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Second-order productivity, second-order payoffs, and the Banzhaf value

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Abstract

First, we suggest and discuss second-order versions of properties for solutions for TU games used to characterize the Banzhaf value, in particular, of standardness for two-player games, of the dummy player property, and of 2-efficiency. Then, we provide a number of characterizations of the Banzhaf value invoking the following properties: (i) [second-order standardness for two-player games or the second-order dummy player property] and 2-efficiency, (ii) standardness for one-player games, standardness for two-player games, and second-order 2-efficiency, (iii) standardness for one-player games, [second-order standardness for two-player games or the second-order dummy player property], and second-order 2-efficiency. These characterizations also work within the classes of simple games, of superadditive games, and of simple superadditive games.

Keywords TU game · Banzhaf value · Second-order marginal contributions · Second-order payoffs · Amalgamation

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1 Introduction

A cooperative game with transferable utility for a finite player set (TU game or simply game) is given by a coalition function that assigns a worth to any coalition (subset of the player set), where the empty coalition obtains zero. (One-point) solutions for TU games assign a payoff to any player in any TU game. Besides the Shapley value (Shapley 1953), the Banzhaf value (Banzhaf 1965; Owen 1975) probably is one of the most eminent one-point solutions for TU games.

Young (1985) characterizes the Shapley value by three properties of solutions: efficiency, symmetry, and marginality. Efficiency: the players' payoffs sum up to the worth generated by the grand coalition. Symmetry: equally productive[1] players obtain the same payoff. Marginality: a player's payoff only depends on her own productivity. In a sense, this characterization indicates that the Shapley value is the unique efficient solution that reflects the players productivities.

Recently, Casajus (2021) suggests a second-order version of Young's (1985) characterization of the Shapley value. This characterization is based on the notions of the players' second-order productivities and second-order payoffs. A player's second-order productivity with respect to another player reflects how the former affects the latter player's productivity (see Footnote 1); a player's second-order payoff with respect to another player reflects how the former affects the latter player's payoff by leaving the game. The Shapley value is the unique solution that satisfies efficiency and second-order versions of symmetry and marginality. Second-order symmetry: players who are equally second-order productive with respect to a third player obtain the same second-order payoff with respect to this third player. Second-order marginality: a player's second-order payoff with respect to another player only depends on her own second-order productivity with respect to this other player. In a sense, this characterization indicates that the Shapley value is the unique efficient solution that reflects the players' second-order productivities in terms of their second-order payoffs.

In this paper, we suggest and discuss second-order characterizations of the Banzhaf value derived from its characterization by Lehrer (1988) and Casajus (2012) using just two properties, standardness for two-player games (Hart and Mas-Colell 1989) or the dummy player property and 2-efficiency (Lehrer 1988). Standardness for two-player games: the payoffs for two-player games coincide with most of the solutions in the literature, in particular, the Shapley value, the Banzhaf value, and the nucleolus (Schmeidler 1969). Dummy player property: if a player's productivity is constant, then her payoff reflects (equals) her productivity. 2-efficiency: A solution is neutral with respect to the payoffs of amalgamated players. More precisely, if two players are amalgamated, then their payoffs sum up to the payoff of the amalgamated

[1] In this paper, a player's productivity in a game refers to her influence on the generation of worth as expressed by her marginal contributions to coalitions not containing her, that is, the differences between the worth generated after she entered such a coalition and the worth generated before she entered.

player. In a sense, this characterization indicates that the Banzhaf value is the unique amalgamation neutral solution that reflects the players' productivities.[2]

In particular, we suggest and discuss second-order versions of standardness for two-player games, the dummy player property, and 2-efficiency. Second-order standardness for two-player games: the second-order payoffs for two-player games coincide with most of the solutions in the literature. Second-order dummy player property: if a player's second-order productivity with respect to another player is constant, then her payoff reflects (equals half of) her second-order productivity. Second-order 2-efficiency: If two players are amalgamated, then their second-order payoffs with respect to a third player sum up the second-order payoff of the amalgamated player with respect to the third player in the amalgamated game.

We obtain a number of characterizations of the Banzhaf value invoking the following properties: (i) [second-order standardness for two-player games or the second-order dummy player property] and 2-efficiency (Theorem 7), (ii) standardness for one-player games,[3] standardness for two-player games, and second-order 2-efficiency (Theorem 10), (iii) standardness for one-player games, [second-order standardness for two-player games or the second-order dummy player property], and second-order 2-efficiency (Theorem 12). Cum grano salis, the latter, for example, indicates that the Banzhaf value is the unique second-order amalgamation neutral solution that reflects the players' second-order productivities. These characterizations also work within the classes of simple games, of superadditive games, and of simple superadditive games (Remarks 9, 11, and 14).

The remainder of this paper is organized as follows. In Sect. 3, we provide basic definitions and notation. In Sect. 4, we survey the characterizations of the Banzhaf value due to Lehrer (1988) and Casajus (2012). In Sect. 5, we survey second-order productivity and second-order payoffs. In Sect. 6, we discuss second-order versions of standardness for two-player games, the dummy player property, and 2-efficiency. In Sect. 7, we provide characterizations of the Banzhaf value using these properties. Some remarks conclude the paper.

