ORIGINAL ARTICLE

Carlos González-Alcón · Peter Borm
Ruud Hendrickx

A composite run-to-the-bank rule for multi-issue allocation situations

Abstract In this paper, we propose a new extension of the run-to-the-bank rule for bankruptcy situations to the class of multi-issue allocation situations. We show that this rule always yields a core element and that it satisfies self-duality. We characterise our rule by means of a new consistency property, issue-consistency.

1 Introduction

In a bankruptcy situation (O'Neill 1982), one has to divide a given amount of money (estate) amongst a set of agents, each of whom has a claim on the estate. The total amount claimed typically exceeds the estate available, so not all the claims of the agents can be fully satisfied. Calleja et al. (2005) extend this model to encompass situations in which the agents can have multiple claims on the estate, each as a result of a particular issue. For such multi-issue allocation (MIA) situations they propose an extension of the run-to-the-bank (RTB) rule of O'Neill as solution for this new class of problems. As is the case for the original rule, this extended RTB rule turns out to coincide with the Shapley value of the corresponding MIA game.

C. González-Alcón (✉)
Departamento de Estadística, I.O. y Computación, Universidad de La Laguna,
38071 La Laguna, Spain
E-mail: cgalcon@ull.es

P. Borm · R. Hendrickx
CentER and Department of Econometrics and Operations Research, Tilburg University,
PO Box 90153, 5000 LE Tilburg, The Netherlands
E-mail: P.E.M.Borm@uvt.nl; ruud@uvt.nl

Contrary to bankruptcy games, however, MIA games need not be convex. Consequently, there exist MIA situations for which the RTB solution is not a core element of the corresponding game. In this paper, we extend the RTB rule in a different way, such that it always yields a core element.

Instead of considering the issues and the players combined, as in Calleja et al. (2005), we propose a two-stage extension: first, we explicitly allocate the estate to the issues (according to a marginal vector), and then, within each issue the money is divided among the agents using the standard RTB rule. An alternative view on composite solution is given in Casas-Méndez et al. (2005).

Based on Aumann and Maschler (1985), we define the concept of self-duality for MIA situations and show that the composite RTB rule is self-dual. Finally, we characterise our composite extension by means of the property of issue-consistency, which generalises the consistency property that was first used by O'Neill (1982).

This paper is organised as follows. In Sect. 2, we present the bankruptcy and MIA models and define the bankruptcy RTB rule. In Sect. 3, we define our composite extension of this rule and show that this rule always yields a core element. In Sect. 4, we define self-duality and prove that the composite RTB rule satisfies this property. Finally, in Sect. 5, we characterise the composite RTB rule by means of issue-consistency and we show that this rule is estate monotonic.

2 Multi-issue allocation situations

A *bankruptcy situation* (O'Neill 1982) is a triple (N, E, c), where N is a finite set of n players, $E \geq 0$ is the estate under contest and $c \in \mathbb{R}_+^N$ is the vector of claims such that $\sum_{i \in N} c_i \geq E$.

With each bankruptcy situation (N, E, c) a *bankruptcy game* can be associated with set of players N and characteristic function $v_{E,c}$, which assigns to each coalition $S \subset N$ the part of the estate that is left for the players in S after the claims of the other players have been satisfied, i.e.,

$$v_{E,c}(S) = \max \left\{ 0, E - \sum_{i \in N \setminus S} c_i \right\}$$

for all $S \subset N$. A nice overview of the bankruptcy literature is provided by Thomson (2003).

A MIA situation (Calleja et al. 2005) is a quadruplet (N, R, E, C), where N, as above, is the set of players, R is the finite set of r issues, $E \geq 0$ is the estate and $C \in \mathbb{R}_+^{R \times N}$ is the matrix of claims. We assume that $\sum_{k \in R, i \in N} c_{ki} \geq E$, $\sum_{k \in R} c_{ki} > 0$ for all $i \in N$ and $\sum_{i \in N} c_{ki} > 0$ for all $k \in R$.

Given a matrix C, we denote by C_k the kth row of C, and by C_{-k} the matrix C without the kth row. Furthermore, we denote $c_{kS} = \sum_{i \in S} c_{ki}$ for $S \subset N$ and $c_{Ki} = \sum_{k \in K} c_{ki}$ for $K \subset R$. In this way, the sum of the components of C_k is denoted by c_{kN}, and the total claim of player $i \in N$ is c_{Ri}.

A permutation τ on R is a bijection $\tau : \{1, \ldots, r\} \to R$, where $\tau(p)$ denotes which element of R is at position p. The set of all $r!$ permutations on R is denoted

by $\Pi(R)$. The reverse permutation of τ, $\tau_{rev} \in \Pi(R)$, is defined by $\tau_{rev}(p) = \tau(r + 1 - p)$ for all $p \in \{1, \ldots, r\}$.

