



# A note on asymptotics of the risk function under confidence region estimation in case of large samples of random size

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## Abstract

The problem of optimal estimation of location and scale parameters of absolutely continuous distributions, by means of two-dimensional confidence regions based on L-statistics, is considered. The case, when the sample size is random and tends to infinity, is studied. The paper can be considered as a supplement to Zaigraev and Alama-Bučko (Metrika 81:283–305, 2018) in case of samples of random size.

**Keywords** Sample of random size · Optimal confidence region · Order statistics · L-statistics

## 1 Introduction

The problem of optimal confidence interval/region estimation is one of the classical problems of Mathematical Statistics. Although, it has mainly been investigated for a single unknown parameter, the multivariate case has also quite a rich history, starting with confidence regions of rectangular shape (see, e.g., Šidák 1967) and ending with those of arbitrary shape (see, e.g., Czarnowska and Nagaev 2001; Alama-Bučko et al. 2006; Zaigraev and Alama-Bučko 2013, 2017, 2018). But in all the cases only samples of non-random size have been taken into account.

Here, the samples of random size are studied and the case of two unknown parameters is considered, where the first parameter  $\vartheta_1 \in \mathbb{R}$  is location and the second parameter  $\vartheta_2 > 0$  is scale. Therefore, two-dimensional confidence regions as estimators of  $\vartheta = (\vartheta_1, \vartheta_2)$  are considered. A solution to the problem of construction of the optimal confidence region for large samples was obtained, e.g., in Zaigraev and Alama-Bučko (2013, 2018). Thus, this note can be considered as a supplement to Zaigraev and Alama-Bučko (2018) in case of samples of random size.

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Traditionally, the size of the available sample is assumed to be deterministic, but in practice the size of the data available for statistical analysis can be often treated as random. Indeed, quite often the number of observations available is unknown until the end of the recording process and also can be treated as an (random) observation. This is the case, for example, in insurance statistics, where in different accounting periods the numbers of insurance claims are different. Or in medical statistics, where the number of patients with a particular disease varies from month to month due, e.g., to seasonal factors or from year to year due, e.g., to some epidemic. In these cases, the number of observations available, as well as the observations themselves, are previously unknown and should be treated as random.

In this note, we discuss the problem described above for samples of random size and compare the quality of estimators constructed from samples of random and non-random size.

We start with notation, assumptions and a review of the construction of the optimal confidence region given in Zaigraev and Alama-Bučko (2018) for samples of non-random size (Sect. 2), while our main result is given in Sect. 3. In that section we essentially use the results from Korolev (2000) and Bening and Korolev (2005).

## 2 Notation and recollections

Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  be a sample from a distribution  $\mathbb{P}_\vartheta$ , that is  $\{x_i\}_{i=1}^n$  are assumed to be independent real-valued random variables having the distribution  $\mathbb{P}_\vartheta$ . Let  $F = F_{(0,1)}$  be the continuous distribution function corresponding to  $\mathbb{P}_{(0,1)}$  and let

$$F^{-1}(p) = \inf\{v \in \mathbb{R} : F(v) \geq p\}, \quad 0 < p < 1$$

be the *quantile function*. The distribution  $\mathbb{P}_{(0,1)}$  is assumed to be absolutely continuous with the density function  $f$ . Let  $x_{m_1:n}, x_{m_2:n}, \dots, x_{m_k:n}$  be the order statistics corresponding to the sample  $\mathbf{x}$ ,  $1 \leq m_1 < m_2 < \dots < m_k \leq n$ , and let  $0 < p_1 < p_2 < \dots < p_k < 1$  be such numbers that  $m_i/n - p_i = o(n^{-1/2})$ , as  $n \rightarrow \infty$ , and  $f(F^{-1}(p_i)) > 0$ ,  $i = 1, \dots, k$ . In what follows we confine ourselves with the case of *central order statistics* (the treatment of the more general case for samples of deterministic size is detailed in Zaigraev and Alama-Bučko (2018)).

