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SHORT NOTE

Consistent estimates of the dynamic figure parameters of the earth

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Abstract The Earth's dynamic figure parameters, namely the principal moments of inertia and dynamic ellipticities of the whole Earth, the fluid outer core and the solid inner core, are fundamental parameters for geodetic, geophysical and astronomical studies. This study aims to re-estimate the mass and the dynamic figure parameters of the Earth on the basis of some global gravity models (EGM2008, EIGEN-6C and EIGEN-6C2) recently released with unprecedented accuracies, as well as an improved value of the gravitational constant G recommended by the Committee on Data for Science and Technology (CODATA). With the potential coefficients of EGM2008, EIGEN-6C and EIGEN-6C2 rescaled to be consistent with the IAU (International Astronomical Union) and IAG (International Association of Geodesy) numerical standards, and other values of relevant parameters also being consistent with those numerical standards, we have obtained consistent estimates of the dynamic figure parameters of the stratified Earth using the theory described in Chen and Shen (J Geophys Res 115:B12419 2010). Our preferred principal moments of inertia for the whole Earth are $A = (80{,}085.1 \pm 9.6) \times 10^{33}$ kg m^2, $B = (80{,}086.8 \pm 9.6) \times 10^{33}$ kg m^2, and $C = (80{,}349.0 \pm 9.6) \times 10^{33}$ kg m^2, respectively, the accuracies being limited by the uncertainties of G and e (dynamic ellipticity of the whole Earth).

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1 Introduction

The earth's mass and dynamic figure parameters (DFPs), including the inertia tensors and dynamic ellipticities of the whole Earth and its internal layers (the fluid outer core and the solid inner core), are fundamental parameters for geodetic, geophysical and astronomical studies, and are essential for studies of rotational motion, structure and dynamics of the earth system [e.g., (Lambeck 1980; Gwinn et al. 1986; Mathews et al. 2002; Dickman 2003; Chen and Shen 2010; Bizouard and Zotov 2013; Chen et al. 2013a, b)].

To obtain these figure parameters, we need five types of inputs as shown in Fig. 1: the degree-2 geopotential coefficients and the dynamic ellipticity (obtainable from astronomical observations) to derive the principal moments of inertia (PMIs) for the whole earth; a spherical Earth model [such as the Preliminary Reference Earth Model PREM, (Dziewonski and Anderson 1981)], the theory of hydrostatic equilibrium and nonhydrostatic corrections to obtain an ellipsoidal Earth model (such as the MHB2000 Earth model to be explained later). As shown by Chen and Shen (2010) (also see Table 9), the PREM-based MHB2000 Earth model significantly overestimates the PMIs. Therefore, we need to use the "observed" PMIs to rescale the MHB2000 model parameters using some assumptions on the earth's internal figure, and then establish an appropriate model for the dynamic figure of the stratified earth. Those are the main procedures adopted by this study to derive the DFPs of the stratified earth.

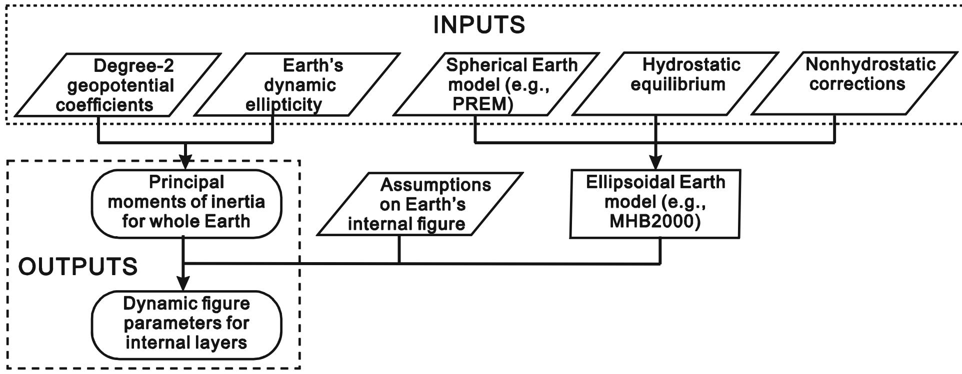

Fig. 1 Flowchart showing procedures to determine the dynamic figure parameters of the stratified earth

Table 1 Data used in relevant geopotential models, where LAGEOS stands for Laser Geodynamics Satellites, DORIS Doppler Orbitography and Radiopositioning Intergrated by Satellite, GRACE Gravity Recovery and Climate Experiment, and GOCE Gravity Field and Steady-State Ocean Circulation Explorer

Model	Year released	Data or model(s) used
JGM-3	1994	JGM-1 gravity model; LAGEOS and DORIS data; TOPEX/Poseidon altimetry
EGM2008	2008	ITG-GRACE03S gravity model (GRACE data only); DTM2006.0 digital topography model; Gravity anomalies obtained from satellite altimetry and terrestrial data
EIGEN-6C	2011	LAGEOS, GRACE and GOCE data; DTU2010 global gravity anomaly obtained from altimetry and gravimetry
EIGEN-6C2	2012	LAGEOS, GRACE and GOCE data (longer time-span); updated DTU2010 global gravity anomaly obtained from altimetry and gravimetry; DTU2010 geoid data over the oceans; EGM2008 geoid heights over the continents

Please refer to Tapley et al. (1996), Pavlis et al. (2012) and Förste et al. (2011, 2012) for more details

Thanks to certain artificial satellites, especially gravity satellites such as LAGEOS, GRACE and GOCE (see the caption of Table 1 for those abbreviations), the global geopotential models (including the degree-2 geopotential coefficients) are gaining higher and higher accuracies as satellite data accumulate. For example, the global geoids determined by some recent models (such as EGM2008, EIGEN-6C and EIGEN-6C2) can reach accuracies of ∼0.2 m or even better (e.g., Förste et al. 2011, 2012; Pavlis et al. 2012; Li et al. 2009, 2013) in contrast to meter-level geoid accuracy of JGM-3 (Tapley et al. 1996), released in 1994 when data were much less abundant and accurate.

