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ORIGINAL ARTICLE

Reduced two-bound core games

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Abstract

This paper studies Davis–Maschler reduced games of two-bound core games and shows that all these reduced games with respect to core elements are two-bound core games with the same pair of bounds. Based on associated reduced game properties, we axiomatically characterize the core, the nucleolus, and the egalitarian core for two-bound core games. Moreover, we show that the egalitarian core for two-bound core games is single-valued and we provide an explicit expression.

Keywords Two-bound core games · Reduced games · Axiomatic analysis

JEL Classification C71

1 Introduction

Cooperative games describe situations where players collaborate in coalitions and generate worth. A pre-imputation allocates the worth of the grand coalition among all players in the game. The main issue is to select reasonable pre-imputations for each game. Among the central solution concepts are the core, the nucleolus (cf. Schmeidler 1969), and the egalitarian core (cf. Arin and Iñarra 2001). The core assigns all pre-imputations that are stable against coalitional deviations. The nucleolus assigns to each

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game with nonempty core the unique core element that lexicographically minimizes the maximal excesses over all coalitions. The egalitarian core assigns all core elements from which no other core element can be obtained by a transfer from a richer to a poorer player.

In this paper, we focus on two-bound core games (cf. Gong et al. 2022), i.e., games where the core is nonempty and can be described by a lower bound and an upper bound on the pre-imputations. Two-bound core games generalize compromise stable games (cf. Quant et al. 2005), which include all games with at most three players and a nonempty core, additive games, unanimity games, bankruptcy games (cf. O'Neill 1982), 1-convex games (cf. Driessen 1985), big boss games (cf. Muto et al. 1988), clan games (cf. Potters et al. 1989), and reasonable stable games (cf. Dietzenbacher 2018).

In particular, we study Davis–Maschler reduced games of two-bound core games and show that all these reduced games with respect to core elements are two-bound core games. Moreover, the core of these reduced games can be described by the same pair of bounds. A solution satisfies the bilateral reduced game property (cf. Davis and Maschler 1965) if each pre-imputation assigned to the original game is consistently assigned to all reduced games with two players. A solution satisfies the converse reduced game property (cf. Davis and Maschler 1965) if each pre-imputation assigned to all reduced games with two players is assigned to the original game. Using the bilateral reduced game property and the converse reduced game property, we axiomatically characterize the core, the nucleolus, and the egalitarian core for two-bound core games. Moreover, we show that the egalitarian core for two-bound core games is single-valued and we provide an explicit expression.

The remainder of this paper is organized as follows. Section 2 introduces preliminary definitions and notations for cooperative games. Section 3 studies Davis-Maschler reduced games of two-bound core games. Section 4 axiomatically characterizes the core, the nucleolus, and the egalitarian core. Section 5 concludes.

2 Preliminaries

Let N be a nonempty and finite set of *players* and let $2^N = \{S \mid S \subseteq N\}$ be the set of all *coalitions*. For all $x \in \mathbb{R}^N$, we denote $x_S = (x_i)_{i \in S}$ for all $S \in 2^N \setminus \{\emptyset\}$. For all $x, y \in \mathbb{R}^N$, we denote $[x, y] = \{z \in \mathbb{R}^N \mid \forall i \in N : x_i \leq z_i \leq y_i\}$.

A *(transferable utility) game* is a pair (N, v), where $v : 2^N \to \mathbb{R}$ assigns to each coalition $S \in 2^N$ its *worth* such that $v(\emptyset) = 0$. The set of all games with player set N is denoted by Γ^N. For simplicity, we write $v \in \Gamma^N$ rather than $(N, v) \in \Gamma^N$.

