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Structure of hybrid censoring schemes and its implications

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Abstract

In this paper, structural properties of (progressive) hybrid censoring schemes are established by studying the possible data scenarios resulting from the hybrid censoring scheme. The results illustrate that the distributions of hybrid censored random variables can be immediately derived from the cases of Type-I and Type-II censored data. Furthermore, it turns out that results in likelihood and Bayesian inference are also obtained directly which explains the similarities present in the probabilistic and statistical analysis of these censoring schemes. The power of the approach is illustrated by applying the approach to the quite complex unified Type-II (progressive) hybrid censoring scheme. Finally, it is shown that the approach is not restricted to (progressively Type-II censored) order statistics and that it can be extended to almost any kind of ordered data.

Keywords Likelihood inference · Exponential distribution · Hybrid censoring · Generalized order statistics · Progressive hybrid censoring · Progressive censoring · Modularization

1 Introduction

Hybrid censoring schemes and related inference have received great attention in the last decade. Various hybrid censoring schemes have been proposed and applied to either order statistics or progressively Type-II censored data. Most papers in this area develop likelihood and Bayesian inference for both a special hybrid censoring scheme and a particular (lifetime) distribution. For a survey on the hybrid censoring schemes proposed and related inferential procedures, we refer to the recent monograph Balakrishnan et al. (2023) and to the review Balakrishnan and Kundu (2013).

While comparing the various publications on (progressive) hybrid censoring, it becomes evident that there exist many structural similarities regarding results, deriva-

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tions, properties, conclusions, etc. A first attempt to understand the structure and concept has been presented in Górny and Cramer (2018b) who proposed the so-called *modularization approach*. The basic idea behind this approach is a decomposition of the joint distribution into some basic modules with an identical probabilistic structure. This paper revisits the approach proposed in Górny and Cramer (2018b), simplifies and refines it, so that it can be easily used for both analysing known hybrid censoring schemes and designing new ones.

In fact, hybrid censoring can be interpreted as a decision rule or design which tells the experimenter when to stop the experiment based on some prefixed time thresholds T_i as well as some involved observed (ordered) failure times $X_{m_i:n}$ (where the parameters m_i are prefixed by the design of the hybrid censoring scheme). For instance, suppose that n objects with lifetimes $X_1, \dots, X_n$ are put on a life test which is subject to unified Type-II hybrid censoring with parameters $k < m$ and $T_1 < T_2$ (details on the four-parameter scheme proposed in Balakrishnan et al. (2008) are presented subsequently). The decision tree generating the unified Type-II hybrid censored sample as well as the respective decision rules are illustrated in Fig. 1. It should be noted that this hybrid censoring scheme ensures a minimum of k measurements. Furthermore, all failures will be observed when the maximum measurement $X_{n:n}$ does not exceed the time threshold T_1.

Considering the ordered sample $X_{1:n} \le \cdots \le X_{n:n}$ of lifetimes, one gets four possible sampling situations due to the design of the hybrid censoring scheme (see Table 1). The test duration is given by the expression

$$(X_{k:n} \vee T_2) \wedge [X_{m:n} \vee (T_1 \wedge X_{n:n})]$$

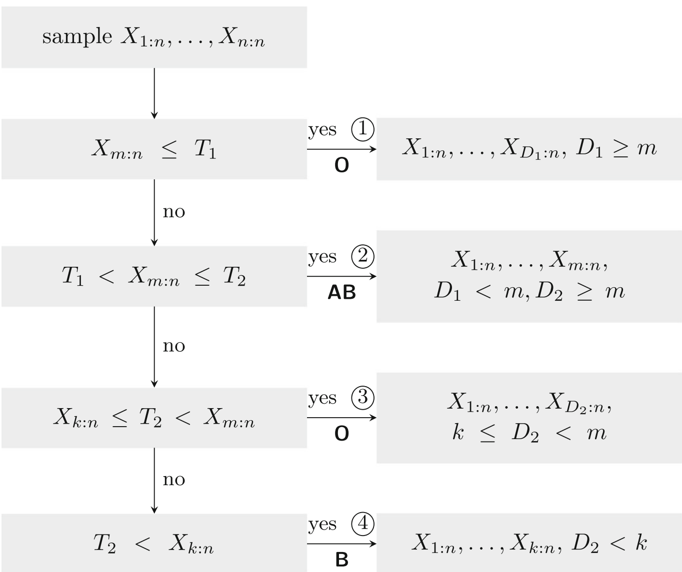

Fig. 1 Decision tree and rules for unified Type-II hybrid censoring with parameters $k < m$ and time thresholds $T_1 < T_2$ (cf. Balakrishnan et al. 2023)

Table 1 Possible sampling situations under unified Type-II hybrid censoring

Scenario no.	Resulting sample	Decision rule		Type
		Counter based	Failure based	
①	$X_{1:n}, \dots, X_{D_1:n}$	$D_1 \geq m$	$X_{m:n} \leq T_1$	I
②	$X_{1:n}, \dots, X_{m:n}$	$D_1 < m \leq D_2$	$X_{k:n} \leq T_1 < X_{m:n}$	II
③	$X_{1:n}, \dots, X_{D_2:n}$	$k \leq D_2 < m$	$X_{k:n} \leq T_2 < X_{m:n}$	I
④	$X_{1:n}, \dots, X_{k:n}$	$D_2 < k$	$T_2 < X_{k:n}$	II

(the min- and max-operators are defined by $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$). Therefore, counting by D_i the failures not exceeding T_i, one gets various sampling situations. In particular, we get four scenarios as illustrated in Table 1. The respective conditions result from the position of the failure times $X_{k:n} < X_{m:n}$ to the thresholds $T_1 < T_2$ (or, equivalently, from the relation of the parameters $k < m$ to the counter values $D_1 \leq D_2$).

Type I means that the sample size is not precisely determined in this case so that it is random. For Type II, the sample size is fixed provided that the respective condition is satisfied. These two situations have to be handled separately in the analysis of the hybrid censored data (and, of course, for statistical inference based on the data).

So far, this modularization concept has mainly been developed to derive the exact distribution of the maximum likelihood estimator in case of exponentially distributed lifetimes. Górny and Cramer (2018b) pointed out that the (conditional) joint cumulative distribution function of hybrid censored data can be decomposed in modules of the same type meaning that these components can be evaluated using the same expressions (for an illustrative complex example, see (Górny and Cramer 2018a)). As a major result, they found that the density functions of the maximum likelihood estimator of the mean can be expressed as linear combination of B-spline functions for any hybrid censoring scheme proposed in the literature so far.

However, as we will show subsequently, the respective similarities are more far-reaching. In fact, we will illustrate that inferential results based on the likelihood function (e.g., maximum likelihood estimators, approximate maximum likelihood estimators) as well as Bayesian inference under any hybrid censoring scheme can be traced back to the well-known cases of Type-I and Type-II censoring, respectively. This particularly applies to both the derivation of the estimators and the related probabilistic analysis. As a matter of fact, the results under hybrid censoring can directly be deduced applying the modularization approach (based on the law of total probability) and the decompositions of the hybrid censoring schemes presented in Tables 3, 4 and 5. Moreover, we illustrate in Sect. 4, that this approach yields a simple method to find the likelihood function for a given hybrid censoring scheme. The results are briefly illustrated for exponentially and Weibull distributed lifetimes. For exponential distributions, it is shown that the maximum likelihood estimator can be expressed in terms of the total time on test of the hybrid censored life testing experiment for any hybrid censoring scheme (of course, provided the maximum likelihood estimator exists). Using the basic conditional distributions of the total time on test given the counter D_i presented in Cramer and Balakrishnan (2013) and Cramer et al. (2016), the distributions can be easily obtained for any hybrid censoring scheme. Finally, we point out in Sect. 5 that our approach can be directly applied to other models of ordered data and particularly to generalized order statistics and related submodels. This means that the presented ideas can be easily used to analyse hybrid censored data from these kind of ordered data. This includes particularly progressive hybrid censoring which has be extensively discussed in the literature. It turns out that the modularization approach yields immediately the desired results by taken into account the results on Type-I and Type-II censoring of progressively Type-II censored data (see Balakrishnan and Cramer 2014, 2023).

Throughout, we illustrate the power of the proposed approach by the unified Type-II hybrid censoring scheme originally introduced by Balakrishnan et al. (2008) (see Table 1 and, e.g., Panahi and Sayyareh 2015; Balakrishnan et al. 2023) since it is one of the more complicated hybrid censoring schemes but has the advantage that a minimum of k observations is guaranteed. This avoids the issue of an empty sample. However, conditioning on the event that at least one failure has been observed, analogous results hold for hybrid censoring schemes with possibly zero observations (see Remark 3.4 for some comments in this direction).