We propose to base the construction of optimal confidence region for  $\vartheta$  on L-statistics  $t_1(\cdot)$  and  $t_2(\cdot)$  being the *asymptotically best linear unbiased estimators* of  $\vartheta_1$  and  $\vartheta_2$ , respectively. As it is well-known (see, e.g., Masoom Ali and Umbach (1998) or Zaigraev and Alama-Bučko (2018)), these estimators look as follows:

$$t_1(\mathbf{x}) = \sum_{j=1}^k a_j x_{m_j:n}, \quad t_2(\mathbf{x}) = \sum_{j=1}^k b_j x_{m_j:n},$$

where

$$\mathbf{a} = (a_1, \dots, a_k) = \frac{\mathbf{u}V^{-1}\mathbf{u}^T\mathbf{1}_kV^{-1} - \mathbf{1}_kV^{-1}\mathbf{u}^T\mathbf{u}V^{-1}}{\mathbf{u}V^{-1}\mathbf{u}^T\mathbf{1}_kV^{-1}\mathbf{1}_k^T - (\mathbf{1}_kV^{-1}\mathbf{u}^T)^2},$$

$$\mathbf{b} = (b_1, \dots, b_k) = \frac{\mathbf{1}_kV^{-1}\mathbf{1}_k^T\mathbf{u}V^{-1} - \mathbf{1}_kV^{-1}\mathbf{u}^T\mathbf{1}_kV^{-1}}{\mathbf{u}V^{-1}\mathbf{u}^T\mathbf{1}_kV^{-1}\mathbf{1}_k^T - (\mathbf{1}_kV^{-1}\mathbf{u}^T)^2},$$

$\mathbf{1}_k = (1, \dots, 1) \in \mathbb{R}^k$ ,  $\mathbf{u} = (F^{-1}(p_1), \dots, F^{-1}(p_k))$ ,  $V = (V_{ij})_{i,j=1}^k$  with

$$V_{ij} = \frac{p_i(1 - p_j)}{f(F^{-1}(p_i))f(F^{-1}(p_j))}, \quad 1 \leq i \leq j \leq k.$$

Note that the well-known result on the form of the asymptotic distribution of selected central order statistics, established by Mosteller (see Mosteller (1946) or David and Nagaraja (2003), Subsection 10.3), is used here.

Let  $\alpha \in (0, 1)$  be a given small number, while  $\mathbf{y} = (y_1, y_2, \dots, y_n)$  be a sample from the distribution  $\mathbb{P}_{(0,1)}$ . Denote  $T_n(\mathbf{y}) = (-t_1(\mathbf{y})/t_2(\mathbf{y}), 1/t_2(\mathbf{y}) - 1)$ . Taking a set  $A_n \in \mathcal{B}^2$  such that  $\mathbb{P}_{(0,1)}(\sqrt{n}T_n(\mathbf{y}) \in A_n) = 1 - \alpha$ , one can obtain

$$\mathbb{P}_{(\theta_1, \theta_2)}\left(\sqrt{n}\left(\frac{\theta_1 - t_1(\mathbf{x})}{t_2(\mathbf{x})}, \frac{\theta_2 - t_2(\mathbf{x})}{t_2(\mathbf{x})}\right) \in A_n\right) = 1 - \alpha$$

(here  $\mathcal{B}^2$  denotes the  $\sigma$ -algebra of Borel subsets of  $\mathbb{R}^2$ ).

Thus, the set

$$B_{A_n}(\mathbf{x}) = (t_1(\mathbf{x}), t_2(\mathbf{x})) + (t_2(\mathbf{x})/\sqrt{n})A_n \tag{1}$$

is a confidence region of level  $1 - \alpha$  for  $\vartheta$ ; its quality can be characterized by the risk function defined as

$$R(\vartheta, B_{A_n}) = E_\vartheta \lambda_2(B_{A_n}(\mathbf{x})) = \lambda_2(A_n)E_\vartheta t_2^2(\mathbf{x})/n, \tag{2}$$

where  $\lambda_2$  is the Lebesgue measure on  $\mathcal{B}^2$ . Under the assumption that the density function  $g_n$  of  $T_n(\mathbf{y})$  is continuous and such that

