A simple construction of complete single-peaked domains by recursive tiling

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Abstract

Single-peakedness was introduced by Black (J Political Econ 56:23–34, 1948) as a sufficient condition to overcome Condorcet paradox. Since then it has been attracting interest from researchers in various fields. In this paper, we propose a simple recursive procedure of constructing complete single-peaked domains of tiling type explicitly for any finite alternative sets, by combining two results published in recent years, and some observations of known results and examples by the author. The underlying basic structure of tiling type and properties of single-peaked domains provided here give a good visualization and make further developments on single-peakedness more easy.

Keywords Condorcet paradox · Single-peaked domain · Rhombus tiling · Bruhat order

Mathematics Subject Classification D71 · C72

1 Introduction

Single-peaked domains, or single-peaked preferences, were introduced by Black (1948) (single peaked curves) as a sufficient condition to overcome Condorcet paradox (Black 1948, 1958), i.e., having a cyclic majority preferences. Given a single-peaked domain, a winner can be determined by a simple majority voting rule. Since then, single-peaked domains and related topics have been extensively researched, see the review paper (Monjardet 2009) and also a recently published paper by Puppe (2018).

As wrote in Fishburn (1997), and mentioned in Danilov et al. (2012): “...it is far from obvious how the restrictions should be selected jointly to produce a large acyclic

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sets”. Our contribution is a neatly simple procedure of constructing complete single-peaked domains of tiling type explicitly for any finite alternative sets. The underlying basic structure and properties of single-peaked domains provided here also give some insights and make further developments on single-peakedness more easy.

Our motivations for this research began with the random assignment problem (Fujishige et al. 2018). Single-peakedness is important to keep required properties for further generalization (Achuthankutty and Roy 2018; Bade 2019; Moulin 1980, 2017; Savaglio and Vannucci 2019). Having a better understanding of the structure of single-peaked domains leads to the simple recursive construction provided in the following. Single-peakedness is also connected to various topics. We give these related discussions in Concluding remarks.

The paper consists of two main sections. Section 2 is divided into two subsections. Section 2.1 gives some basic definitions and recent published results, Sect. 2.2 treats rhombus tilings. Section 3 consists of three subsections, observations, procedure and theorems respectively. They provide (together with validation) the recursive construction of complete single-peaked domains of tiling type. The final section gives concluding remarks.

2 Definitions and preliminaries

2.1 Single-peaked and Condorcet domains

Consider a finite set of alternatives X with a fixed linear order $>$ on it. Let $\mathcal{L}(X)$ denote the set of *all* preferences¹ $<$ on X . A domain $\mathcal{D} \subset \mathcal{L}(X)$ is *single-peaked with respect to the linear order* $>$ on X if, for all (strict) preferences $< \in \mathcal{D}$ and all $x \in X$, the upper contour sets $U_{<}(x) := \{y \in X \mid x < y\} \cup \{x\}$ are connected (‘convex’) viewing from the order $>$. In other words, suppose that $\bar{x} \in X$ is the peak/top of the preference, we have (i) $x' > x \geq \bar{x} \Rightarrow x' < x$, or (ii) $\bar{x} \geq x > x' \Rightarrow x' < x$.

In the following, we denote alternative set as $N = \{1, 2, \dots, n\}$ with the nature order. For $i, j \in N$, we write $i > j$ as ij if without confusion.² From now on, order refers to preference order if without special explanation.

It is known that the single-peaked domain in $\mathcal{L}(N)$ contains at most 2^{n-1} elements, or orderings (Black 1958; Abello and Johnson 1984).

Example 1 For $n = 4$, the single-peaked domain is the set $\{1234, 2134, 2314, 2341, 3214, 3241, 3421, 4321\}$. Note that 2431 is not the element of single-peaked domain. Since $U_{<}(4) = \{2, 4\}$ is not connected, it excludes the middle $\{3\}$. Or, the relation with peak $\{2\}$, $4 > 3 > 2$, but $3 < 4$, violates the definition (i) above.

A *profile* $\pi_m = (<_1, \dots, <_m)$ on domain $\mathcal{D}$ is an element of the Cartesian product $\mathcal{D}^m$ for some number $m \in \mathbb{N}$ of ‘voters’, where $<_\ell$ means the preference of ℓ th voter on alternative set N . For simplicity, suppose m being an odd number.

¹ It also is a linear order here.

² In some literature, ij is expressed in reverse, i is less preferred than j .