2 Preliminaries

We consider a sample of order statistics $X_{1:n}, \ldots, X_{n:n}$ based on an iid sample $X_1, \ldots, X_n$ with common (absolutely continuous) cumulative distribution function F and density function f. Moreover, we consider hybrid censoring schemes with the following scheme parameters

(i) failure numbers $1 \le m_1 < \cdots < m_a \le n$, $a \in \mathbb{N}$,
(ii) time thresholds $T_1 < \cdots < T_b$, $b \in \mathbb{N}$,

which are supposed known (and fixed) for a given hybrid censoring scheme throughout. These parameters are present in the design of the hybrid censoring schemes (see Table 2 for the scheme parameters of the most important hybrid censoring schemes). Notice that a and b may be arbitrary integers. However, so far, no hybrid censoring schemes have been proposed with a, b exceeding 3. The most complex hybrid censoring scheme considered so far seems to be the one discussed in Górny and Cramer (2018a) with $a = b = 3$ (see Table 5).

Time thresholds are present in many hybrid censoring models. In order to avoid trivialities, we assume throughout that such a threshold T_i is included in the support of the baseline cumulative distribution function F, that is, $0 < F(T_i) < 1$. For each threshold T_i, we introduce the random counter $D_i = D(T_i)$ by

$$D_i = \sum_{j=1}^{n} \mathbb{1}_{(-\infty, T_i]}(X_{j:n}), \quad i \in \{1, \ldots, b\}. \tag{2.1}$$

The effectively observed sample size under a hybrid censoring scheme will be denoted by D_{HCS}. For particular hybrid censoring schemes, we introduce a special notation for the involved counter D_{HCS}. For instance, D_{I} and D_{II} denote the corresponding random counters under Type-I and Type-II hybrid censoring, respectively. On the other hand, we will use the same notation for progressive hybrid censoring schemes (see Sect. 5). This reflects the fact that the observed sample size D_{HCS} depends only on the observed failure times and the design of the hybrid censoring scheme but neither on the construction of the data nor on the underlying distributional assumptions. In fact, for a given hybrid censoring scheme, D_{HCS} can be written as some function Υ_{HCS} of D_1 (or D_i if more than one threshold are involved). The function Υ_{HCS} is independent of the particular ordered data and depends only on the design of the hybrid censoring scheme. The random counters introduced in (2.1) will play an important role in the probabilistic analysis of hybrid censoring schemes. In fact, the equivalence

$$D_i \geq \ell \iff X_{\ell:n} \leq T_i \tag{2.2}$$

for $\ell \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, b\}$ will be very useful in the following derivations. Furthermore, it is well-known that D_i has a binomial distribution with support $\{0, \ldots, n\}$ and probability mass function (see, e.g., Balakrishnan et al. 2023)

$$\Pr(D_i = d) = \binom{n}{d} F^d(T_i)(1 - F(T_i))^{n-d}, \quad d \in \{0, \ldots, n\}. \tag{2.3}$$

We denote the joint cumulative distribution function and density function of the first d order statistics $X_{1:n}, \ldots, X_{d:n}$ by $F_{1,\ldots,d:n}$ and $f_{1,\ldots,d:n}$, respectively. The cumulative distribution function and density function of a single order statistic $X_{d:n}$ is denoted by $F_{d:n}$ and $f_{d:n}$, respectively. In case of a parametric family of distributions, we add the parameter as a subscript, that is, $F_{\boldsymbol{\theta}}$, $F_{\boldsymbol{\theta};d:n}$ etc.

3 Structure of hybrid censoring schemes

From a structural point of view, hybrid censoring models depend on both the (complete) sample of order statistics $X_{1:n}, \ldots, X_{n:n}$ and a random counter D_{HCS} which determines the (observed) sample size under the particular hybrid censoring scheme, that is, the effectively observed sample is given by

$$X_{1:n}, \ldots, X_{D_{\mathsf{HCS}}:n} \tag{3.1}$$

with random sample size D_{HCS}. Thus, given $D_{\mathsf{HCS}} = d(> 0)$, the first d failures times $X_{1:n}, \ldots, X_{d:n}$ are observed. Of course, for some hybrid censoring schemes it is possible that no failures are observed, that is, $D_{\mathsf{HCS}} = 0$. This happens when Type-I (or time) censoring with a threshold T is an essential part of the censoring scheme. However, in that case, it is known that all censored lifetimes must exceed T. Thus, the sample may formally be considered as non-empty but filled with constant observations T. In particular, in any hybrid censoring scheme, the sample can be augmented so that the sample has exactly n observations,

$$X_{1:n}, \dots, X_{D_{\mathsf{HCS}}:n}, \underbrace{T, \dots, T}_{n-D_{\mathsf{HCS}}\text{ times}}$$

(see, e.g., Cramer and Balakrishnan 2013). However, although avoiding formally the situation of an empty sample, this does not solve the inferential problems caused by the censoring procedure. Therefore, inference is usually carried out conditionally on the event $\{D_{\mathsf{HCS}} > 0\}$ for those hybrid censoring schemes with $\Pr(D_{\mathsf{HCS}} = 0) > 0$. In is worth mentioning that the hybrid censoring schemes published so far can be categorized in two groups: hybrid censoring schemes with

(i) bounded test duration but possibly sample size equal to zero, that is, $\Pr(D_{\mathsf{HCS}} = 0) > 0$, and
(ii) unbounded test duration but minimum guaranteed sample size, that is, $\Pr(D_{\mathsf{HCS}} > 0) = 1$.

The random counter D_{HCS} of some hybrid censoring schemes depends on the random variables $X_{1:n}, \dots, X_{n:n}$ only via the vector $\mathbf{D}_b = (D_1, \dots, D_b)$ of random counters (see (2.1)) and some deterministic function $\Upsilon_{\mathsf{HCS}} : \{0, \dots, n\}^b \longrightarrow \{0, \dots, n\}$, that is,

$$D_{\mathsf{HCS}} = \Upsilon_{\mathsf{HCS}}(\mathbf{D}_b). \tag{3.2}$$

Interestingly, the duration of the hybrid censored life test is given by a function $\Upsilon^*_{\mathsf{HCS}}$ which results from Υ_{HCS} by merely replacing $D_1, \dots, D_b$ by $T_1, \dots, T_b$ and the parameters $m_1, \dots, m_a$ by the respective order statistics $X_{m_1:n}, \dots, X_{m_a:n}$, respectively. This observation, that is,

$$W_{\mathsf{HCS}} = \Upsilon^*_{\mathsf{HCS}}(T_1, \dots, T_b) \tag{3.3}$$

can be taken from Table 2 by comparing columns three and four. Furthermore, as pointed out in Sect. 5, the functions Υ_{HCS} and $\Upsilon^*_{\mathsf{HCS}}$ depend only on the design of the hybrid censoring procedure but not on the kind of ordered data (except for the assumption that the data is (almost surely strictly) ordered).

Remark 3.1 In order to have a proper definition of Υ_{HCS}, we assume technically throughout that at least one threshold T_1 is involved in the construction of a hybrid censoring scheme. However, practically Υ_{HCS} can be independent of the counter D_i connected to a threshold T_i so that the threshold has no impact on the sample size. For instance, to model a complete sample with sample size n, we choose $b = 1$ and $\Upsilon_{\mathsf{HCS}}(d_1) = n$, that is, Υ_{HCS} is a constant function. For Type-II censoring with m failures to observe, one has $b = 1$ and $\Upsilon_{\mathsf{HCS}}(d_1) = m$.

Basic examples using the threshold explicitly are Type-I censoring ($b = 1$, $\Upsilon_{\mathsf{HCS}}(d_1) = d_1$) and Type-I hybrid censoring ($b = 1$, $\Upsilon_{\mathsf{HCS}}(d_1) = d_1 \wedge m$ where $1 \le m \le n$ is a fixed integer). For other hybrid censoring schemes, the function Υ_{HCS} can be taken from Table 2.