The currently recommended values for the PMIs of the whole Earth by the International Association of Geodesy (IAG) were solved based on the JGM-3 model (Groten 2004). In the latest conventions of the International Earth Rotation and Reference Systems Service [namely the IERS Conventions (2010)], EGM2008 is recommended as the standard global gravity model with the C_{20} value replaced by the Satellite Laser Ranging (SLR) results from Cheng et al. (2011), as the SLR data from LAGEOS contribute most to the determination of C_{20}. The degree-2 coefficients of the revised EGM2008 model (hereafter EGM-IERS model) are much more accurate than those of JGM-3, but no corresponding or consistent values for PMIs are provided. In addition, the uncertainty in the geocentric gravitational constant GM has also been reduced by half [(Ries 2007); also see (Petit and Luzum 2010)]. Thus, it is timely to obtain improved values of the Earth's inertia tensor and then DFPs for the fluid outer core and solid inner core using the theory described in Chen and Shen (2010), to meet the needs of the scientific community.

2 Mass and gravity of the earth

2.1 Earth's mass and its uncertainty

As important astronomical constants, the geocentric gravitational constant GM and equatorial radius a of the earth are recommended to be consistent with the use of geocentric coordinate time (TCG); that is, the IERS values [e.g., (Ries et al. 1992; Ries 2007); also see (Petit and Luzum 2010)]

$$\mathrm{GM} = (398{,}600{,}441.8 \pm 0.4) \times 10^6\ \mathrm{m}^3\ \mathrm{s}^{-2},$$
$$a = 6{,}378{,}136.6 \pm 0.1\ \mathrm{m} \quad (1)$$

are TCG-compatible. However, users usually need values consistent with terrestrial time (TT) as the time coordinate, since almost all the observations are obtained at the Earth's surface. Both TCG and TT are defined in the context of the general theory of relativity, and are respectively valid on the geoid and at the geocenter. Therefore, a TCG-compatible value x_{TCG} can be determined from a TT-compatible value x_{TT} by the rigorous relativistic conversions from the geocenter to the geoid, namely [e.g., (Seidelmann and Fukushima 1992; Petit and Luzum 2010)]

$$x_{\mathrm{TT}} = x_{\mathrm{TCG}}(1 - L_{\mathrm{G}}) \quad (2)$$

where $L_G = 1 - \mathrm{d(TT)/d(TCG)} = 6.969290134 \times 10^{-10}$. Then, we can obtain the TT-compatible values

$$\mathrm{GM} = (398{,}600{,}441.5 \pm 0.4) \times 10^6\ \mathrm{m^3\ s^{-2}},$$
$$a = 6{,}378{,}136.6 \pm 0.1\ \mathrm{m}. \quad (3)$$

One finds that the difference between the TCG- and TT-compatible values for a is much smaller than the uncertainty of a (in fact, the TT-compatible a is about 6,378,136.5955 m).

According to Mohr et al. (2012), the value for the gravitation constant recommended by the Committee on Data for Science and Technology (CODATA), an interdisciplinary Scientific Committee of the International Council for Science (ICSU), reads (CODATA2010 value)

$$G = (66{,}738.4 \pm 8.0) \times 10^{-15}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}. \quad (4)$$

Then it is easy to obtain the mass of the Earth

$$M = (59{,}725.8 \pm 7.2) \times 10^{24}\ \mathrm{kg}. \quad (5)$$

The large uncertainty in M is dominated by the error in G. If we adopt the value $G = (66{,}725.9 \pm 3.0) \times 10^{-15}\ \mathrm{m^3\ kg^{-1}s^{-2}}$ recommended by Groten (2004), $M = (59{,}737.0 \pm 2.7) \times 10^{24}$ kg. Note that the IERS Conventions (2010) recommend the CODATA2006 value $G = (66{,}742.8 \pm 6.7) \times 10^{-15}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$ (then $M = (59{,}721.9 \pm 6.0) \times 10^{24}$ kg) reported by Mohr et al. (2008). One can see that the differences among these G values are even larger than the error of the Groten's (2004) estimate, implying that the earlier uncertainties were probably underestimated and systematic errors exist for various independent determinations of G (Mohr et al. 2008, 2012). This study prefers the CODATA2010 value though IERS Conventions (2010) which have not been updated yet.

