



Investigating support by minimal detectable displacement in confidence region determination and significance test of displacements

Witold Prószyński¹ · Sławomir Łapiński¹

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Abstract

The Minimal Detectable Displacement (MDD) is an important measure of monitoring networks sensitivity to displacements. In addition to the accuracy criteria, it is used as a detectability criterion in the optimal design of such networks. The paper examines whether the MDD provides grounds for verifying the correctness of the confidence, and the significance thresholds applied in the analyses of the determined displacements. According to our knowledge, the task so formulated has not yet been the subject of research presented in the literature in the field of geodetic determination of displacements. Hence, the approach presented here can be regarded as a new proposal extending the application area of the MDD. The investigations are focused on a probabilistic aspect of combining confidence and detectability as well as significance and detectability by the superimposition of the corresponding ellipsoids and their joint analysis. An initial research result is the diagrams showing a significance index and a non-centrality parameter as functions of the rank of the covariance matrix for displacements and also of system redundancy for specified values of Type I and Type II error probabilities. The diagrams, together with the theoretical basis created within the research, made it possible to analyse and evaluate the support by Minimal Detectable Displacement in confidence region determination and significance test of displacements. Based on the analysis of MDD support, two options of modifying the confidence and significance thresholds related to single point displacements are proposed for practical use.

Keywords Ellipsoids superimposition · Error-limit determination · Significance test · MDD support evaluation · Thresholds modification

1 Introduction

The transfer to displacement monitoring of the concept of Minimal Detectable Bias (MDB) introduced by Baarda (1968) was made by Pelzer (1972). The term *network sensitivity with minimal detectable displacement* (MDD) as its measure was introduced and later became an element of network analysis together with accuracy and reliability (Niemeier 1982; Niemeier et al. 1982). The concept of network sensitivity to displacements was elaborated in detail and

extended onto deformation models by Heck (1986). In that paper, the MDD and a sensitivity ellipse for a single network point were first proposed. Some important findings concerning the theory of network sensitivity and datum invariance can be found in, e.g., Even-Tzur (2002, 2010), Xu (1995, 1997). The need for inserting the sensitivity criteria into the network optimization procedures is formulated in (Kuang 1991; Yazji 1997; Alizadeh-Khameneh et al. 2015).

The MDD concept is still commonly used in a priori analyses of monitoring network accuracy and sensitivity. The results of such joint analyses for horizontal networks are presented in the form of a confidence ellipse and a sensitivity ellipse for each network point. However, except for comparing the sizes, no other relationships between these ellipses are considered. Such a limited approach is most certainly due to the lack of a relevant theoretical basis. To our knowledge, in significance tests of computed displacements, the information contained in the sensitivity characteristics has

✉ Witold Prószyński
witold.proszynski@pw.edu.pl

Sławomir Łapiński
slawomir.lapinski@pw.edu.pl

¹ Faculty of Geodesy and Cartography, Warsaw University of Technology, Pl. Politechniki 1, 00-661 Warsaw, Poland

not been taken into account as yet. Hence, the need to examine the possibility of using MDD as a source of supporting information both in a priori analyses of monitoring networks accuracy and in significance tests of displacements. If successful, that might extend the range of applications of MDD.

To respond to the above-mentioned need, the objectives of the present paper are the following:

- to create a theoretical basis that would enable one to investigate the extent of MDD support in the analyses of monitoring network accuracy and also in significance tests of the computed displacements,
- to examine the possibility of modifying the above-mentioned procedures so as to use the supporting information contained in the MDD.

It is necessary to emphasize that the paper concerns only the models with parameters being network points displacements. It does not deal with the models containing also strain parameters as used in network deformation analysis.

2 Brief review of basic concepts and notation

The review concentrates on minimum-constraint datum definition as commonly used in the monitoring of displacements and the measures of network sensitivity to displacements.

2.1 Minimum-constraint datum definitions

Let us consider a displacement monitoring model based on observation differences from two measurement campaigns

$$\mathbf{A} \cdot \mathbf{d} = \Delta \mathbf{I} + \mathbf{v}_{\Delta \mathbf{I}}, \quad \mathbf{C}_{\Delta \mathbf{I}} \tag{1a}$$

$$\mathbf{S} \cdot \mathbf{d} = \mathbf{0} \tag{1b}$$

where $\mathbf{A}(n \times u)$ —design matrix, $\text{rank } \mathbf{A} = u - d$ (d —network defect), $\mathbf{d}(u \times 1)$ —vector of displacements, $\Delta \mathbf{I}(n \times 1)$ —vector of observation differences, $\mathbf{v}_{\Delta \mathbf{I}}(n \times 1)$ —vector of random errors in $\Delta \mathbf{I}$ (with opposite sign), $\mathbf{C}_{\Delta \mathbf{I}}(n \times n)$ —covariance matrix for $\Delta \mathbf{I}$ (positive definite), $\mathbf{S}(d \times u)$ —coefficient matrix in datum constraints.

We shall denote by \mathbf{A}_s and $(\Delta \mathbf{I})_s$ the elements of the standardized model (1) corresponding to \mathbf{A} and $\Delta \mathbf{I}$, respectively.

The models as in (1) having point displacements as parameters are basically equivalent to those with point velocities.

The matrix \mathbf{S} has the following properties

$$\text{rank} \begin{bmatrix} \mathbf{A} \\ \mathbf{S} \end{bmatrix} = u, \quad \text{rank } \mathbf{S} = d \tag{2}$$

The minimum-constraint datum definition eliminates the network defect without causing observation distortions. There are several types of such constraints, e.g., free net constraints.

Below, we present the least squares (LS) solution for the standardized model (1), distinguishing the following two ways in which the result of the global model test can be used in defining the covariance matrix for displacements

- (i) using the a priori variance factor $\sigma_0^2 = 1$, where σ_0^2 is a variance of the standardized observation difference $(\Delta \mathbf{I})_{s,i}$ ($i = 1, \dots, n$);

$$\mathbf{C}_{\hat{\mathbf{d}}} = \sigma_0^2 \mathbf{Q}_{\hat{\mathbf{d}}} = \mathbf{Q}_{\hat{\mathbf{d}}} \tag{3}$$

- (ii) using the estimator (unbiased) of σ_0^2 , i.e., $\hat{\sigma}_0^2$, termed the a posteriori variance factor;

$$\hat{\mathbf{C}}_{\hat{\mathbf{d}}} = \hat{\sigma}_0^2 \mathbf{Q}_{\hat{\mathbf{d}}} \tag{4}$$

$$\text{Ad (i)} \quad \hat{\mathbf{d}} = (\mathbf{A}_s^T \mathbf{A}_s)_s^- \mathbf{A}_s^T (\Delta \mathbf{I})_s; \quad \mathbf{C}_{\hat{\mathbf{d}}} = (\mathbf{A}_s^T \mathbf{A}_s)_s^- \quad \text{rank } \mathbf{C}_{\hat{\mathbf{d}}} = u - d \tag{5}$$

for free net constraints $\mathbf{C}_{\hat{\mathbf{d}}} = (\mathbf{A}_s^T \mathbf{A}_s)_s^- \equiv (\mathbf{A}_s^T \mathbf{A}_s)^+$

$$\text{Ad (ii)} \quad \hat{\mathbf{d}} = (\mathbf{A}_s^T \mathbf{A}_s)_s^- \mathbf{A}_s^T (\Delta \mathbf{I})_s; \quad \hat{\mathbf{C}}_{\hat{\mathbf{d}}} = \hat{\sigma}_0^2 (\mathbf{A}_s^T \mathbf{A}_s)_s^-, \quad \text{rank } \hat{\mathbf{C}}_{\hat{\mathbf{d}}} = \text{rank } \mathbf{C}_{\hat{\mathbf{d}}} \tag{6}$$

In the above formulas, $(\mathbf{A}_s^T \mathbf{A}_s)_s^-$ is a reflexive g -inverse of $\mathbf{A}_s^T \mathbf{A}_s$, such that $\mathbf{S} \cdot (\mathbf{A}_s^T \mathbf{A}_s)_s^- = \mathbf{0}$, $(\mathbf{A}_s^T \mathbf{A}_s)^+$ is a pseudoinverse.