Let $v \in \Gamma^N$. The set of *pre-imputations* is

$$X(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N) \right\},$$

the *core* is

$$C(v) = \left\{ x \in X(v) \,\middle|\, \forall S \in 2^N : \sum_{i \in S} x_i \geq v(S) \right\},$$

and the *egalitarian core* (cf. Arin and Iñarra 2001) is

$$EC(v) = \left\{ x \in C(v) \,\middle|\, \forall i, j \in N : x_i > x_j \Rightarrow s^x_{ij}(v) = 0 \right\},$$

where for all $i, j \in N$ with $i \neq j$ and all $x \in \mathbb{R}^N$,

$$s^x_{ij}(v) = \max_{S \in 2^N : i \in S, j \notin S} \left\{ v(S) - \sum_{k \in S} x_k \right\}.$$

The set of all games with nonempty core and player set N is denoted by Γ^N_b. A game $v \in \Gamma^N$ is *convex* (cf. Shapley 1971) if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \in 2^N$. The set of all convex games with player set N is denoted by Γ^N_c. It is known that $\Gamma^N_c \subseteq \Gamma^N_b$.

A game $v \in \Gamma^N_b$ is a *two-bound core game* (cf. Gong et al. 2022) if there exist $l, u \in \mathbb{R}^N$ such that $C(v) = [l, u] \cap X(v)$, which is equivalent to $C(v) = [l^*(v), u^*(v)] \cap X(v)$, where $l^*_i(v) = \min_{x \in C(v)} x_i$ and $u^*_i(v) = \max_{x \in C(v)} x_i$ for all $i \in N$.[1] The set of all two-bound core games with player set N is denoted by Γ^N_t.

A *solution* φ on a domain of games assigns to each game v in this domain a nonempty set $\varphi(v) \subseteq X(v)$. Note that $\varphi(v) = \{v(N)\}$ for each game v with one player. A solution φ on a domain of games is *single-valued* if $|\varphi(v)| = 1$ for each v in this domain. For a single-valued solution φ on a domain of games and a game v in this domain, $\varphi(v)$ is often identified with its unique element.

The *nucleolus* η (cf. Schmeidler 1969) is a single-valued solution that assigns to each game with nonempty core a unique core element. Maschler et al. (1971) showed that the nucleolus of a convex game $v \in \Gamma^N_c$ is given by

$$\eta(v) = \left\{ x \in X(v) \,\middle|\, \forall i, j \in N, i \neq j : s^x_{ij}(v) = s^x_{ji}(v) \right\}.$$

Gong et al. (2022) showed that the nucleolus of a two-bound core game $v \in \Gamma^N_t$ with $C(v) = [l, u] \cap X(v)$ for $l, u \in \mathbb{R}^N$ is given by

$$\eta_i(v) = \begin{cases} l_i + \min\left\{\frac{1}{2}(u_i - l_i), \lambda\right\} & \text{if } \frac{1}{2}\sum_{i \in N}(u_i + l_i) \geq v(N); \\ l_i + \max\left\{\frac{1}{2}(u_i - l_i), u_i - l_i - \lambda\right\} & \text{if } \frac{1}{2}\sum_{i \in N}(u_i + l_i) \leq v(N), \end{cases}$$

for all $i \in N$, where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} \eta_i(v) = v(N)$. This implies that if two two-bound core games have equal cores, then their nucleoli are equal.

[1] Computational aspects of the bounds l^* and u^* were studied by Bondareva and Driessen (1994).

3 Reduced two-bound core games

In this section, we study Davis–Maschler reduced games of two-bound core games. First, we show that the core of a two-bound core game is equal to the core of a particular convex game, where the worth of each coalition is defined by the minimum total payoff of its members in any pre-imputation between the two bounds.

Theorem 1 *Let $v \in \Gamma_t^N$. Then there exists $\widehat{v} \in \Gamma_c^N$ such that $C(\widehat{v}) = C(v)$.*

Proof Let $l, u \in \mathbb{R}^N$ be such that $C(v) = [l, u] \cap X(v)$. Define $\widehat{v} \in \Gamma^N$ by

$$\widehat{v}(S) = \max\left\{\sum_{i \in S} l_i, v(N) - \sum_{i \in N \setminus S} u_i\right\} \text{ for all } S \in 2^N.$$