$$\lambda_2(\{\mathbf{v} \in \mathbb{R}^2 : g_n(\mathbf{v}) = z\}) = 0 \quad \forall z > 0,$$

the confidence region  $B_{A_n^*}$  with

$$A_n^* = \{\mathbf{v} \in \mathbb{R}^2 : \bar{g}_n(\mathbf{v}) \geq z_\alpha\}$$

is optimal among all the confidence regions of the form (1), that is it has the smallest value of (2) (see Einmahl and Mason (1992)), where  $z_\alpha$  is defined by the equation

$$\int_{A_n^*} \bar{g}_n(\mathbf{v})d\mathbf{v} = 1 - \alpha,$$

and  $\bar{g}_n(\mathbf{v}) = (1/n)g_n(\mathbf{v}/\sqrt{n})$  is the density function of  $\sqrt{n}T_n(\mathbf{y})$ . Moreover, Zaigraev and Alama-Bučko (2017; 2018), have established that  $\sqrt{n}T_n(\mathbf{y})$  has asymptotically, as  $n \rightarrow \infty$ , 2-dimensional normal distribution  $\mathcal{N}_2(0, W)$  and the set  $A_n^*$ , as  $n \rightarrow \infty$ , tends to the ellipse  $A_0 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v}W^{-1}\mathbf{v}^T \leq 2 \ln \alpha^{-1}\}$ , where

$$W = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} V \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}^T = \begin{bmatrix} \mathbf{a}V\mathbf{a}^T & \mathbf{a}V\mathbf{b}^T \\ \mathbf{a}V\mathbf{b}^T & \mathbf{b}V\mathbf{b}^T \end{bmatrix}.$$

Therefore,

$$\lim_{n \rightarrow \infty} \lambda_2(A_n^*) = \lambda_2(A_0) = 2\pi \sqrt{\det W} \ln \alpha^{-1}.$$

Since  $\lim_{n \rightarrow \infty} \mathbb{E}_\vartheta t_2^2(\mathbf{x}) = \theta_2^2 \lim_{n \rightarrow \infty} \mathbb{E}_{(0,1)} t_2^2(\mathbf{y}) = \theta_2^2$ , the order of the risk function  $\mathbb{R}(\vartheta, B_{A_n^*})$  for positive and finite limit value of  $\det W$  is  $1/n$ , as  $n \rightarrow \infty$ .

### 3 Main result

From now on the notation  $\xrightarrow{D}$  means the convergence of random variables or random vectors in distribution. Let  $(N_n)_{n \geq 1}$  be a sequence of integer-valued non-negative random variables such that  $N_n$  and  $x_1, x_2, \dots, x_n$  are independent for any  $n$ . We assume that

(A)  $N_n \rightarrow \infty$  in probability,  $N_n/n \xrightarrow{D} Y$ , as  $n \rightarrow \infty$ , and  $\mathbb{E}N_n = n \forall n \in \mathbb{N}$ , where  $Y$  is a non-degenerate random variable having an absolutely continuous distribution with a distribution function  $G$  and a density function  $g$ .

The next example contains a sequence  $(N_n)_{n \geq 1}$ , for which condition (A) is fulfilled.

**Example** Assume that a random variable  $X$  has a negative binomial distribution with parameters  $r > 0$  and  $p \in (0, 1)$  (denote as  $\mathcal{NB}(r, p)$  distribution), i.e.

$$\mathbb{P}(X = k) = \binom{r+k-1}{k} p^r (1-p)^k, \quad k = 0, 1, 2, \dots$$

with  $\mathbb{E}X = r(1-p)/p$ . If  $r$  is non-integer, then  $\binom{r+k-1}{k}$  is interpreted as

$$\binom{r+k-1}{k} = \frac{\Gamma(r+k)}{k! \Gamma(r)}.$$

Note that  $\mathcal{NB}(1, p)$  is the geometric distribution (denoted further as  $Geo(p)$ ).