For a MIA situation (N, R, E, C), we define a corresponding *MIA game* by assigning to each coalition S the minimum amount they can guarantee themselves if the players in $N \backslash S$ are free to choose an order on the issues and the players, where we assume that an issue cannot be dealt with until the previous one is completed. Given an order on the issues $\tau \in \Pi(R)$, we define the index t_τ by

$$t_\tau = \max \left\{ t \mid \sum_{p=1}^{t} c_{\tau(p)N} \leq E \right\}. \tag{1}$$

So, the issues $\tau(1), \ldots, \tau(t_\tau)$ will be entirely satisfied for all the players, whereas the issue $\tau(t_\tau + 1)$ will only be partially satisfied (if any), with the amount $E_\tau = E - \sum_{p=1}^{t_\tau} c_{\tau(p)N}$. The remaining issues are not handled at all according to τ. In this way, the amount $f_S(\tau)$ which coalition S gets at least equals

$$f_S(\tau) = \sum_{p=1}^{t_\tau} c_{\tau(p)S} + \max\{0, E_\tau - c_{\tau(t_\tau+1),N\backslash S}\}.$$

Then the MIA game is given by the function

$$v_{E,C}(S) = \min_{\tau \in \Pi(R)} f_S(\tau)$$

for all $S \subset N$. Note that this MIA game corresponds to the Q-approach in Calleja et al. (2005).

A bankruptcy situation (N, E, c) can be viewed as a MIA situation (N, R, E, C) in two ways

1. $|R| = 1$ and $C = c$,
2. $R = N$ and $C = \text{diag}(c)$, i.e., the claim matrix is the diagonal matrix with the elements of c on the diagonal.

A *bankruptcy rule* is a function f assigning to every bankruptcy situation (N, E, c) a vector $f(N, E, c) \in \mathbb{R}^N$ such that

1. $0 \leq f_i(N, E, c) \leq c_i$ for all $i \in N$,
2. $\sum_{i \in N} f_i(N, E, c) = E$.

A well-known example of a bankruptcy rule is the RTB rule, introduced by O'Neill (1982), although under a different name (recursive completion). This rule turns out to coincide with the Shapley value of the corresponding bankruptcy game. In Sect. 3 we give a definition of this rule.

A *MIA rule* is a function g assigning to every MIA situation (N, R, E, C) a vector $g(N, R, E, C) \in \mathbb{R}^N$ such that

1. $0 \leq g_i(N, R, E, C) \leq c_{Ri}$ for all $i \in N$,
2. $\sum_{i \in N} g_i(N, R, E, C) = E$.

Calleja et al. (2005) extends the RTB rule to the class of MIA situations and shows that this RTB rule coincides with the Shapley value of the corresponding MIA game.

3 The composite RTB rule

In this section, we extend the RTB rule for bankruptcy situations to the class of MIA situations. Contrary to the extension in Calleja et al. (2005), our rule (mRTB) involves multiple runs to the bank, one by the issues and within each issue by the players.

In order to introduce the mRTB rule, we first define the RTB rule in terms of marginal vectors. Given a cooperative game with player set N and characteristic function v, we define for each permutation $\sigma \in \Pi(N)$ the *marginal vector* $m^\sigma(v)$ by

$$m^\sigma_{\sigma(p)}(v) = v(\{\sigma(1), \ldots, \sigma(p)\}) - v(\{\sigma(1), \ldots, \sigma(p-1)\})$$

for all $p \in \{1, \ldots, n\}$.

The *RTB* rule for bankruptcy situations (cf. O'Neill 1982) coincides with the Shapley value and can thus be expressed as

$$\text{RTB}(N, E, c) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v_{E,c}).$$

Let (N, R, E, C) be a MIA situation. We denote the bankruptcy game corresponding to the situation $(R, E, (c_{kN})_{k \in R})$ by $v^R_{E,C}$.

For $\tau \in \Pi(R)$ and $\sigma \in \Pi(N)$, we define the *composite marginal vector* as

$$mm^{\tau,\sigma}(N, R, E, C) = \sum_{k \in R} m^\sigma(v_{x_k, C_k}),$$

where $x = m^\tau(v^R_{E,C})$.

The set of all composite marginal vectors is a subset of the core of the corresponding MIA game, where the core of a game v is defined by

$$\text{Core}(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \ \forall S \subset N : \sum_{i \in S} x_i \geq v(S) \right\}.$$

Proposition 1 *Let (N, R, E, C) be a MIA situation. Then*

$$mm^{\tau,\sigma}(N, R, E, C) \in \text{Core}(v_{E,C})$$

for all $\tau \in \Pi(R)$, $\sigma \in \Pi(N)$.