2.2 Degree-2 potential coefficients

In this study we will use the degree-2 potential coefficients of EGM-IERS, EGM2008, EIGEN-6C and EIGEN-6C2 (see Table 2) to estimate the DFPs of the stratified Earth on the

Table 2 Degree-2 coefficients of selected global gravity models (zero-tide values; J2000 time system)

	Original ($\times 10^{-11}$)	Error ($\times 10^{-11}$)	Rescaled ($\times 10^{-11}$)
EGM-IERS			
C_{20}	−48,416,948	2[a]	−48,416,945.76
C_{21}	−23.06	1.89[b]	−23.06
S_{21}	140.44	1.94[b]	140.44
C_{22}	243,938.36	0.72	243,938.35
S_{22}	−140,027.37	0.74	−140,027.36
EGM2008			
C_{20}	−48,416,931.74	0.75	−48,416,929.50
C_{21}	−20.66	0.71	−20.66
S_{21}	138.44	0.73	138.44
C_{22}	243,938.36	0.72	243,938.35
S_{22}	−140,027.37	0.74	−140,027.36
EIGEN-6C2			
C_{20}	−48,416,948.94	0.09	−48,416,947.92
C_{21}	−20.85	0.12	−20.85
S_{21}	142.89	0.12	142.89
C_{22}	243,932.92	0.13	243,932.91
S_{22}	−140,026.48	0.13	−140,026.48
EIGEN-6C			
C_{20}	−48,416,941.04	0.02	−48,416,940.01
C_{21}	−19.79	0.02	−19.79
S_{21}	138.21	0.02	138.21
C_{22}	243,935.69	0.02	243,935.68
S_{22}	−140,026.68	0.02	−140,026.67

The tide-free C_{20} values provided by those gravity models are converted to zero-tide ones according to the formula in Chapter 6 of IERS Conventions (2010). All coefficients are valid at epoch 2,000.0 through reductions according to model readme files (for EIGEN-6C and EIGEN-6C2 models, only the linear terms of the coefficients are used while the periodic variations are excluded). All errors are formal errors except for those marked with "a" or "b".

[a] According to Cheng et al. (2011).

[b] Estimated from Eq. (6) using C_{20} error from Cheng et al. (2011) as well as C_{22} and S_{22} errors from EGM2008

basis of the theory described in Sect. 3.1. As stated above, EGM-IERS is a modified version of EGM2008 and serves as the standard global gravity model according to the IERS Conventions (2010). In the EGM-IERS model, the EGM2008 C_{20} is replaced by the SLR results from Cheng et al. (2011) while C_{22} and S_{22} are unchanged. Changes in C_{21} and S_{21} according to Eq. (6.5) of the Conventions [rewritten as Eq. (6) below] should be applied to ensure consistency between the position of the Earth's figure axis and its mean rotation pole. That is, for $(\bar{x}_p, \bar{y}_p)$ denoting the time-varying coordinates of the mean pole position (here, we use the IERS2010 mean pole model described in Section 7.1.4 of the Conventions), we have

$$\begin{aligned} C_{21}^{\mathrm{EGM-IERS}} &= \sqrt{3}\bar{x}_p C_{20}^{\mathrm{EGM-IERS}} - \bar{x}_p C_{22}^{\mathrm{EGM2008}} \\ &\quad + \bar{y}_p S_{22}^{\mathrm{EGM2008}}, \\ S_{21}^{\mathrm{EGM-IERS}} &= -\sqrt{3}\bar{y}_p C_{20}^{\mathrm{EGM-IERS}} - \bar{y}_p C_{22}^{\mathrm{EGM2008}} \\ &\quad - \bar{x}_p S_{22}^{\mathrm{EGM2008}}. \end{aligned} \tag{6}$$

Obviously, EGM-IERS differs slightly from EGM2008 as only C_{20}, C_{21} and S_{21} are different. To obtain estimates consistent with the IERS Conventions (2010), the same degree-2 potential coefficients of EGM-IERS should also be used.

One should note that the formal errors of the degree-2 potential coefficients provided by gravity models are usually significantly underestimated, while the C_{20} error from SLR derived by Cheng et al. (2011) is more reliable (see Table 2). Therefore, we always use the SLR-based C_{20} error to infer the uncertainties of relevant parameters.

While the parameters GM and a serve as scaling constants in the determination of the potential coefficients, the values of GM and a adopted by most gravity models are not consistent with the IAU and IAG numerical standards (see Table 3). Based on GM and a values recommended by the IERS Conventions (2010), we rescaled the potential coefficients of EGM2008, EIGEN-6C and EIGEN-6C2 to ensure those values to be consistent with the IAU and IAG numerical standards (see Table 2). One can see that the differences between the original and rescaled potential coefficients can sometimes exceed their uncertainties, and therefore should not be ignored. We suggest the geopotential model developers always to use the standard values of GM and a to avoid such unnecessary inconsistency.

Table 3 Values of *GM* and *a* for IERS Conventions (2010) and selected global gravity models

	GM ($\times 10^6$ m^3 s^{-2})	a (m)
IERS2010	398,600,441.5	6,378,136.6
EGM2008	398,600,441.5	6,378,136.3
EIGEN-6C2	398,600,441.5	6,378,136.46
EIGEN-6C	398,600,441.5	6,378,136.46

3 Earth's dynamic figure parameters

3.1 Remarks on the relation between the degree-2 geopotential coefficients and dynamic figure parameters

The inertia tensor of the whole Earth is usually derived from the degree-2 potential coefficients available from global gravity models [e.g., (Heiskanen and Moritz 1967; Marchenko and Abrikosov 2001; Marchenko and Schwintzer 2003; Chen and Shen 2010)], which are established in the International Terrestrial Reference Frame [ITRF, (Altamimi et al. 2011)]. The degree-2 gravitational potential can be written as

$$V_2 = \frac{GMa^2}{r^3} \sum_{m=0}^{2} (C_{2m} \cos m\lambda + S_{2m} \sin m\lambda) P_{2m}(\cos\theta), \tag{7}$$

where $C_{2\mathrm{m}}$, $S_{2\mathrm{m}}$ and $P_{2\mathrm{m}}(\cos\theta)$ (m = 0, 1, 2) are degree-2 fully normalized potential coefficients and the associated Legendre function, respectively, while (r, θ, λ) is the spherical coordinate of the field point, with r denoting the distance between the field point and the coordinate origin, θ and λ the co-latitude and longitude, respectively.