Bening and Korolev (2005) show that if the random variable  $N_n$  has  $\mathcal{NB}(m/2, m/(m+2n))$  distribution given  $m > 0, n \in \mathbb{N}$ , then  $N_n/n \xrightarrow{D} U_{m/2}$ , as  $n \rightarrow \infty$ , where  $U_{m/2}$  is a random variable having the gamma distribution  $G(m/2, m/2)$  (it is the scaled  $\chi_m^2$  distribution) with the density

$$d(t) = \frac{(m/2)^{m/2} t^{m/2-1} e^{-mt/2}}{\Gamma(m/2)}, \quad t > 0. \tag{3}$$

In particular, if the random variable  $N_n$  has  $Geo(1/(n + 1))$  distribution,  $n \in \mathbb{N}$ , then  $N_n/n \xrightarrow{D} U_1$  ( $U_1$  has the standard exponential distribution), as  $n \rightarrow \infty$ .

The next result is due to Korolev (2000) and describes the asymptotic distribution of selected central order statistics in case of samples of random size. It can be considered as a generalization of Mosteller’s result.

**Theorem 1** *Assume that assumptions made in Sect. 2 and assumption (A) hold. As  $n \rightarrow \infty$ ,*

$$\sqrt{n}(y_{m_1:N_n} - F^{-1}(p_1), \dots, y_{m_k:N_n} - F^{-1}(p_k)) \xrightarrow{D} Z_{k,V}/\sqrt{Y},$$

where  $Z_{k,V}$  stands for the random vector having the  $k$ -variate normal distribution  $\mathcal{N}_k(0, V)$ . The distribution function and the density of the limit distribution can be written as

$$H(\mathbf{v}) = \int_0^\infty \Phi_V(u^{1/2}\mathbf{v})dG(u), \quad \mathbf{v} \in \mathbb{R}^k,$$

$$h(\mathbf{v}) = \int_0^\infty u^{k/2}\varphi_V(u^{1/2}\mathbf{v})g(u)du, \quad \mathbf{v} \in \mathbb{R}^k,$$

respectively, where  $\Phi_{k,V}$  and  $\phi_{k,V}$  denotes the distribution function and the density corresponding to  $\mathcal{N}_k(0, V)$ , respectively.

It is worth noting that the limit distribution, established in Theorem 1, belongs to the class of elliptical distributions.

The adaptation of Theorem 1 to the random variables  $(N_n)_{n \geq 1}$ , having negative binomial distributions, gives the following result.

**Corollary** *If the random variable  $N_n$  has  $\mathcal{NB}(m/2, m/(m + 2n))$  distribution,  $n \in \mathbb{N}$ , then under the conditions of Theorem 1, as  $n \rightarrow \infty$ ,*

$$\sqrt{n}(y_{m_1:N_n} - F^{-1}(p_1), \dots, y_{m_k:N_n} - F^{-1}(p_k)) \xrightarrow{D} T_k(m, V),$$

where  $T_k(m, V)$  is a random vector having  $k$ -dimensional Student distribution with the density

$$h(\mathbf{v}) = \frac{\Gamma((m + k)/2)}{\Gamma(m/2)m^{k/2}\pi^{k/2}(\det V)^{1/2}} \left(1 + \frac{1}{m}\mathbf{v}V^{-1}\mathbf{v}^T\right)^{-(m+k)/2}, \quad \mathbf{v} \in \mathbb{R}^k. \quad (4)$$

If the random variable  $N_n$  has  $Geo(1/(n + 1))$  distribution,  $n \in \mathbb{N}$ , then the above limit distribution is  $T_k(2, V)$ .

In what follows, let  $\mathcal{G}(s)$ ,  $s > 0$ , be the Laplace transform of the function  $g$ , while  $\mathcal{G}^{-1}$  be the inverse function to  $\mathcal{G}$ , that is

$$s = \mathcal{G}^{-1}(\alpha) \iff \mathcal{G}(s) = \int_0^\infty e^{-su} g(u)du = \alpha.$$

The inverse function exists since the derivative

$$G'(s) = - \int_0^\infty e^{-us} ug(u)du$$

is always negative.

The next result determines the asymptotic of the optimal confidence region in case of samples of random size.

