is available up to degree 300 at least. This modification of Molodensky's problem has transformed it in something different than a pure BVP, where data are given only on the boundary. In spite of this remark, we will continue to call the modified Molodensky problem the GBVP as traditional.

One important achievement has been the recognition that, due to a very close alignment of the normal gravity to the

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normal of the ellipsoid in the topographic layer, up to the Earth surface, there has been some confusion in geodetic literature between two different formulations of the GBVP: the original one in vector form Eremeev et al. (1962) and the other in scalar form Sansò and Sacerdote (1986). Yet, the authors think there is a general consensus with the scalar formulation of the GBVP which in linear form becomes (see Sansò and Sideris 2013, §15.2)

$$\begin{cases} \Delta T = 0 & \text{in } \Omega \equiv \{h \geq H^*_\sigma\} \\ -\frac{\partial T}{\partial h} + \frac{\partial \gamma/\partial h}{\gamma} T = \Delta g & \text{on } S \equiv \{h = H^*_\sigma\} \\ T = 0\left(\frac{1}{r^3}\right) \end{cases} \tag{1.1}$$

where T is the anomalous potential of the gravity field, h the ellipsoidal height, $\sigma = (\lambda, \varphi)$ the ellipsoidal angular coordinates, $\Delta g = g(\sigma) - \gamma(H^*_\sigma, \sigma)$ the free air gravity anomaly and $\{h = H^*_\sigma\}$ the equation of the Marussi telluroid.

Let us remind here that the classical condition on the third Eq. (1.1) means that the zeroth- and first-order degrees of the asymptotic development of T are annihilated. This happens because in the definition of the normal potential U is included the "exact" knowledge of the total mass of the Earth and in fixing the coordinate system to which the normal potential refers, the geometric center of the ellipsoid, equipotential of the normal potential U, is directly placed at the barycenter of the actual gravity field (cf. Sansò and Sideris 2013, §1.3). Contrary to the nonlinear problem, the linearized scalar Molodensky problem can be analyzed in Hilbert spaces of Sobolev type, arriving at a theorem of existence uniqueness and stability of the solution, when the datum $\Delta g(\sigma)$ is in L^2_σ (see Sansò and Venuti 2008 and Sansò and Sideris 2013, §15.3). The proof of the theorem will be reviewed in this paper, yet here we want to underline that the validity of the mentioned theorem relies critically on the satisfaction of the following inequality

$$4J_+^2\left(\frac{\delta R}{R_+} + \frac{2}{L+2}\right) < 1 \tag{1.2}$$

where

- ϑ_+ = maximum inclination of the telluroid with respect to the radial unit vector $\boldsymbol{e}_r$
- $J_+ = \frac{1}{\cos\vartheta_+}$
- $r = R_\sigma$ spherical equation of the telluroid
- $R_+ = \max R_\sigma, \ R_- = \min R_\sigma$
- $\delta R = R_+ - R_-$
- L = max degree of the harmonic coefficients $\{T_{n,m}\}$ that we assume to be known.

As it is apparent also from the symbols used, the inequality (1.2) has been derived starting from a spherical approximation approach. In particular, in the above list the angular coordinates $\sigma = (\varphi, \lambda)$ have to be understood as spherical angles rather than ellipsoidal. The transition from (1.2) to the corresponding inequality for the true linear Molodensky problem produces a small difference with results coming from (1.2), Sansò and Sideris (2013), §15.3.

As we see from (1.2), if we want to treat the GBVP for a telluroid with a strong maximum inclination we need to introduce a larger number of known potential harmonics up to the maximum degree L. Since the corresponding coefficients are imperfectly known, so that the model

$$T_L = \frac{\mu}{R}\sum_{\ell=2}^{L} T_{\ell m} S_{\ell m}(\sigma) \ (\mu = \text{GM}, \ M = \text{Earth mass},$$
$$S_{\ell m} = \text{solid spherical harmonics}) \tag{1.3}$$

has an increasing cumulative error, we would like to keep L as low as possible, compatibly with the value of the geometrical parameter J_+. It is in the purpose of this paper to show that (1.2) can be substituted by the more favorable condition

$$3J_+^2\left(\frac{\delta R}{R_+} + \frac{1}{L+2}\right) < 1, \tag{1.4}$$

implying the knowledge of a global model T_L of much lower maximum degree L, for the same J_+. Therefore, although the paper is certainly a review of many known results, though maybe presenting them with more details and mathematical rigor, it also presents new results discussing their geodetic relevance in the last section.

Remark 1 (Notation and conventions) We will use for any function defined in Ω

$$f'(r,\sigma) \equiv \frac{\partial}{\partial r} f(r,\sigma) \tag{1.5}$$

for the sake of brevity.

For the same reason, we adopt the convention that, if not differently stated, any summation on harmonic degrees and orders (ℓ, m) has to be understood as summation on ℓ in the indicated intervals, while the summation on m is always on its full range, namely $-\ell \leq m \leq \ell$, even if not explicitly written; so, for instance, we have to understand, for any quantity depending on degree and orders

$$\sum_{\ell=2}^{L} A_{\ell m} \equiv \sum_{\ell=2}^{L}\sum_{m=-\ell}^{\ell} A_{\ell m}. \tag{1.6}$$

Moreover, we will use the notation

$$\ell = 0, 1 \ldots, -\ell \leq m \leq \ell, \ S_{\ell m}(r,\sigma) = \left(\frac{R_+}{r}\right)^{\ell+1} Y_{\ell m}(\sigma) \tag{1.7}$$

for the solid spherical harmonics, observing that with this choice the sequence $\{S_{\ell m}\}$ coincides with that of ordinary spherical harmonics on the Brillouin sphere $S_+ \equiv \{r = R_+\}$,

$$S_{\ell m}(R_+, \sigma) \equiv Y_{\ell m}(\sigma). \tag{1.8}$$

Furthermore, we will adopt as measure of the area element of the unit sphere

$$d\sigma = \frac{1}{4\pi} \cos\varphi d\varphi d\lambda, \tag{1.9}$$

so that the surface of the whole sphere is normalized

$$\int d\sigma = 1. \tag{1.10}$$

Yet we have to warn the reader that in this case the volume element and the surface element in spherical coordinates have to be written as

$$d\Omega = 4\pi r^2 dr d\sigma,\ dS = 4\pi J R_\sigma^2 d\sigma, \left(J = \frac{1}{\cos\vartheta}\right) \tag{1.11}$$

Finally, we will slightly modify the classical Lipschitz condition for a function $F(\sigma)$ of the point σ on the unit sphere, by using as definition of the Lipschitz ratio (see Eq. (A21) in Appendix)

$$(\cos\psi_{\sigma\sigma'} = \boldsymbol{e}_\gamma \cdot \boldsymbol{e}_{\gamma'});\ \frac{|F(\sigma) - F(\sigma')|}{\psi_{\sigma\sigma'}} \tag{1.12}$$

instead of

$$\frac{|F(\sigma) - F(\sigma')|}{2\sin\frac{\psi_{\sigma\sigma'}}{2}}, \tag{1.13}$$

where $2\sin\frac{\psi_{\sigma\sigma'}}{2}$ is the Cartesian distance between the two points σ, σ' on the unit sphere. The two definitions are, as a matter of fact, equivalent because the ratio $(\sin\psi/2)/\psi/2$ is positive and bounded above and below in a finite neighborhood of $\psi = 0$.

As for any further doubt about notation, beyond the list Eq. (1.2) and what is presented from Eq. (1.5) to Eq. (1.13), one could consult that of the book (Sansò and Sideris 2013).

The plan of the paper is as follows: After recalling in Sect. 2 many equivalent formulation of the linearized Molodensky problem and proving some technical results in Sect. 3, the main result for the geodetic boundary value problem is fully achieved in Sects. 4 and 5. In Sect. 6, the geodetic significance of the result is discussed in length.

Finally, in Appendix some basic mathematical theory of spaces of harmonic functions are recalled to provide the reader with proper tools necessary to understand the paper.