Proof Let $\tau \in \Pi(R), \sigma \in \Pi(N)$ and let $z = mm^{\tau,\sigma}(N, R, E, C)$. Let x be the marginal vector $m^\tau(v^R_{E,C})$ and $t = t_{\tau_{\text{rev}}}$ [as defined in (1)]. With x_k as estate for issue $k \in R$, we have a collection of bankruptcy situations $\{(N, x_k, C_k)\}_{k \in R}$. However, at most one of them is a nontrivial situation: in the situations $\tau(1), \ldots, \tau(r-t-1)$ the estate equals zero, and in the situations $\tau(r-t+1), \ldots, \tau(r)$ the estate equals the sum of all the claims. Let y be the marginal vector corresponding to σ of the only possible nontrivial bankruptcy situation $(N, x_{\tau(r-t)}, C_{\tau(r-t)})$:

$$y = m^\sigma(v_{x_{\tau(r-t)}, C_{\tau(r-t)}}).$$

We can express z as

$$z = y + \sum_{p=1}^{t} C_{\tau_{rev}(p)}.$$

Obviously, by construction, $\sum_{i \in N} z_i = E = v_{E,C}(N)$. Next, let $S \subset N$. Then $\sum_{i \in S} z_i \geq f_S(\tau_{rev})$, because in issue $\tau(r - t)$, the members of S receive at least $\max\{0, E_{\tau_{rev}} - c_{\tau(r-t), N \setminus S}\}$ according to y. From this, $\sum_{i \in S} z_i \geq v_{E,C}(S)$ follows and hence, $z \in \text{Core}(v_{E,C})$. □

A general relation of inclusion between the set of marginal vectors and the set of composite marginal vectors cannot be established, as it is shown in the following example.

Example 1 Let (N, R, E, C) be the MIA situation with $N = \{1, 2, 3\}$, $R = \{1, 2\}$, estate $E = 10$, and claim matrix $C = \left(\begin{smallmatrix} 9 & 5 & 0 \\ 3 & 7 & 7 \end{smallmatrix}\right)$. The game associated with this situation is

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	N
$v_{E,C}(S)$	0	0	0	3	3	1	10

The sets of marginal and composite marginal vectors can be easily calculated. The results are given in the following tables.

$\sigma \in \Pi(N)$	$m^\sigma(v_{E,C})$	$\tau \in \Pi(R)$	$\sigma \in \Pi(N)$	$mm^{\tau,\sigma}(N, R, E, C)$
123	$(0, 3, 7)$	12	123	$(0, 3, 7)$
132	$(0, 7, 3)$		132	$(0, 7, 3)$
213	$(3, 0, 7)$		213, 231	$(3, 0, 7)$
231	$(9, 0, 1)$		312, 321	$(3, 7, 0)$
312	$(3, 7, 0)$	21	123, 132, 312	$(5, 5, 0)$
321	$(9, 1, 0)$		213, 231, 321	$(9, 1, 0)$

The tables show that $m^{231}(v_{E,C})$ is not a composite marginal vector and that $mm^{21,123}(N, R, E, C)$ does not belong to the set of marginal vectors of the game $v_{E,C}$. □

Now we define the mRTB rule, which extends the RTB rule for bankruptcy situations to the class of MIA situations.

Definition 1 *Let (N, R, E, C) be a MIA situation. The mRTB rule is defined by*

$$m\text{RTB}(N, R, E, C) = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} \text{RTB}(N, m_k^\tau(v_{E,C}^R), C_k). \qquad (2)$$

The mRTB rule can be interpreted as the result of two races: first, the issues "run to the bank" for the money, and next, there are r races among the claimants within each issue. As is the case for the RTB rule for bankruptcy situations, the claims are satisfied as much as possible by the order of arrival.

This mRTB rule first takes the marginal vectors of the "issue game" $v_{E,C}^R$. Associated with each marginal vector $m^\tau(v_{E,C}^R)$ we have r bankruptcy games whose

estates are given by the components of the marginal vector. Next, we take for each player the sum of the RTB solutions of these r situations. Finally, the average among all the marginals is computed. It is readily seen that the mRTB rule can be expressed as

$$mRTB(N, R, E, C) = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \frac{1}{n!} \sum_{\sigma \in \Pi(N)} mm^{\tau,\sigma}(N, R, E, C). \qquad (3)$$

If we start with a bankruptcy situation (N, E, c) and construct one of the two corresponding MIA situations (N, R, E, C) as indicated in Sect. 2, then $\text{RTB}(N, E, c) = mRTB(N, R, E, C)$. So, the mRTB rule is indeed an extension of the RTB rule. It follows immediately from the definition of the Shapley value Sh (cf. Shapley 1953) that

$$mRTB(N, R, E, C) = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} \text{Sh}(N, v_{x_k, C_k}),$$

where $x = m^\tau(v_{E,C}^R)$. However, the mRTB solution of (N, R, E, C) does not in general coincide with the Shapley value of the corresponding game $v_{E,C}$. In fact, the mRTB rule is not even game-theoretic, i.e., two situations leading to the same game might yield different outcomes, as the following example shows.

