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ORIGINAL ARTICLE

A partial ellipsoidal approximation scheme for nonconvex homogeneous quadratic optimization with quadratic constraints

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Abstract
An efficient partial ellipsoid approximation scheme is presented to find a $\frac{1}{\lceil \frac{m}{2} \rceil}$-approximation solution to the nonconvex homogeneous quadratic optimization with m convex quadratic constraints, where $\lceil x \rceil$ is the smallest integer larger than or equal to x. If there is an additional nonconvex quadratic constraint beyond the m convex constraints, we can use the new scheme to find a $\frac{1}{m}$-approximation solution.

Keywords Quadratically constrained quadratic optimization · Ellipsoidal approximation · Approximation algorithm

Mathematics Subject Classification 90C20 · 90C26 · 90C59

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1 Introduction

We investigate nonconvex homogeneous quadratic optimization problems with quadratic constraints, which are expressed as $(\mathrm{Q}_{[m,k]})$:

$$
\begin{aligned}
(\mathrm{Q}_{[m,k]}) \quad & \max_{x\in\mathbb{R}^n} x^T A_0 x \\
\text{s.t.} \quad & x^T A_i x \le 1, \; i = 1, 2, \ldots, m, && (1)\\
& x^T B_j x \le 1, \; j = 1, 2, \ldots, k, && (2)
\end{aligned}
$$

where A_0, A_i, and $B_j \in \mathbb{R}^{n\times n}$ are symmetric matrices, with A_i being positive semi-definite and A_0, B_j being indefinite. $(\mathrm{Q}_{[m,k]})$ has many applications, see Cen et al. (2020), Henrion et al. (2001), Nemirovski et al. (1999), Wolkowicz et al. (2000), Xia (2020) and references therein. Moreover, it can be used to approximately reformulate the general inhomogeneous quadratic optimization:

$$
\begin{aligned}
(\mathrm{GQ}) \quad & \max_{x\in\mathbb{R}^n} \; x^T A_0 x + 2b_0 x \\
\text{s.t.} \quad & x^T A_i x + 2b_i x + c_i \le 1, \; i = 1, 2, \ldots, m, \\
& x^T B_j x + 2d_j x + h_j \le 1, \; j = 1, 2, \ldots, k, && (3)
\end{aligned}
$$

where $b_0, b_i, d_j \in \mathbb{R}^n$ and $c_i, h_j \in \mathbb{R}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, k$. In fact, by introducing a new variable $t \in \mathbb{R}$ with $t^2 \le 1$ and then adding a penalty term $\rho(t^2 - 1)$ (where $\rho \ge 0$ is a sufficient large parameter) to the objective function, we can approximately reformulate (GQ) as the following homogeneous version:

$$
\begin{aligned}
& \max_{x\in\mathbb{R}^n, t\in\mathbb{R}} \; x^T A_0 x + 2tb_0 x + \rho t^2 - \rho \\
\text{s.t.} \quad & x^T A_i x + 2tb_i x + c_i \le 1, \; i = 1, 2, \ldots, m, \\
& x^T B_j x + 2td_j x + h_j \le 1, \; j = 1, 2, \ldots, k, \\
& t^2 \le 1.
\end{aligned}
$$

In particular, if $b_i = 0 = d_j$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, k$, we can simply set $\rho = 0$.

Generally, $(\mathrm{Q}_{[m,k]})$ is NP-hard since it contains the well-known Max-Cut problem as a special case. Only a few cases with tight semidefinite programming (SDP) relaxation (Wang and Kılınç-Karzan 2022; Xia 2020) can be efficiently solved in polynomial time, see details in the next section. So it leads to an increased focus on approximation algorithms. The SDP relaxation approach (Wolkowicz et al. 2000) has been the most popular one due to its ability to solve the Max-Cut problem with a 0.878-approximation bound (Goemans and Williamson 1995). Conversely, there has been limited research on ellipsoid approximation algorithms (Fu et al. 1998; Henrion et al. 2001; Zhang and Xia 2022), which have typically shown poor approximation results and limited applicability, being restricted only to $(\mathrm{Q}_{[m,0]})$. This paper investigates a partial ellipsoid approximation scheme that is more effective in approximating $(\mathrm{Q}_{[m,0]})$, and can be extended to $(\mathrm{Q}_{[m,k]})$ with $k \ge 1$.

The remainder of this paper is structured as follows. In Section 2, we provide a review of polynomially solvable cases of ($Q_{[m,k]}$), and we present two approximate approaches that are used for constructing approximation algorithms in detail. In Section 3, we present the partial ellipsoidal approximation algorithm scheme for ($Q_{[m,k]}$). The relaxation problem can be efficiently solved when $k \leq 1$. In Section 4, we present the results of numerical experiments. Conclusions are provided in Section 5.

Notation. $v(\cdot)$ represents the optimal value of problem $(\cdot)$. $A \succ (\succeq)\ 0$ indicates that the matrix A is positive (semi)definite. $\mathrm{Tr}(A)$ denotes the trace of the matrix A. $\mathrm{Tr}(AB^T) = \Sigma_{i=1}^n \Sigma_{j=1}^n a_{ij} b_{ij}$ represents the inner product of the matrices A and B.