Also known is another datum definition, termed *over-constraint datum* (where $\mathbf{S}(w \times u)$, $\text{rank } \mathbf{S} = w$, $w > d$), fixing some network points. It eliminates the network defect but causes observation distortions. Its use is bounded by the restrictions on mutual displacements of the points being fixed and the types of measured quantities.

2.2 Network sensitivity to displacements

The measures of sensitivity to displacements are derived on the basis of hypotheses testing

$$H_o : \mathbb{E}(\hat{\mathbf{d}}) = \mathbf{0}$$

$$H_a : \mathbb{E}(\hat{\mathbf{d}}) \neq \mathbf{0}$$

where $\hat{\mathbf{d}}$ is the LS estimator of the vector of displacements as in (5) and (6).

If the covariance matrix $\mathbf{C}_{\hat{\mathbf{d}}}$ as in (3) is used, the test statistic is

$$\Phi = \hat{\mathbf{d}}^T \mathbf{C}_d^+ \hat{\mathbf{d}} \tag{7}$$

- under H_0 , Φ has a central χ_h^2 distribution, h is a rank of \mathbf{C}_d^+ ,
- under H_a , Φ has a non-central χ_h^2 distribution, with a non-centrality parameter

$$\lambda = \hat{\mathbf{d}}^T \mathbf{C}_d^+ \hat{\mathbf{d}} \tag{8}$$

A vector $\hat{\mathbf{d}}$ is considered detectable if $\lambda > \lambda_{h,\alpha_o,\beta_o}$, where $\lambda_{h,\alpha_o,\beta_o}$ is a critical value of non-centrality parameter determined for the specified values of Type I and Type II error probabilities α_o and β_o .

If the covariance matrix $\hat{\mathbf{C}}_d$ as in (4) is used, the test statistic is

$$\Phi_F = \frac{\hat{\mathbf{d}}^T \mathbf{Q}_d^+ \hat{\mathbf{d}}}{h \cdot \hat{\sigma}_o^2} \tag{9}$$

where \mathbf{Q}_d is a cofactor matrix, h is a rank of \mathbf{Q}_d , $\hat{\sigma}_o^2$ is the a posteriori variance factor

- under H_0 , Φ_F has a central $F_{h,r}$ distribution (r is a model redundancy; $r = n - u + d$),
- under H_a , Φ_F has a non-central $F_{h,r}$ distribution, with a non-centrality parameter

$$\lambda_F = \frac{\hat{\mathbf{d}}^T \mathbf{Q}_d^+ \hat{\mathbf{d}}}{\hat{\sigma}_o^2} \tag{10}$$

A critical value of the non-centrality parameter is denoted here by $\lambda_{h,r,\alpha_o,\beta_o}$. A vector $\hat{\mathbf{d}}$ is considered detectable if $\lambda_F > \lambda_{h,r,\alpha_o,\beta_o}$, where α_o and β_o as above.

For a single point vector with a non-singular covariance matrix, h is a number of the vector components.

On the basis of the formulas (8) and (10), we can write the equations of h —dimensional sensitivity ellipsoids for the case of \mathbf{C}_d^+ and $\hat{\mathbf{C}}_d$, respectively, i.e.,

$$\mathbf{d}^T \mathbf{C}_d^+ \mathbf{d} = \lambda_{h,\alpha_o,\beta_o} \tag{11}$$

$$\frac{\mathbf{d}^T \mathbf{Q}_d^+ \mathbf{d}}{\hat{\sigma}_o^2} = \lambda_{h,r,\alpha_o,\beta_o} \tag{12}$$

Based on (11) and (12), we get the commonly used sensitivity measure as Minimal Detectable Displacement (MDD), correspondingly for the case of \mathbf{C}_d^+ and $\hat{\mathbf{C}}_d$ (Niemeier 1982, 1985)

$$\|\mathbf{d}\|_{\min} = \sqrt{\frac{\lambda_{h,\alpha_o,\beta_o}}{\Lambda_{\max}}} \quad \|\mathbf{d}\|_{\min} = \hat{\sigma}_o \sqrt{\frac{\lambda_{h,r,\alpha_o,\beta_o}}{\Lambda_{\max}}} \tag{13}$$

where Λ_{\max} is a maximal eigenvalue of the matrix \mathbf{C}_d^+ and \mathbf{Q}_d^+ , $\hat{\sigma}_o$ is a square root of the a posteriori variance factor.

According to (Hsu and Hsiao 2002), both the minimal and the maximal value of MDD should be taken into consideration. The maximal value is based on a minimal eigenvalue Λ_{\min} .

Datum invariance of the sensitivity quadratic form as in (7) is an important issue in the subject of the paper since the MDD defined on its basis is applied in different types of datum definition. It is known that the form is invariant to the choice of the type of minimum-constraint datum definition. A concise proof, complementary to the existing proofs (e.g., Even-Tzur 2010), is presented in “Appendix A” section. It covers all possible types of datum definition.

A non-centrality parameter for a test statistic (10) in the case of the model (1) with weights of observation differences is considered in “Appendix B” section.

3 Investigating MDD support in accuracy analysis and in the testing of displacements

In investigations, we operate with h -dimensional displacement vectors ($h \geq 1$). For simplicity, the corresponding regions of confidence, significance and sensitivity, being line segments ($h = 1$), ellipses ($h = 2$), ellipsoids ($h = 3$) and hyper-ellipsoids ($h > 3$), will be termed for all h the ellipsoids, e.g., (Pope 2008).

The proposal covers the following two task cases:

- Case I Confidence region determination supported by network sensitivity characteristics (variance factor option (i))
- Case II Significance test of computed displacements supported by network sensitivity characteristics (variance factor option (i) and (ii)).

As a final result, the extent of the support is evaluated.

3.1 Creating of the theoretical basis

3.1.1 Variance factor option (i)

A property of the test statistic $\Phi = \hat{\mathbf{d}}^T \mathbf{C}_d^+ \hat{\mathbf{d}}$ under H_0 and H_a , i.e., $\Phi = \lambda$ for $h \geq 1$ (see (7), (8)), as expressed in terms of the corresponding values of Φ and λ (i.e., u and λ , respectively), means the following equality

$$u_{h,\alpha} = \lambda_{h,\alpha,\beta} \tag{14}$$

For a given α , we get β from the above equality. We will say that such β is coordinated with a given α . Assuming an arbitrary value of β not coordinated with a given α and denoted by $\bar{\beta}$ will result in $u_{h,\alpha} \neq \lambda_{h,\alpha,\bar{\beta}}$.

Let critical values of Φ and λ be denoted by u_o and λ_o , where $u_o = u_{h,\alpha_o}$, $\lambda_o = \lambda_{h,\alpha_o,\beta_o}$; α_o and β_o are the values of α and β assumed in u_o and λ_o , respectively. Putting, e.g., $\alpha_o = 0.05$ and $\beta_o = 0.20$, we get from (14) $u_o \neq \lambda_o$, and hence, we can see that $\beta_o = 0.20$ is not coordinated with $\alpha_o = 0.05$. The value of β coordinated with $\alpha_o = 0.05$ will be denoted by $\bar{\beta}$.