Example 2 Let $N = \{1, 2\}$, $R = \{1, 2\}$, $E = 2$, $C = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right)$ and $C' = \left(\begin{smallmatrix} 1 & 3 \\ 1 & 1 \end{smallmatrix}\right)$. Then the two MIA situations (N, R, E, C) and (N, R, E, C') give raise the same MIA game:

S	$\{1\}$	$\{2\}$	N
$v_{E,C}(S)$	0	1	2
$v_{E,C'}(S)$	0	1	2

However, $mRTB(N, R, E, C) = (\frac{1}{2}, \frac{3}{2}) \neq mRTB(N, R, E, C') = (\frac{3}{4}, \frac{5}{4})$. □

The mRTB rule provides a way of obtaining an element of the core of a MIA game without calculating the characteristic function. This is stated in the following theorem.

Theorem 1 *Let (N, R, E, C) be a MIA situation. Then*

$$mRTB(N, R, E, C) \in \text{Core}(v_{E,C}).$$

Proof In Proposition 1, we show that every composite marginal vector lies in the core. The mRTB outcome, being the average of these composite marginals vectors according to Eq. (3), then also is an element of the core, which is a convex set. □

As an alternative to the mRTB rule, another way to extend the RTB rule in a two-stage way would be to apply the RTB rule twice: $\sum_{k \in R} \text{RTB}(N, x_k, C_k)$ with $x = \text{RTB}(R, E, (c_{kN})_{k \in R})$. However, this solution can lie outside the core of the corresponding MIA game, as the next example shows.
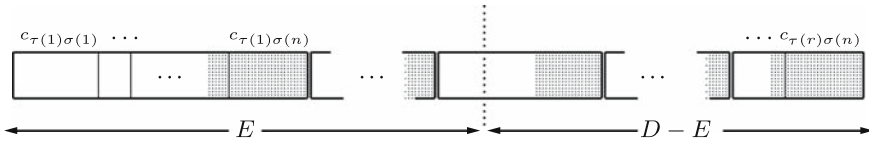
Fig. 1 Proof of Lemma 1

Example 3 Consider the MIA situation (N, R, E, C) with $N = \{1, 2, 3\}$, $R = \{1, 2, 3\}$, estate $E = 51$, and claim matrix

$$C = \begin{pmatrix} 0 & 2 & 6 \\ 0 & 1 & 24 \\ 24 & 2 & 0 \end{pmatrix}.$$

The game associated with this situation is

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	N
$v_{E,C}(S)$	16	3	22	21	46	27	51

We have $x = \text{RTB}(R, E, (c_{kN})_{k \in R}) = (\frac{16}{3}, \frac{67}{3}, \frac{70}{3})$ and $\sum_{k \in R} \text{RTB}(N, x_k, C_k) = (\frac{67}{3}, \frac{5}{2}, \frac{157}{6})$. As $\frac{5}{2} < 3 = v_{E,C}(\{2\})$, this solution is not in the core of $v_{E,C}$. Note that $m\text{RTB}(N, R, E, C) = (\frac{65}{3}, \frac{23}{6}, \frac{153}{6}) \in \text{Core}(v_{E,C})$. □

4 Self-duality

For a MIA situation (N, R, E, C) we denote $D(S) = c_{RS}$, i.e., the total claim of the players in coalition S, and we define $D = D(N)$. Recall that we assume $D \geq E$.

Lemma 1 *Let (N, R, E, C) be a MIA situation. Then*

$$v_{E,C}(S) = v_{D-E,C}(N \backslash S) + D(S) - D + E.$$

Proof To calculate the value of $v_{E,C}(S)$, we must find a permutation on the issues $\tau \in \Pi(R)$ such that the total amount $f_S(\tau)$ assigned to coalition S is minimal. In the definition of $f_S(\tau)$, there is an implicit permutation $\sigma \in \Pi(N)$ on the players, which puts the members of S at the back.

In Fig. 1 we represent all the claims of matrix C in the order indicated by τ and σ, i.e., $c_{\tau(1)\sigma(1)}, c_{\tau(1)\sigma(2)}, \ldots, c_{\tau(r)\sigma(n)}$. The claims of the members of S are shaded. The total of the claims is divided into two parts of lengths E and $D - E$, as the figure shows. From the way in which τ is chosen, the dark zone in the E part is as small as possible, and has length $v_{E,C}(S)$.