To study distributions of the hybrid censored data as well as related statistics, the law of total probability will be utilized in the analysis of hybrid censored samples as

Table 2 Expressions of the function Υ_{HCS} and the test duration W_{HCS} under various hybrid censoring schemes. $\mathbf{d}_s = (d_1, \ldots, d_b)$ where b denotes the number of involved thresholds $T_1, \ldots, T_b$ ($x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$)

Censoring scheme	Scheme parameters m_i, T_i	$\Upsilon_{\mathsf{HCS}}(\mathbf{d}_b)$	Test duration W_{HCS} (as in (3.3))
Complete	n	n	$X_{n:n}$
Type-II	m	m	$X_{m:n}$
Type-I	T_1	d	$X_{n:n} \wedge T_1$
Type-I hybrid	m, T_1	$d_1 \wedge m$	$X_{m:n} \wedge T_1$
Type-II hybrid	m, T_1	$d_1 \vee m$	$X_{m:n} \vee (T_1 \wedge X_{n:n})$
Generalized Type-I hybrid	$k < m, T_1$	$(k \vee d_1) \wedge m$	$(X_{k:n} \vee T_1) \wedge X_{m:n}$
Generalized Type-II hybrid	$m, T_1 < T_2$	$(d_1 \vee m) \wedge d_2$	$[(T_1 \wedge X_{n:n}) \vee X_{m:n}] \wedge T_2$
Unified Type-I hybrid	$k < m, T_1 < T_2$	$(d_1 \wedge m) \vee (d_2 \wedge k)$	$(X_{k:n} \wedge T_2) \vee (X_{m:n} \wedge T_1)$
Unified Type-II hybrid	$k < m, T_1 < T_2$	$(d_1 \vee m) \wedge (d_2 \vee k)$	$(X_{k:n} \vee T_2) \wedge [X_{m:n} \vee (T_1 \wedge X_{n:n})]$
Unified Type-III hybrid	$k < m, T_1 < T_2 < T_3$	$(k \wedge d_3) \vee ((m \vee d_1) \wedge d_2)$	$(X_{k:n} \wedge T_3) \vee [(X_{m:n} \vee [T_1 \wedge X_{n:n}]) \wedge T_2]$
Unified Type-IV hybrid	$k < r < m, T_1 < T_2$	$(r \wedge d_1) \vee ((k \vee d_2) \wedge m)$	$(X_{r:n} \wedge T_1) \vee [(X_{k:n} \vee T_2) \wedge X_{m:n}]$

follows. In order to illustrate the idea, we consider for brevity the case of a single threshold T_1 with corresponding random counter D_1. Using that the events $\{D_1 = d\}$, $d = 0, \ldots, n$, form a decomposition of the space $\Omega = \bigcup_{d=0}^{n}\{D_1 = d\}$, we find

$$\Pr(X_{j:n} \le t_j, 1 \le j \le D_{\mathsf{HCS}}) = \sum_{d=0}^{n} \Pr(X_{j:n} \le t_j, 1 \le j \le \Upsilon_{\mathsf{HCS}}(d), D_1 = d). \tag{3.4}$$

This illustrates that we can combine those probabilities (w.r.t. the values of D_1) where $\Upsilon_{\mathsf{HCS}}(\cdot)$ is constant. Therefore, the image $\Upsilon_{\mathsf{HCS}}(\{0, \ldots, n\}) \subseteq \{0, \ldots, n\}$, that is, the support of D_{HCS}, can be used to identify the different data scenarios.

Example 3.2 For the Type-I and Type-II hybrid censoring scheme, the representation in (3.4) can be written as follows.

(i) For Type-I hybrid censoring, (3.4) is given by

$$\begin{aligned}\Pr(X_{j:n} \le t_j, 1 \le j \le D_{\mathsf{I}}) &= \sum_{d=0}^{m-1} \Pr(X_{j:n} \le t_j, 1 \le j \le d, D_1 = d)\\ &\quad + \Pr(X_{j:n} \le t_m, D_1 \ge m)\\ &= \sum_{d=1}^{m-1} \Pr(X_{j:n} \le t_j, 1 \le j \le d, D_1 = d)\\ &\quad + \Pr(X_{j:n} \le t_m, D_1 \ge m).\end{aligned}$$

Therefore, the Type-I hybrid censoring scheme can be decomposed into the m *parts* $\{D_1 = d\}$, $d = 0, \ldots, m-1$, and $\{D_1 \ge m\}$ corresponding to

$$\Upsilon_{\mathsf{I}}(d) = d, \; d \in \{0, \ldots, m-1\}, \quad \Upsilon_{\mathsf{I}}(d) = m, \; d \in \{m, \ldots, n\}.$$

Note that $D_1 = 0$ is equivalent to $D_{\mathsf{I}} = 0$ which implies an empty sample. This case is excluded in the above probability on the left hand side by the condition $1 \le j \le D_{\mathsf{I}}$. As mentioned above, this illustrates the need for conditional inference in this model (see also Remark 3.4).

(ii) For Type-II hybrid censoring, (3.4) can be written as

$$\begin{aligned}\Pr(X_{j:n} \le t_j, 1 \le j \le D_{\mathsf{II}}) &= \Pr(X_{j:n} \le t_m, D_1 < m)\\ &\quad + \sum_{d=m}^{n} \Pr(X_{j:n} \le t_j, 1 \le j \le d, D_1 = d).\end{aligned}$$

Hence, the Type-II hybrid censoring scheme can be decomposed into the $n-m+1$ *parts* $\{D_1 < m\}$ and $\{D_1 = d\}$, $d = m, \ldots, n$. The function

$$\Upsilon_{\mathsf{II}}(d) = m, \; d \in \{0, \ldots, m-1\}, \quad \Upsilon_{\mathsf{II}}(d) = d, \; d \in \{m, \ldots, n\}.$$

Table 3 Hybrid censoring schemes with a single time threshold and corresponding images of Υ_{HCS}

	Decomposition of image of Υ_{HCS} and module type		
	B $\{D_1 < m_\star\}$	**O** $\{D_1 = d\}$	**A** $\{D_1 \geq m^\star\}$
Censoring scheme	$m_\star$	d	$m^\star$
Complete	–	–	0
Type-II	–	–	0
Type-I	–	$0, \ldots, n-1$	n
Type-I hybrid	–	$0, \ldots, m-1$	m
Type-II hybrid	m	$m, \ldots, n-1$	n
Generalized type-I hybrid	k	$k, \ldots, m-1$	m

Parameters are as in Table 2

Similar (but more complicated) expressions to those presented in Remark 3.2 can be obtained for other hybrid censoring schemes. In order to establish such representations, we decompose the preimage of Υ_{HCS} as follows. For a single threshold T_1, we have the following three different situations:

(i) *single element* type $\{d\}$, that is, $\{D_1 = d\}$;
(ii) *initial part* type $\{0, \ldots, m_\star - 1\}$, that is, $\{D_1 < m_\star\}$;
(iii) *end part* type $\{m^\star, \ldots, n\}$, that is, $\{D_1 \geq m^\star\}$.

The resulting decompositions are summarized in Table 3 for the respective well-known hybrid censoring schemes. As mentioned above, we get for Type-I and Type-II hybrid censoring the following decompositions

$$\begin{aligned} \text{Type-I hybrid censoring: } \Upsilon_{\mathsf{I}}^{-1}(\{0, \ldots, n\}) &= \bigcup_{d=0}^{m} \Upsilon_{\mathsf{I}}^{-1}(\{d\}) \\ &= \Big(\bigcup_{d=0}^{m-1} \{D_1 = d\}\Big) \cup \{D_1 \geq m\}, \\ \text{Type-II hybrid censoring: } \Upsilon_{\mathsf{II}}^{-1}(\{0, \ldots, n\}) &= \bigcup_{d=m}^{n} \Upsilon_{\mathsf{II}}^{-1}(\{d\}) \\ &= \{D_1 < m\} \cup \Big(\bigcup_{d=m}^{n-1} \{D_1 = d\}\Big) \cup \{D_1 \geq n\}. \end{aligned} \tag{3.5}$$

since $\{D_1 = n\} = \{D_1 \geq n\}$ (which means that we observe the complete sample). This representation in (3.5) is more convenient so that we will use it subsequently in order to avoid the discussion of the particular case $\{D_1 = n\}$. It is somewhat different from the other ones since $X_{n:n}$ is the largest observation and, thus, has no successor.

In case of a second threshold T_2, a fourth scenario has to be taken into account:

(iv) *mid part* type $\{m_\star^\star\}$, that is, $\{D_1 < m_\star^\star, D_2 \geq m_\star^\star\}$.

Details for the corresponding hybrid censoring schemes are provided in Table 4. As an example, the preimage of Υ_{uII} can be decomposed as

$$\Upsilon_{\mathsf{uII}}^{-1}(\{0,\dots,n\}) = \bigcup_{d=0}^{m} \Upsilon_{\mathsf{uII}}^{-1}(\{d\})$$
$$= \{D_2 < k\} \cup \left(\bigcup_{d=k}^{m-1}\{D_2 = d\}\right) \cup \{D_1 < m, D_2 \geq m\} \cup \left(\bigcup_{d=m}^{n}\{D_1 = d\}\right) \tag{3.6}$$

By construction of the unified Type-II hybrid censoring scheme, the function Υ_{uII} is given by

$$\begin{aligned}
\Upsilon_{\mathsf{uII}}(d_1, d_2) &= k,\ \ d_2 \in \{0,\dots,k-1\},\\
\Upsilon_{\mathsf{uII}}(d_1, d_2) &= d_2,\ \ d_2 \in \{k,\dots,m-1\},\\
\Upsilon_{\mathsf{uII}}(d_1, d_2) &= m,\ \ d_1 \in \{0,\dots,m-1\}, d_2 \in \{m,\dots,n\},\\
\Upsilon_{\mathsf{uII}}(d_1, d_2) &= d_1,\ \ d_1 \in \{m,\dots,n\}.
\end{aligned}$$

Note that $d_1 \leq d_2$ since $T_1 \leq T_2$.