Fig. 1 The digraph of Bruhat poset for $n = 3$

The (*simple*) *majority relation* associated with a profile π_m is defined by $i \prec_{\pi_m}^{maj} j$ if (and only if) more voters ($> m/2$) prefer j than i . Here is an example of three voter's profile $\pi_3 = (123, 231, 312)$ on alternatives $\{1, 2, 3\}$. It is easy to check

$$1 \prec_{\pi_3}^{maj} 3 \prec_{\pi_3}^{maj} 2 \prec_{\pi_3}^{maj} 1,$$

i.e., a *majority cycle*, $i \prec_{\pi_m}^{maj}, \dots, \prec_{\pi_m}^{maj} i$ for some $i \in N$, occurring.

A domain $\mathcal{D} \subset \mathcal{L}(N)$ without two sub-domains $\{ijk, jki, kij\}$ and $\{kji, jik, ikj\}$ when restricted on any three alternatives i, j, k in N is a *Condorcet domain* (abbreviated as CD).³ Majority circle does not occur in Condorcet domains (Monjardet 2009). A single-peaked domain is the subset of a Condorcet domain.

Condorcet domains are characterized by the never conditions defined as follows. For h in $\{i, j, k\}$ and r in $\{1, 2, 3\}$, the *never conditions*⁴ $hN_{i,j,k}r$ means that h never has rank r . The single-peaked domain corresponds to, 'the middle never ranking the last'. Here 'the middle never ranking the best' is called *pit domain*. These two domains, will be discussed further in the following (sub)sections. For more details about CDs, please see the review paper (Monjardet 2009).

Let $\Omega = \{(i, j) \mid i, j \in N, i < j\}$. For a linear order σ , a pair $(i, j) \in \Omega$ is called an *inversion* (w.r.t. σ) if $j \prec_{\sigma} i$ and $i < j$. The set of inversions for σ is denoted by $\text{Inv}(\sigma)$. For linear orders $\sigma, \tau \in \mathcal{L}(N)$, we write $\sigma \ll \tau$ if $\text{Inv}(\sigma) \subseteq \text{Inv}(\tau)$. $(\mathcal{L}(N), \ll)$ is called the *Bruhat poset*. The *Bruhat digraph* is formed by drawing an arc from σ to τ if and only if τ covers σ .⁵

Figure 1 gives a digraph of Bruhat poset for $n = 3$. The four orders $\{123, 213, 231, 321\}$ on the left path form a single-peaked domain, the four orders $\{123, 132, 321, 312\}$ on the right path from a pit domain.

³ The definition here is also described as a property of majority acyclic in some literature.

⁴ These are also called *value restriction*, or *Latin-square-lessness* by Sen (1966).

⁵ A linear order τ *covers* a linear order σ if $\text{Inv}(\tau)$ has exactly one more inversion than $\text{Inv}(\sigma)$.

Now we introduce a theorem related to single-peaked domains in paper (Puppe 2018) published recently. It is one of two main results which our recursive procedure is based on.

A domain $\mathcal{D} \subset \mathcal{L}(N)$ is called *minimally rich* if for every $i \in N$, there exists an order $\sigma \in \mathcal{D}$ such that σ has i as the top preference. Note that a *maximal width* of the digraph of Bruhat poset means that associated domain $\mathcal{D}$ contains at least one pair of completely reversed orders. For example, $\alpha = 12 \dots (n-1)n$ and $\omega = n(n-1) \dots 21$, with $\text{Inv}(\alpha) = \emptyset$ and $\text{Inv}(\omega) = \Omega$.

Furthermore, the domain of all orders that are single-peakedness is denoted by $SP(N)$. A domain $\mathcal{D}$ is *semi-connected* if it contains two completely reversed orders, and a directed path connecting them in the associated digraph of the Bruhat poset $(\mathcal{D}, \ll)$. The connectedness of the Bruhat digraph is defined in the natural way. Now we are ready to present:

Theorem 1 (Theorem 1, Puppe 2018) *The domain $SP(N)$ of all single-peaked orders is a connected and minimally rich Condorcet domain with maximal width. (In particular, $SP(N)$ is semi-connected.)*

Conversely, if $\mathcal{D} \subset \mathcal{L}(N)$ is a semi-connected and minimally rich Condorcet domain, then $\mathcal{D}$ is single-peaked.

2.2 Representing CDs by rhombus tiling

The characterizations of Condorcet domains by the rhombus tiling given by Danilov et al. in paper (2012) and related papers (Danilov et al. 2007, 2010) develop and unify several related properties known earlier. The representation by the rhombus tiling also admits a simple and good visualization. The notations and contents of this subsection are mainly adopted from the paper (Danilov et al. 2012).