If now we consider the MIA situation $(N, R, D - E, C)$ and we want to calculate $v_{D-E,C}(N \backslash S)$, we must find a permutation $\tau' \in \Pi(R)$ such that the white zone in the $D - E$ segment is minimised. Looking at Fig. 1 from the right hand side, one can see that this minimum is obtained for τ_{rev} (with implicit permutation σ_{rev} on the players).

Furthermore, we have that the E segment is the sum of its white and shaded parts. The white part within E equals the total white zone $D(N \backslash S)$ minus the white

zone in the $D - E$ segment, which has length $v_{D-E,C}(N\backslash S)$. As the shaded part of E has length $v_{E,C}(S)$,

$$E = v_{E,C}(S) + D(N\backslash S) - v_{D-E,C}(N\backslash S).$$

From the equality $D = D(S) + D(N\backslash S)$, we conclude that the statement holds. $\square$

The next lemma gives us the relation between the marginal vectors of the two MIA games with estates E and $D - E$.

Lemma 2 *For the marginal vectors of the games corresponding to the MIA situations (N, R, E, C) and $(N, R, D - E, C)$, we have*

$$m^{\sigma}(v_{E,C}) = ((c_{Ri})_{i \in N}) - m^{\sigma_{\mathrm{rev}}}(v_{D-E,C})$$

for each $\sigma \in \Pi(N)$.

Proof Let $\sigma \in \Pi(N)$ and $p \in \{1, \ldots, n\}$. Let $i = \sigma(p)$ and let S be the coalition $\{\sigma(1), \ldots, \sigma(p - 1)\}$. Then

$$m_i^{\sigma}(v_{E,C}) = v_{E,C}(S \cup \{i\}) - v_{E,C}(S).$$

Using Lemma 1, we have

$$\begin{aligned}
m_i^{\sigma}(v_{E,C}) &= v_{D-E,C}(N\backslash(S \cup \{i\})) + D(S \cup \{i\}) - D + E \\
&\quad - [v_{D-E,C}(N\backslash S) + D(S) - D + E] \\
&= D(\{i\}) + v_{D-E,C}(N\backslash(S \cup \{i\})) - v_{D-E,C}(N\backslash S) \\
&= D(\{i\}) - m_i^{\sigma_{\mathrm{rev}}}(v_{D-E,C}).
\end{aligned}$$

Since $D(\{i\}) = c_{Ri}$, the result follows. $\square$

Following Aumann and Maschler (1985), given a rule f we can define its dual f^* by using f to share not the estate E but the gap $D - E$. So, each player receives his claim (the amount he would receive if the estate were big enough) minus his share (according to f) of the losses:

$$f^*(N, R, E, C) = (c_{Ri})_{i \in N} - f(N, R, D - E, C).$$

A rule is called self-dual if $f^* = f$. The following proposition shows that the mRTB rule is self-dual.

Theorem 2 *The mRTB rule is self-dual.*

Proof Let (N, R, E, C) be a MIA situation. Remember that $v_{E,C}^R$ and $v_{D-E,C}^R$ denote the bankruptcy games corresponding to the bankruptcy situations $(R, E, (c_{kN})_{k \in R})$ and $(R, D - E, (c_{kN})_{k \in R})$, respectively. Then,

$$m\mathrm{RTB}(N, R, E, C) = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} \mathrm{RTB}(N, m_k^{\tau}(v_{E,C}^R), C_k)$$

$$= \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} \mathrm{RTB}(N, c_{kN} - m_k^{\tau\mathrm{rev}}(v_{D-E,C}^R), C_k)$$

$$= \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} \left[C_k - \mathrm{RTB}(N, c_{kN} - c_{kN} + m_k^{\tau\mathrm{rev}}(v_{D-E,C}^R), C_k) \right]$$

$$= \sum_{k \in R} C_k - \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} \mathrm{RTB}(N, m_k^{\tau}(v_{D-E,C}^R), C_k)$$

$$= (c_{Ri})_{i \in N} - m\mathrm{RTB}(N, R, D - E, C),$$

where for the second equality we apply Lemma 2 to the "diagonal" MIA situation corresponding to the bankruptcy situation among the issues, and for the third equality we use self-duality of the RTB rule for bankruptcy situations (cf. Curiel 1988). We conclude that the mRTB rule is self-dual. □

Similarly, the following proposition shows that the RTB rule for MIA situations (cf. Calleja et al. 2005) is self-dual.