2 Basic definitions and notation[4]

Let the universe of players $\mathfrak{U}$ be a countably infinite set, and let $\mathcal{N}$ denote the set of all finite subsets of $\mathfrak{U}$. The cardinalities of $S, T, N \in \mathcal{N}$ are denoted by s, t, and n, respectively. A (finite TU) game for the player set $N \in \mathcal{N}$ is given by a **coalition function** $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, where 2^Ndenotes the power set of N. Subsets of N are called **coalitions**; $v(S)$ is called the worth of coalition S. The set of all games for N is denoted by $\mathbb{V}(N)$; the set of all games is denoted by $\mathbb{V} := \bigcup_{N \in \mathcal{N}} \mathbb{V}(N)$.

2 Alternative characterizations of the Banzhaf value on various domains have been provided, for example, by Dubey and Shapley (1979), Haller (1994), Feltkamp (1995), Nowak (1997), Casajus (2011), and Haimanko (2018).

3 Standardness for one-player games: the payoff equals the worth generated by the single player. This is tantamount to efficiency or the dummy player property for one-player games.

4 This section closely follows Casajus (2020) or Casajus (2021).

For $N \in \mathcal{N}$, $T \subseteq N$, and $v \in \mathbb{V}(N)$, the **subgame** $v|_T \in \mathbb{V}(T)$ is given by $v|_T(S) = v(S)$ for all $S \subseteq T$; for $i \in N$ and $S \subseteq N$, we occasionally write v_{-i} and v_{-S} instead of $v|_{N\setminus\{i\}}$ or $v|_{N\setminus S}$, respectively. For $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$, and $\alpha \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}(N)$ and $\alpha \cdot v \in \mathbb{V}(N)$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. For $N \in \mathcal{N}$, the game $\mathbf{0}^N \in \mathbb{V}(N)$ given by $\mathbf{0}^N(S) = 0$ for all $S \subseteq N$ is called the **null game** for N. For $N \in \mathcal{N}$ and $T \subseteq N$, $T \neq \emptyset$, the game $u_T^N \in \mathbb{V}$ given by $u_T^N(S) = 1$ if $T \subseteq S$ and $u_T^N(S) = 0$ otherwise is called a **unanimity game**. Any $v \in \mathbb{V}(N)$, $N \in \mathcal{N}$, can be uniquely represented by unanimity games. In particular, we have

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T^N, \tag{1}$$

where the coefficients $\lambda_T(v)$ are known as the Harsanyi dividends (Harsanyi 1959), which can be determined recursively by

$$\lambda_T(v) \equiv v(T) - \sum_{S \subsetneq T : S \neq \emptyset} \lambda_S(v). \tag{2}$$

A game $v \in \mathbb{V}(N)$, $N \in \mathcal{N}$, is called **simple** if $v(S) \in \{0, 1\}$ for all $S \subseteq N$; it is called **superadditive** if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$.

A **solution** for $\mathbb{V}$ is an operator that assigns to any $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$ a payoff $\varphi_i(v)$. The **Shapley value** (Shapley 1953) for $\mathbb{V}$, Sh, is given by

$$\mathrm{Sh}_i(v) \equiv \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{t} = \sum_{S \subseteq N\setminus\{i\}} \frac{v(S \cup \{i\}) - v(S)}{n \cdot \binom{n-1}{s}} \tag{3}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$. The **Banzhaf value** (Banzhaf 1965; Owen 1975) for $\mathbb{V}$, Ba, is given by

$$\mathrm{Ba}_i(v) \equiv \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{2^{t-1}} = \sum_{S \subseteq N\setminus\{i\}} \frac{v(S \cup \{i\}) - v(S)}{2^{n-1}} \tag{4}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$.

3 First-order approaches to the Banzhaf value

Lehrer (1988) introduces the notion of amalgamated players/games. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j \in N$, $i \neq j$, the **amalgamated game** $v_{ij} \in \mathbb{V}(N\setminus\{j\})$ is given by

$$v_{ij}(S) = v(S) \quad \text{and} \quad v_{ij}(S \cup \{i\}) = v(S \cup \{i, j\}) \qquad \text{for all } S \subseteq N \setminus \{i, j\}. \tag{5}$$

In the game v_{ij}, players i and j are amalgamated into the (amalgamated) player i. That is, player i behaves as if player j "sits on her shoulders" and brings her productivity with her.[5] Based on this notion, he introduces the following property of solutions.

2-efficiency, 2E. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i,j \in N$, $i \neq j$, we have $\varphi_i(v) + \varphi_j(v) = \varphi_i\left(v_{ij}\right)$.

This property can be paraphrased as follows: a solution is required to be neutral with respect to the payoffs of amalgamated players.

Lehrer (1988, Remark 3) characterizes the Banzhaf value using two properties: 2-efficiency and standardness for two-player games (Hart and Mas-Colell 1989).