2 Preliminary

In this section, we present some known results on specific cases of ($Q_{[m,k]}$) that can be solved polynomially. Subsequently, we discuss two approximate approaches for solving ($Q_{[m,k]}$).

2.1 Polynomial solvable case

As mentioned in Section 1, there exist some cases of ($Q_{[m,k]}$) that can be solved in polynomial time. The partial ellipsoidal approximation scheme is also based on these results, which we discuss before introducing the approximation scheme.

For $m = 1$ and $k = 0$, (GQ) is known as the trust region subproblem (TRS), which is a crucial component of the trust region method for nonlinear programming (Yuan 2015). Despite being nonconvex, TRS has an exact SDP reformulation in its Lagrangian dual, and its SDP relaxation is tight. An optimal solution of TRS can be obtained from an optimal solution of the polynomial solvable SDP problem by performing a matrix rank-one decomposition. Additionally, it has been shown that a TRS can be reformulated as a convex optimization problem, indicating that it has hidden convexity (Xia 2020). Linear-time approximation algorithms for TRS have been proposed in Hazan and Koren (2016); Ho-Nguyen and Kılınç-Karzan (2017), Wang and Xia (2017). In particular, the homogeneous TRS, ($Q_{[1,0]}$), can be viewed as a generalized eigenvalue problem.

With the increasing interest in extending TRS, researchers have explored the addition of another quadratic constraint to this problem. For $m = 2$ and $k = 0$, (GQ) is known as the so-called Celis-Dennis-Tapia subproblem (CDT) (Celis et al. 1985). (GQ) with $m + k \leq 2$ is referred to as the generalized CDT (Peng and Yuan 1997). The additional constraint makes CDT much more difficult to solve than TRS and can lead to a duality gap in general. However, Polyak (1998) proved that the generalized homogeneous CDT, ($Q_{[m,k]}$) with $m + k \leq 2$, has strong duality under mild assumptions, and Ye and Zhang (2003) independently found the globally optimal solution to a homogeneous CDT in polynomial time based on the rank-one decomposition approach. Ai and Zhang (2009) presented an easily verifiable necessary and sufficient condition to determine when CDT and its Lagrangian dual have no duality gap, and if strong duality holds, then an optimal solution of CDT can be obtained from an optimal solution of the SDP relaxation. Bienstock (2016) proved the polynomial-time solvability of the homogeneous CDT and presented an algorithm for general quadratic optimization problems with an arbitrary fixed number of quadratic constraints, while Sakaue et al. (2016) derived another practical algorithm that is guaranteed to find a global solution. Recently, Song et al. (2023) have thoroughly studied local solutions of ($\mathrm{Q}_{[m,k]}$) with $m+k \leq 2$.

Various methods have been used to solve the quadratic problem approximately when there are more than two quadratic constraints in ($\mathrm{Q}_{[m,k]}$), revealing many properties. Luo et al. (2007) and He et al. (2008) discussed a real and complex SDP relaxation for ($\mathrm{Q}_{[m,k]}$), giving a ratio bound between the optimal value of ($\mathrm{Q}_{[m,k]}$) and its SDP relaxation when there is only one indefinite constraint, and A_0 is positive semidefinite. He et al. (2008) also improved the ratio bound given by the approximate S-Lemma of Ben-Tal et al. (2002) for both the real and complex cases when all but one of the constraint matrices are positive semidefinite, while A_0 can be indefinite. To the best of our knowledge, no further investigation has been done on approximate solutions to the general ($\mathrm{Q}_{[m,k]}$).

2.2 Approximation approaches

In this section, we will discuss two different methods for approximately solving ($\mathrm{Q}_{[m,k]}$): SDP relaxation and ellipsoidal approximation.

SDP relaxation is a widely-used approach for approximating ($\mathrm{Q}_{[m,k]}$) with quadratic constraints. The standard Shor's relaxation scheme is used to formulate the SDP relaxation of ($\mathrm{Q}_{[m,k]}$) as follows:

$$\begin{aligned} \text{(SDP)} \quad & \max \operatorname{Tr}(A_0 Y) \\ & \text{s.t.} \quad \operatorname{Tr}(A_i Y) \leq 1, \; i = 1, 2, \ldots, m, && (4) \\ & \qquad \operatorname{Tr}(B_j Y) \leq 1, \; j = 1, 2, \ldots, k, && (5) \\ & \qquad Y \succeq 0, \; Y = Y^T \in \mathbb{R}^{n \times n}, && (6) \end{aligned}$$

where Y is introduced to replace xx^T which certainly satisfies (6), and (4)–(5) follow from (1)–(2), $x^T A_i x = \operatorname{Tr}(A_i x x^T)$ and $x^T B_j x = \operatorname{Tr}(B_j x x^T)$.

Inspired by the 0.878-approximation of the Max-Cut problem by SDP relaxation due to Goemans and Williamson (1995), Nesterov (1998) and Ye (1999) extended the algorithm to the general binary quadratic optimization and box-constrained quadratic optimization, respectively, achieving $\frac{2}{\pi}$-approximation results. For ($\mathrm{Q}_{[m,0]}$), Nemirovski et al. (1999) demonstrated that a $(1 - \frac{1}{(2\ln(2(m+1)\mu))})$-minimizer can be obtained, where $\mu = \min\{m+1, \max_{i=1,\cdots,m} \operatorname{rank}(A_i)\}$. Consequently, SDP relaxation has become a popular and powerful optimization tool, with many studies based on it, not only in mathematics but also in engineering. SDP relaxation has several advantages, including convexity, efficient solvability via interior point methods, and the ability to construct high-quality approximate solutions of the original non-convex problem using its optimal solutions.