In case of more thresholds, additional scenarios are introduced in the same intuitive manner. For unified Type-III hybrid censoring with thresholds $T_1 < T_2 < T_3$ and parameters $k < m$, the decompositions are given in Table 5. Tables 3, 4 and 5 clearly illustrate the design of the hybrid censoring scheme and the respective implications on the probabilistic analysis.

Considering the particular function Υ_{HCS} of an hybrid censoring scheme, we find from the above observations that Eq. (3.4) can be written as a sum of the probabilities

$$\begin{aligned}
\textbf{O:}\ & \Pr(X_{j:n} \leq t_j, 1 \leq j \leq d, D_i = d) = F^{X_{1:n},\dots,X_{d:n},D_i=d}(\boldsymbol{t}_d),\\
\textbf{A:}\ & \Pr(X_{j:n} \leq t_j, 1 \leq j \leq m_\star, D_i \geq m^\star) = F^{X_{1:n},\dots,X_{m^\star:n},D_i\geq m^\star}(\boldsymbol{t}_{m^\star}),\\
\textbf{B:}\ & \Pr(X_{j:n} \leq t_j, 1 \leq j \leq m_\star, D_i < m_\star) = F^{X_{1:n},\dots,X_{m_\star:n},D_i<m^\star}(\boldsymbol{t}_{m_\star}),\\
\textbf{AB:}\ & \Pr(X_{j:n} \leq t_j, 1 \leq j \leq m_\star^\star, D_i < m_\star^\star, D_{i+1} \geq m_\star^\star)\\
& = F^{X_{1:n},\dots,X_{m_\star^\star:n},D_i<m_\star^\star,D_{i+1}\geq m_\star^\star}(\boldsymbol{t}_{m_\star^\star})
\end{aligned}$$

with appropriately chosen values for $d, m_\star^\star, m_\star, m^\star$ (see Tables 3, 4 and 5). Inspired by Górny and Cramer (2018b), these cases are called *module types* ***O***, ***A***, ***B***, ***AB***. Furthermore, notice that

$$\begin{aligned}
\Pr(X_{j:n} \leq t_j, &1 \leq j \leq m_\star, D_i < m_\star^\star, D_{i+1} \geq m_\star^\star)\\
&= \Pr(X_{j:n} \leq t_j, 1 \leq j \leq m_\star, D_{i+1} \geq m_\star^\star)\\
&- \Pr(X_{j:n} \leq t_j, 1 \leq j \leq m_\star, D_i \geq m_\star^\star)\\
&= F^{X_{1:n},\dots,X_{m_\star^\star:n},D_{i+1}\geq m_\star^\star}(\boldsymbol{t}_{m_\star^\star}) - F^{X_{1:n},\dots,X_{m_\star^\star:n},D_i\geq m_\star^\star}(\boldsymbol{t}_{m_\star^\star})
\end{aligned}$$

since $D_i \leq D_{i+1}$ so that module type **AB** can be evaluated using module type **A** with different random counters.

Table 4 Hybrid censoring schemes with two time thresholds and corresponding images of Υ_{HCS}

	Decomposition of image of Υ_{HCS} and module type				
	B $\{D_2 < m_\star\}$ $m_\star$	O $\{D_2 = d\}$ d	**AB** $\{D_1 < m_\star^\star, D_2 \geq m_\star^\star\}$ $m_\star^\star$	**O** $\{D_1 = d\}$ d	**A** $\{D_1 \geq m^\star\}$ $m^\star$
Censoring scheme					
Generalized Type-II hybrid	–	$0, \ldots, m-1$	m	$m, \ldots, n-1$	n
Unified Type-I hybrid	–	$0, \ldots, k-1$	k	$k, \ldots, m-1$	m
Unified Type-II hybrid	k	$k, \ldots, m-1$	m	$m, \ldots, n-1$	n
Unified Type-IV hybrid	k	$k, \ldots, m-1$	m	$m, \ldots, r-1$	r

Parameters are as in Table 2

Table 5 Hybrid censoring schemes with three time thresholds (unified Type-III and scheme taken from Górny and Cramer (2018a)) and corresponding images of Υ_{HCS}

	Decomposition of image of Υ_{HCS} and module type						
	B	**O**	**AB**	**O**	**AB**	**O**	**A**
	$\{D_3 < m_\star\}$	$\{D_3 = d\}$	$\{D_2 < m_\star^\star, D_3 \geq m_\star^\star\}$	$\{D_2 = d\}$	$\{D_1 < m_\star^\star, D_2 \geq m_\star^\star\}$	$\{D_1 = d\}$	$\{D_1 \geq m^\star\}$
	$m_\star$	d	$m_\star^\star$	d	$m^\star$	d	
uType-III	–	$0, \ldots, k-1$	k	$k, \ldots, m-1$	m	$m, \ldots, n-1$	n
GC2018	k	$k, \ldots, m-1$	m	$m, \ldots, r-1$	r	$r, \ldots, n-1$	n

Parameters are as in Table 2

Therefore, in order to evaluate (3.4), it is sufficient to consider the decomposition of the hybrid censoring scheme as given in Tables 3, 4 and 5, the probability mass functions of D_i (see (2.3)) as well as

$$F^{X_{1:n},\dots,X_{d:n},D_j=d},\quad F^{X_{1:n},\dots,X_{m^\star:n},D_j\geq m^\star},\ \text{and}\ F^{X_{1:n},\dots,X_{m_\star:n},D_j<m_\star} \tag{3.7}$$

for D_j, $j=1,\dots,b$, as well as some d, $m_\star$, and $m^\star$, respectively. Finally, it should be mentioned that

$$\begin{aligned}\Pr(X_{j:n}\leq t_j, 1\leq j\leq m_\star, D_i<m_\star) &= \Pr(X_{j:n}\leq t_j, 1\leq j\leq m_\star)\\ &\quad -\Pr(X_{j:n}\leq t_j, 1\leq j\leq m_\star, D_i\geq m_\star)\\ &= F^{X_{1:n},\dots,X_{m_\star:n}}(\boldsymbol{t}_{m_\star})\\ &\quad - F^{X_{1:n},\dots,X_{m_\star:n},D_j\geq m_\star}(\boldsymbol{t}_{m_\star}),\end{aligned}$$

so that only the first two cases in (3.7) remain (see also Cramer et al. 2016). Therefore, the expressions belonging to module types **O** and **A** are sufficient to determine the distribution of the hybrid censored sample given in (3.1).

Now, it has been shown in Cramer and Balakrishnan (2013), that

$$\Pr(X_{j:n}\leq t_j, 1\leq j\leq \ell, D_i\geq \ell) = F_{1,\dots,\ell:n}(\boldsymbol{t}_{\ell-1}, t_\ell\wedge T_i) = F_{1,\dots,\ell:n}(\boldsymbol{t}_\ell\wedge T_i), \tag{3.8}$$

$$\begin{aligned}\Pr(X_{j:n}\leq t_j, 1\leq j\leq \ell, D_i=\ell) &= \mathbb{1}_{[T_i,\infty)}(\min_{\ell+1\leq j\leq m} t_j)\Big\{F_{1,\dots,\ell:n}(\boldsymbol{t}_\ell\wedge T_i)\\ &\quad - F_{1,\dots,\ell+1:n}(\boldsymbol{t}_\ell\wedge T_i, T_i)\Big\}\end{aligned} \tag{3.9}$$

with ℓ chosen appropriately. These expressions illustrate that the joint cumulative distribution function of hybrid censored order statistics can be written in terms of the joint cumulative distribution functions $F_{1,\dots,d:n}$ with some $d\in\{1,\dots,n\}$. As an example, consider unified Type-II hybrid censoring with Υ_{ull} and the respective decomposition given in (3.6). Then, for this relatively complicated hybrid censoring scheme, we find

$$\begin{aligned}\Pr(X_{j:n}\leq t_j, 1\leq j\leq D_{\mathsf{ull}}) &= \Pr(X_{j:n}\leq t_j, 1\leq j\leq k, D_2<k)\\ &+\sum_{d=k}^{m-1}\Pr(X_{j:n}\leq t_j, 1\leq j\leq d, D_2=d)\\ &+\Pr(X_{j:n}\leq t_j, 1\leq j\leq m, D_1<m, D_2\geq m)\\ &+\sum_{d=m}^{n}\Pr(X_{j:n}\leq t_j, 1\leq j\leq d, D_1=d).\end{aligned}$$

Taking into account the identities (3.8) and (3.9) as well as

$$\Pr(X_{j:n}\leq t_j, 1\leq j\leq m, D_1<m, D_2\geq m)$$

$$
\begin{aligned}
&= \Pr(X_{j:n} \le t_j, 1 \le j \le m, D_2 \ge m)\\
&- \Pr(X_{j:n} \le t_j, 1 \le j \le m, D_1 \ge m),
\end{aligned}
$$

we obtain directly the following theorem.