Standardness for two-player games, ST2. For all $i,j \in \mathfrak{U}, i \neq j$ and $v \in \mathbb{V}(\{i,j\})$, we have

$$\varphi_i(v) = v(\{i\}) + \frac{v(\{i,j\}) - v(\{i\}) - v(\{j\})}{2} \overset{(2)}{=} \lambda_{\{i\}}(v) + \frac{\lambda_{\{i,j\}}(v)}{2}.$$

For two-player games, this property requires a solution to assign the standard payoff to the players, that is, the same payoffs as most solutions in the literature, for example, the Banzhaf value itself, the Shapley value, and the nucleolus (Schmeidler 1969). This property indicates that for two-player games a solution reflects the players' productivities as given by their marginal contributions.

Theorem 1 (Lehrer 1988) *The Banzhaf value is the unique solution for* $\mathbb{V}$ *that satisfies standardness for two-player games* (**ST2**) *and 2-efficiency* (**2E**).

Later on, Casajus (2012, Theorem 7) replaces standardness for two-player games with the dummy player property.[6]

For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$, player i is called a **dummy player** in v if

$$v(T \cup \{i\}) - v(T) = v(\{i\}) \qquad \text{for all } T \subseteq N \setminus \{i\}.$$

Dummy, D. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$ such that i is a dummy player in v, we have $\varphi_i(v) = v(\{i\})$.

The dummy player property can be paraphrased as follows: if a player's productivity as given by her marginal contributions is constant, then her payoff should reflect (equal) this constant productivity.

Theorem 2 (Casajus 2012) *The Banzhaf value is the unique solution for* $\mathbb{V}$ *that satisfies the dummy player property* (**D**) *and 2-efficiency* (**2E**).

[5] Originally, Lehrer (1988) amalgamates players i and j into a player denoted by $\overline{\{i,j\}}$. The way he uses amalgamation (p. 96) indicates that $\overline{\{i,j\}} \in \{i,j\}$ as above and in Casajus (2012). In contrast, Nowak (1997) amalgamates the players i and j into the new player $\{i,j\}$. Alonso-Meijide et al. (2012) demonstrate that this mere technicality may matter.

[6] Nowak (1997) characterizes the Banzhaf value using four properties: a version of 2-efficiency (see Footnote 5), the dummy player property, symmetry, and marginality (Young 1985) (called the marginal contributions property by the former). Marginality: for all $N \in \mathcal{N}$, $v,w \in \mathbb{V}(N)$, and $i \in N$ such that $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v) = \varphi_i(w)$.

4 Second-order productivity and second-order payoffs

The marginal contributions of a player $i \in N, N \in \mathcal{N}$ in the game $v \in \mathbb{V}(N)$ given as

$$v(S \cup \{i\}) - v(S), \qquad S \subseteq N \setminus \{i\} \tag{6}$$

indicate her (individual) productivity or contribution to the generation of worth in the game v. Recently, Casajus (2021) introduces the notions of the players' second-order productivities and second-order payoffs. Second-order productivities are conceptualized as second-order marginal contributions: the second-order marginal contributions of player $i \in N$, $N \in \mathcal{N}$, with respect to player $j \in N \backslash \{i\}$ in a game $v \in \mathbb{V}(N)$ are given as

$$\left[v(S \cup \{i,j\}) - v(S \cup \{i\})\right] - \left[v(S \cup \{j\}) - v(S)\right], \qquad S \subseteq N \setminus \{i,j\}. \tag{7}$$

These describe how player i affects the productivity of player j.[7]

The second-order payoff of player $i \in N$, $N \in \mathcal{N}$, with respect to player $j \in N \setminus \{i\}$ in a game $v \in \mathbb{V}(N)$ is given by $\varphi_j(v) - \varphi_j\left(v_{-i}\right)$. It describes how player i affects the payoff of player j by leaving the game.[8]

Using second-order marginal contributions and second-order payoffs, one can define second-order versions of (first-order) properties of solutions (see Casajus 2021).

5 Second-order properties of solutions

In this section, we introduce and discuss second-order versions of standardness for two-player games, the dummy player property, and 2-efficiency.

5.1 Second-order standardness for two-player games

In one player games, (almost) all solutions assign the singleton worth to the player as her (standard) payoff. Standardness for two-player games requires a solution to assign the standard payoffs for two player games. This allows us to translate standardness for two-player games into second-order payoffs.

Second-order standardness for two-player games, 2ST2. For all $i, j \in \mathfrak{U}$, $i \neq j$, and $v \in \mathbb{V}(\{i,j\})$, we have

$$\varphi_i(v) - \varphi_i\left(v_{-j}\right) = \frac{v(\{i,j\}) - v(\{i\}) - v(\{j\})}{2} \overset{(2)}{=} \frac{\lambda_{\{i,j\}}(v)}{2}.$$

[7] The second-order marginal contributions of player i to player j in the game v equal player j's contributions to player i. Often, these are referred to as the second-order derivative of v with respect to i and j (see Owen 1972).

[8] Casajus and Huettner (2018, Definition 9) introduce second-order (and higher-order) payoffs as second-order (and higher-order) contributions.