**Theorem 2** *Under the conditions of Theorem 1,*

1.  $\sqrt{n}T_{N_n}(\mathbf{y}) \xrightarrow{D} Z_{2,W}/\sqrt{Y}$ , as  $n \rightarrow \infty$ , and the density of the limit distribution has the form

$$h(\mathbf{v}) = \int_0^\infty u\varphi_{2,W}(\sqrt{u}\mathbf{v})g(u)du, \quad \mathbf{v} \in \mathbb{R}^2. \tag{5}$$

*In particular, if the random variable  $N_n$  has  $\mathcal{NB}(m/2, m/(m + 2n))$  distribution,  $n \in \mathbb{N}$ , then the random vector  $Z_{2,W}/\sqrt{Y}$  has  $T_2(m, W)$  distribution; if the random variable  $N_n$  has  $\text{Geo}(1/(n + 1))$  distribution,  $n \in \mathbb{N}$ , then the random vector  $Z_{2,W}/\sqrt{Y}$  has  $T_2(2, W)$  distribution.*

2. The set  $A_n^*$ , as  $n \rightarrow \infty$ , tends to the ellipse  $A'_0 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v}W^{-1}\mathbf{v}^T \leq 2G^{-1}(\alpha)\}$ , and

$$\lim_{n \rightarrow \infty} \lambda_2(A_n^*) = \lambda_2(A'_0) = 2\pi \sqrt{\det W}G^{-1}(\alpha).$$

**Proof** The proof of the first part is based on Theorem 1 and the limit distribution of  $\sqrt{n}T_{N_n}(\mathbf{y})$ , that established similarly as the limit distribution of the corresponding statistic for the samples of non-random size (see Corollary 1 of Zaigraev and Alama-Bučko (2018)). As to the proof of the second part, note that the function  $h$ , defined by (5), can be written as

$$\begin{aligned} h(\mathbf{v}) &= \frac{1}{2\pi \sqrt{\det W}} \int_0^\infty \exp\left(-\frac{u}{2}\mathbf{v}W^{-1}\mathbf{v}^T\right) ug(u)du \\ &= -\frac{1}{2\pi \sqrt{\det W}} G'\left(\frac{\mathbf{v}W^{-1}\mathbf{v}^T}{2}\right), \mathbf{v} \in \mathbb{R}^2. \end{aligned}$$

The set  $A_n^*$ , as  $n \rightarrow \infty$ , approximates the set  $A'_0 = \{\mathbf{v} \in \mathbb{R}^2 : h(\mathbf{v}) \geq z'_\alpha\}$ , where  $z'_\alpha$  is defined by the equality

$$\int_{A'_0} h(\mathbf{v})d\mathbf{v} = 1 - \alpha. \tag{6}$$

Since

$$\mathcal{G}''(s) = \int_0^\infty e^{-us} u^2 g(u) du > 0 \quad \forall s > 0,$$

the function  $-\mathcal{G}'$  is monotonically decreasing, that is why

$$h(\mathbf{v}) \geq z'_\alpha \iff \frac{\mathbf{v}W^{-1}\mathbf{v}^T}{2} \leq a \tag{7}$$

for some  $a > 0$  and, moreover,  $\lambda_2(A'_0) = 2a\pi\sqrt{\det W}$ .

Changing the variables in (6):  $v_1 = r \cos \beta, v_2 = r \sin \beta, r \geq 0, \beta \in [0, 2\pi)$ , we obtain

$$\begin{aligned} 1 - \alpha &= \int_{\{\mathbf{v}:h(\mathbf{v})\geq z'_\alpha\}} h(\mathbf{v})d\mathbf{v} \\ &= \frac{1}{2\pi\sqrt{\det W}} \int_0^{2\pi} d\beta \int_0^{\frac{\sqrt{2a}}{\sigma(\beta)}} r dr \int_0^\infty \exp\left(-\frac{u}{2}r^2\sigma^2(\beta)\right) ug(u)du \\ &= \frac{2}{2\pi\sqrt{\det W}} \int_0^{2\pi} \frac{d\beta}{\sigma^2(\beta)} \int_0^{\sqrt{a}} zdz \int_0^\infty e^{-uz^2} ug(u)du, \end{aligned}$$

where  $\sigma^2(\beta) = [\cos \beta \ \sin \beta]W^{-1}[\cos \beta \ \sin \beta]^T$ . Since

$$\int_0^{\sqrt{a}} 2uz e^{-uz^2} dz = 1 - e^{-ua},$$

we obtain

$$1 - \alpha = \frac{1}{2\pi\sqrt{\det W}} \int_0^{2\pi} \frac{d\beta}{\sigma^2(\beta)} \int_0^\infty (1 - e^{-ua})g(u)du. \tag{8}$$