Gong et al. (2022) showed that $C(\widehat{v}) = C(v)$, which implies that $\widehat{v} \in \Gamma_t^N$. For all $S, T \in 2^N$,

$$\begin{aligned}
&\widehat{v}(S) + \widehat{v}(T) \\
&= \max\left\{\sum_{i \in S} l_i, v(N) - \sum_{i \in N \setminus S} u_i\right\} + \max\left\{\sum_{i \in T} l_i, v(N) - \sum_{i \in N \setminus T} u_i\right\} \\
&= \max\left\{\sum_{i \in S} l_i + \sum_{i \in T} l_i, v(N) + \sum_{i \in S} l_i - \sum_{i \in N \setminus T} u_i,\right. \\
&\qquad \left. v(N) + \sum_{i \in T} l_i - \sum_{i \in N \setminus S} u_i, 2v(N) - \sum_{i \in N \setminus S} u_i - \sum_{i \in N \setminus T} u_i\right\} \\
&\leq \max\left\{\sum_{i \in S \cup T} l_i + \sum_{i \in S \cap T} l_i, v(N) + \sum_{i \in S} l_i - \sum_{i \in N \setminus T} u_i + \sum_{i \in S \setminus T} (u_i - l_i),\right. \\
&\qquad \left. v(N) + \sum_{i \in T} l_i - \sum_{i \in N \setminus S} u_i + \sum_{i \in T \setminus S} (u_i - l_i), 2v(N) - \sum_{i \in N \setminus S} u_i - \sum_{i \in N \setminus T} u_i\right\} \\
&= \max\left\{\sum_{i \in S \cup T} l_i + \sum_{i \in S \cap T} l_i, v(N) + \sum_{i \in S \cap T} l_i - \sum_{i \in N \setminus (S \cup T)} u_i,\right. \\
&\qquad \left. v(N) + \sum_{i \in S \cap T} l_i - \sum_{i \in N \setminus (S \cup T)} u_i, 2v(N) - \sum_{i \in N \setminus (S \cup T)} u_i - \sum_{i \in N \setminus (S \cap T)} u_i\right\} \\
&\leq \max\left\{\sum_{i \in S \cup T} l_i, v(N) - \sum_{i \in N \setminus (S \cup T)} u_i\right\} + \max\left\{\sum_{i \in S \cap T} l_i, v(N) - \sum_{i \in N \setminus (S \cap T)} u_i\right\} \\
&= \widehat{v}(S \cup T) + \widehat{v}(S \cap T).
\end{aligned}$$

Hence, $\widehat{v} \in \Gamma_c^N$. □

The following example shows that there exist two-bound core games that are not convex, and that there exist convex games that are not two-bound core games.

Example 1 Let $N = \{1, 2, 3\}$ and let $v \in \Gamma_t^N$ be given by

$$v(S) = \begin{cases} 1 & \text{if } S \in \{\{1,2\},\{1,3\},\{1,2,3\}\}; \\ 0 & \text{otherwise.} \end{cases}$$

Then $C(v) = \{(1, 0, 0)\}$ but $v(\{1, 2\}) + v(\{1, 3\}) > v(\{1\}) + v(\{1, 2, 3\})$. Hence, $v \notin \Gamma_c^N$.

Now, let $N = \{1, 2, 3, 4\}$ and let $v \in \Gamma_c^N$ be given by

$$v(S) = \begin{cases} 2 & \text{if } S = N; \\ 1 & \text{if } S \in \{\{1,2\},\{1,2,3\},\{1,2,4\}\}; \\ 0 & \text{otherwise.} \end{cases}$$

Then $l^*(v) = (0, 0, 0, 0)$, $u^*(v) = (2, 2, 1, 1)$, and $(0, 0, 1, 1) \in [l^*(v), u^*(v)] \cap X(v)$, but $(0, 0, 1, 1) \notin C(v)$. Hence, $v \notin \Gamma_t^N$. △

The *reduced game* (cf. Davis and Maschler 1965) of $v \in \Gamma_t^N$ on $T \in 2^N \setminus \{\emptyset\}$ with respect to $x \in \mathbb{R}^N$, denoted by $v_T^x \in \Gamma^T$, is defined by

$$v_T^x(S) = \begin{cases} v(N) - \sum_{i \in N \setminus T} x_i & \text{if } S = T; \\ \max_{Q \subseteq N \setminus T} \left\{ v(S \cup Q) - \sum_{i \in Q} x_i \right\} & \text{if } S \in 2^T \setminus \{\emptyset, T\}; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

In other words, the worth of a coalition in a reduced game is defined as the maximal remainder in cooperation with any subgroup of players in the original game that are not present in the reduced game. It turns out that all reduced games of two-bound core games with respect to core elements are two-bound core games. Moreover, the core of these reduced games can be described by the same pair of bounds.