We warn the reader that maybe, before attacking the rest of the paper, it could be useful to read Appendix where a number of facts, well known but important, are recalled.

2 Many (almost) equivalent formulations of Molodensky's problem

Here we follow the path of the proof in Sansò and Sideris (2013), §15.3 but for two crucial steps contained in lemmas, in the next section. In particular, the data space will be H_0, which is essentially $L^2(S)$ with an equivalent norm as explained in Appendix, while the solution space H_1 is essentially the Sobolev Space $H^{1,2}(S)$ with a variant closely examined, again in Appendix. We recall only here that having to work with spaces of harmonic functions, we can always identify each function with its trace on S; specifically, we have

$$\| u \|_0^2 \equiv \| u \|_{H_0}^2 = \int u(R_\sigma, \sigma)^2 R_\sigma d\sigma \tag{2.1}$$

and

$$\| u \|_1^2 \equiv \| u \|_{H_1}^2 = \int |\nabla u(R_\sigma, \sigma)|^2 R_\sigma^3 d\sigma. \tag{2.2}$$

Moreover, we have

$$H_1 \subset H_0, \ (\| u \|_1 \leq c \| u \|_0) \tag{2.3}$$

as shown in (A41).

Finally, from Proposition 3, for every u harmonic in Ω, we have equivalently

$$u = O\left(\frac{1}{r^{L+2}}\right) \Leftrightarrow u_{+\ell m} = 0,\ 0 \leq \ell \leq L, \tag{2.4}$$

where

$$u_{+\ell m} = \int u(R_+, \sigma) Y_{\ell m}(\sigma) d\sigma; \tag{2.5}$$

moreover, there are functions $\{Z_{\ell m}\} \in H_0$ that with solid spherical harmonics $\{S_{\ell m}\}$ form a bi-orthogonal system, namely

$$< Z_{\ell m}, S_{jk} >_0 = \delta_{\ell j}\delta_{mk}. \tag{2.6}$$

The span $\{Z_{\ell m}\}$, starting from degree 2, is dense in H_0 and we call

$$W_L = \{Z_{\ell m}, 2 \leq \ell \leq L\}. \tag{2.7}$$

We notice that $Z_{\ell m}$ are useful to represent the harmonic coefficients of any function u harmonic in Ω, referred to S_+ as in (2.5), namely

$$u_{+\ell m} =< Z_{\ell m}, u >_0 . \tag{2.8}$$

Now we rewrite (1.1) adding to the third of (1.1) also the knowledge of some harmonic coefficients up to maximum degree L, all orders. We arrive then to

$$\begin{cases} \Delta T = 0 & \text{in } \Omega \\ -\frac{\partial T}{\partial h} + \frac{\partial \gamma/\partial h}{\gamma} T = \Delta g & \text{on } S \\ T = O\left(\frac{1}{r^3}\right), \\ < Z_{\ell m}, T >_0 = T_{+\ell m}, & 2 \le \ell \le L. \end{cases} \tag{2.9}$$

We can observe that using $\{T_{+\ell m}, 2 \le \ell \le L\}$ as datum is equivalent to considering as known a satellite only model

$$T_L \equiv \frac{\mu}{R} \sum_{\ell=2}^{L} T_{+\ell m} S_{\ell m} \tag{2.10}$$

which can be subtracted to T to arrive at a new reduced problem for the residual potential

$$u = T - T_L, \tag{2.11}$$

as it is a common practice in geodesy.

It is clear that the new problem for u is

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ -\frac{\partial u}{\partial h} + \frac{\partial \gamma/\partial h}{\gamma} u = \Delta g - \Delta g_L & \text{on } S \\ < Z_{\ell m}, u >_0 = 0 & 0 \le \ell \le L, \end{cases} \tag{2.12}$$

where

$$\Delta g_L = \frac{\mu}{R} \sum_{\ell=2}^{L} T_{+\ell m} \left(-\frac{\partial S_{\ell m}}{\partial h} + \frac{\partial \gamma/\partial h}{\gamma} S_{\ell m} \right). \tag{2.13}$$

For reasons that will become soon clear, we rewrite the boundary relation in the form

$$r\frac{\partial u}{\partial h} - r\frac{\partial \gamma/\partial h}{\gamma} u|_S = -r(\Delta g - \Delta g_L)|_S \equiv f(\sigma) \tag{2.14}$$

where $f(\sigma)$ is clearly a known term. It is convenient to introduce a notation for what we call the *Molodensky operator*, also splitting it into a spherical and a perturbative part, in the following way

$$\begin{aligned} Mu &= r\boldsymbol{\nu} \cdot \nabla u - r\frac{\partial \gamma/\partial h}{\gamma} u = \\ &\equiv r\boldsymbol{e}_r \cdot \nabla u + 2u - r(\boldsymbol{e}_r - \boldsymbol{\nu}) \cdot \nabla u - \left(r\frac{\partial \gamma/\partial h}{\gamma} + 2 \right) u \equiv \\ &= M_S u - \delta M u \end{aligned} \tag{2.15}$$

where $\boldsymbol{\nu}$ is the ellipsoidal normal, $\boldsymbol{e}_r$ is the unit radial vector,

$$M_S = r\boldsymbol{e}_r \cdot \nabla + 2 = r\frac{\partial}{\partial r} + 2 \tag{2.16}$$

is the spherical (or simple) Molodensky operator, and

$$\begin{cases} \delta M = r\boldsymbol{\varepsilon} \cdot \nabla + \eta \\ \boldsymbol{\varepsilon} = (\boldsymbol{e}_r - \boldsymbol{\nu}), \ \eta = r\frac{\partial \gamma/\partial h}{\gamma} + 2 \end{cases} \tag{2.17}$$

is the perturbative part of M.

The reason why we call δM a perturbative operator is that at the level of S, as shown in Sansò and Sideris (2013),

$$\varepsilon_+ = \max |\boldsymbol{\varepsilon}| \cong \frac{1}{2} e^2 \tag{2.18}$$

$$\eta_+ = \max |\eta| \cong 2e^2 \tag{2.19}$$

with $e^2 \cong 0.0067$ the squared eccentricity of the ellipsoid, so that δM is a "small" operator. Summarizing, we can reformulate (2.12) as

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ M_S u - \delta M u = f & \text{on } S \\ u \in W_L^\perp & (\text{or } < Z_{\ell m}, u >_0 = 0, \ 2 \le \ell \le L) \end{cases} \tag{2.20}$$

Remark 2 One has to be aware that the third of (2.20) as a matter of fact imposes $(L+1)^2 - 4$ conditions on the solution; so, we may guess that the same number of conditions has to be imposed on the known term $f(\sigma)$ in order to guarantee the existence of a solution. As an alternative, one could introduce an ad hoc function, belonging to some finite-dimensional subspace of H_0, with unknown $(L+1)^2 - 4$ coefficients to be determined from the conditions implied by the third of (2.20). As such, the basis of the subspace used in this way is not fixed on condition that such basis be transversal to $W_L^\perp$; for such a reason, it makes a lot of sense to use as such the subspace W_L itself to reach the right number of dimensions. This has been the suggestion of Hörmander (1976) and we will follow it here, although a choice like span $\{S_{\ell m}; 0 \le \ell \le L\}$ would also be possible.

Let us observe as well that here W_L and $W_L^\perp$ are both considered as spaces for the data and the unknowns: This is possible because the harmonic function u is identified with its trace, as a function of σ, and $u \in H_1 \subset H_0$, to which the datum f is assumed to belong.

So we arrive at the final formulation of the linearized Molodensky problem in the form

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ M_S u - \delta M u = f - \sum_{\ell=2}^{L} a_{\ell m} Z_{\ell m} & \text{on } S \\ < Z_{\ell m}, u >_0 = 0, & 2 \le \ell \le L \end{cases} ; \tag{2.21}$$

of course, the unknowns of this problem become now $u \in W_L^{\perp} \cap H_1$, $\{a_{\ell m}\} \in R^{(L+1)^2-4}$.