The investigation of MDD support requires knowledge on the relationship between the sizes and probability features of the following three concentric and similar h -dimensional ellipsoids, such as sensitivity ellipsoid (E_{sen}), confidence ellipsoid (E_{con}) (also termed as a limit-error ellipsoid) and significance ellipsoid (E_{sig}).

Let us consider equations of these ellipsoids, referring to the expression for a test statistic Φ as in (7). The components of the vector \mathbf{d} will be treated here as algebraic variables.

$$E_{sig} \quad \mathbf{d}^T \mathbf{C}_d^+ \mathbf{d} = u_o; u_o = u_{h,\alpha}; \alpha_o \text{—significance level} \tag{15}$$

u_o —a critical value of the variable Φ such that $P\{\Phi > u_{h,\alpha_o}\} = \alpha_o$

$$E_{con} \quad \mathbf{d}^T \mathbf{C}_d^+ \mathbf{d} = u_{o(\gamma)}; u_{o(\gamma)} = u_{h,\gamma_o}; \gamma_o \text{—confidence level} \tag{16}$$

$u_{o(\gamma)}$ —a critical value of the variable Φ such that $P\{\Phi < u_{h,\gamma_o}\} = \gamma_o$

$$E_{sen} \quad \mathbf{d}^T \mathbf{C}_d^+ \mathbf{d} = \lambda_o; \lambda_o = \lambda_{h,\alpha_o,\beta_o} \text{ (see (11))}$$

λ_o —a critical value of a non-centrality parameter of the variable Φ .

E_{sen} represents a detectability threshold for displacements. It is easy to prove that every vector that satisfies Eq. (11) is an MDD in this direction.

Putting $\gamma_o = 1 - \alpha_o$ in (16), we get $u_{h,\gamma_o} = u_{h,\alpha_o}$ which reduces the number of sizes to be analyzed to two, the corresponding parameters being u_{h,α_o} and $\lambda_{h,\alpha_o,\beta_o}$.

The values of u_{h,α_o} were computed iteratively on the basis of the arguments of cumulative distribution function (CDF) for a χ^2_h —distributed random variable. For the computation of $\lambda_{h,\alpha_o,\beta_o}$ use was made of the efficient and sufficiently accurate numerical program developed by Aydin and Demirel (2005) according to the algorithm and the associated theory given in (Gaida and Koch 1985). The results of the computation are presented in Table 1 and Fig. 1.

The symbol $h^*_{\alpha_o,\beta_o}$ in Table 1 denotes a crossing point of the curves $u_o(h)$ and $\lambda_o(h)$ (see Fig. 1).

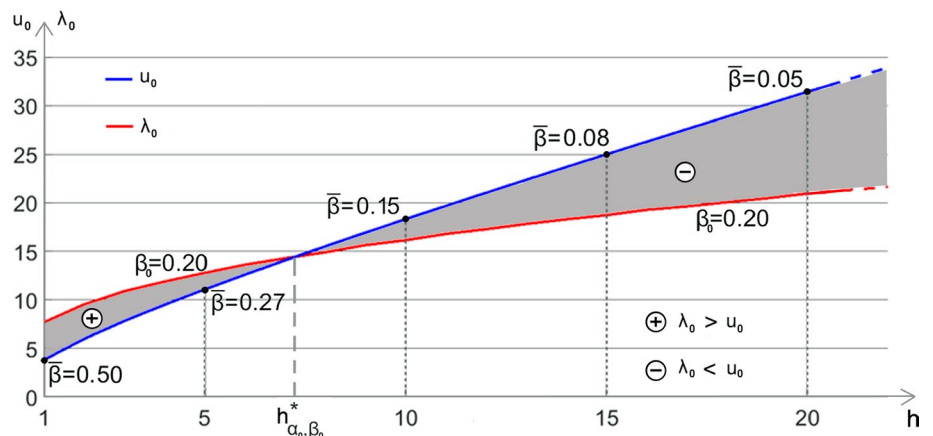
We can see in Fig. 1 that with the increase in h , both the curves increase monotonically, the former one at a higher rate.

At a certain value of h , denoted as $h^*_{\alpha_o,\beta_o}$ (being equal to 7.3), these curves cross each other. For $h < h^*_{\alpha_o,\beta_o}$, we have $u_o < \lambda_o$, whereas for $h > h^*_{\alpha_o,\beta_o}$ we have $u_o > \lambda_o$. For a given h , the symbol $\bar{\beta}$ denotes the value of β coordinated with $\alpha_o = 0.05$. For $u_o < \lambda_o$, we have $\bar{\beta} > \beta_o$ and for $u_o > \lambda_o$, $\bar{\beta} < \beta_o$.

Table 1 Values of u_o and λ_o for different h ($\alpha_o = 0.05$, $\beta_o = 0.20$); $h^*_{\alpha_o,\beta_o} = 7.3$

h	1	2	3	4	5	6	7	8	9	10	20	50
u_o	3.8	6.0	7.8	9.5	11.1	12.6	14.1	15.5	16.9	18.3	31.4	67.5
λ_o	7.8	9.6	10.9	11.9	12.8	13.6	14.4	15.0	15.6	16.2	21.0	30.2

Fig. 1 Parameters u_o and λ_o as functions of h ; $\alpha_o = 0.05$, $\beta_o = 0.20$



We can also notice that the values of $\bar{\beta}$ decrease with the increase in h , which is consistent with the findings in (Aydin 2011) focused on the power of the test.

The properties as presented above for $\alpha_0 = 0.05, \beta_0 = 0.20$ can be observed for other values of α_0 and β_0 .

For single network points, we have $h < h_{\alpha_0, \beta_0}^*$, so the size of E_{sen} is greater than that of E_{sig} and E_{con} (with $\gamma_0 = 1 - \alpha_0$). For networks taken as a whole or for the subsets of their points, such that $h > h_{\alpha_0, \beta_0}^*$, we have the opposite situation.

3.1.2 Variance factor option (ii)

From the formulas (9) and (10), it follows that the property analogous to that in the variance factor option (i), i.e., $\Phi = \lambda$, does not hold directly for Φ_F . We can get this effect by introducing a rescaled variable Φ_F , defined by

$$\Psi = h \cdot \Phi_F \tag{17}$$

Hence,

$$\Psi = \frac{\hat{\mathbf{d}}^T \mathbf{Q}_d^+ \hat{\mathbf{d}}}{\hat{\sigma}_0^2} \tag{18}$$

The values of Ψ will be denoted by “ w ”.

The variable does not have an identifiable probability distribution but has a property useful for the present derivations, i.e.,

Property *If $r \rightarrow \infty$, the probability distribution of the variable $\Psi \rightarrow \chi_h^2$ (the property is well known in mathematical statistics).*

So, since with $r \rightarrow \infty \hat{\sigma}_0^2 \rightarrow \sigma_0^2$, the variable $\Psi \rightarrow \frac{\hat{\mathbf{d}}^T \mathbf{Q}_d^+ \hat{\mathbf{d}}}{\sigma_0^2} = \hat{\mathbf{d}}^T \mathbf{C}_d^+ \hat{\mathbf{d}} \equiv \Phi$.

As was assumed in Sect. 2.1, $\sigma_0^2 = 1$.