The degree-2 coefficients are directly related to the inertia tensor $\mathbf{I} = \mathrm{diag}\{I_{ij}, i, j = 1, 2, 3\}$ $(I_{ij} = I_{ji})$, namely (e.g., Heiskanen and Moritz 1967)

$$\begin{aligned} C_{20} &= -\frac{I_{33} - (I_{11} + I_{22})/2}{\sqrt{5}Ma^2}, \\ C_{21} &= \frac{I_{13}}{\sqrt{15}Ma^2}, \quad S_{21} = \frac{I_{23}}{\sqrt{15}Ma^2}, \\ C_{22} &= \frac{\sqrt{15}(I_{22} - I_{11})}{10Ma^2}, \quad S_{22} = \frac{\sqrt{15}I_{12}}{5Ma^2}. \end{aligned} \tag{8}$$

Wherein, the diagonal elements I_{ij} $(i = j)$ are termed as moments of inertia, while others are called products of inertia. If we let the reference frame axes be aligned with the three eigenvectors of $\mathbf{I}$, $\mathbf{I}$ can be diagonalized to

$$\mathbf{I} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}. \tag{9}$$

Those reference frame axes are called principal axes while A, B, and C are termed PMIs (here, we assume A and B are equatorial PMIs while C is the polar PMI). Comparing Eqs. (8) and (9), one can easily find that C_{21}, S_{21} and S_{22} will vanish, while C_{20} and C_{22} will remain in the Principal Axial Frame (PAF; however, new symbols A_{20} and A_{22} must be introduced to avoid ambiguity). That is, in the PAF, the degree-2 potential reduces to

$$V_2 = \frac{GMa^2}{r_I^3}\left[A_{20}P_{20}(\cos\theta_I) + A_{22}\sin 2\lambda_I P_{22}(\cos\theta_I)\right] \quad (10)$$

with

$$A_{20} = -\frac{C-(A+B)/2}{\sqrt{5}Ma^2}, \qquad A_{22} = \frac{\sqrt{15}(B-A)}{10Ma^2}. \quad (11)$$

If the Earth were a rotationally symmetric body (namely $A = B < C$), A_{22} will vanish, too. Therefore, we will soon find that the ITRF is not a PAF and the Earth should be a triaxial body ($A \neq B$; we assume $A < B$) since all the four coefficients C_{2m}, S_{2m} ($m = 1, 2$) are non-zero. Thus, the $I_{ij}(i = j)$ derived from C_{20} and C_{22} are only moments of inertia, not PMIs, and A_{20} (not C_{20}) is actually the dynamic figure factor.

3.2 Dynamic figure factor of the Earth

By solving an eigenvalue–eigenvector problem as discussed by Marchenko and Abrikosov (2001), Marchenko and Schwintzer (2003) and Chen and Shen (2010), we find A_{20}, A_{22} can be derived from C_{2m}, S_{2m} ($m = 0, 1, 2$):

$$\begin{cases} \bar{A}_{2,0} = \frac{\sqrt{3}\varepsilon_3}{2} \\ \bar{A}_{2,2} = \varepsilon_1 + \frac{\varepsilon_3}{2}, \end{cases} \quad (12)$$

where

$$\begin{cases} \varepsilon_1 = 2\sqrt{\frac{p}{3}}\sin(\frac{\varphi+\pi}{3}) \\ \varepsilon_2 = -2\sqrt{\frac{p}{3}}\sin\frac{\varphi}{3}, \\ \varepsilon_3 = 2\sqrt{\frac{p}{3}}\sin(\frac{\varphi-\pi}{3}) \end{cases}$$
$$\begin{cases} p = \bar{C}_{2,0}^2 + \bar{C}_{2,1}^2 + \bar{S}_{2,1}^2 + \bar{C}_{2,2}^2 + \bar{S}_{2,2}^2 \\ q = \frac{2\bar{C}_{2,0}^3}{3\sqrt{3}} + \frac{\bar{C}_{2,0}}{\sqrt{3}}(\bar{C}_{2,1}^2 + \bar{S}_{2,1}^2 - 2\bar{C}_{2,2}^2 - 2\bar{S}_{2,2}^2) \\ \quad +\bar{C}_{2,2}(\bar{C}_{2,1}^2 - \bar{S}_{2,1}^2) + 2\bar{C}_{2,1}\bar{S}_{2,1}\bar{C}_{2,2}. \\ \varphi = \arcsin\left(\frac{3\sqrt{3}}{2}p^{-3/2}q\right) \quad (-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}) \end{cases} \quad (13)$$

According to Eqs. (12) and (13), as well as values of the relevant parameters listed in Tables 2 and 3, we obtain the degree-2 potential coefficients for the above models valid in the Principal Axes Frame as listed in Table 4. Among those values, we recommend the ones together with their uncertainties corresponding to the EGM-IERS model:

$$A_{20} = (4{,}841{,}694.6 \pm 0.2) \times 10^{-10}, \qquad A_{22} = (28{,}127.1 \pm 0.5) \times 10^{-10}, \quad (14)$$

Table 4 Degree-2 potential coefficients for selected global gravity models valid in the Principal Axes Frame (raw results from MatLab outputs)