Theorem 3.3 *Consider the unified Type-II hybrid censoring scheme with function Υ_{uII} and respective decomposition as given in (3.6). Then, for $t_1, \dots, t_n \in \mathbb{R}$, the joint cumulative distribution function of the data is given by*

$$
\begin{aligned}
&\Pr(X_{j:n} \le t_j, 1 \le j \le D_{\mathsf{uII}}) = F_{1,\dots,k:n}(\boldsymbol{t}_k) - F_{1,\dots,k:n}(\boldsymbol{t}_k \wedge T_2)\\
&\quad + \sum_{d=k}^{m-1} \mathbb{1}_{[T_2,\infty)}\left(\min_{d+1 \le j \le m} t_j\right)\left\{F_{1,\dots,d:n}(\boldsymbol{t}_d \wedge T_2) - F_{1,\dots,d+1:n}(\boldsymbol{t}_d \wedge T_2, T_2)\right\}\\
&\quad + F_{1,\dots,m:n}(\boldsymbol{t}_m \wedge T_2) - F_{1,\dots,m:n}(\boldsymbol{t}_m \wedge T_1)\\
&\quad + \sum_{d=m}^{n-1} \mathbb{1}_{[T_1,\infty)}\left(\min_{d+1 \le j \le n} t_j\right)\left\{F_{1,\dots,d:n}(\boldsymbol{t}_d \wedge T_1) - F_{1,\dots,d+1:n}(\boldsymbol{t}_d \wedge T_1, T_1)\right\}\\
&\quad + F_{1,\dots,n:n}(\boldsymbol{t}_n \wedge T_1).
\end{aligned}
\tag{3.10}
$$

Remark 3.4 In cases where $\Pr(D_{\mathsf{HCS}} = 0) > 0$, that is, the hybrid censored experiment may terminate without observing a failure, the distribution will be obtained conditionally on the event $\{D_{\mathsf{HCS}} > 0\}$ (see, e.g., Type-I (hybrid) censoring or generalized Type-II hybrid censoring). Therefore, instead of (3.4), we consider

$$
\begin{aligned}
&\Pr(X_{j:n} \le t_j, 1 \le j \le D_{\mathsf{HCS}} \mid D_{\mathsf{HCS}} > 0)\\
&= \frac{1}{\Pr(D_{\mathsf{HCS}}) > 0)} \Pr(X_{j:n} \le t_j, 1\\
&\le j \le D_{\mathsf{HCS}})
\end{aligned}
$$

which obviously can be evaluated by analogy with (3.4).

4 Joint density functions and likelihood functions

An important application of the representations obtained in Sect. 3 is the derivation of the joint density function of the hybrid censored sample $X_{1:n}, \dots, X_{D_{\mathsf{HCS}}:n}$ and the (random) sample size $D_{\mathsf{HCS}} = \Upsilon_{\mathsf{HCS}}(\mathbf{D}_b)$. Notice that, once the hybrid censored sample is observed, one can reconstruct the particular data situation. This means that we exactly know which censoring scenario has led to the observed measurements. In order to find the respective density functions, we need only to consider the censoring scenario and the respective cumulative distribution function.

For illustration, we consider unified Type-II hybrid censoring which has two thresholds T_1 and T_2. Let $D_{\mathsf{uII}} = \ell$. First, without loss of generality, we can assume $t_1 < \cdots < t_\ell$ since the density function will be zero otherwise. For the cases $T_2 < t_k$, $T_1 < t_m \le T_2$, and $t_n \le T_1$, we get directly the corresponding density function of the first order statistics from the representation in (3.8) (see (4.1)). Suppose $t_\ell \le T_2 < t_{\ell+1}$ for some $\ell \in \{k, \ldots, m-1\}$. Then, $D_2 = \ell$ and, according to (3.9), the corresponding part of the cumulative distribution function is given by

$$F_{1,\ldots,\ell:n}(\boldsymbol{t}_\ell) - F_{1,\ldots,\ell+1:n}(\boldsymbol{t}_\ell, T_2).$$

Then, using the marginal density functions of order statistics, it is easy to see that the corresponding density function is given by

$$\frac{n!}{(n-\ell)!}\left(\prod_{j=1}^{\ell} f(t_j)\right)\overline{F}^{n-\ell}(T_2).$$

The same argument can be used in the case $t_\ell \le T_1 < t_{\ell+1}$ for some $\ell \in \{m, \ldots, n-1\}$. Summing up, we get the following expression for the joint density function of the unified Type-II hybrid censored sample from Theorem 3.3.

Theorem 4.1 *Consider the unified Type-II hybrid censoring scheme. Then, the joint density function of the measurements $X_{j:n}$, $1 \le j \le D_{\mathsf{uII}}$, and the (random) sample size D_{uII} is given by*

$$f^{X_{j:n},1\le j\le D_{\mathsf{uII}},D_{\mathsf{uII}}}(\boldsymbol{t}_\ell, \ell)$$
$$= \begin{cases} f_{1,\ldots,\ell:n}(\boldsymbol{t}_\ell), & T_2 < t_\ell \\ \frac{n!}{(n-\ell)!}\prod_{j=1}^{\ell} f(t_j)\overline{F}^{n-\ell}(T_2) & t_\ell \le T_2 < t_{\ell+1} \textit{ for some } \ell \in \{k, \ldots, m-1\} \\ f_{1,\ldots,\ell:n}(\boldsymbol{t}_\ell), & T_1 < t_\ell \le T_2 \\ \frac{n!}{(n-\ell)!}\prod_{j=1}^{\ell} f(t_j)\overline{F}^{n-\ell}(T_1) & t_\ell \le T_1 < t_{\ell+1} \textit{ for some } \ell \in \{m, \ldots, n-1\} \\ f_{1,\ldots,\ell:n}(\boldsymbol{t}_\ell), & t_\ell \le T_1 \textit{ or } \ell = n \\ 0, & \textit{otherwise} \end{cases}. \tag{4.1}$$

For the other hybrid censoring schemes, the joint density function can be obtained in the same manner so that we do not present details here. Respective expressions can be found in Balakrishnan et al. (2023).

The preceding discussion illustrates that the joint density function of a hybrid censored sample can be easily obtained along the same lines as shown above for the unified Type-II hybrid censoring scheme by considering the parameters and module types given in Tables 3, 4 and 5 and the ordered arguments $t_1 < \cdots < t_\ell$ with $D_{\mathsf{HCS}} = \ell$. Notice that for hybrid censoring schemes with $\Pr(D_{\mathsf{HCS}} = 0) > 0$, one has to consider conditional density functions, that is, $f^{X_{j:n},1\le j\le D_{\mathsf{HCS}},D_{\mathsf{HCS}}|D_{\mathsf{HCS}}>0}$ (see comments in Remark 3.4).

Clearly, (4.1) yields the likelihood function. As a consequence, we obtain directly from Theorem 4.1 the following corollary.

Corollary 4.2 *Given a parametric model with density functions $f_{\boldsymbol{\theta}}$, $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^p$, and an observed sample $x_1, \ldots, x_\ell$, ℓ, the likelihood function is given by*

$$\mathscr{L}(\boldsymbol{\theta} \mid x_1, \ldots, x_\ell, \ell)$$
$$= \begin{cases} f_{\boldsymbol{\theta};1,\ldots,\ell:n}(\boldsymbol{x}_\ell), & T_2 < x_\ell \\ \frac{n!}{(n-\ell)!}\prod_{j=1}^{\ell} f_{\boldsymbol{\theta}}(x_j)\overline{F}_{\boldsymbol{\theta}}^{n-\ell}(T_2) & x_\ell \le T_2 < x_{\ell+1} \textit{ for some } \ell \in \{k, \ldots, m-1\} \\ f_{\boldsymbol{\theta};1,\ldots,\ell:n}(\boldsymbol{x}_\ell), & T_1 < x_\ell \le T_2 \\ \frac{n!}{(n-\ell)!}\prod_{j=1}^{\ell} f_{\boldsymbol{\theta}}(x_j)\overline{F}_{\boldsymbol{\theta}}^{n-\ell}(T_1) & x_\ell \le T_1 < x_{\ell+1} \textit{ for some } \ell \in \{m, \ldots, n-1\} \\ f_{\boldsymbol{\theta};1,\ldots,\ell:n}(\boldsymbol{x}_\ell), & x_\ell \le T_1 \textit{ or } \ell = n \end{cases} \tag{4.2}$$

Introducing the notation W_{HCS} and w for the test duration and observed test duration (see Table 2), respectively, we find the following compact form of the likelihood function of hybrid censored order statistics.