Now, we find the critical value of Φ for a given significance level α . Denoting the values of Φ_F by “ z ”, we shall write

$$P\{\Phi_F > z_{h,r,\alpha}\} = \alpha$$

Substituting (17) and duly modifying the internal inequality, we obtain finally

$$P\{\Psi > w_{h,r,\alpha}\} = \alpha \tag{19}$$

where $w_{h,r,\alpha} = h \cdot z_{h,r,\alpha}$.

The variable Ψ , being compatible with Φ as regards properties and scale, satisfies the equality $\Psi = \lambda_F$ for $h \geq 1$ and $r \geq 1$. So, the values of Ψ and λ_F , i.e., w and λ_F are bound by the equality

$$w_{h,r,\alpha} = \lambda_{h,r,\alpha,\beta} \tag{20}$$

In analogy to (14), β is coordinated with a given value of α .

Let critical values of Ψ and λ_F be denoted by w_0 and $\lambda_{F,0}$, where $w_0 = w_{h,r,\alpha_0}, \lambda_{F,0} = \lambda_{h,r,\alpha_0,\beta_0}$ ($\alpha_0 = 0.05, \beta_0 = 0.20$). We may check that $w_0 \neq \lambda_{F,0}$, and hence, $\beta_0 = 0.20$ is not coordinated with $\alpha_0 = 0.05$. The value of β coordinated with $\alpha_0 = 0.05$ will be denoted by $\bar{\beta}$ (for simplicity of notation, the same symbol as for variance factor option (i) is used).

For Case II (variance factor option (ii)), we need to acquire knowledge on the relationship between the sizes of the corresponding significance ellipsoid and the sensitivity ellipsoid. We shall denote these ellipsoids by \hat{E}_{sig} and \hat{E}_{sen} , respectively. We write their equations without additional explanations

$$\hat{E}_{sig} \frac{\mathbf{d}^T \mathbf{Q}_d^+ \mathbf{d}}{\hat{\sigma}_0^2} = w_0; \quad w_0 = w_{h,r,\alpha_0} \tag{21}$$

$$\hat{E}_{sen} \frac{\mathbf{d}^T \mathbf{Q}_d^+ \mathbf{d}}{\hat{\sigma}_0^2} = \lambda_{F,0}; \quad \lambda_{F,0} = \lambda_{h,r,\alpha_0,\beta_0} \tag{22}$$

To the above-mentioned property of Ψ concerned with r increasing to infinity, we may add the following

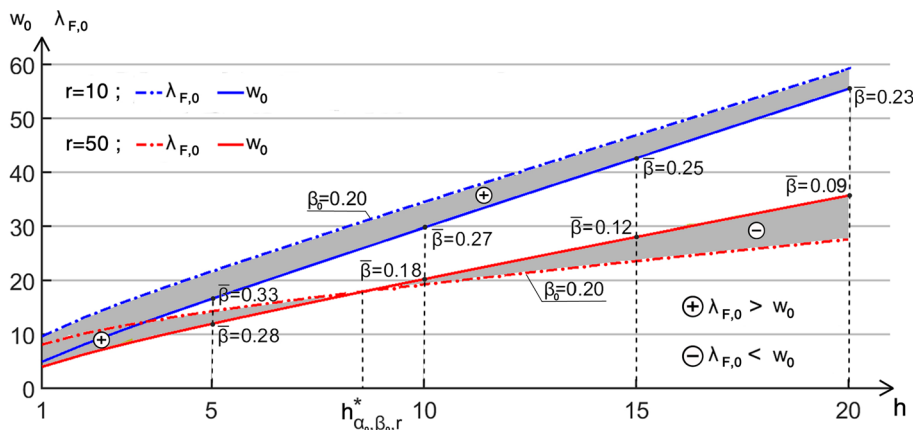
- since with $r \rightarrow \infty w_{h,r,\alpha} \rightarrow u_{h,\alpha}$ and $\lambda_{h,r,\alpha,\beta} \rightarrow \lambda_{h,\alpha,\beta}$, so $\hat{E}_{sig} \rightarrow E_{sig}$ and $\hat{E}_{sen} \rightarrow E_{sen}$.

Table 2 Values of w_0 and $\lambda_{F,0}$ for different h and r ($\alpha_0 = 0.05, \beta_0 = 0.20$); $h_{\alpha_0, \beta_0, r}^*$

r	h	1	2	3	4	5	6	7	8	9	10	20	50	$h_{\alpha_0, \beta_0, r}^*$
5	w_0	6.6	11.6	16.2	20.8	25.3	29.7	34.1	38.5	43.0	47.4	91.2	222.2	x
	$\lambda_{F,0}$	12.3	18.7	24.5	30.1	35.7	41.2	46.7	52.1	57.5	62.9	116.8	277.9	
10	w_0	5.0	8.2	11.1	13.9	16.6	19.3	21.9	24.6	27.2	29.8	55.5	131.9	49.9
	$\lambda_{F,0}$	9.7	13.2	16.2	19.0	21.7	24.3	26.9	29.5	32.0	34.5	59.1	131.8	
20	w_0	4.4	7.0	9.3	11.5	13.6	15.6	17.6	19.6	21.5	23.5	42.5	98.3	12.0
	$\lambda_{F,0}$	8.7	11.2	13.3	15.1	16.8	18.4	20.0	21.5	23.0	24.4	38.4	78.4	
100	w_0	3.9	6.2	8.1	9.9	11.5	13.1	14.7	16.3	17.8	19.3	33.5	73.9	7.9
	$\lambda_{F,0}$	8.0	9.9	11.3	12.5	13.6	14.5	15.4	16.2	17.0	17.8	24.3	40.1	

x—value difficult to determine

Fig. 2 Parameters w_0 and $\lambda_{F,0}$ as functions of h , for $r=10$ and $r=50$ ($\alpha_0=0.05, \beta_0=0.20$)



The values w_0 were obtained by rescaling the values z_{h,r,α_0} , computed iteratively based on the arguments of CDF for $F_{h,r}$ —distributed variable. Like the values of λ_0 in option (i), the values of $\lambda_{F,0}$ were computed with the use of a numerical program developed by Aydin (2011). The results of the computation are presented in Table 2 and Fig. 2.

The symbol $h^*_{\alpha_0, \beta_0, r}$ in Table 2 denotes a crossing point of the curves $w_0(h)$ and $\lambda_{F,0}(h)$ for some r (see Fig. 2).

Figure 2 presents the curves $w_0(h)$ and $\lambda_{F,0}(h)$ for $r=10$ and $r=50$. The curves are generally similar to their equivalents in Fig. 1. We can see in Table 2 and Fig. 2 that the values of h where these curves intersect (denoted here as $h^*_{\alpha_0, \beta_0, r}$) increase with the decrease in r . For $r=10$, the intersection of the curves falls outside the diagram.

Interpretation of the cases $h < h^*_{\alpha_0, \beta_0, r}$ and $h > h^*_{\alpha_0, \beta_0, r}$ is like that given for option (i). The values of $\bar{\beta}$ decrease with the increase in h .

3.2 Analysis for Case I

3.2.1 Variance factor option (i)

Usually, both the confidence ellipse and the sensitivity ellipse are presented together as a result of network quality analysis. We provide the description of characteristic displacement vectors and some probabilistic features of each of the ellipsoids (here—ellipsoids).