By the parallel property of arcs of rhombus tiles, tilings can be represented via a vector summation. Precisely, let $\xi_i \in \mathbb{R} \times \mathbb{R}_{>0}$ ($i \in N$) be vectors in the upper half-plane. The sum of segments $[0, \xi_i]$ of $(\xi_1, \xi_2, \dots, \xi_n)$ forms a *zonogon*, denoted by $Z = Z_n$. This is the symmetric $2n$ -gon formed by the points $\sum_{i=1}^n a_i \xi_i$ over all $0 \leq a_i \leq 1$ ($1 \leq i \leq n$), with the *source* vertex (point $(0, 0)$) at the bottom, and the *sink* vertex (point $\xi_1 + \xi_2 + \dots + \xi_n$) at the top. A rhombus congruent to the sum of two segments $[0, \xi_i]$ and $[0, \xi_j]$ is called an *ij-tile*, or simply a *tile*, where $1 \leq i < j \leq n$.

A *rhombus tiling* (or simply a *tiling*), denoted by T , is a subdivision of a zonogon into a set of *rhombus tiles* satisfying the following condition: if two tiles intersect, then their intersection consists of a single vertex or a single edge. Figure 2 shows two examples of rhombus tilings. We label the arcs of tilings with corresponding alternative $i \in N$ (same labels of parallel arcs are omitted if they are obvious to readers).

Using the language of graph in the following, we associate to a tiling T with the planar directed graph $G_T = (V_T, A_T)$ whose vertices and edges are those occurring in the tile T . Note that graph G_T , or tiling T is an acyclic graph since all arcs are oriented upward. Figure 2, as well as Figs. 4 and 5 in Sect. 3, all are examples of graphic tilings.

An maximal directed path is a path going from the source to the sink by the acyclicity of G_T . We call such path a *snake* of T . The labels on the directed path of each snake

Fig. 2 Two tilings of the zonogon of size 3

form a linear order, or a permutation of alternatives of N (Elnitsky 1997; Danilov et al. 2012). The zonogon, or the summation of upward vector segments in a plane, is bounded by two special snakes, the left boundary $\text{lbd}(Z)$ and right boundary $\text{rbd}(Z)$ of Z , the former gives a linear order α , and the latter gives ω .

For a directed path of G_T , or a linear order, if instead of two arcs of a tile on the path, by adding the other two arcs of the tile, we get a new path that differs one inversion in the linear order. Hence we can view a tile as a permutation of two adjacent, or consecutive alternatives. Moreover, right snakes have more inversion pairs than left ones, and the set of snakes is a Bruhat poset. And all snakes of a tiling also form a lattice. See more details in papers (Elnitsky 1997; Danilov et al. 2012).

Let us denote the set of linear orders associated with the snakes in tiling T by $\Sigma(T)$. In our examples shown above, $\Sigma(T)$ of tiling T in Fig. 2 associates with the set of four orders on the left path of Fig. 1; and $\Sigma(T')$, the four orders on the right path of Fig. 1.

The preference domain represented by a tiling is majority acyclic, i.e., a Condorcet domain. Since restricting every three alternatives leads to two tiles show in Fig. 2, they are peak-pit domains. Formally, a set $\mathcal{D} \subset \mathcal{L}(N)$ is called *peak-pit* domain if for each triple $i < j < k$ in N satisfies never conditions: j never ranking the last, best, or both (Danilov et al. 2012).

We say that domain $\mathcal{D}$ is *complete* if it is inclusion-wise maximal, i.e., adding to $\mathcal{D}$ any new linear order violates the majority acyclicity.

Theorem 2 (Theorem 1, Danilov et al. 2012) *The set $\Sigma(T)$ of tiling T is a peak-pit Condorcet domain and also complete.*

Proof of the converse of Theorem 2 is less trivial. The never conditions of Condorcet domain defined in Section 2.1 relate to the separation theorems in algebra, via the reduced words of permutations, or the transpositions of symmetric groups, and Plücker functions (Leclerc and Zelevinsky 1998), also see papers (Danilov et al. 2007, 2010). The following result also has been shown in the paper (Danilov et al. 2012).

Theorem 3 (Theorem 2, Danilov et al. 2012) *Each peak-pit domain is a domain of tiling type.*

Fig. 3 Tilings for single-peaked domains from previous results

3 A recursive construction of complete single-peaked domains by tiling

In this section, we provide our main result, an explicit construction of complete single-peaked domains of tiling type of any cardinality size, based on the results introduced in last section and some observations shown in the following.