Proposition 2 *The* RTB *rule for MIA situations (cf. Calleja et al. 2005) is self-dual.*

Proof Calleja et al. (2005) show that the RTB rule coincides with the Shapley value of the associated cooperative game. So, for a MIA situation (N, R, E, C),

$$\mathrm{RTB}(N, R, E, C) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v_{E,C}).$$

From Lemma 2 it then follows that

$$\frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v_{E,C}) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} [(c_{Ri})_{i \in N} - m^{\sigma\mathrm{rev}}(v_{D-E,C})]$$

$$= (c_{Ri})_{i \in N} - \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v_{D-E,C})$$

$$= (c_{Ri})_{i \in N} - \mathrm{RTB}(N, R, D - E, C).$$

This shows that the RTB rule is self-dual. □

As a result of this proposition, the RTB rule is self-dual for bankruptcy situations as well, which was first proved by Curiel (1988).

In a similar way, one can show that the composite rule indicated at the end of Sect. 3, which applies the RTB rule twice, is self-dual.

5 Issue consistency and monotonicity

In this section we characterise the mRTB rule as a consistent extension of the RTB rule for bankruptcy situations to the class of MIA situations. This so-called issue-consistency, which is inspired by O'Neill's claim-consistency, allows us to easily establish estate monotonicity of the mRTB rule.

Definition 2 *A bankruptcy rule* f *is called claim-consistent (cf. O'Neill 1982) if for each bankruptcy situation* (N, E, c) *the following relation holds*:

$$f_i(N, E, c) = \frac{1}{n} \left[\min\{E, c_i\} + \sum_{j \in N \setminus \{i\}} f_i(N \setminus \{j\}, \max\{E - c_j, 0\}, (c_t)_{t \in N \setminus \{j\}}) \right] \tag{4}$$

for all $i \in N$.

If a rule is claim-consistent, the solution can be viewed as an average of n payoffs. Each payoff is calculated by fixing a player $j \in N$ and giving him as much as possible, $\min\{E, c_j\}$; then, the remaining $\max\{E - c_j, 0\}$ is shared among the other players. Claim-consistency determines a unique rule for bankruptcy situations, which is the RTB rule.

For MIA situations we define a new kind of consistency. A rule is issue-consistent if it can be expressed as an average of payoffs too, but now the payoffs are calculated by fixing an issue $k \in R$ and allocating to it the amount $\min\{E, c_{kN}\}$, while the part of the estate that is left is shared among the remaining issues.

Definition 3 *A MIA rule* f *is called issue-consistent if for each MIA situation* (N, R, E, C) *the following relation holds*:

$$f(N, R, E, C) = \frac{1}{r} \sum_{k \in R} \left[f(N, \{k\}, \min\{E, c_{kN}\}, C_k) \right.$$
$$\left. + f(N, R \setminus \{k\}, \max\{E - c_{kN}, 0\}, C_{-k}) \right]. \tag{5}$$

Issue-consistency allows us to extend any rule defined for bankruptcy situations to MIA situations: the first term of the summation in (5) applies the rule f to a (perhaps trivial) bankruptcy situation, while the second term applies f to a MIA situation with $r - 1$ issues, so the expression can be recursively expanded until f is used only on bankruptcy situations (i.e., MIA situations with only one issue). Analogous to claim-consistency, every bankruptcy rule has a unique issue-consistent extension. The next theorem shows that the mRTB rule is the issue-consistent extension of the RTB rule.

Theorem 3 *The* mRTB *rule is the only MIA rule that satisfies issue-consistency and coincides with the* RTB *rule for bankruptcy problems*.

Proof Let (N, R, E, C) be a MIA situation. Then

$$\gamma = m\text{RTB}(N, R, E, C) = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} \text{RTB}(N, m_k^\tau(v_{E,C}^R), C_k).$$

For each permutation τ on the issues, we split the second summation into one term corresponding to its last element $\tau(n)$, and the terms associated with the remaining elements

$$\gamma = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \left[\text{RTB}(N, m_{\tau(n)}^\tau(v_{E,C}^R), C_{\tau(n)}) \right.$$
$$\left. + \sum_{k \in R \setminus \{\tau(n)\}} \text{RTB}(N, m_k^\tau(v_{E,C}^R), C_k) \right].$$

Since $m_{\tau(n)}^\tau(v_{E,C}^R) = \min\{E, c_{\tau(n)N}\}$ and there are $(r-1)!$ permutations $\tau \in \Pi(R)$ with $\tau(n) = k$, we can write

$$\gamma = \frac{1}{r!}(r-1)! \sum_{k \in R} \text{RTB}(N, \min\{E, c_{kN}\}, C_k)$$

$$+ \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R \setminus \{\tau(n)\}} \text{RTB}(N, m_k^\tau(v_{E,C}^R), C_k).$$

The mRTB rule coincides with RTB if there is only one issue, i.e.,

$$\text{RTB}(N, \min\{E, c_{kN}\}, C_k) = m\text{RTB}(N, \{k\}, \min\{E, c_{kN}\}, C_k).$$