Despite its many advantages, SDP relaxation has some non-negligible disadvantages. Not all types of optimization problems can be approximately solved using SDP relaxation, and the computational cost of SDP increases as the problem scales up. In particular, Wu et al. (2018) showed that SDP relaxation is misleading in approximating the weighted maximin dispersion problem over an ℓ_p-ball.

Various methods have been proposed for approximating a bounded subset of $\mathbb{R}^n$ with an ellipsoid, which can be used for approximation. The subset can be enclosed by or enclose an ellipsoid from either the inside or the outside. Given a feasible set $\mathscr{S}$ of $\mathbb{R}^n$, an ellipsoid can be constructed such that:

$$\mathscr{E}_{in} \subseteq \mathscr{S} \subseteq \mathscr{E}_{out}.$$

By optimizing the same objective function over $\mathscr{E}_{in}$ and $\mathscr{E}_{out}$, lower and upper bounds on the optimal value of the objective function over $\mathscr{S}$ can be obtained, respectively.

There are two distinct approaches for constructing ellipsoids. Tarasov, Khachiyan, and Erlikh proposed an approach where they construct an inner ellipsoid that has the largest volume and is fully enclosed by the feasible set (Tarasov et al. 1988). In contrast, Nemirovski and Yudin constructed an outer ellipsoid that has the smallest volume and completely encloses the feasible set (Nemirovski and Yudin 1983). By using a pair of ellipsoids with identical parameters, an approximation bound $(r/R)^2$ can be obtained, where r and R represent the radius of the inner and outer ellipsoids, respectively (Fu et al. 1998). The quality of the approximations can be assessed based on the volume or the radius of the ellipsoid.

Due to the different approaches in ellipsoid construction, we have two distinct methods for ellipsoidal approximation. One method involves constructing an inner ellipsoid and then expanding it to cover the entire subset. The other method involves constructing an outer ellipsoid that fully encloses the feasible set and then shrinking it until it is entirely contained within the subset. We outline the ellipsoidal approximation algorithm for the general (GQ) problem with convex constraints as below:

Algorithm 1 Ellipsoidal Approximation Algorithm for (GQ)

Input: data of (GQ) with $k = 0$.

(a) Construct an inner or outer ellipsoid corresponding to the feasible set.

(b) Solve the relaxation problem consisting of the origin objective function and the constructed ellipsoid, which is a TRS subproblem.

Output: The optimal value of the TRS subproblem is an approximate result for (GQ).

The Löwner–John ellipsoid Schrijver (1986) is a minimal-volume ellipsoid used to approximate polyhedra, denoted by the union or sum of ellipsoids, $\mathscr{E}_{lj}$. For a polyhedron set S of full dimension, a Löwner–John concentric ellipsoid pair $\mathscr{E}_{lj}$ and $\tau\mathscr{E}_{lj}$, where $\tau = \frac{1}{n}$ (Boyd and Vandenberghe 2004), can bound S as follows: $\tau\mathscr{E}_{lj} \subseteq S \subseteq \mathscr{E}_{lj}$. Without additional assumptions on the convex set, the approximation bound $\frac{1}{n}$ cannot be improved. However, if the set is point-symmetric, the ratio can be tightened to $\frac{1}{\sqrt{n}}$ (Boyd and Vandenberghe 2004). Lovász (1986) showed that a weak Löwner–John ellipsoid pair with $\tau = \frac{1}{((n+1)\sqrt{n})}$ and $r = 1$ can be computed in polynomial time for S.

The Dikin ellipsoid, another well-known inner ellipsoid, is derived from the definition of the barrier function and analytic center of a convex set (Boyd and El Ghaoui 1993). The construction process of the Dikin ellipsoid is as follows. Let Ω be the set defined by the m ellipsoidal constraints (3).

The logarithmic barrier function $L(x)$ for Ω is defined as:

$$L(x) = -\sum_{i=1}^{m} \log\left(-x^T A_i x - 2b_i^T x - c_i\right).$$

It can be verified that $L(x)$ goes to infinite as x gets closer to the boundary of Ω. Moreover, $L(x)$ is analytic and strictly convex when $x \in \Omega$, and has a unique minimizer denoted by x^*, which is defined as the analytic center of Ω.