	A_{20}	A_{22}
EGM-IERS	−4.841694575932572e−04	2.812713623260785e−06
EGM2008	−4.841692949601702e−04	2.812713623263007e−06
EIGEN-6C2	−4.841694791845976e−04	2.812662094771952e−06
EIGEN-6C	−4.841694000907875e−04	2.812687075829106e−06

which implies that

$$C-(A+B)/2 = -\sqrt{5}Ma^2A_{20} = (26{,}304.6 \pm 7.0) \times 10^{31}\ \mathrm{kg\ m^2},$$
$$B - A = 10/\sqrt{15}Ma^2A_{22} = (17{,}645.3 \pm 4.7) \times 10^{29}\ \mathrm{kg\ m^2}, \quad (15)$$

Obviously, A_{20} reflects the difference between the polar principal inertial moment C and the mean equatorial one $(A + B)/2$, while A_{22} relates to the difference between the two equatorial principal inertial moments A and B.

3.3 Dynamic figure parameters for the whole earth

Based on Eq. (15) and given the value for the Earth's geophysical dynamic ellipticity

$$e = \frac{C-(A+B)/2}{(A+B)/2} \quad (16)$$

or the astronomical dynamical ellipticity

$$H = \frac{C-(A+B)/2}{C}, \quad (17)$$

we can obtain the PMIs in the form

$$\begin{cases} A = -\sqrt{5}Ma^2\left(\frac{A_{20}}{e} + \frac{A_{22}}{\sqrt{3}}\right) \\ B = -\sqrt{5}Ma^2\left(\frac{A_{20}}{e} - \frac{A_{22}}{\sqrt{3}}\right) \\ C = -\sqrt{5}Ma^2\frac{1+e}{e}A_{20} \end{cases} \quad (18)$$

or

$$\begin{cases} A = \sqrt{5}Ma^2\left[(1-\frac{1}{H})A_{20} - \frac{A_{22}}{\sqrt{3}}\right] \\ B = \sqrt{5}Ma^2\left[(1-\frac{1}{H})A_{20} + \frac{A_{22}}{\sqrt{3}}\right]. \\ C = -\sqrt{5}Ma^2\frac{A_{20}}{H} \end{cases} \quad (19)$$

Equations (18) and (19) are equivalent to each other due to the fact $H = e/(1+e)$ or $e = H/(1-H)$. The polar dynamic ellipticity for the whole Earth is defined by Eq. (16) while the equatorial one can be written as $e' = (B - A)/A$; those for the fluid outer core and solid inner core are denoted with the subscripts "f" and "s", respectively.

Mathews et al. (1991) provided hydrostatic equilibrium values of the DFPs corresponding to the Preliminary Reference Earth Model (PREM, Dziewonski and Anderson 1981). To take into account the non-equilibrium correction, Mathews et al. (2002) further revised the dynamic ellipticities for the whole Earth and the fluid outer core (e and e_f) based on least squares fit of the Very Long Baseline Interferometry (VLBI) observations, and then established the MHB2000 (or IAU2000A) precession–nutation model based on those parameters. Let us call that set of DFPs the MHB2000 Earth model hereafter for convenience (see Table 9). Since PREM provides no error estimates of its parameters, the MHB2000 parameters have no uncertainties either, except for e and e_f which are obtained from VLBI data (see Tables 5 and 9). Thus, it is impossible for this study to infer the uncertainties for the DFPs for the fluid outer core and solid inner core (see Tables 8 and 9).

Adopting the astronomical dynamical ellipticity $H = (327{,}376.34 \pm 0.32) \times 10^{-8}$ (equivalent to $e = (328{,}451.61 \pm 0.32) \times 10^{-8}$) from Marchenko and Abrikosov (2001), Chen and Shen (2010) derived the Earth's polar and equatorial ellipticities based on the EGM2008 model, and then used them as constraints to revise or infer the polar and equatorial ellipticities of the core on the basis of the MHB2000 model. As the e value from MHB2000 might be more consistent with the IERS Conventions, it is adopted to recalculate the other ellipticities (see Table 5) using the theory and method described in [Chen and Shen (2010); the basic idea will be explained in Sect. 3.4].

Table 5 Polar and equatorial dynamic ellipticities for the triaxially stratified earth: the MHB2000 model for the earth's dynamic figure parameters

Parameter		MHB2000[a]
Polar ellipticity of the whole earth ($\times 10^{-3}$)	e	3.2845479 ± 0.0000012
Equatorial ellipticity of the whole earth ($\times 10^{-5}$)	e'	2.2033215
Polar ellipticity of the fluid core ($\times 10^{-3}$)	e_f	2.6456 ± 0.0020
Equatorial ellipticity of the fluid core ($\times 10^{-5}$)	e'_f	1.7747062
Polar ellipticity of the solid core ($\times 10^{-3}$)	e_s	2.422
Equatorial ellipticity of the solid core ($\times 10^{-5}$)	e'_s	1.6247121

[a] The equatorial dynamic ellipticities for the MHB2000 model are calculated by this study using the method described in Appendix A of Chen and Shen (2010)