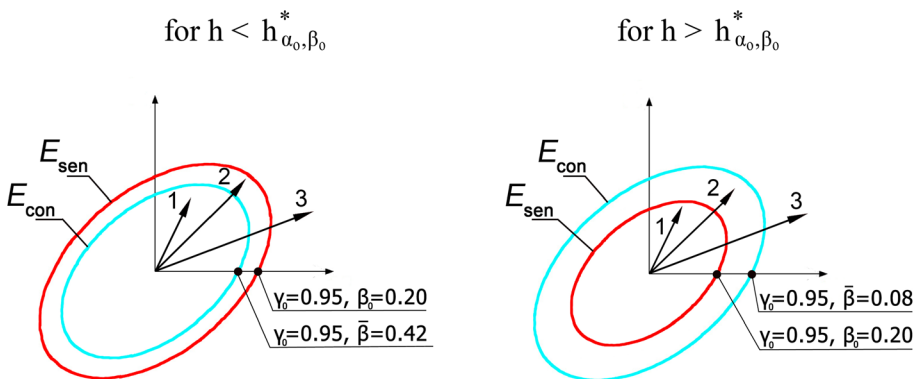
The values of $\bar{\beta}$ in Fig. 3 are computed for $h=2$ ($h < h^*_{\alpha_0, \beta_0}$), $h=15$ ($h > h^*_{\alpha_0, \beta_0}$).

Since we consider the a priori accuracy analysis, the vectors 1, 2, 3 represent potential (not determined) displacement vectors.

Deterministic description of the characteristic vectors:

for $h < h^*_{\alpha_0, \beta_0}$	for $h > h^*_{\alpha_0, \beta_0}$
1. error-affected displacements displ. under detectability threshold	1. displ. under detectability threshold error-affected displacements
2. error-free displacements displ. under detectability threshold	2. detectable displacements error-affected displacements
3. error-free displacements detectable displacements	3. error-free displacements detectable displacements

Fig. 3 Presentation of ellipsoids for analysis in Case I (1, 2, 3—vectors of characteristic location, $\gamma_0 = 1 - \alpha_0$)



For a zone between the ellipsoids (see vector 2), we observe some inconsistency in the above description. It is due to the fact that the confidence related and the sensitivity related qualifications concern different assumptions (i.e., hypothesis H_0 and hypothesis H_a), with Type I and Type II error probabilities not mutually coordinated. The detectability threshold λ_0 is different from the error-limit $u_{o(\gamma)}$.

3.3 Analysis for Case II

3.3.1 Variance factor option (i)

For the analysis of significance combined with sensitivity, we present the significance ellipsoid and the sensitivity ellipsoid together with the description of characteristic displacement vectors and some probabilistic features of each of the ellipsoids.

The values of $\bar{\beta}$ in Fig. 4 are computed for $h=2$ ($h < h_{\alpha_0, \beta_0}^*$), $h=15$ ($h > h_{\alpha_0, \beta_0}^*$).

The deterministic description is as follows

for $h < h_{\alpha_0, \beta_0}^*$	for $h > h_{\alpha_0, \beta_0}^*$
1. insignificant displacement	1. displ. under detectability threshold
2. significant displacement displ. under detectability threshold	2. detectable displacement insignificant displacement
3. significant displacement detectable displacement	3. detectable displacement significant displacement

Like in Case I, for a zone between the ellipsoids (see vector 2), we observe some inconsistency in the above description. The cause of inconsistency is as that explained in Case I. Here, the detectability threshold λ_0 is different from the significance threshold u_0 .

We may readily classify the computed h -dimensional displacement vector $\hat{\mathbf{d}}$ either as 1, 2 or 3, respectively, by means of the following checks based on the values u of the test statistic Φ , i.e.,

$$\text{for } h < h_{\alpha_0, \beta_0}^* : u \leq u_0, u_0 < u \leq \lambda_0, u > \lambda_0 \tag{23}$$

$$\text{for } h > h_{\alpha_0, \beta_0}^* : u \leq \lambda_0, \lambda_0 < u \leq u_0, u > u_0 \tag{24}$$

Fig. 4 Presentation of ellipsoids for analysis in Case II, option (i) (1, 2, 3—vectors of characteristic location)

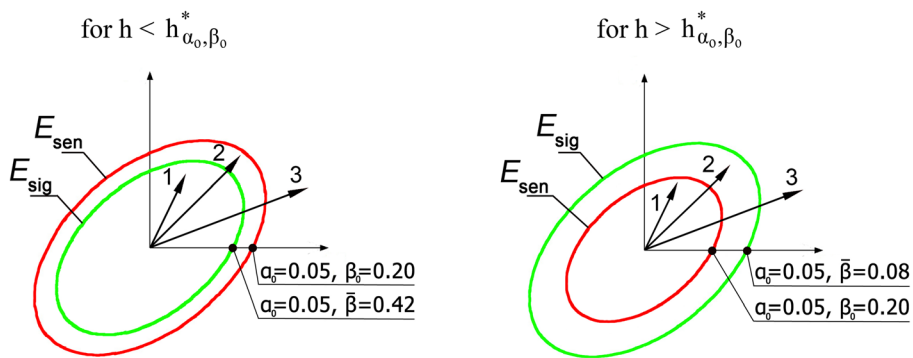
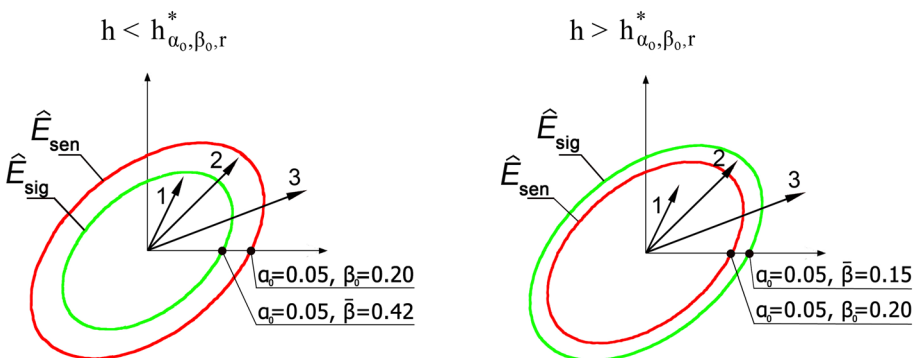


Fig. 5 Presentation of ellipsoids for analysis in Case II, option (ii) (1, 2, 3—vectors of characteristic location)



3.3.2 Variance factor option (ii)

Since the deterministic description of the displacement vectors is analogous to that in option (i), we only present both the ellipsoids together with their probabilistic features (Fig. 5).

To classify the computed h -dimensional displacement vector $\bar{\mathbf{d}}$ either as 1, 2 or 3, respectively, we use here the following checks based on the values w of the variable Ψ and λ_F as in (21) and (22)

$$\text{for } h < h_{\alpha_0, \beta_0, r}^* : w \leq w_0, w_0 < w \leq \lambda_{F,0}, w > \lambda_{F,0} \quad (25)$$

$$\text{for } h > h_{\alpha_0, \beta_0, r}^* : w \leq \lambda_{F,0}, w_{f,r,\lambda} < w < w_{f,r,\alpha}, w > w_0 \quad (26)$$

For this variance factor option, the greater the system redundancy, the lower is the detectability threshold, and the smaller are the displacement vectors that can be considered as insignificant. It is therefore recommended to always ensure an appropriately high redundancy level.

The values of β in Fig. 5 are computed for $r=20$ and $h=2$ ($h < h_{\alpha_0, \beta_0, r}^*$), $h=20$ ($h > h_{\alpha_0, \beta_0, r}^*$).