Denoting by $v^{R \setminus \{k\}}$ the bankruptcy game associated with the situation $(R, \max\{E - c_{kN}, 0\}, (c_{\ell N})_{\ell \in R \setminus \{k\}})$, we have

$$\gamma = \frac{1}{r} \sum_{k \in R} m\text{RTB}(N, \{k\}, \min\{E, c_{kN}\}, C_k)$$

$$+ \frac{1}{r} \sum_{k \in R} \left[\frac{1}{(r-1)!} \sum_{\tau \in \Pi(R \setminus \{k\})} \sum_{\ell \in R \setminus \{k\}} \text{RTB}(N, m_\ell^\tau(v^{R \setminus \{k\}}), C_\ell) \right].$$

Comparing the expression in brackets with the definition of mRTB given in Eq. (2) yields

$$m\text{RTB}(N, R, E, C) = \frac{1}{r} \sum_{k \in R} m\text{RTB}(N, \{k\}, \min\{E, c_{kN}\}, C_k)$$

$$+ \frac{1}{r} \sum_{k \in R} m\text{RTB}(N, R \setminus \{k\}, \max\{E - c_{kN}, 0\}, C_{-k}).$$

This shows that the mRTB rule is issue-consistent. This, together with the uniqueness of issue-consistent extension, proves the result. □

In the following example, we show how issue-consistency can be used to compute the mRTB solution of a MIA situation.

Example 4 Consider again the MIA situation of Example 3. As a first step in computing the mRTB solution we follow O'Neill's representation of the recursive completion method (which yields the RTB solution) and construct the tree of reduced problems shown in Fig. 2.

For each of the three issues we construct the reduced problem (in the second column of matrices) in which this issue is fully satisfied (with the corresponding payoffs stated above the reduced matrix), and the remaining estate (stated in front of the matrix) has to be divided among the remaining issues. For each of the three reduced problems, we again construct two reduced problems in the same way (third column).

Next, we use issue-consistency to construct the mRTB solution, starting with the reduced problems that have only one issue left. For these situations, mRTB coincides with RTB and we can apply the latter to obtain the payoffs. For instance, in the

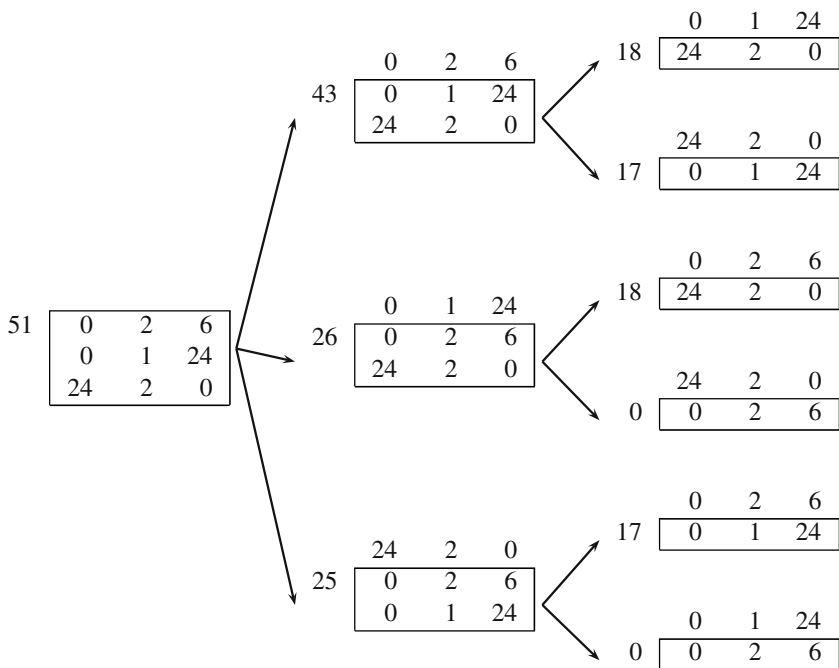**Fig. 2** Forward step for Example 4

situation at the top of the third column, we get $\text{RTB}(N, 18, (24, 2, 0)) = (17, 1, 0)$, which is stated below the matrix, in Fig. 3.

Having done this for all one-issue situations, we apply issue-consistency to construct the mRTB solutions of the bigger problems. Looking at the top problem in the second column, issue-consistency requires that the mRTB allocation (again stated below the matrix) is given by averaging over two vectors: one in which the first issue is fully dealt with [yielding $(0, 1, 24) + (17, 1, 0) = (17, 2, 24)$] and the other in which the second issue is fully satisfied [yielding $(24, 2, 0) + (0, \frac{1}{2}, 16\frac{1}{2}) = (24, 2\frac{1}{2}, 16\frac{1}{2})$]. Performing these computations for all remaining matrices (see Fig. 3) results in $m\text{RTB}(N, E, C) = (\frac{130}{6}, \frac{23}{6}, \frac{153}{6})$.