Using the potential coefficients in Table 4, the MHB2000 e value and Eq. (18), we can obtain the PMIs corresponding to the selected gravity models as listed in Table 6, among which we again recommend the ones together with their uncertainties corresponding to the EGM-IERS model (see the Appendix for the derivation of the uncertainties of A, B and C):

$$\begin{aligned} A &= (80{,}085.1 \pm 9.6) \times 10^{33}\ \text{kg m}^2, \\ B &= (80{,}086.8 \pm 9.6) \times 10^{33}\ \text{kg m}^2, \\ C &= (80{,}349.0 \pm 9.6) \times 10^{33}\ \text{kg m}^2, \\ I &= (A + B + C)/3 = (80{,}173.6 \pm 9.6) \times 10^{33}\ \text{kg m}^2. \end{aligned} \tag{20}$$

Compared to the values recommended by Groten (2004) $G = 6.67259 \times 10^{-11}\ \text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$ and $e = 3.2845161 \times 10^{-3}$ are adopted; please refer to Case IV in Table 8): $A = (80{,}101 \pm 2) \times 10^{33}$ kg m^2, $B = (80{,}102 \pm 2) \times 10^{33}$ kg m^2, $C = (80{,}365 \pm 2) \times 10^{33}$ kg m^2, one can see significant differences and that the uncertainties for our estimates are larger than Groten's since the G uncertainty has increased but is more reliable now, according to the CODATA2010 standards reported by Mohr et al. (2012).

On the other hand, the orientations of the principal axes, characterized by corresponding longitudes and co-latitudes, can be expressed as

$$\begin{cases} \lambda_i = \arctan \frac{m_i}{l_i} \\ \theta_i = \arctan \frac{n_i}{\sqrt{l_i^2 + m_i^2}} \end{cases} \quad (i = A,\ B \text{ and } C), \tag{21}$$

where

$$\begin{cases} u_i = \frac{\bar{C}_{2,1}\bar{S}_{2,1} + \bar{S}_{2,2}(\varepsilon_i - 2\bar{C}_{2,0}/\sqrt{3})}{\bar{S}_{2,1}\bar{S}_{2,2} + \bar{C}_{2,1}(\varepsilon_i + \bar{C}_{2,2} + \bar{C}_{2,0}/\sqrt{3})} \\ n_i = \left[1 + u_i^2 + \left(\frac{\bar{S}_{2,1}u_i - \varepsilon_i + 2\bar{C}_{2,0}/\sqrt{3}}{\bar{C}_{2,1}}\right)^2\right]^{-1/2} \\ m_i = u_i n_i \\ l_i = \frac{n_i}{\bar{C}_{2,1}}\left(\varepsilon_i - \frac{2\bar{C}_{2,0}}{\sqrt{3}} - \bar{S}_{2,1}u_i\right) \end{cases} \tag{22}$$

Table 6 Principal moments of inertia derived from selected global gravity models ($\times 10^{37}$ kg m^2)

	A	B	C
EGM-IERS	8.008508198529732	8.008684651715571	8.034901043692734
EGM2008	8.008505508432828	8.008681961618667	8.034898344760078
EIGEN-6C2	8.008508557286135	8.008685007239379	8.034901402005884
EIGEN-6C	8.008507248220102	8.008683699740512	8.034900089426317

Listed are raw data from Matlab outputs

Table 7 Orientations of earth's principal axes (unit: degree)

	Groten (2004)	EGM-IERS	EGM2008	EIGEN-6C2	EIGEN-6C
λ_A	-14.9291 ± 0.0010	-14.92851 ± 0.00020	-14.928509	14.928706	14.928583
θ_A	Not provided	0.0000398 ± 0.0000013	0.00003788	0.00003879	0.00003727
λ_B	Not provided	75.07149 ± 0.00020	75.071491	75.071294	75.071417
θ_B	Not provided	0.0000890 ± 0.0000013	0.00008805	0.00009096	0.00008806
λ_C	Not provided	-80.81 ± 0.77	-81.65	81.84	82.00
θ_C	Not provided	89.9999025 ± 0.0000013	89.99990414	89.99990111	89.99990439

Table 8 Dynamic figure parameters for the triaxially stratified earth

Parameter	Case I	Case II	Case III	Case IV
Principal moments of inertia of the whole Earth ($\times 10^{37}$ kg m^2)				
A	8.0085082	8.0085857	8.0100085	8.0100860
B	8.0086847	8.0087622	8.0101849	8.0102625
C	8.0349010	8.0349786	8.0364063	8.0364838
Principal moments of inertia of the mantle ($\times 10^{37}$ kg m^2)				
A_m	7.0971636	7.0972323	7.0984931	7.0985619
B_m	7.0973239	7.0973926	7.0986535	7.0987222
C_m	7.1211386	7.1212073	7.1224726	7.1225413
Principal moments of inertia of the core ($\times 10^{36}$ kg m^2)				
A_c	9.1134460	9.1135342	9.1151533	9.1152415
B_c	9.1136076	9.1136959	9.1153149	9.1154032
C_c	9.1376245	9.1377127	9.1393363	9.1394245
Principal moments of inertia of the fluid outer core ($\times 10^{36}$ kg m^2)				
A_f	9.0549367	9.0550244	9.0566330	9.0567207
B_f	9.0550794	9.0551851	9.0567937	9.0568814
C_f	9.0789730	9.0790607	9.0806738	9.0807615
Principal moments of inertia of the solid inner core ($\times 10^{34}$ kg m^2)				
A_s	5.8509312	5.8509878	5.8520272	5.8520839
B_s	5.8510262	5.8510829	5.8521223	5.8521790
C_s	5.8651498	5.8652064	5.8662485	5.8663052
Polar ellipticity of the whole Earth ($\times 10^{-3}$)				
e	3.2845479	3.2845161	3.2845479	3.2845161
Equatorial ellipticity of the whole Earth ($\times 10^{-5}$)				
e'	2.2033215	2.2033002	2.2033215	2.2033002
Polar ellipticity of the fluid core ($\times 10^{-3}$)				
e_f	2.6456000	2.6455744	2.6456000	2.6455744
Equatorial ellipticity of the fluid core ($\times 10^{-5}$)				
e'_f	1.7747062	1.7746890	1.7747062	1.7746890
Polar ellipticity of the solid core ($\times 10^{-3}$)				
e_s	2.4220000	2.4219766	2.4220000	2.4219766
Equatorial ellipticity of the solid core ($\times 10^{-5}$)				
e'_s	1.6247121	1.6246964	1.6247121	1.6246964