Theorem 4.3 *Consider a hybrid censoring scheme applied to order statistics' data. Then, given hybrid censored data $x_1, \ldots, x_\ell$ with $D_{\mathsf{HCS}} = \ell$, the likelihood function can be written as*

$$\mathscr{L}(\boldsymbol{\theta} \mid x_1, \ldots, x_\ell, \ell) = \frac{n!}{(n-\ell)!}\left(\prod_{j=1}^{\ell} f_{\boldsymbol{\theta}}(x_j)\right)\overline{F}_{\boldsymbol{\theta}}^{n-\ell}(w), \tag{4.3}$$

where w denotes the observed test duration induced by the applied hybrid censoring scheme.

Notice that the mathematical structure of this function does not depend on the particular (hybrid) censoring scheme. Of course, the realizations of both the sample size $D_{\mathsf{HCS}} = \ell$ and the test duration $W_{\mathsf{HCS}} = w$ depend on the structure of the (hybrid) censoring scheme. But, once the data has been observed, the mathematical treatment is the same for all hybrid censoring schemes. Therefore, independently of the (hybrid) censoring scheme, the derivation of maximum likelihood estimates is along the same lines.

4.1 Exponentially distributed lifetimes

Suppose that the lifetimes are iid exponentially distributed random variables with mean $\vartheta > 0$. Then, from Theorem 4.3, the log-likelihood function for given hybrid censored data $x_1, \ldots, x_\ell, \ell$ is given by

$$\begin{aligned} \mathscr{L}^*(\vartheta \mid x_1, \ldots, x_\ell, \ell) &= \ln \frac{n!}{(n-\ell)!} - \ell \ln \vartheta - \frac{1}{\vartheta}\sum_{j=1}^{\ell} x_j + \frac{(n-\ell)w}{\vartheta} \\ &= \ln \frac{n!}{(n-\ell)!} - \ell \ln \vartheta - \frac{1}{\vartheta}\mathrm{TTT}_\ell \end{aligned} \tag{4.4}$$

with test duration w under the hybrid censoring scheme and total time on test statistic

$$\mathtt{TTT}_\ell = \sum_{j=1}^{\ell} x_j + (n-\ell)w. \tag{4.5}$$

Using standard arguments (see, e.g., Balakrishnan and Cramer 2014, Section 12.1), the maximum likelihood estimator of ϑ can be easily obtained from (4.4) for any given hybrid censoring scheme.

Theorem 4.4 *Given a hybrid censored sample of exponentially distributed lifetimes with mean ϑ, the maximum likelihood estimator of ϑ is given by*

$$\widehat{\vartheta}_{HCS} = \frac{1}{D_{HCS}} \left(\sum_{j=1}^{D_{HCS}} X_{j:n} + (n - D_{HCS}) W_{HCS} \right) = \frac{TTT_{D_{HCS}}}{D_{HCS}} \tag{4.6}$$

provided $D_{HCS} > 0$. $TTT_{D_{HCS}} = \sum_{j=1}^{D_{HCS}} X_{j:n} + (n - D_{HCS}) W_{HCS}$ denotes the total time on test under the hybrid censoring scheme as given in (4.5).

In order to find the distribution of the maximum likelihood estimator, we can again utilize the modularization approach. In case of unified Type-II hybrid censoring, the result given in (3.6) leads to the following expression for the cumulative distribution function of the maximum likelihood estimator

$$\begin{aligned}
F_\vartheta^{\widehat{\vartheta}_{\mathsf{ull}}}(t) &= \Pr_\vartheta(\widehat{\vartheta}_{\mathsf{ull}} \le t) = \Pr_\vartheta(\widehat{\vartheta}_{\mathsf{ull}} \le t, D_2 < k) + \sum_{d=k}^{m-1} \Pr_\vartheta(\widehat{\vartheta}_{\mathsf{ull}} \le t, D_2 = d) \\
&\quad + \Pr_\vartheta(\widehat{\vartheta}_{\mathsf{ull}} \le t, D_1 < m, D_2 \ge m) + \sum_{d=m}^{n} \Pr_\vartheta(\widehat{\vartheta}_{\mathsf{ull}} \le t, D_1 = d) \\
&= \Pr_\vartheta(\mathtt{TTT}_k \le kt, D_2 < k) + \sum_{d=k}^{m-1} \Pr_\vartheta(\mathtt{TTT}_d \le dt, D_2 = d) \\
&\quad + \Pr_\vartheta(\mathtt{TTT}_m \le mt, D_1 < m, D_2 \ge m) + \sum_{d=m}^{n} \Pr_\vartheta(\mathtt{TTT}_d \le t, D_1 = d).
\end{aligned}$$

Using the same ideas leading to the expression (3.10), the density function of the maximum likelihood estimator can now be obtained from the conditional density functions $f_\vartheta^{\mathrm{TTT}_\ell | D_i \ge \ell}$, $f_\vartheta^{\mathrm{TTT}_\ell | D_i < \ell}$ and $f_\vartheta^{\mathrm{TTT}_\ell | D_i = \ell}$. As illustrated in Balakrishnan et al. (2023), these conditional density functions can be expressed compactly in terms of B-spline functions. Expressions for the conditional density functions can be found in Balakrishnan et al. (2023, Theorems 5.7 and 5.21). Hence, the density function of $\widehat{\vartheta}_{\mathsf{ull}}$ (and more general of $\widehat{\vartheta}_{\mathsf{HCS}}$) can be written as a linear combination of B-spline functions. For brevity, we do not present details here.

Furthermore, the above results can be used to conduct Bayesian inference. Suppose that ϑ has an inverse gamma prior $\mathsf{I}\Gamma(\lambda, \beta)$ with density function

$$\pi(\vartheta) = \frac{\lambda^\beta}{\Gamma(\beta)} \vartheta^{-(\beta+1)} \mathrm{e}^{-\lambda/\vartheta}, \quad \vartheta > 0, \tag{4.7}$$

where $\beta > 0$ and $\lambda > 0$ are the hyper-parameters. Using the general expression of the likelihood function in (4.3) for the exponential distribution, the posterior density function $p(\cdot \mid \text{data})$ of ϑ becomes

$$p(\vartheta \mid \text{data}) = \frac{(\mathtt{TTT}_d + \lambda)^{d+\beta}}{\Gamma(d+\beta)} \vartheta^{-(d+\beta+1)} \mathrm{e}^{-(\mathtt{TTT}_d + \lambda)/\vartheta}, \quad \vartheta > 0, \tag{4.8}$$

where the total time on test $\mathtt{TTT}_d$ is defined in (4.5). Since the posterior density function is the density function of an inverse gamma distribution with parameters $d + \beta > 2$ and $\mathtt{TTT}_d + \lambda$, the Bayesian estimator of ϑ under squared-error loss, being the posterior mean, is simply obtained as

$$\widehat{\vartheta}_{\mathsf{B;\,HCS}} = \frac{\mathtt{TTT}_{D_{\mathsf{HCS}}} + \lambda}{D_{\mathsf{HCS}} + \beta - 1} \tag{4.9}$$

provided $D_{\mathsf{HCS}} + \beta > 1$. For particular hybrid censoring schemes, we refer to Draper and Guttman (1987), Kundu (2007), Kundu and Pradhan (2009), Kundu et al. (2013), Bayoud (2014), and Balakrishnan et al. (2023). Of course, the results can also be utilized for two-parameter exponential distributions (see Chan et al. 2015).

4.2 Weibull distributed lifetimes

As a second example, we consider briefly Weibull distributed lifetimes with cumulative distribution function $F_{\vartheta,\beta}(t) = 1 - \exp(-(t/\vartheta)^\beta)$, $t \geq 0$. Then, we find from (4.3) in Theorem 4.3 the log-likelihood function

$$\begin{aligned}\mathscr{L}^*(\vartheta, \beta \mid x_1, \ldots, x_\ell, \ell) = \ln \frac{n!}{(n-\ell)!} - \ell \log(\beta/\vartheta) + (\beta - 1) \sum_{j=1}^{\ell} \log x_j \\ - \frac{1}{\vartheta} \sum_{j=1}^{\ell} x_j^\beta - \frac{n-\ell}{\vartheta} w^\beta.\end{aligned} \tag{4.10}$$

Clearly, this function has the same structure as the log-likelihood function under Type-I and Type-II censoring. This finding leads us directly to the following theorem by applying the results of Balakrishnan and Kateri (2008) to the present situation.