For two-player games, this property requires the second-order payoffs of a solution to coincide with the standard second-order payoffs, that is, the second-order payoffs of most solutions in the literature. For the Banzhaf value and the Shapley value, this is immediate from (4) and (3), respectively. This property indicates that for two-player games a solution reflects the players' second-order productivities as given by their second-order marginal contributions.

Remark 3 Both the Banzhaf value and the Shapley value satisfy second-order standardness for two-player games (**2ST2**).

We conclude this subsection by clarifying the relation between standardness for two-player games and second-order standardness for two-player games: they do not imply each other. Consider the solutions $\mathrm{Ba}^{\div 2}$ and Two for $\mathbb{V}$ given by

$$\mathrm{Ba}_i^{\div 2}(v) \equiv \frac{n}{2} \cdot \mathrm{Ba}_i(v) \qquad \text{for all } N \in \mathcal{N},\ v \in \mathbb{V}(N), \text{ and } i \in N,$$

and

$$\mathrm{Two}_i(v) \equiv \sum_{\ell \in N \setminus \{i\}} \frac{\lambda_{\{i,\ell\}}(v)}{2} \qquad \text{for all } N \in \mathcal{N},\ v \in \mathbb{V}(N), \text{ and } i \in N. \tag{8}$$

The solution $\mathrm{Ba}_i^{\div 2}$ satisfies standardness for two-player games but not second-order standardness for two-player games. The solution Two satisfies second-order standardness for two-player games but not standardness for two-player games.[9]

5.2 The second-order dummy player property

A player is a dummy player in a game if all her marginal contributions equal her marginal contribution to the empty set. Analogously, a player is a second-order dummy player with respect to another player if all her second-order marginal contributions with respect to this player equal her second-order marginal contribution with respect to this player to the empty set. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j \in N$, $i \neq j$, player i is called a **second-order dummy player with respect to** j in v if

$$\begin{aligned}&\left[v(T \cup \{i,j\}) - v(T \cup \{i\})\right] - \left[v(T \cup \{j\}) - v(T)\right]\\ &\quad = \left[v(\{i,j\}) - v(\{i\})\right] - \left[v(\{j\}) - v(\emptyset)\right] \overset{(2)}{=} \lambda_{\{i,j\}}(v)\end{aligned}$$

for all $T \subseteq N \setminus \{i,j\}$.[10]

[9] In (8), we use the standard convention that the sum over an empty set is zero. We are grateful to an anonymous referee for suggesting to clarify this point.

[10] Besner (2022) introduces the related notion of disjointly productive players. In our parlance, such players would be second-order null players to each other: for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j \in N$, $i \neq j$, player i is called a **second-order null player with respect to** j in v if $\left[v(T \cup \{i,j\}) - v(T \cup \{i\})\right] - \left[v(T \cup \{j\}) - v(T)\right] = 0$ for all $T \subseteq N \setminus \{i,j\}$.

Note that this definition is symmetric with respect to i and j. That is, player i is a second-order dummy player with respect to j, whenever j is a second-order dummy player with respect to i. Hence, one could say that they are **second-order dummies to each other**. This is also reflected by the following characterization of second-order dummy players in terms of Harsanyi dividends.[11]

Lemma 4 *For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i,j \in N$, $i \neq j$, player i is a second-order dummy player with respect to j in v if and only if $\lambda_{T\cup\{i,j\}}(v) = 0$ for all $T \subseteq N\backslash\{i,j\}$, $T \neq \emptyset$.*

Proof The claim trivially holds true for $n \leq 2$. Let now $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i,j \in N$ be as in the lemma and $n > 2$. By Casajus (2021, Equation 7), we have

$$\left[v(T\cup\{i,j\}) - v(T\cup\{j\})\right] - [v(T\cup\{i\}) - v(T)] = \sum_{S\subseteq T} \lambda_{S\cup\{i,j\}}(v).$$

for $T \subseteq N \setminus \{i,j\}$. The claim now easily follows by induction on t. □

The dummy player property requires a dummy player's payoff to equal her marginal contribution to the empty set. Whereas marginal contributions can be attributed to a single player, second-order marginal contributions can be equally attributed to both players involved. Taking this into account, the second-order version of the dummy player property requires the second-order payoff of a second-order dummy player with respect to another player to equal *half* of her second-order contribution with respect to this player to the empty set.[12] Another way to motivate the division by two is as follows. The dummy player property requires a dummy player to obtain her standard payoff in the restriction of the game to this player. Analogously, the second-order dummy player property requires two players who are dummies to each other to "obtain" their standard second-order payoffs in the restriction of the game to the two players.

Second-order dummy, 2D. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i,j \in N$, $i \neq j$ such that i is a second-order dummy player with respect to j in v, we have

$$\varphi_j(v) - \varphi_j\left(v_{-i}\right) = \frac{\left[v(\{i,j\}) - v(\{i\})\right] - \left[v(\{j\}) - v\left(\emptyset\right)\right]}{2} \overset{(2)}{=} \frac{\lambda_{\{i,j\}}(v)}{2}. \tag{9}$$

It is well-known that both the Shapley value and the Banzhaf value satisfy the dummy player property. They also satisfy the second-order dummy player property.