3.4 Evaluating the MDD support in task Cases I and II

The MDD support is understood here as providing grounds for verifying the correctness of the significance (or confidence) threshold. The support is the greater the closer is the sensitivity ellipsoid to significance (or confidence) ellipsoid.

The value of $\bar{\beta}$ obtained from the equality (14) with $\alpha_0=0.05$ (or $\gamma_0 = 1 - \alpha_0$) depends except h only on the value of α_0 . Hence, $\bar{\beta}$ is only partially connected with the detectability threshold λ_0 , which is defined on both α_0 and β_0 . The difference $|\lambda_0 - u_0| \neq 0$ is a main cause of inconsistency in the description of vector qualification discussed in Sect. 3.2 (Fig. 3) and Sect. 3.3 (Figs. 4 and 5). The smaller is $\bar{\beta}$, and the smaller is $|\lambda_0 - u_0|$, the greater is MDD support. The analysis of the above-mentioned relationships led to introducing of the following directly-proportional measure of MDD support, defined by

$$\text{for } h < h_{\alpha_0, \beta_0}^* \quad Q\left(\frac{\beta}{\bar{\beta}}, \frac{u_0}{\lambda_0}\right) \quad (27)$$

$$\text{for } h > h_{\alpha_0, \beta_0}^* \quad Q\left(\frac{\beta}{\bar{\beta}}, \frac{\lambda_0}{u_0}\right) \quad (28)$$

Table 3 Values of Q for some h ($\alpha_0=0.05, \beta_0=0.20$); italics concerns $h > h_{\alpha_0, \beta_0}^*$

h	1	2	3	4	5	6	7	8	9	10	20	30	40	50
$\bar{\beta}$	0.50	0.58	0.64	0.69	0.73	0.76	0.79	0.81	0.84	0.86	0.96	0.98	0.995	0.998
$\frac{u_0, \lambda_0}{\lambda_0, u_0}$	0.49	0.62	0.72	0.79	0.86	0.92	0.98	0.97	0.92	0.89	0.67	0.56	0.49	0.45

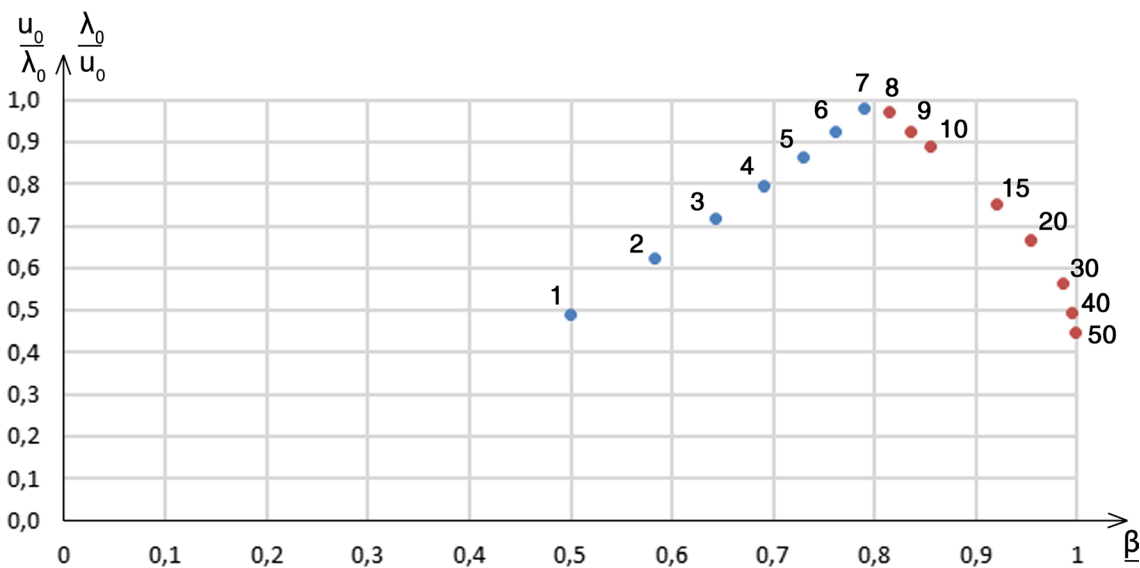


Fig. 6 Plot of Q values for some h ($\alpha_0=0.05, \beta_0=0.20$); Q points are labeled with the values of h ; $h < h_{\alpha_0, \beta_0}^*$ in blue, $h > h_{\alpha_0, \beta_0}^*$ in red

where $\underline{\beta} = 1 - \bar{\beta}$ is a power of the test for $u_o = \lambda_o$, the second components in Q are equal to 1 for $h = h_{\alpha_o, \beta_o, r}^*$.

Since both the parameters in Q are mutually incompatible, in the case of opposite trends in their functions of h , ranking of Q will not be possible. Therefore, the ranking of the MDD support will then have to be carried out with respect to each of the parameters separately.

The measure of MDD support as in (27) and (28) is constructed on the basis of parameters used in Table 1 and diagrams in Fig. 1. Hence, variability of Q values is some other representation of the above-mentioned diagrams.

The values of Q for some h are presented in Table 3 and Fig. 6.

The results in Table 3 and Fig. 6 apply to Task Case I and Task Case II (variance factor option (i)). We may conclude that the MDD support is the smallest for single point displacements in 1D networks ($\underline{\beta} = 0.5, \frac{u_o}{\lambda_o} = 0.49$) and becomes successively greater for 2D ($\underline{\beta} = 0.58, \frac{u_o}{\lambda_o} = 0.62$) and 3D networks ($\underline{\beta} = 0.64, \frac{u_o}{\lambda_o} = 0.72$). For $h > h_{\alpha_o, \beta_o}^*$ (i.e., for global network displacements), in terms of $\underline{\beta}$ the support is increasing, whereas in terms of $\frac{u_o}{\lambda_o}$ it is decreasing. For $h = 50$, $\underline{\beta}$ is approaching 1 while $\frac{u_o}{\lambda_o}$ falls down to 0.45.

In analogy to the definition in (27) and (28), the measure of MDD support for Task Case II (variance factor option (ii)) is defined as follows

$$\text{for } h < h_{\alpha_o, \beta_o, r}^* \quad Q\left(\underline{\beta}, \frac{w_o}{\lambda_{F,o}}\right) \tag{29}$$

$$\text{for } h > h_{\alpha_o, \beta_o, r}^* \quad Q\left(\underline{\beta}, \frac{\lambda_{F,o}}{w_o}\right) \tag{30}$$

where $\underline{\beta} = 1 - \bar{\beta}$ is a power of the test for $\lambda_F = w_o$, $\bar{\beta}$ is obtained from the equality (20), i.e., $w_{h,r,\alpha_o} = \lambda_{h,r,\alpha_o,\bar{\beta}}$, the second components in Q are equal to 1 for $h = h_{\alpha_o, \beta_o, r}^*$. The MDD support is presented in Table 4 and Fig. 7.

Like in Fig. 6, the MDD support is the smallest for single point displacements in 1D networks and becomes successively greater for 2D and 3D networks. And also, for $h > h_{\alpha_o, \beta_o, r}^*$ (i.e., for global network displacements), in terms of $\underline{\beta}$ the support is increasing, whereas in terms of $\frac{\lambda_o}{u_o}$ it is decreasing.

According to the property presented in Sect. 3.1 (variance factor option (ii)), with the increase in r , the MDD support becomes closer and closer to that in Fig. 6 for Task Case II (variance factor option (i)).