Theorem 4.5 *Consider Weibull distributed lifetimes with cumulative distribution function as given above. Then, the maximum likelihood estimators $\widehat{\vartheta}$ and $\widehat{\beta}$ of ϑ and β exist uniquely for every hybrid censoring scheme (provided $D_{\mathsf{HCS}} = \ell > 0$). In particular,*

$$\widehat{\vartheta} = \widehat{\vartheta}(\widehat{\beta}) = \frac{1}{\ell}\left(\sum_{j=1}^{\ell} x_j^{\widehat{\beta}} + (n-\ell)w^{\widehat{\beta}}\right)$$

and $\widehat{\beta}$ is the unique solution of the equation

$$0 = \frac{\ell}{\beta} - \frac{\sum_{j=1}^{\ell} x_j^{\beta}\log x_j + (n-\ell)w^{\beta}\log w}{\widehat{\vartheta}(\beta)} + \sum_{j=1}^{\ell}\log x_j.$$

Furthermore, well-known numerical procedures can be used to compute the maximum likelihood estimates (see, e.g., Rinne 2008 and references given therein).

Relevant references on hybrid censored Weibull data are, e.g., Kundu (2007), Banerjee and Kundu (2008), Habibi Rad and Yousefzadeh (2014).

The result in (4.10) can also be used to derive directly approximate maximum likelihood estimates as has been done under Type-II censoring by Balakrishnan and Varadan (1991) and under Type-I and Type-II (hybrid) censoring by Kundu (2007) and Banerjee and Kundu (2008), respectively. Similar comments as those for the exponential distribution apply to Bayesian inference for hybrid censored Weibull data (see, e.g., Zhu 2020).

4.3 Lifetimes having other distributions

In the preceding two sections on exponentially and Weibull distributed lifetimes, we have illustrated that the results obtained under Type-I and Type-II censoring can be directly applied to determine the maximum likelihood estimates, approximate maximum likelihood estimates, and Bayesian estimates under an arbitrary hybrid censoring scheme. The same comment is true for any other distribution! In this regard, we have to determine only the respective estimates under Type-I and Type-II censoring. Once these are available, we can use the same methodology (e.g., solving the likelihood equations), by considering the values of the sample size D_{HCS} and the test duration W_{HCS} of the particular hybrid censoring scheme. As a consequence, we get the same expressions (or equations to be solved) for any hybrid censoring scheme choosing these quantities appropriately. As examples, one may have a look at, e.g., Sen et al. (2018), Zhu et al. (2019), Arabi Belaghi and Noori Asl (2019) for log-normal, Birnbaum-Saunders, and Burr XII distributions. For a survey on related references dealing with both various hybrid censoring schemes and miscellaneous distributions, we refer to the extensive bibliography provided in Balakrishnan et al. (2023).

5 Beyond order statistics: application to other kinds of ordered data

In the preceding sections, we have considered order statistics from an iid sample $X_1, \ldots, X_n$. However, a given hybrid censoring scheme can be applied to any (almost surely strictly) ordered data $X_{(1)} < \cdots < X_{(n)}$. Therefore, in this section we assume that the underlying data $X_{(1)}, \ldots, X_{(n)}$ is only (almost surely) strictly ordered. For example, choosing suitable distributional assumptions, this approach includes a variety of models of ordered random variables like (upper) record data, minimal repair data, generalized order statistics, jump times of some continuous time stochastic processes (non-homogeneous Poisson processes, birth-death processes, renewal processes),…. It is worth mentioning that there is no restriction imposed on the dependence structure. Therefore, the following discussion also includes order statistics from dependent random variables $X_1, \ldots, X_n$.

In fact, the definition of the sample size and the test duration is identical to the order statistics' case, that is, (2.1) is generally given as

$$D_i = \sum_{j=1}^{n} \mathbb{1}_{(-\infty,T_i]}(X_{(j)}). \tag{5.1}$$

This yields the same expressions as in Table 2 with $X_{j:n}$ replaced by $X_{(j)}$, $j \in \{1, \ldots, n\}$. In particular, we get the same function Υ_{HCS} as in (3.2), that is $D_{\mathsf{HCS}} = \Upsilon_{\mathsf{HCS}}(\mathbf{D}_b)$, since Υ_{HCS} depends only on the structure of the hybrid censoring scheme but neither on the lifetimes nor on their distribution. Therefore, we can directly use the modularization approach leading to the same decomposition and module types. Notice that we use in the subsequent derivations only that the random variables are ordered.

As an example, we consider again unified Type-II hybrid censoring but now for the data $X_{(1)}, \ldots, X_{(n)}$. Then, by analogy with (5.2), we get the joint cumulative distribution function of $X_{(1)}, \ldots, X_{(D_{\mathsf{uII}})}$

$$\begin{aligned}
&\Pr(X_{(j)} \le t_j, 1 \le j \le D_{\mathsf{uII}}) = \Pr(X_{(j)} \le t_j, 1 \le j \le k, D_2 < k)\\
&\quad + \sum_{d=k}^{m-1} \Pr(X_{(j)} \le t_j, 1 \le j \le d, D_2 = d)\\
&\quad + \Pr(X_{(j)} \le t_j, 1 \le j \le m, D_1 < m, D_2 \ge m)\\
&\quad + \sum_{d=m}^{n} \Pr(X_{(j)} \le t_j, 1 \le j \le d, D_1 = d)\\
&= F_{(1,\ldots,k)}(\boldsymbol{t}_k) - F_{(1,\ldots,k)}(\boldsymbol{t}_k \wedge T_2)\\
&\quad + \sum_{d=k}^{m-1} \mathbb{1}_{[T_2,\infty)}\Big(\min_{d+1\le j\le m} t_j\Big)\Big\{F_{(1,\ldots,d)}(\boldsymbol{t}_d \wedge T_2) - F_{(1,\ldots,d+1)}(\boldsymbol{t}_d \wedge T_2, T_2)\Big\}\\
&\quad + F_{(1,\ldots,m)}(\boldsymbol{t}_m \wedge T_2) - F_{(1,\ldots,m)}(\boldsymbol{t}_m \wedge T_1)\\
&\quad + \sum_{d=m}^{n-1} \mathbb{1}_{[T_1,\infty)}\Big(\min_{d+1\le j\le n} t_j\Big)\Big\{F_{(1,\ldots,m)}(\boldsymbol{t}_d \wedge T_1) - F_{(1,\ldots,d+1)}(\boldsymbol{t}_d \wedge T_1, T_1)\Big\}\\
&\quad + F_{(1,\ldots,n)}(\boldsymbol{t}_n \wedge T_1),
\end{aligned} \tag{5.2}$$

where $F_{(1,\ldots,\ell)}$ denotes the joint cumulative distribution function of $X_{(1)},\ldots,X_{(\ell)}$. The similarity of the expressions in (5.2) and (3.10) of Theorem 3.3 is striking. This enables us to state the following general theorem.

Theorem 5.1 *Consider a hybrid censoring scheme applied to (almost strictly) ordered data $X_{(1)} < \cdots < X_{(n)}$ where $F_{(1,\ldots,\ell)}$ denotes the joint cumulative distribution function of the random variables $X_{(1)},\ldots,X_{(\ell)}$, $1 \le \ell \le n$.*

Then, the joint distribution of the hybrid censored data and the corresponding counter D_{HCS} has the same structure as that obtained for hybrid censored order statistics' data. In order to get the corresponding cumulative distribution function, the cumulative distribution function $F_{1,\ldots,\ell:n}$ of the first ℓ order statistics has to be replaced by $F_{(1,\ldots,\ell)}$ only for a given index ℓ.

Moreover, the derivation of the joint density function can also be carried out as for order statistics provided that the joint density function has a special form. In particular, if $X_{(1)},\ldots,X_{(n)}$ are generalized order statistics based on some absolutely continuous cumulative distribution function F with density function f and parameters $\gamma_1,\ldots,\gamma_n > 0$, that is, the joint density function is given by (see Kamps 1995; Cramer 2006)

$$f_{(1,\ldots,n)}(\boldsymbol{t}_n) = \left(\prod_{j=1}^{n}\gamma_j\right)\prod_{j=1}^{n} f(t_j)\overline{F}^{\gamma_j-\gamma_{j+1}-1}(t_j), \quad t_1 < \cdots < t_n.$$

First, the marginal density function of the first ℓ generalized order statistics is given by ($\gamma_{n+1} \equiv 0$)

$$f_{(1,\ldots,\ell)}(\boldsymbol{t}_\ell) = \left(\prod_{j=1}^{\ell}\gamma_j\right)\left(\prod_{j=1}^{\ell-1} f(t_j)\overline{F}^{\gamma_j-\gamma_{j+1}-1}(t_j)\right) f(t_\ell)\overline{F}^{\gamma_\ell-1}(t_\ell).$$

Then, as in Sect. 4, we get for $t_\ell \le T_2 < t_{\ell+1}$ that

$$F_{(1,\ldots,\ell)}(\boldsymbol{t}_\ell) - F_{(1,\ldots,\ell+1)}(\boldsymbol{t}_\ell, T_2)$$

has the density function

$$\left(\prod_{j=1}^{\ell}\gamma_j\right)\left(\prod_{j=1}^{\ell} f(t_j)\overline{F}^{\gamma_j-\gamma_{j+1}-1}(t_j)\right)\overline{F}^{\gamma_{\ell+1}}(T_2).$$

Thus, for example, we get under unified Type-II censoring, the joint density function presented in the following theorem (cf. (4.1)).