[11] Besner (2022, Lemma 3.3) provides a related characterization of disjointly productive players (see Footnote 10).

[12] Besner (2022) introduces the related disjointly productive players property. In our parlance, this would be the **second-order null player property**: for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i,j \in N$, $i \neq j$ such that i is a second-order null player with respect to j in v (see Footnote 10), we have $\varphi_j(v) - \varphi_j\left(v_{-i}\right) = 0$. That is, second-order null players "obtain" a zero second-order payoff.

Lemma 5 *Both the Banzhaf value and the Shapley value satisfy the second-order dummy player property* (**2D**).

Proof For $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i,j \in N, i \neq j$, we obtain

$$\begin{aligned} &\mathrm{Ba}_i(v) - \mathrm{Ba}_i\left(v_{-j}\right) \\ &\overset{(4)}{=} \sum_{S \subseteq N \setminus \{i,j\}} \frac{\left[v(S \cup \{i,j\}) - v(S \cup \{j\})\right] - \left[v(S \cup \{i\}) - v(S)\right]}{2^{n-1}}. \end{aligned} \tag{10}$$

By Casajus (2021, Proof of Proposition 5), we also have

$$\begin{aligned} &\mathrm{Sh}_j(v) - \mathrm{Sh}_j\left(v_{-i}\right) \\ &= \sum_{S \subseteq N \setminus \{i,j\}} \frac{\left[v(S \cup \{i,j\}) - v(S \cup \{i\})\right] - \left[v(S \cup \{j\}) - v(S)\right]}{n \cdot \binom{n-1}{s+1}}. \end{aligned} \tag{11}$$

One easily checks that the coefficients of the second-order marginal contributions in (10) and (11) sum up to 1/2. This implies that both Sh and Ba satisfy **2D**. $\square$

We conclude this subsection by clarifying the relation between the dummy player property and the second-order dummy player property: they do not imply each other. Consider the solutions One and Ba^{+1} for $\mathbb{V}$ given by

$$\mathrm{One}_i(v) \equiv v(\{i\}) \overset{(2)}{=} \lambda_{\{i\}}(v) \qquad \text{for all } N \in \mathcal{N},\ v \in \mathbb{V}(N), \text{ and } i \in N, \tag{12}$$

and

$$\mathrm{Ba}_i^{+1}(v) \equiv \mathrm{Ba}_i(v) + 1 \qquad \text{for all } N \in \mathcal{N},\ v \in \mathbb{V}(N), \text{ and } i \in N. \tag{13}$$

The solution One satisfies the dummy player property but not the second-order dummy player property. The solution Ba_i^{+1} satisfies the second-order dummy player property but not the dummy player property.

5.3 Second-order 2-efficiency

2-efficiency can be translated into second-order payoffs in a straightforward way.

Second-order 2-efficiency, 22E. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i,j,k \in N$, $i \neq j \neq k \neq i$, we have

$$\varphi_k\left(v_{ij}\right) - \varphi_k\left(\left(v_{ij}\right)_{-i}\right) = \varphi_k(v) - \varphi_k\left(v_{-i}\right) + \varphi_k(v) - \varphi_k\left(v_{-j}\right).$$

When two players are amalgamated, then their second-order payoffs with respect to a third player sum up to the second-order payoff of the amalgamated player with respect to this third player in the amalgamated game. This property can be paraphrased as follows: a solution is neutral with respect to the second-order payoffs of amalgamated players.

In contrast to the Banzhaf value, the Shapley value fails 2-efficiency. The same holds true for second-order 2-efficiency. In order to see that the Shapley value fails second-order 2-efficiency consider $v = u_N^N$ for $N \in \mathcal{N}$ with $n > 2$. In contrast, the Banzhaf value also satisfies second-order 2-efficiency.

Lemma 6 *The Banzhaf value satisfies second-order* 2*-efficiency* (**22E**).

Proof For $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j, k \in N$, $i \neq j \neq k \neq i$, we have

$$\begin{aligned} &\mathrm{Ba}_k\left(v_{ij}\right) - \mathrm{Ba}_k\left(\left(v_{ij}\right)_{-i}\right) \\ &\quad \overset{(10),(5)}{=} \sum_{S \subseteq N \setminus \{i,j,k\}} \frac{\left[v(S \cup \{i,j,k\}) - v(S \cup \{i,j\})\right] - \left[v(S \cup \{k\}) - v(S)\right]}{2^{n-2}} \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\mathrm{Ba}_k(v) - \mathrm{Ba}_k\left(v_{-i}\right) \\ &\quad \overset{(10)}{=} \sum_{S \subseteq N \setminus \{i,j,k\}} \frac{\left[v(S \cup \{i,k\}) - v(S \cup \{i\})\right] - \left[v(S \cup \{k\}) - v(S)\right]}{2^{n-1}} \\ &\qquad + \sum_{S \subseteq N \setminus \{i,j,k\}} \frac{\left[v(S \cup \{i,j,k\}) - v(S \cup \{i,j\})\right] - \left[v(S \cup \{j,k\}) - v(S \cup \{j\})\right]}{2^{n-1}}. \end{aligned} \tag{15}$$