We can obtain the gradient and Hessian of $L(x)$ from the definition:

$$\nabla L(x) = \sum_{i=1}^{m} \frac{2(b_i + A_i x)}{-x^T A_i x - 2b_i^T x - c_i},$$

$$\nabla^2 L(x) = \sum_{i=1}^{m} \left(\frac{4(b_i + A_i x)(b_i + A_i x)^T}{(-x^T A_i x - 2b_i^T x - c_i)^2} + \frac{2A_i}{-x^T A_i x - 2b_i^T x - c_i} \right).$$

For any $r \geq 0$ and interior point z in Ω, we define an ellipsoid centered at z as:

$$\mathscr{E}(z; r) := \{x \in \mathbb{R}^n : (x - z)^T \nabla^2 L(z)(x - z) \leq r^2\}. \tag{7}$$

The ellipsoid $\mathscr{E}(z; 1)$ is called the Dikin ellipsoid at z, and it is known that $\mathscr{E}(z; 1) \subseteq \Omega$ for all interior points z of Ω (Nesterov and Nemirovski 1994). Then we have

$$\mathscr{E}(x^*; 1) \subseteq \Omega \subseteq \mathscr{E}(x^*; \sqrt{m^2 + m}), \tag{8}$$

which was first derived in Fu et al. (1998) and later corrected in Zhang and Xia (2022).

Henrion et al. (2001) showed that if all the constraints are homogeneous (i.e., $b_i = 0$ for $i = 1, 2, \ldots, m$), then $x^* = 0$ and (8) can be strengthened to

$$\mathscr{E}(0; 1) \subseteq \Omega \subseteq \mathscr{E}(0; \sqrt{m}).$$

3 Partial ellipsoidal approximation scheme

In this section, we present the new partial ellipsoidal approximation scheme for $(\mathrm{Q}_{[m,k]})$.

Consider Ω as the feasible region of $(\mathrm{Q}_{[m,k]})$. Note that Ω is not empty as it contains the origin as a feasible point.

Algorithm 2 Partial Ellipsoidal Approximation Algorithm for ($Q_{[m,k]}$)

Input: data of ($Q_{[m,k]}$).
(a) Construct an inner or outer ellipsoid corresponding to all the convex constraints.
(b) Solve the relaxation problem consisting of the original objective function, original indefinite quadratic constraints and the constructed ellipsoid.
Output: The optimal value of the relaxation problem is an approximate result for ($Q_{[m,k]}$).

The partial ellipsoidal approximation algorithm enables the approximation of ($Q_{[m,k]}$) with a number of indefinite quadratic constraints, which is not achievable by the classical ellipsoidal approximation algorithms. Moreover, this algorithm can significantly improve the previous approximate results for ($Q_{[m,k]}$) with $k = 0$ provided by the ellipsoidal approximation algorithm.

3.1 Partial ellipsoidal approximation for ($Q_{[m,1]}$) and extension

We first consider the special case ($Q_{[m,1]}$). Throughout this section, we make the following assumption:

Assumption 1 There is at least one quadratic constraint with A_i positive definite, $i \in \{1, 2, \ldots, m\}$.

We will prove that the partial ellipsoidal approximation scheme can approximate ($Q_{[m,1]}$) by the generalized homogeneous CDT. We define the feasible set of ($Q_{[m,1]}$), (1)–(2), as Ω_H. By separating the constraints that define Ω_H into convex and non-convex groups, we define two sets as:

$$\Omega_{HC} := \{x \in \mathbb{R}^n : x^T A_i x \le 1, \ i = 1, 2, \ldots, m\},$$
$$\Omega_{HN} := \{x \in \mathbb{R}^n : x^T B_1 x \le 1\}.$$

The analytic center for Ω_{HC} is clearly 0. According to (7), the Dikin ellipsoid of Ω_{HC} is simplified to

$$\mathscr{E}_H^r := \{x \in \mathbb{R}^n : x^T A_H x \le r^2\},$$

where

$$A_H = 2 \sum_{i=1}^{m} A_i.$$

Since $0 \in \mathscr{E}_H^1$ and $0 \in \Omega_{HN}$ hold simultaneously, we have $\mathscr{E}_H^1 \bigcap \Omega_{HN} \neq \emptyset$. Moreover, we have the following result.

Theorem 2 *For the feasible set* Ω_H*, we have an enclosure and inclusion by*

$$\mathscr{E}_H^{\sqrt{2}} \bigcap \Omega_{HN} \subseteq \Omega_H \subseteq \mathscr{E}_H^{\sqrt{2m}} \bigcap \Omega_{HN}. \tag{9}$$

Proof We first prove the inclusion relation between $\mathscr{E}_H^{\sqrt{2}} \bigcap \Omega_{HN}$ and Ω_H. By definition, a vector $x \in \mathscr{E}_H^{\sqrt{2}} \bigcap \Omega_{HN}$ if and only if it satisfies

$$x^T A_H x \le 2,\ x^T B_1 x \le 1.$$

Since $A_i \succeq 0$, we can be deduced that

$$x^T A_i x \le \sum_{i=1}^m x^T A_i x = \frac{1}{2} x^T A_H x \le 1.$$

Hence, $x \in \Omega_H$.