3 Two technical lemmas

In this section, we prove two technical lemmas. We have not placed them into Appendix because the first leads to a substantial change with respect to the proof in Sansò and Sideris (2013), the second because it is a fundamental tool in the theory of geodetic BVPs.

Lemma 1 *Assume*

$$u \in W_L^{\perp} \cap H_1; \tag{3.1}$$

then,

$$\| u \|_0 \le c_L J_+ \| u \|_1 \tag{3.2}$$

with

$$c_L = \frac{\delta R}{R_+} + \frac{1}{L+2}. \tag{3.3}$$

Proof Let us call

$$u_+(\sigma) = u(R_+, \sigma). \tag{3.4}$$

We can write

$$\| u \|_0 \le \| u_+ - u \|_0 + \| u_+ \|_0 . \tag{3.5}$$

On the other hand, in $\Omega_+ \equiv \{r \ge R_+\}$, including S_+, we know that we can use the convergent series

$$u(r, \sigma) \equiv \sum_{\ell=L+1}^{+\infty} u_{+\ell m} S_{\ell m}(r, \sigma). \tag{3.6}$$

So we can write

$$\begin{aligned} \| u_+ \|_0^2 &= \int \mathrm{d}\sigma \, R_\sigma u^2(R_+, \sigma) \\ &\le R_+ \sum_{\ell=L+1}^{+\infty} u_{+\ell m}^2 \le \frac{R_+}{L+2} \sum_{\ell=L+1}^{+\infty} (\ell+1) u_{+\ell m}^2 . \end{aligned} \tag{3.7}$$

But we know that

$$\begin{aligned} \int_{S_+} -u' u \mathrm{d}S_+ &\equiv 4\pi R_+^2 \int -u'(R_+, \sigma) u(R_+, \sigma) \mathrm{d}\sigma \\ &= 4\pi R_+ \sum_{\ell=L+1}^{+\infty} (\ell+1) u_{+\ell m}^2 . \end{aligned} \tag{3.8}$$

So we can write, by applying the Gauss theorem through the surface S_+ bounding Ω_+,

$$\begin{aligned} \| u_+ \|_0^2 &\le \frac{1}{4\pi} \frac{1}{L+2} \int_{S_+} -u' u \mathrm{d}S_+ \\ &\equiv \frac{1}{4\pi} \frac{1}{L+2} \int_{\Omega_+} \nabla u^2 \mathrm{d}\Omega . \end{aligned} \tag{3.9}$$

Coming to the difference $u_+ - u$, we have

$$\begin{aligned} |u_+(\sigma) - u(R_\sigma, \sigma)|^2 &= \left| \int_{R_\sigma}^{R_+} u' \mathrm{d}s \right|^2 \\ &\le \int_{R_\sigma}^{R_+} u'^2 s^2 \mathrm{d}s \int_{R_\sigma}^{R_+} \frac{\mathrm{d}s}{s^2} \\ &\equiv \frac{\delta R}{R_+ R_\sigma} \int_{R_\sigma}^{R_+} u'^2 s^2 \mathrm{d}s . \end{aligned} \tag{3.10}$$

We multiply (3.10) by R_σ and integrate in $\mathrm{d}\sigma$, to get, also recalling (1.11),

$$\begin{aligned} \| u_+ - u \|_0^2 &= \int R_\sigma |u_+ - u|^2 \mathrm{d}\sigma \\ &\le \frac{\delta R}{R_+} \int \mathrm{d}\sigma \int_{R_\sigma}^{R_+} u'^2 s^2 \mathrm{d}s \\ &\le \frac{\delta R}{R_+} \frac{1}{4\pi} \int_{\Omega \setminus \Omega_+} \mathrm{d}\Omega |\nabla u|^2 . \end{aligned} \tag{3.11}$$

Now we use (3.8) and (3.4) in (3.5) to obtain

$$\begin{aligned}\| u \|_0 &\le \left[\frac{\delta R}{R_+}\frac{1}{4\pi}\int_{\Omega\setminus\Omega_+}|\nabla u|^2 d\Omega\right]^{1/2} \\ &\quad + \left[\frac{1}{L+2}\frac{1}{4\pi}\int_{\Omega_+}|\nabla u|^2 d\Omega\right]^{1/2} \\ &\le c_L^{1/2}\frac{1}{\sqrt{4\pi}}\left[\int_\Omega |\nabla u|^2 d\Omega\right]^{1/2} \\ &= \frac{c_L^{1/2}}{\sqrt{4\pi}}\left[\int_S -u u_n dS\right]^{1/2} \\ &\le \frac{c_L^{1/2}}{\sqrt{4\pi}}\left[\int |u||\nabla u| 4\pi J R_\sigma^2 d\sigma\right]^{1/2} \\ &\le c_L^{1/2}[J_+ \| u \|_0 \| u \|_1]^{1/2} \end{aligned} \tag{3.12}$$

In the last step of (3.12), we have used Schwartz' inequality and the fact that $R_\sigma^2 \equiv R_\sigma^{3/2}\cdot R_\sigma^{1/2}$ and denoted by u_n the external normal derivative of u on S.

Squaring (3.12) and simplifying by $\| u \|_0$ we get (3.2). □

Remark 3 We remind that in Appendix we have already found a majorization of the type (3.2), yet there, since we wanted to prove only that $H_1 \subset H_0$, we have arrived at the inequality (A41) which is valid $\forall u \in H_0$. In Lemma (1), however, we have restricted this evaluation to functions that are in $W_L^\perp$; whence, we got the much tighter inequality (3.2) where c_L is expected to be significantly smaller than 1.

It is clear that a good step forward has been done in the use of inequality (3.7) which causes a gain by almost a factor 2 with respect to previous work. Still this inequality seems very rough and maybe one could do better in future.

Lemma 2 (Energy integral identity) *Assume that $u \in H_1$ and put*

$$M_S u = r u' + 2u = v \in H_0; \tag{3.13}$$

then, the following identity holds

$$4\pi \| u \|_1^2 \equiv \int_S (-3u + 2v) u_n dS. \tag{3.14}$$

Proof We start from the differential identity, valid for $\forall u \in H_1$,

$$\nabla\cdot(r u' + 2u)\nabla u = \frac{1}{2} r \frac{\partial}{\partial r}|\nabla u|^2 + 3|\nabla u|^2, \tag{3.15}$$

which is easily verified expressing all operators in Cartesian coordinates and recalling that u has to be harmonic.

Integrating (3.15) on Ω, recalling (1.11), applying Gauss theorem and using the identity

$$\begin{aligned}\frac{1}{2}\int_{R_\sigma}^{+\infty} r\frac{\partial}{\partial r}|\nabla u|^2 r^2 dr &= -\frac{1}{2}R_\sigma^3|\nabla u(R_\sigma,\sigma)|^2 \\ &\quad -\frac{3}{2}\int_{R_\sigma}^{+\infty}|\nabla u|^2 r^2 dr\end{aligned} \tag{3.16}$$

we find

$$-\int_S v u_n dS \equiv \frac{3}{2}\int_\Omega |\nabla u|^2 d\Omega - \frac{1}{2}\cdot 4\pi\int_S |\nabla u|^2 R_\sigma^3 d\sigma. \tag{3.17}$$

Multiplying by 2, reordering and using Gauss theorem again, we find (3.14), which is an identity using only u, or some of its functionals, at the level of S. □

Remark 4 We underline that there are families of energy integral identities (see Sansò and Sideris 2013), §15.3. In this case, (3.14) is tailored to the GBVP, involving the harmonic function v given by (3.13).

4 The simple Molodensky problem (SMP) and the Prague method

By definition, the simple Molodensky problem is just (2.21) without the perturbative term $\delta M u$, namely

$$\begin{cases}\Delta u = 0 & \text{in } \Omega \\ M_s u = f - \sum_{\ell=0}^L a_{\ell m} Z_{\ell m} & \text{on } S \\ < Z_{\ell m}, u >_0 = 0, & 0 \le \ell \le L.\end{cases} \tag{4.1}$$

This is a relatively easy problem and its solution is classical, dating back to the times of Moritz (1965) and Krarup (2006). What is not classical is the proof that the solution satisfies the regularity and stability relations

$$\| u \|_1 \le D_L \| f \|_0 \tag{4.2}$$

$$\| \sum_{\ell=0}^L a_{\ell m} Z_{\ell m} \|_0 \le \| f \|_0, \tag{4.3}$$

for a certain specific constant D_L.