3.1 Some observations of known results and examples

We first define an associated tiling for a preference domain. For a (complete peak-pit) domain $\mathcal{D} \subset \mathcal{L}(N)$, let us denote the associated tiling T such that $\Sigma(T) = \mathcal{D}$ by $T_{\mathcal{D}}$.

Observation 1 Suppose that a domain $\mathcal{D}$ is single-peaked and $T_{\mathcal{D}}$ is its associated tiling. Then, for alternative set $N = \{1, \dots, n\}$, the minimally rich condition given in Theorem 1 means that there should be n arcs connected to the source of $T_{\mathcal{D}}$. On the other hand, by the never condition of single-peaked domains, ‘middle can not be the least preference’, only the first and the last, $\{1, n\}$, of alternative set N can be the last element of any sink-peaked liner order. Hence, the arcs connected to the sink of $T_{\mathcal{D}}$ are two alternatives only, $\{n\}$ and $\{1\}$. We summarize and illustrate these facts in Fig. 3.

The above conclusions, i.e., the results have been known and obtained, do not define what are the exact tilings for single-peaked domains, and how to build these tilings. We will approach the problem with tilings of lower cardinality size of single-peaked domains.

We have seen an associated tiling $T_{\mathcal{D}}$ for single-peaked domain $\mathcal{D}$ with $n = 3$, i.e., T in Fig. 2. The tiling for Example 1 given in Sect. 2.1 is drawn in Fig. 4. We also add two trivial cases, $n = 1$ and $n = 2$ for the completeness of the discussions in the following.

Fig. 4 Three tilings of single-peaked domains for $n = 1$, $n = 2$ and $n = 4$

Fig. 5 The tiling of single-peaked domain for $n = 5$

For the single-peaked domain of size 5, it is the tiling⁶ introduced by Elnitsky (1997) (Fig. 8 of the paper) as a tiling example, we redraw it in Fig. 5 here for comparing and better understanding.⁷

3.2 Recursive procedure

Our construction needs notations specifically indexed by recursive procedure steps $n \geq 1$.

Denote a *single-peaked domain* (of all orders) of tiling type with respect to alternative set $N = \{1, \dots, n\}$ by $\mathcal{SP}_n \subset \mathcal{L}(N)$. Note that we have $\Sigma(T_{\mathcal{SP}_n}) = \mathcal{SP}_n$. We use T_{RP_n} ($n \geq 1$) for the tilings obtained from Recursive Procedure and in the proof of Theorem 4.

⁶ The associated order set of the example, $\Sigma(T)$, is also known as higher, or second, Bruhat order (Galambos and Reiner 2008; Ziegler 1993).

⁷ Actually, the author gets the hint from these examples.

Fig. 6 Step 1 of the recursive procedure

Recursive Procedure

Step 1 T_{RP_1} , or T_{SP_1} has two vertices, the source and the sink, and one *vertical* arc from the source and the sink (see the left diagram of Fig. 6).

To construct T_{RP_2} , or T_{SP_2} , first, add another vertical arc which parallels to, and has the same direction, length and label as T_{SP_1} . Then, the end of the new vertical arc becomes the new sink of tiling, and the original source remains to be the source of new tiling (see the middle diagram of Fig. 6).

Draw two other parallel arcs in the same direction, both are labeled by alternative $\{2\}$, one links from the source of T_{SP_1} to the head of the new vertical arc, while the other links the ends of two vertical arcs (see the right diagram of Fig. 6).

Step $n - 1$ Identify $\text{rbd}(T_{RP_{n-1}})$, the right boundary of $T_{RP_{n-1}}$ constructed in the previous procedure (see the left diagram of Fig. 7). To construct T_{RP_n} , first, add another path which parallels to and has the same direction, length and labels as $\text{rbd}(T_{RP_{n-1}})$. Then, the end of the new path becomes the new sink, and the original source remains to be the source of new tiling (see the middle diagram of Fig. 7).

Draw n other paralleling arcs in the same direction, all are labeled by alternative $\{n\}$, the first arc links from the original source to the head of the new path, while every other arc links the ends of the two paralleling arcs between $\text{rbd}(T_{RP_{n-1}})$ and the new path (see the right diagram of Fig. 7).