Next, we prove the inclusion relation between Ω_H and $\mathscr{E}_H^{\sqrt{2m}} \bigcap \Omega_{HN}$. For any $x \in \Omega_H$, we have $x^T B_1 x \le 1$ and

$$x^T A_i x \le 1,\ i = 1, 2, \ldots, m.$$

By summing up the inequalities above, we obtain

$$x^T A_H x \le 2 \sum_{i=1}^m x^T A_i x \le 2m.$$

which implies that $x \in \mathscr{E}_H^{\sqrt{2m}}$. Hence, we have $\Omega_H \subseteq \mathscr{E}_H^{\sqrt{2m}}$. Therefore, we obtain the desired result (9). □

Theorem 2 presents a $\frac{1}{m}$ ellipsoidal approximation bound for $(\mathrm{Q}_{[m,1]})$, which can be achieved by solving the generalized homogeneous problem CDT:

$$\begin{aligned} \text{(GCDT)} \quad & \max\ x^T A_0 x \\ & \text{s.t.}\ \ x^T A_H x \le 2m, \\ & \qquad\ x^T B_1 x \le 1. \end{aligned}$$

Corollary 3 *It holds that*

$$\frac{1}{m} v(\text{GCDT}) \le v(\mathrm{Q}_{[m,1]}) \le v(\text{GCDT}). \tag{10}$$

Remark 1 He et al. (2008) proposed an SDP relaxation-based method for approximating $(\mathrm{Q}_{[m,1]})$, which provides the following approximation bound:

$$\frac{1}{2\log(174m\mu)} v(\text{SDP}) \le v(\mathrm{Q}_{[m,1]}) \le v(\text{SDP}),$$

where $\mu = \min\{m, \max_i \operatorname{rank}(A_i \hat{X})\}$ and $\hat{X}$ is the optimal solution of SDP relaxation corresponding to $(\mathrm{Q}_{[m,1]})$. Notice that bound (10) is more tight than than the bound provided by He et al.'s bound for small values of the number of convex constraints m. Moreover, (10) is exact for the case $m = 1$.

Polyak (1998) showed that (GCDT) can be globally solved. Consider the optimization problem:

$$\begin{aligned}(\mathrm{Q}_{[2,0]})\quad \max\ & x^T A_0 x\\ \text{s.t.}\ & x^T A_1 x \le \alpha_1,\\ & x^T A_2 x \le \alpha_2.\end{aligned}$$

The Lagrangian dual problem of $(\mathrm{Q}_{[2,0]})$ reads as

$$\begin{aligned}(\mathrm{D})\quad \min\ & -\lambda_1\alpha_1 - \lambda_2\alpha_2,\\ \text{s.t.}\ & A_0 + \lambda_1 A_1 + \lambda_2 A_2 \succeq 0,\\ & \lambda_1, \lambda_2 \ge 0.\end{aligned}$$

Under mild assumptions, strong duality holds for $(\mathrm{Q}_{[2,0]})$ and (D).

Theorem 4 (Polyak 1998) *For $n \ge 3$, if there exist nonnegative $\mu_0, \mu_1, \mu_2 \in \mathbb{R}$ satisfy that*

$$\mu_0 A_0 + \mu_1 A_1 + \mu_2 A_2 \succ 0,$$

and if there exists $x_0 \in \mathbb{R}^n$ such that

$$x_0^T A_1 x_0 < \alpha_1,\ x_0^T A_2 x_0 < \alpha_2,$$

the strong duality holds between $(Q_{[2,0]})$ and its dual (D) and the SDP relaxation is tight, which can be represented by the following equality.

$$v(\mathrm{Q}_{[2,0]}) = v(\mathrm{D}) = v(\mathrm{SDP}).$$

With Theorem 4, we can obtain the optimal value of (GCDT) in polynomial time by solving the SDP relaxation. To recover the solution of $(\mathrm{Q}_{[2,0]})$, one can apply the rank-one decomposition approach, see details in Ye and Zhang (2003). Then we can obtain an approximate value of $(\mathrm{Q}_{[m,1]})$. To solve (GCDT) with nonhomogeneous objective function instead of homogeneous objective function, Sakaue et al. (2016) presented a polynomial time algorithm.

The partial ellipsoidal approximation process presented above only considers a single indefinite constraint. Moreover, the approach can be extended to handle $(\mathrm{Q}_{[m,k]})$ with a fixed number of indefinite constraints, which can be relaxed to a homogeneous quadratic optimization as follows:

$$\begin{aligned}(\mathrm{HQP})\quad \max\ & x^T A_0 x\\ \text{s.t.}\ & x^T A_H x \le 2m,\end{aligned}$$

$$x^T B_j x \leq 1,\ \ j = 1, 2, \ldots, k.$$

A $\frac{1}{m}$ ellipsoidal approximation bound can also be achieved with this relaxation. Moreover, if k is fixed, Bienstock (2016) developed a polynomial-time (though not efficient) algorithm to solve (HQP), enabling to compute an approximation of the optimal value of $(\mathrm{Q}_{[m,k]})$.

In the partial ellipsoidal approximation process of the above problems, the Dikin ellipsoid is used as an example to obtain the approximation bound related to the number of convex quadratic constraints. The Löwner–John ellipsoid can also be constructed as an ellipsoidal relaxation to obtain an approximation bound that is related to the dimensionality of the problem instead of the number of convex quadratic constraints. In specific applications, when the dimensionality of the problem is large, but the number of convex quadratic constraints is small, the Dikin ellipsoid is selected. Otherwise, the Löwner–John ellipsoid is a better alternative.

3.2 Improved partial ellipsoidal approximation for $(\mathrm{Q}_{[m,0]})$ and extension

In this subsection, we focus on $(\mathrm{Q}_{[m,0]})$, which have convex quadratic constraints only. We improve the partial ellipsoidal approximation algorithm by dividing the convex constraints into T groups and then prove that the approximation bound can be significantly strengthened.