Theorem 2 *Let $v \in \Gamma_t^N$, $T \in 2^N \setminus \{\emptyset\}$, $x \in C(v)$, and let $l, u \in \mathbb{R}^N$ be such that $C(v) = [l, u] \cap X(v)$. Then*

$$C(v_T^x) = [l_T, u_T] \cap X(v_T^x).$$

Proof Let $y \in C(v_T^x)$. Then

$$\sum_{i \in T} y_i + \sum_{i \in N \setminus T} x_i = v_T^x(T) + \sum_{i \in N \setminus T} x_i = v(N) - \sum_{i \in N \setminus T} x_i + \sum_{i \in N \setminus T} x_i = v(N).$$

Let $S \in 2^N$. If $S \cap T = \emptyset$, then

$$\sum_{i \in S \cap T} y_i + \sum_{i \in S \setminus T} x_i = \sum_{i \in S} x_i \geq v(S).$$

If $S \cap T = T$, then

$$\sum_{i \in S \cap T} y_i + \sum_{i \in S \setminus T} x_i = v_T^x(T) + \sum_{i \in S \setminus T} x_i = v(N) - \sum_{i \in N \setminus T} x_i + \sum_{i \in S \setminus T} x_i = \sum_{i \in S} x_i \geq v(S).$$

If $S \cap T \notin \{\emptyset, T\}$, then

$$\begin{aligned} \sum_{i \in S \cap T} y_i + \sum_{i \in S \setminus T} x_i &\geq v_T^x(S \cap T) + \sum_{i \in S \setminus T} x_i \\ &= \max_{Q \subseteq N \setminus T} \left\{ v((S \cap T) \cup Q) - \sum_{i \in Q} x_i \right\} + \sum_{i \in S \setminus T} x_i \\ &\geq v(S) - \sum_{i \in S \setminus T} x_i + \sum_{i \in S \setminus T} x_i \\ &= v(S). \end{aligned}$$

This means that $(y, x_{N \setminus T}) \in C(v)$, so $(y, x_{N \setminus T}) \in [l, u] \cap X(v)$, which implies that $y \in [l_T, u_T] \cap X(v_T^x)$. Hence, $C(v_T^x) \subseteq [l_T, u_T] \cap X(v_T^x)$.

Let $y \in [l_T, u_T] \cap X(v_T^x)$. Then $(y, x_{N \setminus T}) \in [l, u] \cap X(v)$, so $(y, x_{N \setminus T}) \in C(v)$. Let $S \in 2^T \setminus \{\emptyset, T\}$. For all $Q \subseteq N \setminus T$,

$$\sum_{i \in S} y_i = \sum_{i \in S} y_i + \sum_{i \in Q} x_i - \sum_{i \in Q} x_i \geq v(S \cup Q) - \sum_{i \in Q} x_i,$$

so

$$\sum_{i \in S} y_i \geq \max_{Q \subseteq N \setminus T} \left\{ v(S \cup Q) - \sum_{i \in Q} x_i \right\} = v_T^x(S).$$

This implies that $y \in C(v_T^x)$. Hence, $[l_T, u_T] \cap X(v_T^x) \subseteq C(v_T^x)$. □

4 The core, the nucleolus, and the egalitarian core

In this section, we axiomatically characterize the core, the nucleolus, and the egalitarian core for two-bound core games using Davis-Maschler reduced game properties. A solution satisfies the *bilateral reduced game property* if the restriction of each pre-imputation assigned to the original game is consistently assigned to all reduced games with two players. The *converse reduced game property* requires that all pre-imputations for which each two-player restriction is assigned to the corresponding reduced game are assigned to the original game.

Bilateral reduced game property (cf. Davis and Maschler 1965) For all $v \in \Gamma_t^N$, all $T \in 2^N$ with $|T| = 2$, and all $x \in \varphi(v)$, we have $v_T^x \in \Gamma_t^T$ and $x_T \in \varphi(v_T^x)$.

