Nash blocks

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Abstract

A product set of pure strategies is a Nash block if it contains all best replies to the Nash equilibria of the game in which the players are restricted to the strategies in the block. This defines an intermediate block property, between curb (Basu and Weibull, Econ Lett 36(2):141–146, https://doi.org/10.1016/0165-1765(91)90179-O, http://www.sciencedirect.com/science/article/pii/016517659190179O, 1991) and coarse tenability (Myerson and Weibull (2015) Econometrica 83(3):943–976, https://doi.org/10.3982/ECTA11048). While the new concept is defined without reference to the consideration-set framework that defines tenability, the framework can be used to characterize Nash blocks in terms of potential conventions when large populations of individuals recurrently interact. Although weaker than curb, Nash blocks nevertheless maintain several robustness properties of curb sets. For example, every Nash block contains an essential component and is robust against payoff perturbations.

Keywords Minimal Nash block · Set-valued solution concept · Index theory

1 Introduction

In Nash's mass-action interpretation (Nash 1950), individuals are randomly and repeatedly drawn from large populations to play a game against each other. It is assumed that an individual's behavior cannot influence the future behavior of other individuals, implying the absence of supergame effects. A Nash equilibrium is then seen as a stationary state in population frequencies over the individual's pure strategies. In

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such interactions, conventions for how to play the underlying game often emerge.[1] These conventions not only help the individuals to coordinate but also simplify their decision-making (see, e.g., Schelling 1960; Lewis 1969). However, as is well-known, simple examples show that many Nash equilibria are implausible as such conventions. The completely mixed Nash equilibrium in the following coordination game is a striking example[2]

$$
\begin{array}{cc}
 & L \quad R \\
L & 1, 1 \quad 0, 0 \\
R & 0, 0 \quad 1, 1
\end{array}
$$

In this game, it seems unreasonable that the individuals would not be able to over time coordinate their expectations to settle on one of the strict equilibria.

In the spirit of Myerson and Weibull (2015), this paper develops a set-valued concept that identifies Nash equilibria that are compatible with potential conventions in finite normal-form games. In such a game, a *block* is a nonempty set of pure strategies for each player, and a *block game* is a game where each player is restricted to using strategies with support in the associated block. Myerson and Weibull (2015) interpret blocks as the basis for potential conventions, that is, sets of strategies individuals take into consideration when called to play the game in their player role. For such conventions to be self-enforcing, no individual should be able to do better by using strategies outside of the convention. This notion can be formalized in several ways. An elegant and operational formulation is due to Basu and Weibull (1991): a block is *closed under rational behavior*, or curb, if it contains all best replies to the mixed-strategy profiles with support in the block.

As noted by Myerson and Weibull (2015), curb blocks are sometimes large and depend on features of the game that may be regarded as strategically inessential. Therefore, the authors weaken the above robustness requirement to only hold when the *overall population play constitutes a Nash equilibrium*. They formalized this within a framework in which every player role in a game is represented by a large population of boundedly rational individuals. Using this framework, the authors elaborate two block properties called *coarse* and *fine tenability*, respectively. Coarse tenability satisfies the weaker robustness requirement by requiring that the block contains at least one best reply to any population Nash equilibrium. Fine tenability relaxes this requirement by restricting the population distribution of boundedly rational individuals to be biased towards more rationality types. *Settled equilibria* are Nash equilibria with support in minimal coarsely and finely tenable blocks.

In this paper, I explore an intermediate block property, between curb and coarse tenability, called *Nash block*. A Nash block contains all best replies to all Nash equilibria of its block game. Thus, every Nash equilibrium of a Nash-block game is a Nash equilibrium of the full game. A *Nash-block settled equilibrium*, or NBE, is any

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[1] Here, a convention is best described by the words of Peyton Young: "A convention is a pattern of behavior that is customary, expected, and self-enforcing" (Young 1993, p. 57).

[2] Note that this equilibrium satisfies most refinements in the literature, such as perfection (Selten 1975), properness (Myerson 1978), strategic stability (Kohlberg and Mertens 1986) and essentiality (Wu and Jiang 1962).

Nash equilibrium with support in some *minimal* Nash block. Every finite game has at least one such equilibrium.[3] Minimality (in terms of set inclusion) is here motivated by the fact that it simplifies the players' decision-making by reducing the number of strategies they have to consider.

The main advantage of the Nash block concept is that it offers a simple way to decompose games into self-contained block games that can be studied in isolation from the full game. To find a Nash block, a simple algorithm suffices which *does not* require finding all Nash equilibria nor to perturb the data of the game: (1) Start with any block and compute all Nash equilibria of the associated block game, (2) add a (weakly) better reply for one of the players to one of the Nash equilibria from the previous step, (3) repeat steps 1 and 2 until no more strategies are added. This process stops after a finite number of repetitions and the resulting block is a Nash block.[4] The point is that if the analyst (or the players) has a reasonable prior of what the starting block should be, e.g., the strategies that have been observed to be used in the past, there is no need to consider the whole game.

To further see the benefits of the concept, consider an analyst studying a situation as described by the mass-action interpretation. When the analyst specifies a model for the analysis, instead of ad-hoc selecting the strategies that she deems "essential" for the problem at hand, the Nash block concept provides a systematic approach to select candidate blocks for the analysis to focus on.[5]

For Nash equilibria of the full game with support in some Nash block, only analyzing the Nash-block game is without loss of generality in the following sense: global properties of an equilibrium component– such as robustness against payoff perturbations– are determined by local properties of the same component in the corresponding Nash-block game.[6] I establish this using index theory (Ritzberger 1994).