In the sum of (15) and the analogous expression for j, the "right" terms cancel out or double up, respectively, and one obtains (14). This shows that Ba satisfies **22E**. □

We conclude this section by clarifying the relation between second-order 2-efficiency and 2-efficiency. Second-order 2-efficiency and 2-efficiency do not imply each other. The solution One in (12) satisfies second-order 2-efficiency but not 2-efficiency. The solution E for $\mathbb{V}$ given by

$$\mathrm{E}_i(v) \equiv \frac{v(N)}{2^{n-1}} \qquad \text{for all } N \in \mathcal{N},\ v \in \mathbb{V}(N), \text{ and } i \in N, \tag{16}$$

satisfies 2-efficiency but not second-order 2-efficiency.

6 Second-order approaches to the Banzhaf value

In this section, we provide our second-order approaches to the Banzhaf value. Based on the second-order properties introduced in the previous section, we provide partial and full second-order characterizations of the Banzhaf value. In particular, we replace one or both properties in Theorems 1 and 2 with their second-order versions, respectively.

6.1 Partial second-order approaches with 2-efficiency

Theorems 1 and 2 remain true if one replaces standardness for two-player games with second-order standardness for two-player games or the dummy player property with the second-order dummy player property, respectively. This indicates that the Banzhaf value is the unique 2-efficient solution that reflects the players' second-order productivities.

Theorem 7

(i) *The Banzhaf value is the unique solution for $\mathbb{V}$ that satisfies second-order standardness for two-player games* (**2ST2**) *and* 2-*efficiency* (**2E**).
(ii) *The Banzhaf value is the unique solution for $\mathbb{V}$ that satisfies the second-order dummy player property* (**2D**) *and* 2-*efficiency* (**2E**).

These characterizations are non-redundant. The solution Two in (8) satisfies second-order standardness but not 2-efficiency. The solution E in (16) satisfies 2-efficiency but not second-order standardness. The Shapley value satisfies the second-order dummy player property but not 2-efficiency. The solution Zero for $\mathbb{V}$, given by

$$\mathrm{Zero}_i(v) \equiv 0 \qquad \text{for all } N \in \mathcal{N},\ v \in \mathbb{V}(N), \text{ and } i \in N$$

satisfies 2-efficiency but not the second-order dummy player property.

We prepare the proof of the theorem by a lemma.

Lemma 8

(i) *If a solution for $\mathbb{V}$ satisfies second-order standardness for two-player games* (**2ST2**) *and* 2-*efficiency* (**2E**), *then it satisfies standardness for two player games* (**ST2**).
(ii) *If a solution for $\mathbb{V}$ satisfies the second-order dummy player property* (**2D**) *and* 2-*efficiency* (**2E**), *then it satisfies standardness for two player games* (**ST2**).

Proof (i) Let the solution φ for $\mathbb{V}$ satisfy **2ST2** and **2E**. Fix $i, j \in \mathfrak{U}$, $i \neq j$. First, we have

$$\varphi_i\left(\mathbf{0}^{\{i,j\}}\right) \overset{\mathbf{2ST2}}{=} \varphi_i\left(\mathbf{0}^{\{i\}}\right) \overset{\mathbf{2E}}{=} \varphi_i\left(\mathbf{0}^{\{i,j\}}\right) + \varphi_j\left(\mathbf{0}^{\{i,j\}}\right).$$

This implies (a) $\varphi_j\left(\mathbf{0}^{\{j\}}\right) = 0$. Analogously, we obtain (b) $\varphi_i\left(\mathbf{0}^{\{i\}}\right) = 0$. Second, for all $\alpha \in \mathbb{R}$, we have

$$\varphi_i\left(\alpha \cdot u_{\{i,j\}}^{\{i,j\}}\right) \overset{\text{(b)}}{=} \varphi_i\left(\alpha \cdot u_{\{i,j\}}^{\{i,j\}}\right) - \varphi_i\left(\mathbf{0}^{\{i\}}\right) \overset{\mathbf{2ST2}}{=} \frac{\alpha}{2} \overset{\mathbf{2ST2},\text{(b)}}{=} \varphi_j\left(\alpha \cdot u_{\{i,j\}}^{\{i,j\}}\right) \tag{17}$$

and

$$\varphi_i\left(\alpha \cdot u_{\{i\}}^{\{i\}}\right) \overset{\mathbf{2E}}{=} \varphi_i\left(\alpha \cdot u_{\{i,j\}}^{\{i,j\}}\right) + \varphi_j\left(\alpha \cdot u_{\{i,j\}}^{\{i,j\}}\right) \overset{(17)}{=} \alpha. \tag{18}$$

Finally, for all $v \in \mathbb{V}(\{i,j\})$, we have

$$\varphi_i(v) \overset{\mathbf{2ST2}}{=} \frac{\lambda_{\{i,j\}}(v)}{2} + \varphi_i\left(\lambda_{\{i\}}(v) \cdot u_{\{i\}}^{\{i\}}\right) \overset{(18)}{=} \frac{\lambda_{\{i,j\}}(v)}{2} + \lambda_{\{i\}}(v),$$

which concludes the proof of part (i).