As an example of how issue-consistency can be used, we show that the mRTB rule is estate monotonic. A rule is called estate monotonic if no player gets less when the estate increase.

Definition 4 *A MIA rule f is estate monotonic if for every pair of MIA situations (N, R, E, C) and (N, R, E', C) with $E' \geq E$ we have that*

$$f_i(N, R, E', C) \geq f_i(N, R, E, C)$$

for all $i \in N$.

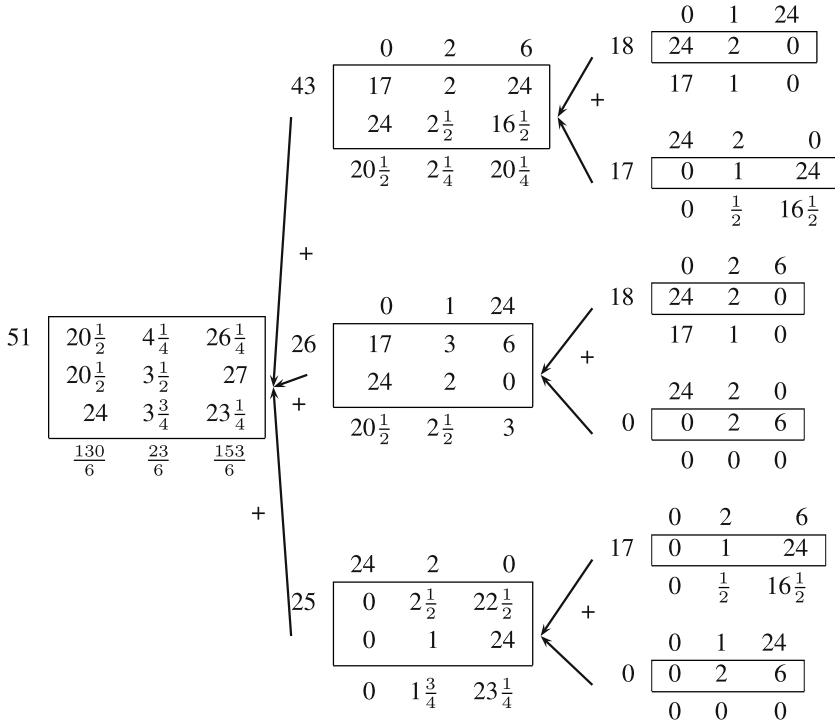**Theorem 4** *The mRTB rule is estate monotonic.*

Fig. 3 Backward step for Example 4

Proof We show that mRTB rule is monotonic by induction on the number of issues r. If $r = 1$ then mRTB coincides with RTB and this rule is monotonic on the class of bankruptcy games (cf. Curiel 1988).

Next, assume that mRTB is monotonic for situations with $r - 1$ issues. Let C be a claim matrix with r rows. By issue-consistency we have

$$m\text{RTB}(N, R, E, C) = \frac{1}{r} \sum_{k \in R} \left[m\text{RTB}(N, \{k\}, \min\{E, c_{kN}\}, C_k) \right.$$

$$\left. + m\text{RTB}(N, R \setminus \{k\}, \max\{E - c_{kN}, 0\}, C_{-k}) \right].$$

In the first term inside the brackets we actually apply the RTB rule to a bankruptcy situation. So, by monotonicity of the RTB rule, this term increases if the estate is raised. The second term is the application of mRTB to a $(r - 1)$-issue allocation situation, which by the induction hypothesis satisfies estate monotonicity. Adding up all terms, we have that mRTB is estate monotonic. □

References

Aumann R, Maschler M (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. J Econ Theory 36:195–213

Calleja P, Borm P, Hendrickx R (2005) Multi-issue allocation situations. Eur J Oper Res 164:730–747

Casas-Méndez B, Borm P, Carpente L, Hendrickx R (2005) The constrained equal awards rule for bankruptcy problems with a priori unions. Ann Oper Res 137:211–227

Curiel I (1988) Cooperative game theory and applications. PhD Thesis, Nijmegen University, Nijmegen, The Netherlands

O'Neill B (1982) A problem of rights arbitration from the Talmud. Math Soc Sci 2:345–371

Shapley L (1953) A value for *n*-person games. In: Kuhn H, Tucker A (eds) Contributions to the theory of games II, annals of mathematics studies, vol 28. Princeton University Press, Princeton, pp 307–317

Thomson W (2003) Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math Soc Sci 45:249–297