Theorem 5.2 *Consider the unified Type-II hybrid censoring scheme. Then, the joint density function of the hybrid censored generalized order statistics $X_{(j)}$, $1 \le j \le D_{\mathsf{uII}}$, and the (random) sample size D_{uII} is given by*

$$f^{X_{(j)},1\le j\le D_{ull},D_{ull}}(\boldsymbol{t}_\ell,\ell) = \begin{cases} f_{(1,\dots,\ell)}(\boldsymbol{t}_\ell), & T_2 < t_\ell \\ \left(\prod_{j=1}^{\ell}\gamma_j\right)\left(\prod_{j=1}^{\ell} f(t_j)\overline{F}^{\gamma_j-\gamma_{j+1}-1}(t_j)\right)\overline{F}^{\gamma_{\ell+1}}(T_2) & t_\ell \le T_2 < t_{\ell+1}, \ell\in\{k,\dots,\dots,m-1\} \\ f_{(1,\dots,\ell)}(\boldsymbol{t}_\ell), & T_1 < t_\ell \le T_2 \\ \left(\prod_{j=1}^{\ell}\gamma_j\right)\left(\prod_{j=1}^{\ell} f(t_j)\overline{F}^{\gamma_j-\gamma_{j+1}-1}(t_j)\right)\overline{F}^{\gamma_{\ell+1}}(T_1) & t_\ell \le T_1 < t_{\ell+1}, \ell\in\{m,\dots,n-1\} \\ f_{(1,\dots,\ell)}(\boldsymbol{t}_\ell), & t_\ell \le T_1 \text{ or } \ell = n \end{cases} \tag{5.3}$$

As for order statistics, we notice that, in general, we have only two types of density functions. Thus, given an observed sample $x_1,\dots,x_\ell,\ell$ and choosing w as the test duration given in Table 2 of the particular hybrid censoring scheme, we get the following result as a generalization of Theorem 4.3.

Theorem 5.3 *Consider a parametric model with density functions $f_{\boldsymbol{\theta}}$, $\boldsymbol{\theta}\in\Theta\subseteq\mathbb{R}^p$. Suppose that $X_{(1)},\dots,X_{(n)}$ are generalized order statistics based on the absolutely continuous cumulative distribution function $F_{\boldsymbol{\theta}}$ with density function $f_{\boldsymbol{\theta}}$ and parameters $\gamma_1,\dots,\gamma_n>0$.*

Then, given an observed hybrid censored sample $x_1,\dots,x_\ell,\ell$ of the generalized order statistics and the counter of the hybrid censoring scheme, the likelihood function is given by

$$\mathscr{L}(\boldsymbol{\theta}\mid x_1,\dots,x_\ell,\ell) = \left(\prod_{j=1}^{\ell}\gamma_j\right)\left(\prod_{j=1}^{\ell} f_{\boldsymbol{\theta}}(x_j)\overline{F}_{\boldsymbol{\theta}}^{\gamma_j-\gamma_{j+1}-1}(x_j)\right)\overline{F}_{\boldsymbol{\theta}}^{\gamma_{\ell+1}}(w), \tag{5.4}$$

where w denotes the test duration induced by the hybrid censoring scheme.

Notice that the expression in (5.4) equals the density function $f_{\boldsymbol{\theta};(1,\dots,\ell)}(\boldsymbol{x}_\ell)$ when $w=x_\ell$. This case corresponds to the marginal distribution of the first ℓ generalized order statistics. In case w equals a threshold T_i, the above density function can be interpreted as joint density function of a Type-I censored sample of generalized order statistics and $D_i=\ell$ with D_i given in (5.1). It seems that this model has not been discussed in the literature so far. But, the derivation of the joint density function can be directly taken from the results in Burkschat et al. (2016) who considered Type-I hybrid censored sequential order statistics (from exponential distributions).

In case of exponential lifetimes, we get from Theorem 5.3 a similar expression as in the case of order statistics, that is, the log-likelihood function for given hybrid censored data $x_1,\dots,x_\ell,\ell$ is given by

$$\mathscr{L}^*(\vartheta\mid x_1,\dots,x_\ell,\ell) = \ln\prod_{j=1}^{\ell}\gamma_j - \ell\ln\vartheta - \frac{1}{\vartheta}\mathtt{TTT}_\ell \tag{5.5}$$

with test duration w induced by the applied hybrid censoring scheme and total time on test statistic

$$\mathtt{TTT}_\ell = \sum_{j=1}^{\ell}(\gamma_j - \gamma_{j+1})x_j + \gamma_{\ell+1}w. \tag{5.6}$$

Then, the maximum likelihood estimator of ϑ can be established for any given hybrid censoring scheme (cf. Kamps 1995; Cramer and Kamps 2003).

Theorem 5.4 *Given a hybrid censored sample of generalized order statistics based on an exponential distribution with mean ϑ and parameters $\gamma_1, \ldots, \gamma_n$, the maximum likelihood estimator of ϑ is given by*

$$\widehat{\vartheta}_{HCS} = \frac{1}{D_{HCS}}\left(\sum_{j=1}^{D_{HCS}}(\gamma_j - \gamma_{j+1})X_{j:n} + \gamma_{D_{HCS}+1}W_{HCS}\right) = \frac{TTT_{D_{HCS}}}{D_{HCS}}$$

provided $D_{HCS} > 0$.

Similar comments as given in Sects. 4.1–4.3 apply to these data, too. Here, it should be mentioned that the distribution of the random counter D_i given in (5.1) is of great interest. Of course, it is no longer binomially distributed as in the order statistics' case. In particular, one has to calculate its probability mass function, that is, for $d \in \{0, \ldots, n\}$,

$$\begin{aligned}
\Pr(D_i = d) &= \begin{cases} \Pr(T_i < X_{(1)}), & d = 0 \\ \Pr(X_{(d)} \le T_i < X_{(d+1)}), & d \in \{1, \ldots, n-1\} \\ \Pr(X_{(n)} \le T_i), & d = n \end{cases} \\
&= \begin{cases} 1 - F_{(1)}(T_i), & d = 0 \\ F_{(d)}(T_i) - F_{(d+1)}(T_i), & d \in \{1, \ldots, n-1\} \\ F_{(n)}(T_i), & d = n \end{cases}.
\end{aligned}$$

Therefore, we need expressions for the one dimensional marginal distribution functions $F_{(d)}$ of generalized order statistics to compute these probabilities. Such expressions can be found in, e.g., Cramer and Kamps (2003).

In order to give a brief review of some results available in the literature, we provide some additional information on particular models like progressive Type-II censoring and record values.

Remark 5.5 (i) Hybrid censoring of progressively Type-II censored data has been widely discussed in the literature. Since it can be seen as a particular case of generalized order statistics, the above considerations apply directly to so-called progressive hybrid censoring. A detailed review of respective results can be found in Chapter 6 of Balakrishnan et al. (2023). In particular, it turns out that, using the modularization approach, the inferential and probabilistic results obtained under progressive hybrid censoring parallel those available under hybrid censoring. Of course, the results are somewhat more complicated since the distribution of progressively Type-II censored order statistics is more complicated than that of order statistics (see Balakrishnan and Cramer 2014). However, the structure of the results relies only on the structure of hybrid censoring scheme, that is, it depends on the random counter and the function Υ_{HCS}. In this regard, many results (like (conditional) distribution of the maximum likelihood estimation for exponentially distributed lifetimes) can be easily adapted using the respective distributions, probability mass functions, etc.

(ii) Epstein (1954) considered a Type-I hybrid censoring model called '*hybrid censoring with replacement*'. A closer look at this model shows that this model equals Type-I censoring of record values (for details on record values, see Arnold et al. 1998 or Nevzorov 2001). The model has been further studied in, e.g., Ebrahimi (1986). From the preceding comments, it is clear that inferential and probabilistic results can be directly obtained using the results for Type-I and Type-II censored record values. First results along the lines of the present study have recently been reported in Berzborn and Cramer (2023). Further research on this model is ongoing and will be presented in the near future.

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Conflict of interest The author declares that he has no conflict of interest.

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