Converse reduced game property (cf. Davis and Maschler 1965) For all $v \in \Gamma_t^N$ and all $x \in X(v)$, if $v_T^x \in \Gamma_t^T$ and $x_T \in \varphi(v_T^x)$ for all $T \in 2^N$ with $|T| = 2$, then $x \in \varphi(v)$.

By requiring the solution to assign the core to all games with two players, Peleg (1986) characterized the core using the bilateral reduced game property and the converse reduced game property. We obtain a similar axiomatic characterization of the core for two-bound core games.

Unanimity (cf. Peleg 1986) For all $v \in \Gamma_t^N$ with $|N| = 2$, we have $\varphi(v) = \{x \in X(v) \mid \forall i \in N : x_i \geq v(\{i\})\}$.

Theorem 3 *The core is the unique solution for two-bound core games satisfying unanimity, the bilateral reduced game property, and the converse reduced game property.*

Proof Clearly, the core satisfies unanimity. To prove that the core satisfies the bilateral reduced game property, let $v \in \Gamma_t^N$, let $T \in 2^N$ with $|T| = 2$, let $x \in C(v)$, and let $l, u \in \mathbb{R}^N$ be such that $C(v) = [l, u] \cap X(v)$. By Theorem 2, $C(v_T^x) = [l_T, u_T] \cap X(v_T^x)$. In view of $x_T \in [l_T, u_T] \cap X(v_T^x)$, this implies that $v_T^x \in \Gamma_t^T$ and $x_T \in C(v_T^x)$. Hence, the core satisfies the bilateral reduced game property.

To prove that the core satisfies the converse reduced game property, let $v \in \Gamma_t^N$ and let $x \in X(v)$ be such that $v_T^x \in \Gamma_t^T$ and $x_T \in C(v_T^x)$ for all $T \in 2^N$ with $|T| = 2$. Let $S \in 2^N \setminus \{\emptyset, N\}$ and let $j \in N \setminus S$. For all $i \in S$,

$$x_i \geq v_{\{i,j\}}^x(\{i\}) = \max_{Q \subseteq N \setminus \{i,j\}} \left\{ v(\{i\} \cup Q) - \sum_{k \in Q} x_k \right\} \geq v(S) - \sum_{k \in S \setminus \{i\}} x_k,$$

so $\sum_{i \in S} x_i \geq v(S)$. This implies that $x \in C(v)$. Hence, the core satisfies the converse reduced game property.

To prove uniqueness, let φ be a solution for two-bound core games satisfying unanimity, the bilateral reduced game property, and the converse reduced game property. We show that $\varphi(v) = C(v)$ for all $v \in \Gamma_t^N$. By unanimity, $\varphi(v) = C(v)$ for all $v \in \Gamma_t^N$ with $|N| \leq 2$. Let $v \in \Gamma_t^N$ with $|N| \geq 3$.

Let $x \in \varphi(v)$. By the bilateral reduced game property of φ, $v_T^x \in \Gamma_t^T$ and $x_T \in \varphi(v_T^x)$ for all $T \in 2^N$ with $|T| = 2$, so $x_T \in C(v_T^x)$ for all $T \in 2^N$ with $|T| = 2$. By the converse reduced game property of the core, this implies that $x \in C(v)$. Hence, $\varphi(v) \subseteq C(v)$.

Let $x \in C(v)$. By the bilateral reduced game property of the core, $v_T^x \in \Gamma_t^T$ and $x_T \in C(v_T^x)$ for all $T \in 2^N$ with $|T| = 2$, so $x_T \in \varphi(v_T^x)$ for all $T \in 2^N$ with

$|T| = 2$. By the converse reduced game property of φ, this implies that $x \in \varphi(v)$. Hence, $C(v) \subseteq \varphi(v)$. □

By requiring the solution to assign the nucleolus to all two-bound core games with two players, we obtain an axiomatic characterization of the nucleolus for two-bound core games using the bilateral reduced game property.