Algorithm 3 Improved Partial Ellipsoidal Approximation Algorithm

Input: data of $(\mathrm{Q}_{[m,0]})$.
(a) Divide the convex constraints into T groups.
(b) For the convex constraints in each group, construct an inner or outer ellipsoid.
(c) Solve the relaxation problem consisting of the origin objective function, origin indefinite quadratic constraints and the T newly constructed ellipsoids.
Output: The optimal value of the relaxation problem is an approximate result for $(\mathrm{Q}_{[m,k]})$.

For simplicity, we consider $(\mathrm{Q}_{[m,0]})$ under Assumption 1. We define the feasible set in $(\mathrm{Q}_{[m,0]})$, (1), as Ω_e and split the constraints into two halves based on their order (i.e., $T = 2$), and define the two sets as follows:

$$\Omega_1 := \{x \in \mathbb{R}^n : x^T A_i x \leq 1,\ i = 1, 2, \ldots, \lceil \frac{m}{2} \rceil\},$$
$$\Omega_2 := \{x \in \mathbb{R}^n : x^T A_i x \leq 1,\ i = \lceil \frac{m}{2} \rceil + 1, \lceil \frac{m}{2} \rceil + 2, \ldots, m\}.$$

Based on the definitions of Ω_e, Ω_1, and Ω_2, it is evident that the analytic centers of these three sets are all at $x^* = 0$. The Dikin ellipsoids defined in (7) of Ω_1 and Ω_2 can be expressed as follows:

$$\mathscr{E}_{D1}^r := \{x \in \mathbb{R}^n : x^T A_{D1} x \leq r^2\},$$
$$\mathscr{E}_{D2}^r := \{x \in \mathbb{R}^n : x^T A_{D2} x \leq r^2\},$$

where the two matrices are given by

$$A_{D1} = 2\sum_{i=1}^{\lceil \frac{m}{2} \rceil} A_i, \ A_{D2} = 2\sum_{i=\lceil \frac{m}{2} \rceil+1}^{m} A_i.$$

Theorem 5 *The convex feasible set Ω_e is included and encloses as:*

$$\mathscr{E}_{D1}^{\sqrt{2}} \bigcap \mathscr{E}_{D2}^{\sqrt{2}} \subseteq \Omega_e \subseteq \mathscr{E}_{D1}^{\sqrt{2\lceil \frac{m}{2} \rceil}} \bigcap \mathscr{E}_{D2}^{\sqrt{2\lceil \frac{m}{2} \rceil}}.$$

Proof We prove the two inclusion relations in order. First, by the definition of Dikin ellipsoids, the vector x belongs to $\mathscr{E}_{D1}^{\sqrt{2}} \bigcap \mathscr{E}_{D2}^{\sqrt{2}}$ if and only if

$$x^T A_{D1} x \leq 2, \ x^T A_{D2} x \leq 2.$$

Dividing both sides of the inequalities by 2, we get

$$\sum_{i=1}^{\lceil \frac{m}{2} \rceil} x^T A_i x \leq 1, \ \sum_{i=\lceil \frac{m}{2} \rceil+1}^{m} x^T A_i x \leq 1.$$

Since $A_i \succeq 0$ for $i = 1, 2, \ldots, m$, we have

$$x^T A_i x \leq \sum_{i=1}^{\lceil \frac{m}{2} \rceil} x^T A_i x \leq 1, \ i = 1, 2, \ldots, \lceil \frac{m}{2} \rceil,$$
$$x^T A_i x \leq \sum_{i=\lceil \frac{m}{2} \rceil+1}^{m} x^T A_i x \leq 1, \ i = \lceil \frac{m}{2} \rceil + 1, \lceil \frac{m}{2} \rceil + 2, \ldots, m.$$

Combining the above two inequalities yields that

$$x^T A_i x \leq 1, \ i = 1, 2, \ldots, m,$$

which means $x \in \Omega_e$ and the first inclusion relation is proved.

Next, suppose that $x \in \Omega_e$, so that $x^T A_i x \leq 1$ holds for $i = 1, 2, \ldots, m$. Summing up these inequalities separately, we obtain

$$\sum_{i=1}^{\lceil \frac{m}{2} \rceil} x^T A_i x \leq \lceil \frac{m}{2} \rceil, \ \sum_{i=\lceil \frac{m}{2} \rceil+1}^{m} x^T A_i x \leq \lceil \frac{m}{2} \rceil.$$

Therefore, we have

$$x^T A_{D1} x \leq 2\lceil \frac{m}{2} \rceil, \ x^T A_{D2} x \leq 2\lceil \frac{m}{2} \rceil.$$

The second inclusion relation is proved. The proof is complete. □

Then, according to Theorem 5 and based on the following homogeneous relaxation:

$$\begin{aligned}(\text{HCDT})\quad & \max\ x^T A_0 x \\ & \text{s.t. } x^T A_{D1} x \le \lceil \frac{m}{2} \rceil, \\ & \quad\ \ x^T A_{D2} x \le \lceil \frac{m}{2} \rceil,\end{aligned}$$

we obtain an ellipsoidal approximation bound of $\frac{1}{\lceil \frac{m}{2} \rceil}$ for ($\mathrm{Q}_{[m,0]}$). The relaxation (HCDT) is a homogeneous CDT and can be efficiently solved via SDP as shown in Theorem 4.