Table 4 Values of Q for some h and $r=20$ ($\alpha_o=0.05, \beta_o=0.20$); italics concerns $h > h_{\alpha_o, \beta_o, r}^*$

h	1	2	3	4	5	6	7	8	9	10	20	30	40	50
$\underline{\beta}$	0.51	0.58	0.63	0.67	0.70	0.72	0.74	0.76	0.77	0.78	0.85	0.88	0.89	0.90
$\frac{w_o, \lambda_{F,o}}{\lambda_{F,o}, w_o}$	0.50	0.62	0.70	0.76	0.81	0.85	0.88	0.91	0.94	0.96	0.90	0.85	0.82	0.80

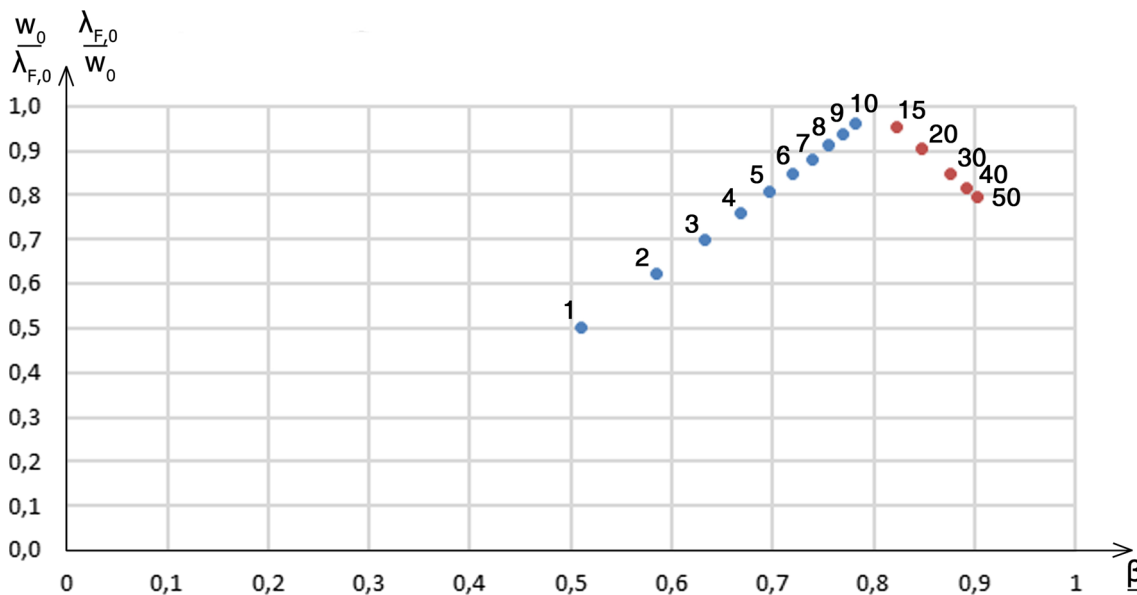


Fig. 7 Plot of Q values for some h and $r=20$ ($\alpha_o=0.05, \beta_o=0.20$); Q points are labeled with the values of h ; $h < h_{\alpha_o, \beta_o, r}^*$ in blue, $h > h_{\alpha_o, \beta_o, r}^*$ in red

The diagrams in Figs. 6 and 7 confirm the theory in Sect. 3.1 in that the MDD support for both the variance factor options increases up to the intersection point (h_{α_o, β_o}^* or $h_{\alpha_o, \beta_o, r}^*$), and behind this point it increases in terms of the first parameter but decreases in terms of the second parameter. Since the smaller is r (variance factor option (ii)), the intersection point falls for greater h , a zone of the increasing MDD support becomes longer. This effect was observed on a diagram for $r=10$ where $h_{\alpha_o, \beta_o, r}^*=49.9$, not shown in the present paper.

3.5 Investigating the possibility of thresholds modification

The question now arises whether the MDD support gives grounds for advantageous modification of the thresholds in the significance test of displacements (or confidence region determination) and detectability for $h=1, 2, 3$. The following two possibilities can be taken into account:

- lowering the detectability threshold while keeping the sensitivity (or confidence) threshold fixed, i.e., E_{sen} is brought toward E_{sig} in terms of size,
- raising the sensitivity (or confidence) threshold while keeping the detectability threshold fixed, i.e., E_{sig} is brought toward E_{sen} in terms of size.

From the analyses carried out for the first possibility, it follows that such a modification would result in an unacceptable increase in the probability of the Type II error. Maximum values of this probability, expressed in terms of the power of the test β are presented in Table 3 and Fig. 4. So, we pass to the second possibility.

Below, we present the analysis for two reasonable options of modifying the significance threshold for both the variance options, i.e., (i) and (ii). In Option 1, we assume $u_{o,m} = 0.5(u_o + \lambda_o)$, while in Option 2 $u_{o,m} = \lambda_o$, where the subscript

“m” denotes the modified quantity. In Option 1, $E_{sig,m}$ lies in between E_{sig} and E_{sen} , while in Option 2 $E_{sig,m} \equiv E_{sen}$.

The data for the analysis are presented in Tables 5 and 6. The symbol $\bar{\beta}_m$ denotes β coordinated with $\alpha_{o,m}$, and as in Sect. 3.1 (variance factor option (i)), the symbol $\bar{\beta}$ denotes β coordinated with α_o (here $\alpha_o = 0.05$).

Ad option 1 (variance factor option (i))

For $h < h_{\alpha_o, \beta_o}^*$, the Type I error probability α is decreased from the level of 0.05 to an average level of 0.02 (for $h=1, 2, 3$), whereas the coordinated Type II error probability β displays a slight increase of about 0.01 from the average level of about 0.43. For $h > h_{\alpha_o, \beta_o}^*$, the Type I error probability α is increased over the level of 0.05, whereas the coordinated Type II error probability β is decreased below 0.20.

Ad option 2 (variance factor option (i))

The changes, being analogous to those in Option 1, are greater. For $h < h_{\alpha_o, \beta_o}^*$, the probability α is decreased to an average level of 0.01, while the coordinated Type II error probability β displays a slight increase of about 0.02.

Based on the above results, it seems reasonable to apply modification only for $h < h_{\alpha_o, \beta_o}^*$, and thus for single point displacements. Either of the Options, as advantageous in terms of testing errors probabilities, is acceptable. Instead of the diversified level of $\alpha_{o,m}$ for different h , the averaged common values 0.02 and 0.01 can be recommended for Option 1 and Option 2, respectively. However, since it was found in computations that for $h=3$ in Option 2 (with $\alpha_{o,m} = 0.01$) $E_{sig,m}$ lies slightly outside E_{sen} , the significance level $\alpha_{o,m} = 0.015$ is proposed for Option 2. With this level $E_{sig,m}$ lies inside E_{sen} , i.e., $u_{o,m} < \lambda_o$.

It was verified that the above conclusions also apply to the variance factor option (ii) the more, the greater the r . The computations showed that for $h=3$ in Option 2 with $\alpha_{o,m} = 0.015$ $\hat{E}_{sig,m}$ lies inside \hat{E}_{sen} (i.e., $w_{o,m} < \lambda_{F,o}$) for $r \geq 23$. For $h=2$ and $h=1$, we have $r \geq 10$ and $r \geq 7$, respectively.