Standardness (cf. Aumann and Maschler 1985) For all $v \in \Gamma_t^N$ with $N = \{i, j\}$, we have

$$\varphi_i(v) = v(\{i\}) + \frac{1}{2}\left(v(N) - v(\{i\}) - v(\{j\})\right).$$

Lemma 1 (cf. Peleg 1986) *Let $v \in \Gamma^N$, let $T \in 2^N \setminus \{\emptyset\}$, let $i, j \in T$ with $i \neq j$, and let $x \in X(v)$. Then*

$$s_{ij}^{x_T}(v_T^x) = s_{ij}^x(v).$$

Theorem 4 *The nucleolus is the unique solution for two-bound core games satisfying standardness and the bilateral reduced game property.*

Proof It is known that the nucleolus satisfies standardness. To prove that the nucleolus satisfies the bilateral reduced game property and the converse reduced game property (used in the uniqueness part), let $v \in \Gamma_t^N$ and let $x \in X(v)$. By Theorem 1, there exists $\widehat{v} \in \Gamma_c^N$ such that $C(\widehat{v}) = C(v)$. This implies that $\eta(\widehat{v}) = \eta(v)$. By Lemma 1, $s_{ij}^{x_{\{i,j\}}}\left(\widehat{v}_{\{i,j\}}^x\right) = s_{ij}^x(\widehat{v})$ for all $i, j \in N$ with $i \neq j$. This implies that $x = \eta(\widehat{v})$ if and only if $\widehat{v}_T^x \in \Gamma_t^T$ and $x_T = \eta\left(\widehat{v}_T^x\right)$ for all $T \in 2^N$ with $|T| = 2$. By Theorem 2, if $x \in C(v)$ and $C(v) = C(\widehat{v})$, then $C\left(\widehat{v}_T^x\right) = C\left(v_T^x\right)$ for all $T \in 2^N$ with $|T| = 2$. By Theorem 3, $x \in C(v)$ if and only if $v_T^x \in \Gamma_t^T$ and $x_T \in C\left(v_T^x\right)$ for all $T \in 2^N$ with $|T| = 2$. Together, this implies that $x = \eta(v)$ if and only if $v_T^x \in \Gamma_t^T$ and $x_T = \eta\left(v_T^x\right)$ for all $T \in 2^N$ with $|T| = 2$. Hence, the nucleolus satisfies the bilateral reduced game property and the converse reduced game property.

To prove uniqueness, let φ be a solution for two-bound core games satisfying standardness and the bilateral reduced game property. We show that $\varphi(v) = \eta(v)$ for all $v \in \Gamma_t^N$. By standardness, $\varphi(v) = \eta(v)$ for all $v \in \Gamma_t^N$ with $|N| \leq 2$. Let $v \in \Gamma_t^N$ with $|N| \geq 3$ and let $x \in \varphi(v)$. By the bilateral reduced game property of φ, $v_T^x \in \Gamma_t^T$ and $x_T \in \varphi\left(v_T^x\right)$ for all $T \in 2^N$ with $|T| = 2$, so $x_T = \eta\left(v_T^x\right)$ for all $T \in 2^N$ with $|T| = 2$. By the converse reduced game property of the nucleolus, this implies that $x = \eta(v)$. Hence, $\varphi(v) = \eta(v)$. □

By Theorem 1, the core of a two-bound core game is equal to the core of a particular convex game. It can be shown that if two games have equal cores, then the games have equal egalitarian cores. Arin and Iñarra (2001) showed that the egalitarian core is single-valued for convex games. Together, this implies that the egalitarian core for two-bound core games is single-valued. We provide an explicit expression.

Theorem 5 *The egalitarian core of a two-bound core game $v \in \Gamma_t^N$ is given by*

$$EC_i(v) = \begin{cases} l_i^*(v) & \text{if } \lambda \leq l_i^*(v); \\ \lambda & \text{if } l_i^*(v) \leq \lambda \leq u_i^*(v); \\ u_i^*(v) & \text{if } \lambda \geq u_i^*(v), \end{cases}$$

for all $i \in N$, where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} EC_i(v) = v(N)$.