Corollary 6 *It holds that*

$$\frac{1}{\lceil \frac{m}{2} \rceil} v(\text{HCDT}) \le v(\mathrm{Q}_{[m,0]}) \le v(\text{HCDT}). \tag{11}$$

The approximation bound $\frac{1}{\lceil \frac{m}{2} \rceil}$ is nearly twice as large as the bound $\frac{1}{m}$ reported in Henrion et al. (2001). Moreover, the bound is tight when $m = 2$.

Consider ($\mathrm{Q}_{[m,k]}$). If we divide the convex constraints into $T \ge 2$ groups, by solving the quadratic optimization relaxation with T convex constraints and k nonconvex constraints, the similar approach leads to the approximation bound of $\frac{1}{\lceil \frac{m}{T} \rceil}$, which improves Corollaries 3 and 6. The cost is at the global solvability of the relaxation problem. If $T + k$ is fixed, there is a polynomial-time (though not efficient) algorithm for globally solving the relaxation problem, see Bienstock (2016).

4 Numerical experiment

To compare the approximation bounds calculated by the partial ellipsoidal scheme and the algorithm based on SDP relaxation, we have performed a series of numerical experiments with convex constraints. All of the numerical experiments were conducted using MATLAB R2015b on a laptop with a 2.2 GHz processor and 32 GB of RAM. We used the SDPT3 algorithm to solve $v(SDP)$ within the CVX toolbox (Grant and Boyd 2013). $v(Q_{[m,0]})$ and $v(CDT)$ are solved by the MATLAB function fmincon.

Example 1 Consider a special case of ($\mathrm{Q}_{[2,0]}$), referred to as (TTRS) in Burer and K (2013). It can be approximately reformulated as the following homogeneous quadratic programming:

$$\begin{aligned}(\text{TTRS})\ & \max\ -x^T Q x - c^T x t \\ & \text{s.t. } x^T x \le 1, \\ & \quad\ \ x^T H x \le 1, \\ & \quad\ \ t^2 \le 1.\end{aligned}$$

With the matrices and vector

$$H = \frac{1}{2}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

one can verify that $v(TTRS) = 4$ and $v(SDP) = 4.25$ by computation. Applying He et al's approach, we can estimate the lower and upper bound based on SDP relaxation:

$$0.0764v(SDP) \leq v(TTRS) \leq v(SDP).$$

Adding the first and second constraint together, we obtain a (CDT) relaxation for (TTRS):

$$\begin{aligned} \text{(CDT)} \ \max \ & -x^T Qx - c^T xt \\ \text{s.t.} \ & x^T(I + H)x \leq 2, \\ & t^2 \leq 1. \end{aligned}$$

The optimal value $v(CDT) = 4.25$. Applying the improved partial ellipsoidal approximation approach yields the estimated lower and upper bound based on ellipsoidal relaxation:

$$0.5v(CDT) \leq v(TTRS) \leq v(CDT).$$

It is worth noting that the lower bound estimated by the improved ellipsoid method is significantly tighter than the bound given by He et al., while the optimal value of the CDT relaxation remains the same as the SDP relaxation.

Example 2 We used MATLAB scripts to generate the input data with a fixed random seed in ($Q_{[m,0]}$), where each matrix of the objective function and quadratic constraints is randomly generated:

```
rand('state',1);
```

The indefinite matrix A_0 in the objective function is generated with the MATLAB scripts:

```
A_0=rand(n,n); A_0=(A_0+A_0')/2;
```

The semidefinite matrices $A_i, i = 1, \ldots, m$ in the constraints are generated with the MATLAB scripts:

```
D=diag(rand(n,1)); U=orth(rand(n,n)); A(:,:,i)=U'*D*U;
```

We began by setting the dimension n to 10, 50, 100, and 150 in turn, and the number of convex constraints m varying from 1 to 70. The reported theoretical results are presented in Fig. 1, which are consistent with the theoretical deductions. We can observe that both approximation ratios decrease as the number of constraints m increases, and the two curves intersect at around $m = 50$. It suggest that the derived ellipsoidal