(ii) Let the solution φ for $\mathbb{V}$ satisfy **2D** and **2E**. Fix $i,j \in \mathfrak{U}, i \neq j$. First, we have

$$\varphi_i\left(\mathbf{0}^{\{i\}}\right) \overset{\mathbf{2E}}{=} \varphi_i\left(\mathbf{0}^{\{i,j\}}\right) + \varphi_j\left(\mathbf{0}^{\{i,j\}}\right) \overset{\mathbf{2D}}{=} \varphi_i\left(\mathbf{0}^{\{i\}}\right) + \varphi_j\left(\mathbf{0}^{\{j\}}\right).$$

This implies (c) $\varphi_j\left(\mathbf{0}^{\{j\}}\right) = 0$. Analogously, we obtain (d) $\varphi_i\left(\mathbf{0}^{\{i\}}\right) = 0$. Second, for all $\alpha \in \mathbb{R}$, we have

$$\varphi_i\left(\alpha \cdot u_{\{i\}}^{\{i\}}\right) \overset{\mathbf{2E}}{=} \varphi_i\left(\alpha \cdot u_{\{i,j\}}^{\{i,j\}}\right) + \varphi_j\left(\alpha \cdot u_{\{i,j\}}^{\{i,j\}}\right) \overset{\mathbf{2D}}{=} \alpha + \varphi_i\left(\mathbf{0}^{\{i\}}\right) + \varphi_j\left(\mathbf{0}^{\{j\}}\right) \overset{\text{(c),(d)}}{=} \alpha. \tag{19}$$

Finally, for all $v \in \mathbb{V}(\{i,j\})$, we have

$$\varphi_i(v) \overset{\mathbf{2D}}{=} \frac{\lambda_{\{i,j\}}(v)}{2} + \varphi_i\left(\lambda_{\{i\}}(v) \cdot u_{\{i\}}^{\{i\}}\right) \overset{(19)}{=} \frac{\lambda_{\{i,j\}}(v)}{2} + \lambda_{\{i\}}(v),$$

which concludes the proof of part (ii). □

Proof of Theorem 7 Existence: By Remark 3 and Lemma 5, the Banzhaf value satisfies **2ST2** and **2D**. By Lehrer (1988, Remark 3), it satisfies **2E**. Uniqueness: Let φ be a solution for $\mathbb{V}$ that satisfies **2ST2** or **2D** and **2E**. By Lemma 8, the solution φ satisfies **ST**. By Lehrer (1988, Remark 3), the solution φ coincides with the Banzhaf value. □

Remark 9 Lemma 8 and Theorem 7 hold true within the classes of simple games, of superadditive games, and of simple superadditive games. Both the amalgamation of players and the standard removal of players from a game yields simple games from simple games and superadditive games from superadditive games. Hence, the proofs of Lemma 8, Lehrer (1988, Remark 3), and Theorem 12 work within these classes, respectively.

6.2 Partial second-order approaches with second-order 2-efficiency

Theorems 1 and 2 do not remain true if one replaces 2-efficiency with second-order 2-efficiency (see the non-redundancy argument below). Adding standardness for one-player games to the list of properties yields a characterization of the Banzhaf value.[13] Cum grano salis, this indicates that the Banzhaf value is the unique second-order 2-efficient solution that reflects the players' productivities.

Standardness for one-player games, ST1. For all $i \in \mathfrak{U}$ and $v \in \mathbb{V}(\{i\})$, we have $\varphi_i(v) = v(\{i\}) \overset{(2)}{=} \lambda_{\{i\}}(v)$.

Theorem 10 *The Banzhaf value is the unique solution for* $\mathbb{V}$ *that satisfies standardness for one-player games* (**ST1**), *standardness for two-player games* (**ST2**), *and second-order* 2*-efficiency* (**22E**).

This characterization is non-redundant. The Shapley value satisfies all properties but second-order 2-efficiency. The solution One in (12) satisfies all properties but standardness for two-player games. The solution Ba^0 for $\mathbb{V}$ given by

$$\mathrm{Ba}_i^0(v) \equiv \begin{cases} \mathrm{Ba}_i(v), & n > 1, \\ 0, & n = 1 \end{cases} \qquad \text{for all } N \in \mathcal{N},\ v \in \mathbb{V}(N), \text{ and } i \in N$$

satisfies all properties but standardness for one-player games.

Proof Existence: By (4), the Banzhaf value trivially **ST1** and **ST2**. By Remark 3, it satisfies **22E**.

Uniqueness: Let the solutions φ and ψ for $\mathbb{V}$ satisfy **ST1**, **ST2**, and **22E**. We show $\varphi = \psi$ by induction on n.

Induction basis: For $n \leq 2$, the claim is immediate from **ST1** and **ST2**.

Induction hypothesis (IH): Suppose $\varphi(v) = \psi(v)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$ such that $n \leq \theta$.