Proof Let $v \in \Gamma_t^N$. Then $|EC(v)| = 1$. Define $x \in \mathbb{R}^N$ by $x_i = \min\{\max\{l_i^*(v), \lambda\}, u_i^*(v)\}$ for all $i \in N$, where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} x_i = v(N)$. Then $x \in [l^*(v), u^*(v)] \cap X(v)$, so $x \in C(v)$. Suppose for the sake of contradiction that $x \notin EC(v)$. Then there exist $i, j \in N$ such that $x_i > x_j$ and $s_{ij}^x(v) \neq 0$. Then $x_i = l_i^*(v)$ or $x_j = u_j^*(v)$. Moreover, $s_{ij}^x(v) < 0$, so $v(S) < \sum_{k \in S} x_k$ for all $S \in 2^N$ with $i \in S$ and $j \notin S$. Let $0 < \varepsilon < -s_{ij}^x(v)$. Define $x' \in \mathbb{R}^N$ by $x_i' = x_i - \varepsilon$, $x_j' = x_j + \varepsilon$, and $x_k' = x_k$ for all $k \in N \setminus \{i, j\}$. Then $x' \in C(v)$, which contradicts the definition of $l_i^*(v)$ or $u_j^*(v)$. Hence, $x \in EC(v)$. □

By requiring the solution to assign the egalitarian core to all games with two players, Arin and Iñarra (2001) characterized the egalitarian core for games with a nonempty core using the bilateral reduced game property and the converse reduced game property. We obtain a similar axiomatic characterization of the egalitarian core for two-bound core games without requiring the converse reduced game property.

Constrained egalitarianism (cf. Dutta 1990) For all $v \in \Gamma_t^N$ with $N = \{i, j\}$, we have

$$\varphi_i(v) = \begin{cases} \max\left\{v(\{i\}), \tfrac{1}{2}v(N)\right\} & \text{if } v(\{i\}) \geq v(\{j\}); \\ v(N) - \varphi_j(v) & \text{if } v(\{i\}) < v(\{j\}). \end{cases}$$

Theorem 6 *The egalitarian core is the unique solution for two-bound core games satisfying constrained egalitarianism and the bilateral reduced game property.*

Proof It is known that the egalitarian core satisfies constrained egalitarianism. To prove that the egalitarian core satisfies the bilateral reduced game property and the converse reduced game property (used in the uniqueness part), let $v \in \Gamma_t^N$ and let $x \in X(v)$. By Theorem 3, $x \in C(v)$ if and only if $v_T^x \in \Gamma_t^T$ and $x_T \in C\left(v_T^x\right)$ for all $T \in 2^N$ with $|T| = 2$. By Lemma 1, $s_{ij}^{x_{\{i,j\}}}\left(v_{\{i,j\}}^x\right) = s_{ij}^x(v)$ for all $i, j \in N$ with $i \neq j$. Together, this implies that $x = EC(v)$ if and only if $v_T^x \in \Gamma_t^T$ and $x_T = EC\left(v_T^x\right)$ for all $T \in 2^N$ with $|T| = 2$. Hence, the egalitarian core satisfies the bilateral reduced game property and the converse reduced game property.

To prove uniqueness, let φ be a solution for two-bound core games satisfying constrained egalitarianism and the bilateral reduced game property. We show that $\varphi(v) = EC(v)$ for all $v \in \Gamma_t^N$. By constrained egalitarianism, $\varphi(v) = EC(v)$ for all $v \in \Gamma_t^N$ with $|N| \leq 2$. Let $v \in \Gamma_t^N$ with $|N| \geq 3$ and let $x \in \varphi(v)$. By the bilateral reduced game property of φ, $v_T^x \in \Gamma_t^T$ and $x_T \in \varphi\left(v_T^x\right)$ for all $T \in 2^N$ with $|T| = 2$, so $x_T = EC\left(v_T^x\right)$ for all $T \in 2^N$ with $|T| = 2$. By the converse reduced game property of the egalitarian core, this implies that $x = EC(v)$. Hence, $\varphi(v) = EC(v)$. □

5 Concluding remarks

In this paper, we axiomatically characterized the core, the nucleolus, and the egalitarian core for two-bound core games using the Davis–Maschler reduced game properties. In fact, it can be shown that these solutions satisfy the stronger reduced game property which requires that the restriction of each pre-imputation assigned to the original game is consistently assigned to all reduced games (not only with two players), but the weaker bilateral reduced game property suffices in the axiomatic characterizations. To show that the properties in these axiomatic characterizations are independent, we introduce the following additional solutions.