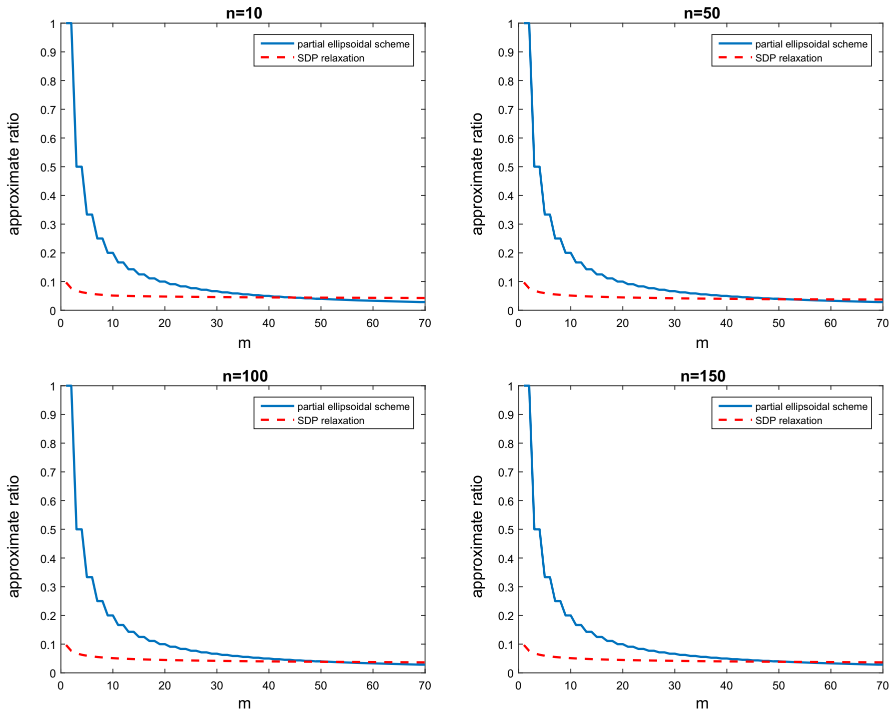

Fig. 1 Approximate ratios when $n = 10, 50, 100, 150$

approximation algorithm is a promising alternative to the SDP relaxation algorithm, particularly for problems with a relatively small number of constraints.

Next, we set the number of convex constraints m to 10, 30, 50, and 70 in turn, and the dimension n varied from 10 to 150. The results of these experiments are presented in Fig. 2, which indicate that the derived approximation bound remains relatively stable as the dimension n increases. By contrast, the approximation ratio calculated by the SDP relaxation algorithm decreases when $n < m$ and remains constant after $n = m$. These observations highlight the superior scalability of the derived ellipsoidal approximation scheme, which is able to maintain a high level of accuracy even in high-dimensional settings.

5 Conclusion

We present a partial ellipsoidal approximation scheme for approximating the nonconvex quadratic optimization with m convex quadratic constraints and k nonconvex quadratic constraints, denoted by $(\mathrm{Q}_{[m,k]})$. By efficiently solving a generalized homogeneous relaxation problem CDT, we can establish a $\frac{1}{m}$ approximation bound for $(\mathrm{Q}_{[m,1]})$. The derived ellipsoidal approximation bound is tighter than that of the SDP relaxation when m is relatively small.

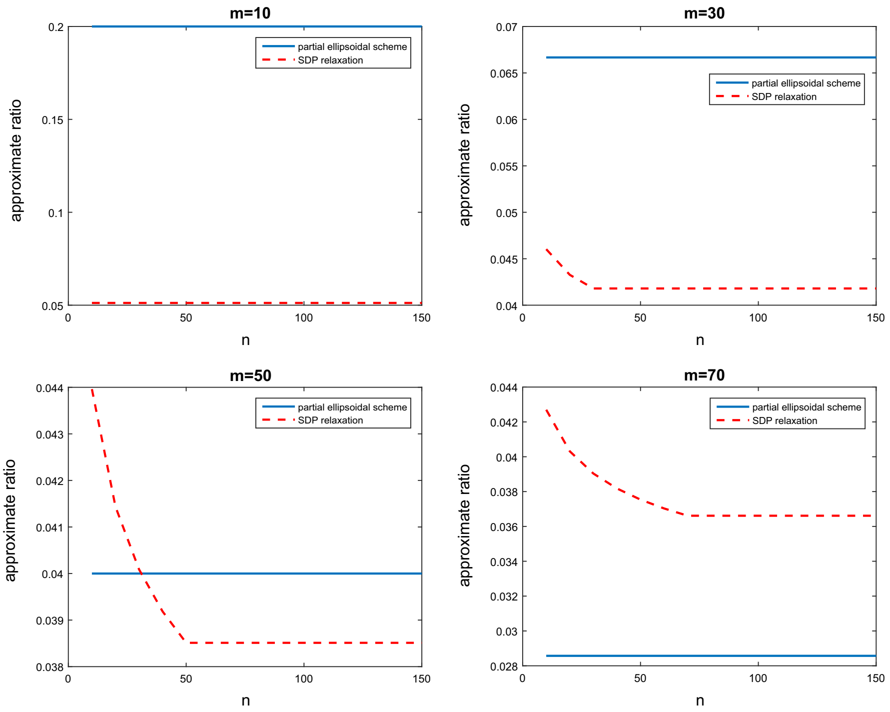

Fig. 2 Approximate ratios when $m = 10, 30, 50, 70$

Then we introduce the grouping approach for further improvement. In particular, for $(\mathrm{Q}_{[m,0]})$, we can solve a homogeneous relaxation problem CDT to obtain an approximation bound of $\frac{1}{\lceil \frac{m}{2} \rceil}$, which is nearly twice as large as the previous ellipsoid approximation bound. Numerical experiments demonstrate the superiority of derived ellipsoidal bound when m is relatively small, say $m \leq 50$. We have extended the approach to the general $(\mathrm{Q}_{[m,k]})$. Moreover, we can improve the approximation bound by increasing the number of constraint groups, although this also increases the complexity of the relaxation problem.

In future research, our attention will be strengthening the partial ellipsoidal relaxation. It is unknown whether there is a better approximation bound by combining the partial ellipsoidal relaxation with the SDP relaxation.

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Declarations

Conflict of interest The authors declare that they have no conflict of interests.

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