We observe already at this point that the parametric function $\sum_{\ell=2}^L a_{\ell m} Z_{\ell m}$ being finite-dimensional, since $\{Z_{\ell m},\ 2 \le \ell \le L\}$ are linearly independent, has a squared norm

$$\| \sum_{\ell=j}^L a_{\ell m} Z_{\ell m} \|_0^2 \equiv \sum_{\ell,j=2}^L a_{\ell m} < Z_{\ell m}, Z_{jk} >_j a_{jk} \tag{4.4}$$

that is equivalent to the squared Cartesian norm

$$|\{a_{\ell m}\}|^2 \equiv \sum_{\ell=2}^{L} a_{\ell m}^2. \tag{4.5}$$

The method of analysis adopted is what Krarup has called the Prague method, which is essentially the spherical version of the Cartesian approach to the oblique BVP by Giraud (1934), Miranda (1970). It is based on properties of the operator

$$M_s = r\frac{\partial}{\partial r} + 2 = \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} + 2 \tag{4.6}$$

that is summarized by the following proposition.

Proposition 1 *Properties of M_S*

(a) *$M_S : H_1 \to H_0$, implying that $v = M_S u$ is harmonic*
(b)

$$v_{+\ell m} = -(\ell - 1)u_{+\ell m} \tag{4.7}$$

(c) *M_S is invertible $W_L^\perp \cap H_1 \to W_L^\perp \subset H_0$, $L > 1$, with continuous inverse.*

Proof a) that M_S is continuous $H_1 \to H_0$ is obvious, once we have proved that

$$v = O\left(\frac{1}{r^3}\right) \tag{4.8}$$

and

$$\Delta v = 0 \ \text{ in } \Omega. \tag{4.9}$$

As for (4.8), we simply have (recall that u is very smooth in Ω)

$$u = O\left(\frac{1}{r^3}\right) \Rightarrow v = rO\left(\frac{1}{r^4}\right) + O\left(\frac{1}{r^3}\right) = O\left(\frac{1}{r^3}\right). \tag{4.10}$$

Now from the identity

$$\Delta v = \Delta M_S u = (M_S + 2)\Delta u, \tag{4.11}$$

we see that $\Delta u = 0 \Rightarrow \Delta v = 0$;

b) since u, v are harmonic in Ω, they both have the converging representation in Ω_+

$$u = \sum_{\ell=2}^{+\infty} u_{+\ell m} S_{\ell m}, \quad v = \sum_{\ell=2}^{+\infty} v_{+\ell m} S_{\ell m}, \tag{4.12}$$

so that (4.7) is just the classical Stokes relation;

c) from (4.7), we see that

$$\ell \geq 2, \ u_{+\ell m} = -\frac{v_{+\ell m}}{\ell - 1} \tag{4.13}$$

so that given v in Ω_+ we find one and only one u harmonic in this set. However, by the unique continuation property u, which is harmonic in the whole Ω, is also univocally determined there. Following an alternative more classical reasoning, we can even determine the explicit form of M_S^{-1}, in fact it is well known that $v = M_S u$ implies, (see Heiskanen and Moritz 1967),

$$u(r, \sigma) = -\frac{1}{r^2}\int_r^{+\infty} v(s, \sigma)s\,ds. \tag{4.14}$$

From (4.14), we see that given any $v \in H_0$, ($v = O\left(\frac{1}{r^3}\right)$), u can be computed because the integral is converging. This means that M_S is injective and surjective on H_0, and then, the inverse M_S^{-1} is a bounded operator by the open mapping theorem Yosida (1980), i.e., (4.2) holds for $L = 2$ and some constant D_L. The same reasoning can be applied to $W_L^\perp$ for any $L > 2$. □

Let us underline that in this way we have not determined the constant D_L; this will be done in Theorem 2.

Proposition 2 *The problem (4.1) is equivalent to a Dirichlet problem for*

$$v = M_S u, \tag{4.15}$$

namely

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v|_S = f - \sum_{\ell=0}^{L} a_{\ell m} Z_{\ell m} & \text{on } S \\ < Z_{\ell m}, v >= 0 & 2 \leq \ell \leq L. \end{cases} \tag{4.16}$$

Proof Let u, $\{a_{\ell m}, 0 \leq \ell \leq L\}$ be a solution of (4.1), then v is harmonic in Ω by Proposition 1,a) and

$$v|_S = M_S u|_S = f - \sum_{\ell=2}^{L} a_{\ell m} Z_{\ell m}. \tag{4.17}$$

Moreover, $v \in W_L^\perp$ by Proposition 1,c), so that the third of (4.16) is satisfied too. Vice versa, let v, $\{a_{\ell m}, 2 \leq \ell \leq L\}$ be a solution of (4.16); then, u given by (4.14) is harmonic and

$$M_S u = v \tag{4.18}$$

so that

$$M_S u|_S \equiv f - \sum_{\ell=0}^{L} a_{\ell m} Z_{\ell m}. \tag{4.19}$$

Moreover, always by Proposition 1,c),

$$u = O\left(\frac{1}{r^{L+2}}\right) \Rightarrow u \in W_L^{\perp},$$

so that the third of (4.1) is satisfied too. □

Thanks to Proposition 1,c) and Proposition 2, we know that if we can find a solution $v, \{a_{\ell m}\}$, of (4.16) in H_0 then there is a solution $u, \{a_{\ell m}\}$ of (4.1) with u belonging to H_1, though the constant D_L in (4.2) is not yet known. The solution of (4.16) is provided by the next theorem.

Theorem 1 *Given any $f \in H_0$, there is one and only one solution $v, \{a_{\ell m}, 2 \le \ell \le L\}$ of (4.16), with $v \in H_0$ too, and such that*

$$\| v \|_0 \le \| f \|_0 \tag{4.20}$$

$$\| \sum_{\ell=0}^{L} a_{\ell m} Z_{\ell m} \|_0 \le \| f \|_0 . \tag{4.21}$$

Proof Let us first consider the simple Dirichlet problem

$$\begin{cases} \Delta w = 0 \text{ in } \Omega \\ w = f \quad \text{on } S; \end{cases} \tag{4.22}$$

we know that this problem has one and only one solution and if $f \in L^2_\sigma$, also $w \in L^2_\sigma$. Moreover, if $f \in H_0$ by hypothesis then

$$w_{+jk} = 0 \quad j = 0, 1 \tag{4.23}$$

namely $w \in H_0$ and it is isometrically related to f.

Now we put

$$\sum_{\ell=2}^{L} a_{\ell m} Z_{\ell m} = P_{W_L} w, \tag{4.24}$$

where we have introduced P_{W_L}, the symbol of he H_0-orthogonal projector on W_L. Such equation is clearly equivalent to the system

$$\begin{aligned} 2 \le j \le L; \quad & \sum_{\ell=2}^{L} a_{\ell m} < Z_{\ell m}, Z_{jk} >_0 \\ & = < w, Z_{jk} >_0 \equiv < f, Z_{jk} >_0, \end{aligned} \tag{4.25}$$

which univocally determines $\{a_{\ell m}\}$ because $\{Z_{\ell m}\}$ are linearly independent.

We can therefore define a v by

$$v = w - \sum_{\ell=2}^{L} a_{\ell m} Z_{\ell m} = (I - P_{W_L}) w \equiv P_{W_L^{\perp}} w. \tag{4.26}$$

Of course, v is harmonic in Ω and

$$\begin{aligned} 2 \le j \le L; \quad & v_{+jk} = < w, Z_{jk} >_0 \\ & - \sum_{\ell=2}^{L} a_{\ell m} < Z_{\ell m}, Z_{jk} >_0 = 0; \end{aligned} \tag{4.27}$$

furthermore,

$$v|_S = f - \sum_{\ell=2}^{L} a_{\ell m} Z_{\ell m}, \tag{4.28}$$

in other words v is a solution of (4.16).