Case I: $G = 6.67384 \times 10^{-11}$ m^3 kg^{-1} s^{-2} and $e = 3.2845479 \times 10^{-3}$ are adopted
Case II: $G = 6.67384 \times 10^{-11}$ m^3 kg^{-1} s^{-2} and $e = 3.2845161 \times 10^{-3}$ are adopted
Case III: $G = 6.67259 \times 10^{-11}$ m^3 kg^{-1} s^{-2} and $e = 3.2845479 \times 10^{-3}$ are adopted
Case VI: $G = 6.67259 \times 10^{-11}$ m^3 kg^{-1} s^{-2} and $e = 3.2845161 \times 10^{-3}$ are adopted

Table 9 Parameters for the rotationally symmetric Earth (here the mean equatorial PMI A_x is equivalent to $(A_x + B_x)/2$ in Table 8, x = null, m, c, f, or s)

Parameter	MHB2000	Case I	Case II	Case III	Case IV
A $(\times 10^{37}$ kg m$^2)$	8.0115	8.0085964	8.0081460	8.0100967	8.0101743
C $(\times 10^{37}$ kg m$^2)$	8.0378	8.0349010	8.0349786	8.0364063	8.0364838
A_m $(\times 10^{37}$ kg m$^2)$	7.0999	7.0972437	7.0968446	7.0985733	7.0986420
C_m $(\times 10^{37}$ kg m$^2)$	7.1237	7.1211386	7.1212073	7.1224726	7.1225413
A_c $(\times 10^{36}$ kg m$^2)$	9.1168	9.1135268	9.1130142	9.1152341	9.1153223
C_c $(\times 10^{36}$ kg m$^2)$	9.1410	9.1376245	9.1377127	9.1393363	9.1394245
A_f $(\times 10^{36}$ kg m$^2)$	9.0583	9.0550170	9.0545078	9.0567133	9.0568010
C_f $(\times 10^{34}$ kg m$^2)$	9.0823	9.0789730	9.0790607	9.0806738	9.0807615
A_s $(\times 10^{34}$ kg m$^2)$	5.8531	5.8509787	5.8506496	5.8520748	5.8521314
C_s $(\times 10^{34}$ kg m$^2)$	5.8673	5.8651498	5.8652064	5.8662485	5.8663052
e $(\times 10^{-3})$	3.2845479	3.2845479	3.2845161	3.2845479	3.2845161
$e_f (\times 10^{-3})$	2.6456	2.6456000	2.6455744	2.6456000	2.6455744
$e_s (\times 10^{-3})$	2.422	2.4220000	2.4219766	2.4220000	2.4219766

The four cases are the same as those in Table 8

All the listed values of MHB2000 Earth model are from Mathews et al. (1991) except for those of e and e_f, which are fitted from the VLBI nutation data by Mathews et al. (2002)

Using Eqs. (21) and (22), as well as the potential coefficients in Table 2, we can derive the orientations of the principal axes as listed in Table 7. We also recommend those values together with their uncertainties corresponding to the EGM-IERS model. The uncertainties for other models are not shown since they are not realistic.

3.4 Dynamic figure parameters for the earth's internal layers

With the PMIs for the whole Earth determined, we use them as constraints, revise the MHB2000 Earth model, and obtain the DFPs for the fluid outer core and solid inner core using the method described in Appendix A of Chen and Shen (2010). The basic idea can be stated as follows: since the polar dynamic ellipticity decreases as the depth (measured from the Earth's surface) increases according to the hydrostatic equilibrium theory (nonhydrostatic corrections do not change that fact), we assume that the equatorial ellipticity decreases by the same fraction as the polar one with respect to the increasing depth. That is just the assumption on Earth's internal figures mentioned in Fig. 1. Then, we can use the "observed" PMIs to rescale the MHB2000 model listed in Table 9 to obtain the DFPs for the fluid outer core and solid inner core, relying on that assumption. More details can be found in Chen and Shen (2010).

In Table 8, we only provide the DFPs corresponding to the EGM-IERS model, but with different choices of G and e. Taking into account the fact that the triaxialities of the Earth are not important in many cases [for example, the geophysical excitations of polar motion as discussed in Chen et al. (2013b)], we also provide a corresponding version for the rotationally symmetric stratified Earth as listed in Table 9, where the values for mean equatorial PMIs of the whole Earth and the fluid core are provided. In Tables 8 and 9, the values for Case I are the most consistent with the CODATA and IAU/IAG numerical standards, and thus are preferred by this study.

One finds that the values of PMIs increase if e increases, but decrease as G increases; the perturbations from changes in G on the PMIs are greater than those from changes in e. These results imply that the DFPs are significantly affected by the systematic errors in G and e.