Table 5 Data for Option 1; variance factor option (i)

h	$u_{o,m}$	λ_o	$\alpha_{o,m}$	$\bar{\beta}_m$	$\bar{\beta}$	h	$u_{o,m}$	λ_o	$\alpha_{o,m}$	$\bar{\beta}_m$	$\bar{\beta}$
1	5.85	7.85	0.016	0.500	0.500	10	17.27	16,24	0.069	0.138	0.145
2	7.81	9.64	0.020	0.427	0.417	20	26.19	20,96	0.160	0.032	0.045
3	9.36	10.90	0.025	0.370	0.358	30	34.16	24.55	0.274	0.007	0.014

Table 6 Data for Option 2; variance factor option (i)

h	$u_{o,m}$	$\alpha_{o,m}$	$\bar{\beta}_m$	$\bar{\beta}$	h	$u_{o,m}$	$\alpha_{o,m}$	$\bar{\beta}_m$	$\bar{\beta}$
1	7.85	0.005	0.500	0.500	10	16.24	0.093	0.130	0.145
2	9.64	0.008	0.435	0.417	20	20.96	0.399	0.019	0.045
3	10.90	0.012	0.379	0.358	30	24.55	0.747	0.002	0.014

Table 7 Data for Option 2, variance factor option (i), $\alpha_{o,m}=0.015$

h	u_o	$u_{o,m}$	λ_o	$\alpha_{o,m}$	$\bar{\beta}_m$	$\bar{\beta}$
1	3.84	5.92	7.85	0.015	0.500	0.500
2	5.99	8.40	9.64	0.015	0.430	0.417
3	7.81	10.46	10.90	0.015	0.377	0.358

If the above-mentioned restrictions on the value of r are acceptable, the significance level $\alpha_{o,m}=0.015$ can be proposed for both options (i) and (ii) in Option 2. Otherwise, the value of $\alpha_{o,m}$ for option (ii) should be increased accordingly. The data for variance factor option (i) are presented in Table 7.

A similar modification can be carried out for confidence area determination (variance factor option (i)). The modified confidence level $\gamma_{o,m}$ would be 0.98 for Option 1 and 0.985 for Option 2.

The choice of Options for both Task Cases I and II can be made depending on the type of practical applications.

The determined measures of MDD support and the proposed threshold modifications have a general character and are valid for any realization of Task Case I or Task Case II. Therefore, enclosing a numerical example did not seem to be necessary.

4 Concluding remarks

Theoretical findings of the research can be summarized as follows:

- determining Type II error probability $\bar{\beta}$ corresponding to Type I error probability α_o ,
- introducing a rescaled Fisher-distributed test statistic which made it possible to construct a significance ellipsoid comparable with the sensitivity ellipsoid for the case when the a posteriori variance factor is used,
- proposing a 2-parameter measure Q of MDD support in the accuracy analysis and significance test of displacements.

The computations show that the MDD support both in confidence region determination and significance test of single point displacements is the smallest for 1D networks. It increases successively for 2D and 3D networks. For global network displacements, with the increase in h , the MDD support increases with regard to $\bar{\beta}$ much more than for single point displacements but decreases with regard to the second parameter in Q . Based on the analysis of MDD support, two options of modifying the significance and confidence thresholds for single point displacements are proposed. The modified thresholds being closer to the detectability threshold, correspond

to significance level $\alpha=0.02$ or $\alpha=0.015$ and the confidence level $\gamma=0.98$ or $\gamma=0.985$. The advantage of these modifications is that the proposed decrease in the significance level α (or increase in confidence level γ) results in a negligibly small increase in the Type II error probability β .

In future research, it is planned to consider other types of displacement monitoring models and also, based on publications such as, e.g., (Xu et al. 2000) to examine the possibility of extending the findings of the present paper onto geometrical deformation approach.

Appendix A

Another proof for datum invariance of sensitivity quadratic form

Thesis A necessary and sufficient condition for datum invariance of the form Φ in a model (1) is that the coefficient matrix S in minimum-constraint datum definition eliminating the model defect is of the rank equal to that defect.

Proof Substituting \hat{d} and C_d as in (5) into the formula $\Phi = \hat{d}^T C_d^+ \hat{d}$ (as in (7)), we get

$$\begin{aligned} \Phi &= (\Delta I)_s^T A_s (A_s^T A_s)_s^- [(A_s^T A_s)_s^-]^+ (A_s^T A_s)_s^- A_s^T (\Delta I)_s \\ &= (\Delta I)_s^T A_s (A_s^T A_s)_s^- A_s^T (\Delta I)_s \end{aligned} \tag{31}$$

According to (Rao and Mitra 1971; Rao 1973), the expression $A(A^T A)^- A^T$, where $(A^T A)^-$ denotes a generalized inverse of $A^T A$, does not depend on the choice of the generalized inverse.

Hence, for any given $(\Delta I)_s$, excluding $(\Delta I)_s \in \text{Ker}(A_s^T)$, i.e., $A_s^T (\Delta I)_s = 0$, the form Φ will have the same value for any type of the inverse $(A_s^T A_s)_s^-$ used. This means any type of minimum-constraint datum definition [see (2)].□

It follows immediately from the proof that for any over-constraint datum definition (i.e., where $\text{rank } S > d$) the form Φ is not datum invariant.

Appendix B

Non-centrality parameter for the model (1) with weights of observation differences

The displacement vector $\hat{\mathbf{d}}$ and its covariance matrix $\mathbf{C}_{\hat{\mathbf{d}}}$ (and $\hat{\mathbf{C}}_{\hat{\mathbf{d}}}$) are as follows

$$\begin{aligned}\hat{\mathbf{d}} &= (\mathbf{A}^T \mathbf{P}_{\Delta} \mathbf{A})_{\mathbf{S}}^{-1} \mathbf{A}^T \mathbf{P}_{\Delta} \Delta \\ \mathbf{C}_{\hat{\mathbf{d}}} &= \sigma_{o,w}^2 (\mathbf{A}^T \mathbf{P}_{\Delta} \mathbf{A})_{\mathbf{S}}^{-1} \\ \hat{\mathbf{C}}_{\hat{\mathbf{d}}} &= \hat{\sigma}_{o,w}^2 (\mathbf{A}^T \mathbf{P}_{\Delta} \mathbf{A})_{\mathbf{S}}^{-1}\end{aligned}$$

where \mathbf{P}_{Δ} —weight matrix (pos. definite), $\sigma_{o,w}^2$ —a priori variance of the observation difference of unit weight, $\hat{\sigma}_{o,w}^2$ —estimator (unbiased) of $\sigma_{o,w}^2$.

Denoting $\hat{\mathbf{C}}_{\hat{\mathbf{d}}}$ as $\hat{\mathbf{C}}_{\hat{\mathbf{d}}} = \hat{\sigma}_{o,w}^2 \mathbf{G}$, where $\mathbf{G} = (\mathbf{A}^T \mathbf{P}_{\Delta} \mathbf{A})_{\mathbf{S}}^{-1}$, the test statistic Ψ corresponding to that in (18), will be

$$\Psi = \frac{\hat{\mathbf{d}}^T \mathbf{G} \hat{\mathbf{d}}}{\hat{\sigma}_{o,w}^2}$$

Due to the property that $\Psi = \lambda_F$ (as in Sect. 3.1, variance factor option (ii)), we get the non-centrality parameter for the analyzed option of the model (1).

We can check the same convergence of Ψ to Φ , i.e., since with $r \rightarrow \infty$ $\hat{\sigma}_{o,w}^2 \rightarrow \sigma_{o,w}^2$, so $\Psi \rightarrow \frac{\hat{\mathbf{d}}^T \mathbf{G} \hat{\mathbf{d}}}{\sigma_{o,w}^2} \equiv \Phi$.

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Author contributions W.P. theory, analyses and text; S.Ł. computation, tables and figures.

Data availability All data generated during the current study are available in this published article.

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