The uniqueness of w implies that of v.

Finally, the two definition (4.24) and (4.26) show that

$$\| v \|_0 \le \| w \|_0 = \| f \|_0 \tag{4.29}$$

$$\| \sum_{\ell=2}^{L} a_{\ell m} Z_{\ell m} \|_0 \le \| w \|_0 \equiv \| f \|_0 . \tag{4.30}$$

□

The final step of the section is the determination of D_L.

Theorem 2 *Assume that*

$$3 J_+^2 c_L < 1, \tag{4.31}$$

then the solution $u, \{a_{\ell m}\}$ of (4.1) satisfies

$$\| u \|_1 \le D_L \| f \|_0 \tag{4.32}$$

with

$$D_L = \frac{2 J_+}{1 - 3 J_+^2 c_L}. \tag{4.33}$$

Proof We already know that u exists and is in H_1 by Proposition 1,c); so, we can apply the Lemma 2 and Eq. (3.14). Recalling that $\mathrm{d}S = 4\pi J R_\sigma^2 \mathrm{d}\sigma$, we find

$$\begin{aligned} 4\pi & \| u \|_1^2 \\ & \le 4\pi \int_S |2v - 3u||u_n| J R_\sigma^2 \mathrm{d}\sigma \le \\ & \le 4\pi J_+ \left[\int |2v - 3u|^2 R_\sigma \mathrm{d}\sigma \right]^{1/2} \left[\int |u_n|^2 R_\sigma^3 \mathrm{d}\sigma \right]^{1/2} \\ & \le 4\pi J_+ \| 2v - 3u \|_0 \| u \|_1 \\ & \le 4\pi J_+ (2 \| v \|_0 + 3 \| u \|_0) \cdot \| u \|_1 . \end{aligned} \tag{4.34}$$

Simplifying by $4\pi \parallel u \parallel_1$, we get

$$\parallel u \parallel_1 \leq 2J_+ \parallel v \parallel_0 + 3J_+ \parallel u \parallel_0 . \quad (4.35)$$

But by Lemma 1 (see Eq. (3.2))

$$\parallel u \parallel_0 \leq J_+ c_L \parallel u \parallel_1 \quad (4.36)$$

that, inserted into (4.35), observing that $\parallel v \parallel_0 \leq \parallel f \parallel_0$, gives

$$(I - 3J_+^2 c_L) \parallel u \parallel_1 \leq 2J_+ \parallel f \parallel_0, \quad (4.37)$$

so proving (4.31), (4.32), (4.33). □

5 Solution of the linearized GBVP

Let us write the linearized GBVP, modified as in (2.21) in the form

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ M_S u = (f + \delta M u) - \sum_{\ell=2}^{L} a_{\ell m} Z_{\ell m} & \text{on } S \\ < Z_{\ell m}, u >_0 = 0 & 2 \leq \ell \leq L. \end{cases} \quad (5.1)$$

On the basis of the analysis of the SMP, we know that if a solution of (5.1) exists, and if

$$3J_+^2 c_L < 1, \quad (5.2)$$

u has to satisfy, recalling (4.32) and (3.2)

$$\begin{aligned} \parallel u \parallel_1 &\leq D_L \parallel f + \delta M u \parallel_0 \\ &\leq D_L \parallel f \parallel_0 + D_L(\parallel R_\sigma \boldsymbol{\varepsilon} \cdot \nabla u \parallel_0 + \parallel \eta u \parallel_0) \\ &\leq D_L \parallel f \parallel_0 + D_L(\varepsilon_+ \parallel u \parallel_1 + \eta_+ \parallel u \parallel_0) \\ &\leq D_L \parallel f \parallel_0 + D_L(\varepsilon_+ + \eta_+ J_+ c_L) \parallel u \parallel_1 . \end{aligned} \quad (5.3)$$

As we see, we obtain once more a majorization of the type

$$\parallel u \parallel_1 \leq K_L \parallel f \parallel_0 \quad (5.4)$$

with

$$K_L = \frac{D_L}{1 - D_L(\varepsilon_+ + \eta_+ J_+ c_L)}, \quad (5.5)$$

if the condition

$$D_L(\varepsilon_+ + \eta_+ J_+ c_L) < 1 \quad (5.6)$$

is satisfied. These inequalities clearly call for the application of a simple iterative scheme to prove the existence of the solution. This is done in the next theorem.

Theorem 3 *Let us assume that the condition (5.6) is satisfied; then the iterative scheme*

$$\begin{cases} \Delta u_{n+1} = 0 \\ M_S u_{n+1} = f + \delta M u_n - \sum_{\ell=2}^{L} a_{\ell m}^{(n+1)} Z_{\ell m} \\ u_{n+1} \in W_L^\perp \end{cases} \quad (5.7)$$

is convergent in $H_1 \otimes R^{(L+1)^2-4}$ to the solution u, $\{a_{\ell m}, 2 \leq \ell \leq L\}$, of the linearized GBVP (2.21).

Proof Let us recall that, according to the analysis of the SMP, we can split (5.7) into two problem, namely:

$$\begin{cases} \Delta u_{n+1} = 0 \\ M_S u_{n+1} = P_{W_L^\perp}(f + \delta M u_n) \end{cases} \quad (5.8)$$

and

$$\sum_{\ell=2}^{L} a_{\ell m}^{(n+1)} Z_{\ell m} = P_{W_L}(f + \delta M u_n). \quad (5.9)$$

We can note that indeed (5.8) has a solution automatically belonging to $W_L^\perp$; subtracting two steps in the iteration, we get

$$\begin{cases} \Delta(u_{n+1} - u_n) = 0 \\ M_S(u_{n+1} - u_n) = P_{W_L^\perp} \delta M(u_n - u_{n-1}), \end{cases} \quad (5.10)$$

implying, repeating the reasoning in (5.3),

$$\parallel u_{n+1} - u_n \parallel_1 \leq D_L(\varepsilon_+ + \eta_+ J_+ c_L) \parallel u_n - u_{n-1} \parallel_1 . \quad (5.11)$$

Therefore, it is clear that, if (5.6) is satisfied, the sequence $\{u_n\}$ converges to some u in $H_1 \cap W_L^\perp$. So we have as well

$$M_S u_n \underset{H_0}{\rightarrow} M_S u, \ \delta M u_n \underset{H_0}{\rightarrow} \delta M u, \quad (5.12)$$

and u is part of the solution of the linearized GBVP, i.e.,

$$\begin{cases} \Delta u = 0 \\ M_S u = P_{W_L^\perp}(f - \delta M u). \end{cases} \quad (5.13)$$

Coming to the sequence $\{a_{\ell m}^{(n)}\}$, we write (5.9) in the equivalent form

$$\begin{aligned} 2 \leq j \leq L; \quad & \sum_{\ell=2}^{L} a_{\ell m}^{(n+1)} < Z_{\ell m}, Z_{jk} >_0 = < f, Z_{jk} >_0 \\ & + < \delta M u_n, Z_{jk} >_0 . \end{aligned} \quad (5.14)$$

Taking the limit of the relation (5.14), since everything is finite-dimensional and $\delta M u_n \underset{H_0}{\to} \delta M u$, we see that also $\{a_{\ell m}^{(n)}\}$ is convergent to some $\{a_{\ell m}\}$ such that

$$2 \le k \le L; \quad \sum_{\ell=2}^{L} a_{\ell m} < Z_{\ell m}, Z_{jk} >_0 \\ =< f, Z_{jk} >_0 + < \delta M u, Z_{jk} >_0 \tag{5.15}$$

or

$$\sum_{\ell=2}^{L} a_{\ell m} Z_{\ell m} = P_{W_L}(f + \delta M u) \tag{5.16}$$

Putting (5.13) and (5.16) together, we see that u, $\{a_{\ell m}\}$ give the complete solution of the linearized GBVP. □

6 Discussion and questions

Two points related to the choice of the data space H_0 and solution space H_1 are worth discussing. The first is qualitative; as for irregularity of their functions, H_0 is essentially $L^2(S)$ and H_1 is essentially the Sobolev space $H^{1,2}(S)$. The fact that the solution is in $H^{1,2}$ then means that its gradient at the boundary is in L^2. In geodetic terms, this means that if $\Delta g(\sigma)$ is an L^2 function on the boundary, then the deflection of the vertical $\boldsymbol{\delta} = \boldsymbol{n} - \boldsymbol{\nu}$ is also in L^2 and we are justified to compute it on S. In other words, Vening Meinesz formulas are valid, even without a particular regularization of the datum Δg, which is discrete in nature, other than a simple least-squares interpolation.