From (7) it follows that taking  $z'_\alpha = 0$  we get  $a = +\infty$ , and the same reasoning as above leads us to the formula

$$\frac{1}{2\pi\sqrt{\det W}} \int_0^{2\pi} \frac{d\beta}{\sigma^2(\beta)} = 1. \tag{9}$$

Substituting (9) in (8), we have

$$1 - \alpha = 1 - \int_0^\infty e^{-ua} g(u)du = 1 - \mathcal{G}(a) \iff a = \mathcal{G}^{-1}(\alpha)$$

and  $\lambda_2(A'_0) = 2\pi\sqrt{\det W}\mathcal{G}^{-1}(\alpha)$ . □

**Remark 1** The value  $\lambda_2(A'_0) = 2\pi \sqrt{\det W} \mathcal{G}^{-1}(\alpha)$  from Theorem 2 is larger than  $\lambda_2(A_0) = 2\pi \sqrt{\det W} \ln \alpha^{-1}$  from Sect. 2. Indeed, since  $\mathbb{E}Y = 1$ , from Jensen’s inequality it follows that  $\mathbb{E}\alpha^Y > \alpha^{\mathbb{E}Y} = \alpha$ , i.e.

$$\int_0^\infty \alpha^u g(u) du = \mathcal{G}(\ln \alpha^{-1}) > \alpha \iff \ln \alpha^{-1} < \mathcal{G}^{-1}(\alpha).$$

**Example (continuation)** If the random variable  $N_n$  has  $\mathcal{NB}(m/2, m/(m + 2n))$  distribution,  $n \in \mathbb{N}$ , then the function  $h$  has the form (see (4) for  $k = 2$ ):

$$h(\mathbf{v}) = \frac{\Gamma(m/2 + 1)}{\Gamma(m/2)m\pi\sqrt{\det W}} \left(1 + \frac{1}{m} \mathbf{v}W^{-1}\mathbf{v}^T\right)^{-m/2-1}, \mathbf{v} \in \mathbb{R}^2.$$

In this case the function  $g$  is defined by (3) and

$$\mathcal{G}(s) = \int_0^\infty e^{-su} g(u) du = \left(1 + \frac{2s}{m}\right)^{-m/2}, \quad s > 0,$$

while

$$\mathcal{G}^{-1}(\alpha) = \frac{m}{2} \left(\alpha^{-2/m} - 1\right), \quad \alpha \in (0, 1).$$

Therefore,  $A'_0 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v}W^{-1}\mathbf{v}^T \leq m(\alpha^{-2/m} - 1)\}$ , and

$$\lim_{n \rightarrow \infty} \lambda_2(A_n^*) = \lambda_2(A'_0) = \pi m \left(\alpha^{-2/m} - 1\right) \sqrt{\det W}.$$

Evidently,  $\lambda_2(A'_0) > \lambda_2(A_0) \iff$

$$m(\alpha^{-2/m} - 1) > 2 \ln \alpha^{-1} \quad \forall m > 0 \quad \forall \alpha \in (0, 1).$$

The last inequality is a consequence of the known inequality:  $x - 1 > \ln x \quad \forall x > 1$ . Moreover, note that  $m(\alpha^{-2/m} - 1) \rightarrow 2 \ln \alpha^{-1}$ , as  $m \rightarrow \infty$ .

**Remark 2** If the distribution of  $Y$  is degenerate and  $N_n/n \rightarrow 1$  in probability, as  $n \rightarrow \infty$ , then the results for samples of random size do not differ from those, obtained when the sample size is non-random (Sect. 2).

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