In the above Step $n - 1$, note that we can also change the $\text{rbd}(T_{RP_{n-1}})$ to $\text{lbd}(T_{RP_{n-1}})$, the left boundary of $T_{RP_{n-1}}$, and add $\text{lbd}(T_{RP_{n-1}})$ to the left side of $T_{RP_{n-1}}$. T_{RP_n} ($n > 1$) are symmetric, $\text{rbd}(T_{RP_n})$, or $\text{lbd}(T_{SP_n})$, is not crucial here.

3.3 Validation and properties of Recursive Procedure

In this subsection, we show that for any given alternative set $N = \{1, \dots, n\}$, T_{RP_n} obtained from Recursive Procedure satisfies

$$\Sigma(T_{RP_n}) = \mathcal{SP}(N) \quad (n \geq 1).$$

Fig. 7 Step $n - 1$ of recursive procedure

The claim is shown by two theorems, we also discuss some properties.

Theorem 4 For any alternative sets $N = \{1, 2, \dots, n\}$,⁸ the tiling $T_{\mathcal{SP}_n}$ of single-peaked domains $\mathcal{SP}_n \subset \mathcal{L}(N)$ can be explicitly constructed for every integer $n \geq 1$ in a recursive process.

Proof A construction procedure has been provided in Sect. 3.2. We will show that T_{RP_n} obtained from Recursive Procedure being single-peaked domains, i.e., $\Sigma(T_{RP_n}) = \Sigma(T_{\mathcal{SP}_n})$ for any integers $n \geq 1$.

Since our Recursive Procedure is the CD domains of tiling type, by Theorem 2 we have $\Sigma(T_{RP_n})$ for $n \geq 1$ are peak-pit domains,

We now prove that the tilings T_{RP_n} being semi-connected. It is clear that each T_{RP_n} contains two boundaries, $\text{lbd}(T_{RP_n})$ and $\text{rbd}(T_{RP_n})$, which are completely reversed. The connection paths between $\text{lbd}(T_{RP_n})$ and $\text{rbd}(T_{RP_n})$ in the digraph induced by Bruhat poset can be obtained according to a similar discussion given in Elnitsky (1997). If a snake, or a preference order, say A , is not $\text{rbd}(T_{RP_n})$, there exists at least one tile in the right of snake A , having two arcs of T_{RP_n} , say i and j ($i < j$), common with the snake A . Replacing two arcs of ij tile in snake A with two other arcs of the tile, we obtain a new snake, say B , with $\text{Inv}(B) = \text{Inv}(A) \cup (i, j)$. Hence the order induced by B covers the order by A , and there is an arc from A to B in the Bruhat digraph. Beginning with $\text{lbd}(T_{RP_n})$, continuing the above process until we reach $\text{rbd}(T_{RP_n})$. Therefore, tiling T_{RP_n} is semi-connected.

It is easy to see that each step of Recursive Procedure increases one arc adjacent to the source. Beginning from Step 1, there are n arcs adjacent to the source of T_{RP_n} ($n \geq 1$). Recall Observation 1, we know that the domain $\Sigma(T_{RP_n})$ satisfying minimally rich condition.⁹

Now all the conditions for single-peakedness needed by Theorem 1 have been satisfied.

⁸ Here, the alternative set N can be replaced by any linear order set X after some slight refinements.

⁹ Note also that only $\{1, n\}$ are adjacent to the sink of T_{RP_n} for $n > 1$.

By definition, the single-peaked domain $\Sigma(T_{\mathcal{SP}_n})$ is a peak-pit domain. It can be contained in a domain of tiling type as indicated in Theorem 3, which is exactly the tiling T_{RP_n} obtained from our Recursive Procedure for any integers, i.e., T_{RP_n} coinciding with $T_{\mathcal{SP}_n}$ for $n \geq 1$. $\square$

Remark 1 The directed graph $G_{T_{RP_n}}$, or $G_{T_{\mathcal{SP}_n}}$ obtained from Recursive Procedure is consist of $1 + n + (n - 1) + \cdots + 1 = 1 + \frac{n(n+1)}{2}$ vertices and $n \times n$ arcs. From the proof of Theorem 4, we know that the number of tiles in $T_{\mathcal{SP}_n}$ is the diameter of the Bruhat digraphs associated with single-peaked domains $\mathcal{SP}_n$ for $n > 1$. The different tiles intersecting two arcs with snake A chosen in the proof of Theorem 4 leads to different sets of orders (see also Elnitsky 1997).

Here is a theorem about the completeness of $\mathcal{SP}_n$.