The second point is more technical and refers to the fact that H_0 and hence H_1 too are defined in such a way as to contain only functions T for which $T_{+\ell m} = 0$, $(\ell = 0, 1)$, namely orthogonal to $Z_{\ell m} (\ell = 0, 1)$. In this way, the non-uniqueness of the inversion of the simple Molodensky operator, which has a zero space spanned by first degree harmonics, is hidden. For this matter, we could have used larger spaces and discuss this non-uniqueness, yet we have preferred to follow the geodetic tradition according to which the non-uniqueness is eliminated a priori in the definition of the anomalous potential, by a suitable choice of the reference system. As for the geodetic relevance of the results obtained in this theoretical review, we can compare the maximum degree L of the harmonic coefficients, assumed to be known, determined according to the requirement (5.6), versus the same L determined by the previous proof Sansò and Sideris (2013), §15.3, at least for some significant values of the maximum inclination ϑ_+.

Table 1 L coming from old formula (6.5) and from new formula (6.3), in function of ϑ_+

ϑ_+	60°	65°	70°	75°	80°
L old (6.5)	37	57	103	292	n.a
L new (6.3)	12	17	29	58	203

To this aim, recalling also (3.3), we can solve the inequality (5.6) obtaining

$$L > \frac{3 + 2\eta_+}{\cos^2 \vartheta_+ - 2\varepsilon_+ \cos \vartheta_+ - \frac{\delta T}{R_+}(3 + 2\eta_+)} - 2, \tag{6.1}$$

that with the numerical values

$$3 + 2\eta_+ = 3.0268 \text{ , } 2\varepsilon_+ = 0.0067 \text{ , } \frac{\delta R}{R_+} = 0.0047 \tag{6.2}$$

becomes

$$L > \frac{3.0268}{\cos^2 \vartheta_+ - 0.0067 \cos \vartheta_+ - 0.0142} - 2. \tag{6.3}$$

The same variable L for the old solution can be determined by the inequality (see Sansò and Sideris 2013)

$$\frac{2R}{R_+} + \frac{2}{L+1} > \frac{1 - e^2 J_+}{4 J_+^2 (1 + e^2)} \equiv \frac{\cos \vartheta_+ - 0.0067 \cos \vartheta_+}{4.0268}. \tag{6.4}$$

Solving with respect to L, we find

$$L > \frac{8.0536}{\cos^2 \vartheta_+ - 0.0067 \cos \vartheta_+ - 0.0378} - 2. \tag{6.5}$$

Computing (6.3), (6.5) for a few large values of ϑ_+ we get Table 1.

As we can see the value $\vartheta_+ = 80°$ has to be excluded for obvious reasons. In fact, we have to realize that incorporating into the GBVP data coming from harmonic coefficients $\{T_{+\ell m}, 2 \le \ell \le L\}$ as derived from satellite observations only, which are directly connected to the development of T outside a Brillouin sphere, implies the introduction of a so-called *cumulative error* into the final solution. The higher L, the larger is the cumulative error $\mathcal{E}_c(L)$, in terms of geoid (see, for instance, Sansò and Sideris 2013, §3.8). Computing $\mathcal{E}_c(L)$ from the model GOCO 06 for some values of L, we get Table 2.

One could decide that an admissible mean square error could be of the order of 0.1 mm, given the foundational role of the actual result for geodesy and considering that, due to the distribution of errors certainly far from normal on the

Table 2 Commission error in geoid from the model GOCO 06, for some values of L (in mm)

L	20	40	60	80	100	120	140	160	180	200
$\mathcal{E}_c(L)$	$3 \cdot 10^{-2}$	$4 \cdot 10^{-2}$	$6 \cdot 10^{-2}$	0.12	0.27	0.64	1.61	3.75	7.55	14.60

Table 3 Percentage of points of with pointwise inclination larger than $tg\vartheta_+$ for the test area $(11 \cdot 10^6)$ points

ϑ_+	60°	65°	70°	75°
$\%(tg\vartheta > tg\vartheta_+)$	0.13	0.03	0.01	(10^{-5})

tails, pointwise errors can be dozens of times larger than their r.m.s.

In this case, as we see, we should have limited the maximum inclination of geoid to $\vartheta_+ = 60°$ with the old formula, but $\vartheta_+ = 75°$ with the new one. To understand what is the significance of such constraints in a realistic scenario, for instance, in areas of rough topography like the Alps, the Andes or the Himalaya, we have analyzed a digital terrain model, with a 90 m resolution and only altitudes larger than 500 m, in the alpine area $45° \leq \varphi \leq 48°$; $6° \leq \lambda \leq 10^0$. Of course, the telluroid is not identical to the digital terrain model, but we know it follows it closely. So we have counted the percentage of points where the terrain inclination, measured by $tg\vartheta$, was exceeding $tg\vartheta_+$ for some values of ϑ_+, getting the result displayed in Table 3.

This gives an idea of the degree of smoothing of the telluroid, that one has to introduce in order to be compliant with the constraints imposed by the analysis of the GBVP.

All in all, we believe that a model with numbers

$$L = 58 \quad \mathcal{E}_c = 0.06 \text{ mm} \quad \vartheta_+ = 75°$$

can provide an acceptable framework in which the real situation of the Earth can be comprised, although the choice

$$L = 29 \quad \mathcal{E}_c = 0.035 \text{ mm} \quad \vartheta_+ = 70°$$

could also be a reasonable compromise.

A last comment is on the question whether the present estimates could be further improved by passing to an ellipsoidal approximation instead of the spherical one. This would decrease by a factor of 5 the constant $\frac{\delta R}{R_+}$. The matter is stimulating from the methodological point of view, yet a first analysis of the problem shows that in front of a more difficult analytical apparatus, the improvement in terms of lowering L and raising ϑ_+ seems to be tiny and maybe not worth the effort at least from a practical point of view.

Appendix A Some spaces of harmonic function, spherical harmonics and the trace operator

We are given a bounded domain B, with boundary S, a star-shaped Lipschitz surface. Ω is the open set exterior to S. S_+ is a Brillouin sphere, with radius close to the lower bound, e.g., for the Earth $R_+ \cong 6, 385$ m, basically the equatorial radius of the Earth plus the height of the Chimborazo mountain. We denote with $\overline{\mathcal{H}}(\Omega)$ the linear space of functions harmonic in Ω

$$\overline{\mathcal{H}}(\Omega) = \{u;\ \Delta u = 0 \text{ in } \Omega\}. \tag{A1}$$

$\overline{\mathcal{H}}(\Omega)$ is a Frechet space, i.e., a linear topological vector space, complete with respect to the topology induced by the semi norms

$$p_K(u) = \sup_{x \in K} |u(x)|; \quad K \text{ compact } \subset \Omega \tag{A2}$$

(see Yosida 1980,I,9, Krarup 2006; Sansò and Venuti 2005). Let us recall that

$$\overline{\mathcal{H}}(\Omega) \subset C^\infty(\Omega) \tag{A3}$$

namely every $u \in \overline{\mathcal{H}}(\Omega)$ has continuous derivatives of any order in every compact domain $K \in \Omega$, and in particular, it is bounded on K, the imbedding (A3) being continuous. We now consider a subspace of $\overline{\mathcal{H}}(\Omega)$ namely that of harmonic functions regular at infinity defined as follows

$$\mathcal{H}(\Omega) \equiv \{u \in \overline{\mathcal{H}}(\Omega);\ \sup_{\Omega} r|u(x)| < +\infty\} \tag{A4}$$