A solution that satisfies unanimity and the converse reduced game property, but not the bilateral reduced game property, is the solution $\widehat{X}$, which is for all $v \in \Gamma_t^N$ defined by

$$\widehat{X}(v) = \begin{cases} C(v) & \text{if } |N| \leq 2; \\ X(v) & \text{if } |N| \geq 3. \end{cases}$$

A solution that satisfies unanimity and the bilateral reduced game property, but not the converse reduced game property, is the solution $\widehat{C}$, which is for all $v \in \Gamma_t^N$ defined by

$$\widehat{C}(v) = \begin{cases} C(v) & \text{if } |N| \leq 2; \\ \eta(v) & \text{if } |N| \geq 3. \end{cases}$$

A solution that satisfies standardness, but not the bilateral reduced game property, is the solution $\widehat{\eta}$, which is for all $v \in \Gamma_t^N$ defined by

$$\widehat{\eta}(v) = \begin{cases} \eta(v) & \text{if } |N| \leq 2; \\ X(v) & \text{if } |N| \geq 3. \end{cases}$$

A solution that satisfies constrained egalitarianism, but not the bilateral reduced game property, is the solution $\widehat{EC}$, which is for all $v \in \Gamma_t^N$ defined by

$$\widehat{EC}(v) = \begin{cases} EC(v) & \text{if } |N| \leq 2; \\ X(v) & \text{if } |N| \geq 3. \end{cases}$$

An overview of these solutions, their properties, and the axiomatic characterizations is presented in the following table. Here, + indicates that the rule satisfies the property,

	C	η	EC	$\widehat{X}$	$\widehat{C}$	$\widehat{\eta}$	$\widehat{EC}$
Unanimity	$+^*$	$-$	$-$	$+$	$+$	$-$	$-$
Standardness	$-$	$+^*$	$-$	$-$	$-$	$+$	$-$
Constrained egalitarianism	$-$	$-$	$+^*$	$-$	$-$	$-$	$+$
Bilateral reduced game property	$+^*$	$+^*$	$+^*$	$-$	$+$	$-$	$-$
Converse reduced game property	$+^*$	$+$	$+$	$+$	$-$	$+$	$+$

$-$ indicates that the rule does not satisfy the property, and $*$ indicates the axiomatic characterizations.

Hence, the properties in Theorems 3, 4, and 6 are independent.

References

Arin J, Iñarra E (2001) Egalitarian solutions in the core. Int J Game Theory 30(2):187–193
Aumann R, Maschler M (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. J Econ Theory 36(2):195–213
Bondareva O, Driessen T (1994) Extensive coverings and exact core bounds. Games Econ Behav 6(2):212–219
Davis M, Maschler M (1965) The kernel of a cooperative game. Naval Res Logist Q 12(3):223–259
Dietzenbacher B (2018) Bankruptcy games with nontransferable utility. Math Soc Sci 92:16–21
Driessen T (1985) Properties of 1-convex n-person games. OR Spectr 7(1):19–26
Dutta B (1990) The egalitarian solution and reduced game properties in convex games. Int J Game Theory 19(2):153–169
Gong D, Dietzenbacher B, Peters H (2022) Two-bound core games and the nucleolus. Ann Oper Res. https://doi.org/10.1007/s10479-022-04949-0
Maschler M, Peleg B, Shapley L (1971) The kernel and bargaining set for convex games. Int J Game Theory 1(1):73–93
Muto S, Nakayama M, Potters J, Tijs S (1988) On big boss games. Econ Stud Q 39(4):303–321
O'Neill B (1982) A problem of rights arbitration from the Talmud. Math Soc Sci 2(4):345–371
Peleg B (1986) On the reduced game property and its converse. Int J Game Theory 15(3):187–200
Potters J, Poos R, Tijs S, Muto S (1989) Clan games. Games Econ Behav 1(3):275–293
Quant M, Borm P, Reijnierse H, van Velzen B (2005) The core cover in relation to the nucleolus and the Weber set. Int J Game Theory 33(4):491–503
Schmeidler D (1969) The nucleolus of a characteristic function game. SIAM J Appl Math 17(6):1163–1170
Shapley L (1971) Cores of convex games. Int J Game Theory 1(1):11–26