Theorem 5 *For any alternative sets $N = \{1, 2, \dots, n\}$, if $\mathcal{SP}_n$ ($n \geq 1$) are the single-peaked domains obtained from Recursive Procedure, then we have $\mathcal{SP}_n = \mathcal{SP}(N)$.*

Proof We have shown in Theorem 4 that T_{RP_n} ($n \geq 1$) obtained by Recursive Procedure are single-peaked domains $\mathcal{SP}_n$ of tiling type. By Theorem 3, we know that the tiling type of CD domains is peak-pit complete. Since single-peaked domains $\mathcal{SP}_n$ are peak-pit domains. We have $\mathcal{SP}_n$ being complete, i.e., $\mathcal{SP}_n = \mathcal{SP}(N)$ for all $n \geq 1$. $\square$

Remark 2 From the proof of Theorem 5, we know that $\mathcal{SP}_n$ ($n \geq 1$) are not only complete in single-peakedness, but also complete on peak-pit domains. Let σ be a linear order, $\mathcal{SP}_n \cup \{\sigma\}$ is cyclic if σ is a pit order.¹⁰ But this does not include general CD domains. Actually, for a given set of alternatives, the maximal cardinality of a CD domain is in general strictly larger than the cardinality of the single-peaked domain.

Finally, let us see how the single-peaked preferences are constructed recursively from Recursive Procedure.

Proposition 1 *For each alternative set $N = \{1, 2, \dots, n\}$ with $n > 2$, the following three types of single-peaked preferences are created recursively by $T_{\mathcal{SP}_n}$ in Recursive Procedure:*

- (i) Join arc labelled by the new alternative from sink to new sink
- (ii) Move from source immediately to the added “rbd-copy”
- (iii) Move to “rbd-copy” later via “rbd”.

Moreover, the number of orderings of type (i) is equal to $|\Sigma(T_{\mathcal{SP}_{n-1}})|$.

Proof For type (i), it is clear from the construction. The operation creates all the orderings that have the new alternative at the bottom.

For types (ii) and (iii), the key point is: for each snake of $T_{\mathcal{SP}_n}$, or single-peaked preference, once the snake entering “rbd-copy” newly added in Step $n - 1$, it ends within “rbd-copy”.

¹⁰ Note $\mathcal{SP}_n \cup \{\sigma\} = \mathcal{SP}_n$ if σ is a peak order.

Precisely, type (ii) creates the (unique) ordering that puts the new alternative on top. For type (iii), after each snake entering from a vertex of “rbd” of $T_{\mathcal{SP}_{n-1}}$ via the arc labelled by the new alternative, it just follows the vertex of “rbd-copy” continuously. Hence, we can simply view the operation as inserting a new alternative to the orderings intersection with “rbd” in $\mathcal{SP}_{n-1}$. Hence, type (iii) creates an ordering that puts the new alternatives somewhere in the middle.

For the part of “Moreover”, since each snake, or the ordering of single-peaked of tiling type, ends at the sink. $\square$

4 Concluding remarks

In this paper, we have shown a simple recursive procedure of constructing complete single-peaked domains of tiling type explicitly for any finite alternative sets. The underlying basic structure and properties of single-peaked domains provided here give some insights and make further developments on single-peakedness more easy.

Acyclic and single-peaked domains have been researched from different subjects. Beside of majority acyclic properties, strategy-proofness is another main subject of single-peakedness, as shown in papers early (Moulin 1980) and recently (Achuthankutty and Roy 2018; Bade 2019; Moulin 2017; Savaglio and Vannucci 2019). Single-peaked domain, as a subset of Condorcet domain of preference orders, it also bears some more properties related to topology. Tiling representation presented here, paper (Danilov et al. 2012), and the arrangement of pseudolines (Galambos and Reiner 2008; Danilov et al. 2012) are some of them. The higher Bruhat orders mentioned in Fig. 5 are related to oriented matroids (Björner et al. 2008; Ziegler 1993) with additional general structures.

As mentioned in the Introduction, our work is motivated by random assignment problems (Fujishige et al. 2018; Moulin 2017) related to strategy-proofness. The recent paper given by Puppe (2018) also hints a relation to the structure of the base polyhedra of submodular system (Fujishige 2005; Zhan 2005), which our extended assignment problems are based on Fujishige et al. (2018).

The authors of Theorems 2 and 3 also published other related papers (Danilov et al. 2007, 2010) on Plücker functions and their basis, and the separated set-systems. These also are mentioned in oriented matroids, and seem to have some relations with the recursive structure provided here.

Exploring the relation among these mentioned above will be our future work.

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