Since any function harmonic in Ω is also harmonic in the spherical domain $\Omega_+ \equiv \{r \geq R_+\}$, we know that $\forall u \in \mathcal{H}(\Omega)$, and in $\{r \geq R_+\}$, we can write

$$u = \sum_{\ell,m=0}^{+\infty} u_{+\ell m} S_{\ell m}(r, \sigma)$$
$$\left(S_{\ell m} = \left(\frac{R_+}{r} \right)^{\ell+1} Y_{\ell m}(\sigma) \right) \tag{A5}$$

and that

$$u_{+\ell m} = \int_{S_+} u Y_{\ell m}(\sigma) \mathrm{d}\sigma = \int u(R_+, \sigma) Y_{\ell m}(\sigma) \mathrm{d}\sigma. \tag{A6}$$

Since $S_+\{r = R_+\}$ is a compact set contained in Ω, if $u^{(N)} \to u$ in $\mathcal{H}(\Omega)$ we have that $u^{(N)}(R_+, \sigma) \to u(R_+, \sigma)$ uniformly in σ, so we have

$$\lim_{N\to\infty} u_{+\ell m}(u^{(N)}) = u_{+\ell m}(u); \tag{A7}$$

in other words, $u_{+\ell m}(u)$ given by (A6) is a continuous linear functional on $\mathcal{H}(\Omega)$. It follows that the space

$$\mathcal{H}^{(L)}(\Omega) = \{u \in \mathcal{H}(\Omega);\ u_{+\ell m}(u) = 0, \quad 0 \le \ell \le L\} \tag{A8}$$

is a proper closed subspace of $\mathcal{H}(\Omega)$.

Proposition 3 *$\mathcal{H}^{(L)}(\Omega)$ is characterized by the condition*

$$u \in \mathcal{H}(\Omega),\ u = O\left(\frac{1}{r^{L+2}}\right). \tag{A9}$$

Proof

$$\begin{aligned} & u \in \mathcal{H}^{(L)}(\Omega) \\ & \Rightarrow u = \sum_{\ell=L+1}^{+\infty} a_{+\ell m} \left(\frac{R_+}{r}\right)^{\ell+1} Y_{\ell m}(\sigma) \text{ in } \Omega_+ \\ & \Rightarrow \sup_r r^{L+2}|u| = \left| \sum_{m=-(L+1)}^{L+1} u_{+(L+1,m)} R_+^{L+2} Y_{\ell m}(\sigma) \right| \le c \end{aligned} \tag{A10}$$

because it is a finite sum of finite numbers. So

$$u \in \mathcal{H}^{(L)}(\Omega) \Rightarrow u = O\left(\frac{1}{r^{L+2}}\right). \tag{A11}$$

Vice versa, if $u = O\left(\frac{1}{r^{L+2}}\right)$ we have

$$j \le L, \quad \lim_{r\to\infty} r^j |u| = \left| \sum_{m=-j}^{j} u_{+jm} R_+^{j+1} Y_{\ell m}(\sigma) \right| = 0 \tag{A12}$$

which implies that

$$u_{+jm} = 0 \quad j \le L, \tag{A13}$$

because $\{Y_{jm}(\sigma)\}$ are linearly independent. □

Remark 5 Thanks to the work (Sansò and Sacerdote 2008), we know that $\forall u \in \mathcal{H}(\Omega)$ and we can define a trace operator, γ_S, according to the following procedure:

- first we define a sequence S_m of C^∞ surfaces

  $$S_m = \{r = R_{m\sigma}\} \tag{A14}$$

  such that

  $$R_{m\sigma} > R_\sigma, \ \lim_{m\to\infty} R_{m\sigma} = R_\sigma \tag{A15}$$

  uniformly in σ. This is obtained by the usual mollifier technique (see the discussion on radial basis functions in Freeden et al. (2009)); even more, since R_σ is Lipschitz, $R_{m\sigma}$ can be taken to satisfy uniformly the Lipschitz condition with the same constant,
- then we take the sequence

  $$u_m(\sigma) = u(R_{m\sigma}, \sigma); \tag{A16}$$

  indeed, $u_m(\sigma)$ uniformly identifies the function $u(r, \sigma)$ in the whole Ω, and when $u(r, \sigma) \in H^s_{loc}(\Omega), 3/2 > s > 1/2$, case in which one can define the ordinary trace operator, one has (see McLean 2000)

  $$u_m(\sigma) \to u(\sigma) \text{ in } H^{s-1/2}(S). \tag{A17}$$

- In reality, as proved in Sansò and Sacerdote (2008), the $-1/2$ rule holds $\forall s \in R$, namely even for $s \le 1/2$, when we work with spaces of harmonic functions, even for the whole $\mathcal{H}(\Omega)$ one can define the $\gamma_S(u)$ operator proving the one-to-one correspondence between

  $$u \leftrightarrow \gamma_S(u). \tag{A18}$$

  Yet the characterization of this space of traces (that is larger than the space of distributions $\mathcal{D}(S)^*$) goes beyond the scope of this paper where only traces in $L^2(S)$ or $H^{1,2}(S)$ are needed. Similar, but not identical, results are known in mathematical literature (see Shlapunov 2002).

Fundamental is to understand what happens when $u_n(\sigma)$ is a bounded sequence in L^2_σ or in $L^2(S)$. First of all, the two norms are equivalent because

$$\begin{aligned} \| u|_S \|^2_{L^2_\sigma} &= \int u^2(R_\sigma, \sigma)\mathrm{d}\sigma; \\ \| u|_S \|^2_{L^2(S)} &= \int u^2(R_\sigma, \sigma)\mathrm{d}S \end{aligned} \tag{A19}$$

and the area element $\mathrm{d}S$ can be written

$$\mathrm{d}S = 4\pi J R^2_\sigma \mathrm{d}\sigma \tag{A20}$$

with $J = (\cos\vartheta)^{-1}$, $\cos\vartheta = \boldsymbol{n} \cdot \boldsymbol{e}_r$ and $\boldsymbol{n}$ the normal unit vector at $\mathrm{d}S$. Let us recall that requiring that S is star-shaped and Lipschitz in the variable σ, which means that

$$S \equiv \{r = R_\sigma\};\ |R_\sigma - R_{\sigma'}| \leq K\psi_{\sigma\sigma'} \\ (\cos\psi_{\sigma\sigma'} = \boldsymbol{e}_\sigma \cdot \boldsymbol{e}_{\sigma'}), \tag{A21}$$

for some constant K.

Whence,

$$|\nabla_\sigma R_\sigma| \leq K \tag{A22}$$

for almost all σ. Since

$$\cos\vartheta = \frac{1}{\sqrt{1 + tg^2\vartheta}} = \frac{1}{\sqrt{1 + \frac{|\nabla_\sigma R_\sigma|^2}{R_\sigma^2}}} \tag{A23}$$

recalling (A22), one has

$$R_\sigma^2 \leq J R_\sigma^2 \leq R_\sigma\sqrt{R_\sigma^2 + K^2} \tag{A24}$$

that further used with (A20) gives

$$4\pi R_-^2 \mathrm{d}\sigma \leq \mathrm{d}S \leq 4\pi R_+\sqrt{R_+^2 + k^2}\mathrm{d}\sigma. \tag{A25}$$

This proves the equivalence of the two norms in (A19) or better the equivalence with any other norm

$$\| u|_S \|_W^2 = \int u^2(R_\sigma, \sigma) w(\sigma) \mathrm{d}\sigma \tag{A26}$$

with the weight function $w(\sigma)$ positively bounded above and below.

As a matter of fact, one can prove that if $\{u_m(\sigma)\}$, given by (A16), is bounded in L_σ^2 it is also convergent in this space and one can put

$$\gamma_S(u) = u(R_\sigma, \sigma) = \lim_{m\to\infty} u_m(\sigma). \tag{A27}$$

(see Theorem 8, Corollary 3 in Sansò and Sideris (2013) §13.5, or Cimmino (1955)).