4 Conclusions and discussions

In this study, we have discussed the relation between the degree-2 potential coefficients of global gravity model and the DFPs of the stratified Earth. Based on potential coefficients rescaled using GM and a values recommended by the IERS Conventions (2010), we have provided new estimates of the Earth's mass and DFPs, which are consistent with the CODATA and IAU/IAG numerical standards. We find that those estimates are significantly affected by the systematic errors of gravitational constant (G) and Earth's dynamic ellipticity (e or H), according to Tables 8 and 9. If we choose $G = (66{,}738.4 \pm 8.0) \times 10^{-15}$ m^3 kg^{-1} s^{-2} (from CODATA2010, Mohr et al. 2012) and $e = (3{,}284{,}547.9 \pm 1.2) \times 10^{-9}$ (from Mathews et al. 2002), the PMIs of the whole Earth are $A = (80{,}085.1 \pm 9.6) \times 10^{33}$ kg m^2, $B = (80{,}086.8 \pm 9.6) \times 10^{33}$ kg m^2, and $C = (80{,}349.0 \pm 9.6) \times$

10^{33} kg m^2, respectively, which might be the best estimates currently available.

Finally, let us briefly discuss the importance of these new estimates.

Accurate estimates of the earth's mass and PMIs can place strong constraints on earth's internal structure and density distribution (the mass M and the mean PMI $I = (A+B+C)/3$ are important input data to invert a spherical earth model such as PREM). Therefore, improved estimates of the earth's mass and PMIs should be helpful to develop some new earth models more accurate than PREM.

Changes in values of PMIs will affect models for the Earth's librations, nutations and polar motion excitations. Their impacts on nutations are significant (mostly caused by the Earth's triaxiality) and should not be ignored according to Chen (2011), who derived corrections to the IAU2000A$_{R06}$ nutation model due to the Earth's triaxiality and found 13 nutation terms exceeding 0.001 mas (milli-arc-second). They might also be important to polar motion excitations since they will lead to 1 ∼ 2 % differences (about 1 ∼ 2 mas at most) in the estimates of the geophysical excitations (Dickman 2003; Chen et al. 2013a, b). However, the impacts on librations are quite small and negligible, on the order of magnitude of 10^{-5} mas and 10^{-6} ms (milli-second) for the polar motion and spin librations, respectively, according to Eqs. (10) and (8) in Chao et al. (1991). In sum, improved PMI estimates are critical to studies and applications on space geodesy and geophysics.

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Appendix: Uncertainty of the Earth's Principal moments of inertia

Let us introduce the parameter

$$K = \sqrt{5}Ma^2 \tag{23}$$

then the total differential of K can be written as

$$\mathrm{d}K = \sqrt{5}(a^2\mathrm{d}M + 2Ma\,\mathrm{d}a) \tag{24}$$

which implies that the uncertainty of K has the form

$$u_K = \sqrt{5a^4u_M^2 + 20M^2a^2u_a^2} \tag{25}$$

Based on the values and uncertainties of M and a, we obtain

$$K = (54{,}329.4 \pm 6.5) \times 10^{34}\ \mathrm{kg\ m^2} \tag{26}$$

Using Eq. (23), we can rewrite Eq. (18) as

$$\begin{cases} \sqrt{3}Ae = -K(\sqrt{3}A_{2,0} + A_{2,2}e) \\ \sqrt{3}Be = -K(\sqrt{3}A_{2,0} - A_{2,2}e) \\ Ce = -K(1+e)A_{2,0} \end{cases} \tag{27}$$

The total differential of the first equation of Eq. (27) is

$$\sqrt{3}A\mathrm{d}e + \sqrt{3}e\mathrm{d}A = -(\sqrt{3}A_{2,0} + A_{2,2}\mathrm{d}e) \\ -K(\sqrt{3}\mathrm{d}A_{2,0} + e\mathrm{d}A_{2,2} + A_{2,2}\mathrm{d}e) \tag{28}$$

then the uncertainty of A should satisfy

$$3e^2u_A^2 = (\sqrt{3}A_{2,0} + A_{2,2}e)^2u_K^2 \\ +K^2(3u_{A_{2,0}}^2 + e^2u_{A_{2,2}}^2 + A_{2,2}^2u_e^2) \\ +3A^2u_e^2 \approx 3A_{2,0}^2u_K^2 \tag{29}$$

when noting that the term $3A_{2,0}^2u_K^2$ is at least 6 orders of magnitude larger than other terms in the right-hand side of Eq. (29). It is not difficult to find whether B has the same uncertainty as A, hence

$$u_A = u_B \approx \left|\frac{A_{2,0}}{e}\right| u_K \tag{30}$$

Through a similar process, we have

$$C\mathrm{d}e + e\mathrm{d}C = -(1+e)A_{2,0}\mathrm{d}K - KA_{2,0}\mathrm{d}e - K(1+e)\mathrm{d}A_{2,0} \tag{31}$$

according to the third equation of Eq. (27), and thus

$$e^2u_C^2 = (1+e)^2A_{2,0}^2u_K^2 + K^2A_{2,0}^2u_e^2 \\ +K^2(1+e)^2u_{A_{2,0}}^2 + C^2u_e^2 \tag{32}$$

Then we get

$$u_C \approx \left|\frac{A_{2,0}}{e}\right| u_K \tag{33}$$

Using numerical values of A_{20}, e and u_K, it is easy to obtain

$$u_A = u_B = u_C = 9.6 \times 10^{33}\ \mathrm{kg\ m^2} \tag{34}$$

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