Induction step: Let now $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$ be such that $n = \theta + 1 \geq 3$. For $i \in N$ choose $j, k \in N \backslash \{i\}$, $j \neq k$. Now, we have

$$\begin{aligned} 2 \cdot \varphi_i(v) &\overset{\mathbf{22E}}{=} \varphi_i\left(v_{jk}\right) - \varphi_i\left(\left(v_{jk}\right)_{-j}\right) + \varphi_i\left(v_{-j}\right) + \varphi_i\left(v_{-k}\right) \\ &\overset{IH}{=} \psi_i\left(v_{jk}\right) - \psi_i\left(\left(v_{jk}\right)_{-j}\right) + \psi_i\left(v_{-j}\right) + \psi_i\left(v_{-k}\right) \overset{\mathbf{22E}}{=} 2 \cdot \psi_i(v), \end{aligned}$$

which concludes the proof. □

Theorem 2 does not remain true if one replaces 2-efficiency with second-order 2-efficiency even if one adds standardness for one-player games. The solution One in (12) satisfies standardness for one-player games, the dummy player property, and second-order 2-efficiency.

Remark 11 Theorem 10 holds true within the classes of simple games, of superadditive games, and of simple superadditive games. The reasoning is as in Remark 9.

[13] Standardness for one-player games is equivalent to efficiency for one-player games or the dummy player property for one-player games. Efficiency: for all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$, we have $\sum_{i \in N} \varphi_i(v) = v(N)$.

6.3 Full second-order approaches

Replacing both properties in Theorems 1 and 2 with their second-order counterparts does not yield characterizations of the Banzhaf value (see the non-redundancy argument below). Adding standardness for one-player games to the list of properties yields a characterization of the Banzhaf value. Cum grano salis, this indicates that the Banzhaf value is the unique second-order 2-efficient solution that reflects the players' second-order productivities.

Theorem 12 *The Banzhaf value is the unique solution for* $\mathbb{V}$ *that satisfies standardness for one-player games* (**ST1**), [*second-order standardness for two-player games* (**2ST2**) *or the second-order dummy player property* (**2D**)], *and second-order* 2*-efficiency* (**22E**).

These characterizations are non-redundant. The Shapley value satisfies all properties but second-order 2-efficiency. The solution One in (12) satisfies all properties but second-order standardness for two-player games and the second-order dummy player property. The solution Ba^{+1} in (13) satisfies all properties but standardness for one-player games.

We prepare the proof of the theorem by a lemma.

Lemma 13 *If a solution for* $\mathbb{V}$ *satisfies standardness for one-player games* (**ST1**) *and* [*second-order standardness for two-player games* (**2ST2**) *or the second-order dummy player property* (**2D**)], *then it satisfies standardness for two-player games* (**ST2**).

Proof Let the solution φ for $\mathbb{V}$ satisfy **ST1** and [**2ST2** or **2D**]. Let $i, j \in \mathfrak{U}$, $i \neq j$, and $v \in \mathbb{V}(\{i, j\})$. Trivially, players i and j are second-order dummies to each other in v. Hence, we obtain

$$\varphi_i(v) \overset{\textbf{2ST2}\text{ or }\textbf{2D}}{=} \frac{\lambda_{\{i,j\}}(v)}{2} + \varphi_i\left(v_{-j}\right) \overset{\textbf{ST1}}{=} \frac{\lambda_{\{i,j\}}(v)}{2} + \lambda_{\{i\}}(v),$$

which concludes the proof. □

Proof of Theorem 12 Existence: By (4), the Banzhaf value trivially satisfies **ST1**. By Remark 3 and Lemmas 5 and 6, it satisfies **2ST2**, **2D**, and **22E**.

Uniqueness: Let φ and ψ be solutions for $\mathbb{V}$ that satisfy **ST1**, [**2ST2** or **2D**], and **22E**. We show $\varphi = \psi$ by induction on n.

Induction basis: For $n = 1$, the claim is immediate from **ST1**. For $n = 2$, the claim follows from **ST1**, [**2ST2** or **2D**], and Lemma 13.

Induction hypothesis (IH): Suppose $\varphi(v) = \psi(v)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$ such that $n \leq \theta$.

Induction step: As in the proof of Theorem 10. □

Remark 14 Lemma 13 and Theorem 12 hold true within the classes of simple games, of superadditive games, and of simple superadditive games. The reasoning is as in Remark 9.

7 Concluding remarks

In this paper, we suggest a number of characterizations of the Banzhaf value indicating that the latter is the unique (second-order) 2-efficient solution that reflects the players' (second-order) productivities in terms of their second-order payoffs. Casajus (2020, Appendix A) suggests and discusses higher-order marginal contributions and higher-order payoffs and related properties of solutions, higher-order symmetry and higher-order marginality, with respect to the Shapley value. Whereas the Shapley value satisfies these higher-order properties, it is not the unique efficient solution that satisfies these higher-order properties.

For higher-order versions of the characterizations in this paper, there would be additional difficulties. Second-order standardness for two-player games and the second-order dummy player property refer to the players' standard payoffs in one-player games and in two-player games. For games with three and more players, there are no standard payoffs. Hence, there is not even a unique or standard way to define higher-order versions of standardness for games with more than two players and the dummy player property.

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