So we can consider the subspace of $\mathcal{H}(\Omega)$ of all u endowed with an L_σ^2 trace on S. One can show that this is a Hilbert space under the scalar product

$$< u, v >_{L_\sigma^2} = \int \gamma_S(u)\gamma_S(v)\mathrm{d}\sigma. \tag{A28}$$

In mathematical literature, this is called a *Hardy Space*, h_2, (see Axler et al. 1992) and it is indeed isometric with L_σ^2. To be useful to the development of this paper, we will slightly change the norm definition in h_2 with the equivalent norm (see (A26))

$$\| u \|_{h_2}^2 \equiv \int \gamma_S(u)^2 R_\sigma \mathrm{d}\sigma. \tag{A29}$$

Finally, since as discussed in the paper the data function $f(\sigma)$ needs to be the trace of a function $v \in \mathcal{H}^{(1)}(\Omega)$, namely missing zero and first degree harmonics, we can introduce the data space H_0 as a subspace of h_2, namely

$$H_0 \equiv \{v \in \mathcal{H}^{(1)}(\Omega);\ u_{+jm}(v) = 0\ j \leq 1; \\ \int \gamma_S(v)^2 R_\sigma \mathrm{d}\sigma < +\infty\}. \tag{A30}$$

Clearly H_0 is a proper subspace of h_2 and therefore a Hilbert space itself, with norm given by (A30). It is easy to see, by using a Green's function representation as in Sansò and Sideris (2013), §13.5, that in h_2 and then in H_0 too, for every compact $K \subset \Omega$ one has

$$\sup_K |u(x)| \leq c' \| u \|_{h_2} \leq c \| u \|_0 \tag{A31}$$

for suitable constants c', c. It follows, by taking $K = S_+$, that all the harmonic coefficients

$$\forall n > 1, |m| \leq n,\ u_{+nm}(u) = \int u(R_+, \sigma) Y_{nm}(\sigma)\mathrm{d}\sigma \tag{A32}$$

are continuous linear functionals of u in H_0. So by the Riesz theorem, there is a sequence $\{Z_{nm}\} \subset H_0$ such that

$$u_{+nm}(u) =< Z_{nm}, u >_0 . \tag{A33}$$

Let us observe that, by our definition, the functions $S_{nm}(r, \sigma) = (R_+/r)^{n+1} Y_{nm}(\sigma)$ have traces on S, $S_{nm}(R_\sigma, \sigma) = (R_+/R_\sigma)^{n+1} Y_{nm}(\sigma)$. It is then obvious that, since R_σ is positively bounded below and above, $S_{nm} \in H_0$, for $n > 1$. It is well known that $\{S_{nm}\}$ is not orthonormal in H_0, unless S is itself a sphere. However, it is proved in Sansò and Sideris (2013) that $\{S_{nm}\}$ are always linearly independent and complete, or total, in H_0.

It is enough to take $u = S_{nm}(r, \sigma)$ in (A33) to realize that

$$< Z_{nm}, S_{jk} >_0 = \delta_{uj}\delta_{mk}, \tag{A34}$$

namely $\{Z_{nm}, S_{jk}\}$ is a bi-orthogonal system. Hence, $\{Z_{nm}\}$ is a system of linearly independent functions, complete too in H_0. Relevant to the matter of this paper is the following decomposition: Let us call

$$V_L = span\{S_{\ell m};\ 2 \leq \ell \leq L\} \tag{A35}$$

$$W_L = span\{Z_{\ell m};\ 2 \leq \ell \leq L\}; \tag{A36}$$

then, we can write, $\forall v \in H_0$,

$$\begin{cases} v = v_L + v^L \\ v_L = \sum_{\ell=2}^{L} < v, Z_{\ell m} > S_{\ell m} \in V_L \\ v_L = v - v_L \in W_L^\perp; \end{cases} \tag{A37}$$

the last of (A37) comes from the bi-orthogonality relation (A34), namely

$$2 \le j \ \le L, < Z_{jk}, v^L >=< Z_{jk}, v > - < Z_{jk}, v_L >= 0 . \tag{A38}$$

Therefore, we have

$$v^L \in W_L^{\perp} \Rightarrow v^L = O\left(\frac{1}{r^{L+2}}\right), \tag{A39}$$

according to Proposition 3.

Now we have to define still another Hilbert space of harmonic functions, where we want to look for a solution of our boundary value problem. Namely we define a space H_1 as

$$H_1 \equiv \{u, u \in H_0;\ \| \ u\ \|_1^2 = L = \| \ R_\sigma |\nabla u|\ \|_0^2 = \int |\nabla u|^2 R_\sigma^3 \mathrm{d}\sigma \ < +\infty\}. \tag{A40}$$

We have to prove that:

a) the $\| \ u \ \|_1$ in (A40) is a true norm, i.e strictly positive when $u \neq 0$,
b) that H_1 has a stronger topology than that in H_0, namely that

$$\| \ u \ \|_0 \le c \ \| \ u \ \|_1 \tag{A41}$$

for a suitable constant c.

The point a) is trivial because

$$u \in H_0,\ \| \ u \ \|_1 = 0 \Rightarrow R_\sigma |\nabla u||_S = 0 \Rightarrow u_n|_S = 0, \tag{A42}$$

with $u_n|_S$ the normal derivative at the boundary, and the homogeneous exterior Neumann problem has only the trivial solution $u = 0$ in spaces of harmonic functions regular at infinity. In our case, as a matter of fact $u = O\left(\frac{1}{r^3}\right)$.

As for point b) a standard reasoning runs as follows: From

$$u(R, \sigma) = -\int_{R_\sigma}^{+\infty} u'(s, \sigma)\mathrm{d}s \equiv -\int_{R_\sigma}^{+\infty} su' \frac{1}{s}\mathrm{d}s, \tag{A43}$$

noticing that $su' \cong O\left(\frac{1}{s^3}\right)$ and is therefore square integrable in ds, we get by Schwartz inequality

$$|u(r, \sigma)|^2 \le \frac{1}{R_\sigma} \int_{R_\sigma}^{+\infty} u'^2 s^2 \mathrm{d}s. \tag{A44}$$

We multiply (A44) by R_σ, integrate in dσ, apply Gauss theorem and recall (A20), to find

$$\begin{aligned} \| \ u \ \|_0^2 &\le \int_\Omega u'^2 \mathrm{d}\Omega \le \int_\Omega |\nabla u|^2 \mathrm{d}\Omega \\ &\equiv -\int_S u u_n \mathrm{d}S \le 4\pi J_+ \int |u||u_n| R_\sigma^2 \mathrm{d}\sigma \\ &\le 4\pi R_+ J_+ \ \| \ u \ \|_0 \| \ R_\sigma |u_n| \ \|_0 \\ &\le 4\pi R_+ J_+ \ \| \ u \ \|_0 \| \ u \ \|_1 \ . \end{aligned} \tag{A45}$$

Simplifying (A45) by $\| \ u \ \|_0$, we get (A41) with $c = 4\pi J_+$. A much better inequality is discussed in Sect. 3, Lemma 2 if we assume that $u \in W_L^{\perp}$.

Remark 6 One might wonder whether H_1 so defined is the same as the Sobolev space $H^{1,2}(S)$ with norm

$$\| \ u \ \|_{H^{1,2}(S)}^2 = \int_S [u^2(R_\sigma, \sigma) + |\nabla_\sigma u(R_\sigma, \sigma)|^2] \mathrm{d}S; \tag{A46}$$

note that this is one of the possible many equivalent norms in $H^{1,2}(S)$. The answer is that in reality

$$H_1 \subset H^{1,2}(S); \tag{A47}$$

in fact one can prove that the norm

$$\| \ \tilde{u} \ \|_1^2 = \| \ u \ \|_{H^{1,2}(S)}^2 + \| \ R_\sigma u' \ \|_0^2 \tag{A48}$$

is equivalent to $\| \ u \ \|_1$; whence, (A47) immediately follows.

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