I show by the way of examples that the solution concept developed in this paper offers an operational approach to equilibrium selection that has good cutting power in games in which standard solution concepts are known to perform poorly.[7] These include games with cheap talk or signaling, which are typically associated with a plethora of Nash equilibria, many of which constitute implausible predictions.

As hinted by its definition, the Nash block concept is closely related to the curb concept. I show that most of the latter's robustness properties are inherited by Nash blocks. In particular, a strategy profile constitutes a singleton Nash block if and only if it is a strict Nash equilibrium. Moreover, every Nash block contains the support of an essential component (Jiang 1963)—thus also a proper equilibrium (Myerson

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[3] In the above coordination game, it is easy to verify that the two strict Nash equilibria constitutes the two minimal Nash blocks, $\{L\} \times \{L\}$ and $\{R\} \times \{R\}$ (this is also true for the other just-mentioned concepts).

[4] One could add all better replies in step 2 so that the resulting block is independent on the particular order that strategies are added. To find minimal Nash blocks, one can reapply the algorithm for all Nash equilibria with support in a Nash block while adding strategies in different orders.

[5] See Sect. 7 for a discussion and references to the literature.

[6] Thus, it is possible to combine the NBE concept with, e.g., strategic stability (Kohlberg and Mertens 1986; Mertens 1989, 1991) to obtain a solution concept defined by strategically stable sets of NBE, while maintaining existence in every finite game.

[7] Additional examples to the ones in the text can be found in the Appendix.

1978) and a strategically stable subset. Despite these similarities, the Nash and the curb concept differ on an open set of games.

The Nash block concept is also related to coarse tenability. Although the former has a succinct definition without reference to population play, it can be characterized within Myerson and Weibull (2015) population framework. This characterization provides a micro foundation for the concept and highlights its relationship with coarse tenability. While the latter requires that no individual does *strictly better* using strategies outside the convention (in equilibrium), the Nash block concept strengthens this condition to require that any individual does *strictly worse* doing so. I show by way of example that coarsely (and finely) tenable blocks lack some of the robustness properties that Nash blocks inherit from the curb concept.

Finally, I discuss a property that minimal Nash blocks share with minimal tenable blocks (but not curb blocks), namely, that such blocks are *not* invariant under the deletion of strictly dominated strategies. However, in contrast to minimal tenable blocks, minimal Nash blocks are invariant under the addition of such strategies. Therefore, in any given game it is easy to identify the set of such blocks that are invariant under both the removal and addition of strictly dominated strategies.

The rest of the paper is organized as follows. Section 2 contains a motivating example. In Sect. 3, notation and definitions are provided. The Nash block concept is introduced in Sect. 4, where some of its fundamental properties are provided and its cutting power in important classes of games is highlighted. Section 5 discusses the concept's relationship with coarse tenability. In Sect. 6, index theory is used to study properties of Nash equilibrium components with support in the same Nash block. In Sect. 7, it is shown that minimal Nash blocks may contain strictly dominated strategies. Finally, related set-valued concepts are discussed in Sects. 8 and 9 contains a discussion of the concept's relationship with explicitly modeled dynamics.

2 Motivating example

Although the coordination game in the introduction suggests that the curb concept can be useful in refining the set of Nash equilibria in some games, the robustness properties of curb blocks are not necessarily related to the Nash equilibrium concept per se.[8] This may have adverse effects on the selectiveness of the concept, as suggested by the following simple example. Consider the generic perfect information extensive-form game given in Fig. 1.

Its normal form is given by

	L	R		L	R
U	3, 3, 3	3, 3, 3	U	1, 1, 1	1, 1, 1
D	4, 0, 0	2, 2, 2	D	4, 0, 0	2, 2, 2

$$W \qquad\qquad\qquad E$$

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[8] For example, a Nash equilibrium refinement can be defined by considering Nash equilibria with support in *minimal* curb blocks.
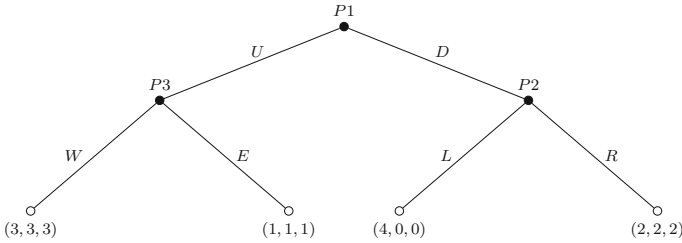
Fig. 1 A generic perfect information extensive-form game

This game has two Nash equilibrium components (disjoint, closed and connected sets of Nash equilibria): the component where 1 plays U, 2 assigns sufficiently little weight to L and 3 plays W; and the component where 1 plays D, 2 plays R and 3 assigns sufficiently little weight to W. The subgame perfect equilibrium (SPE) is (U, R, W).

In this game, player 2 and 3's (pure) strategy in the SPE weakly dominates the other strategy, respectively. Therefore, since a curb block includes all best replies to the strategy profiles with support in the block, it must include the SPE. As 2 is indifferent between her strategies in this equilibrium, both L and R must be included in the block. In fact, the unique curb block is the whole strategy space as D is optimal for 1 if 2 plays L and 3 plays W and E is optimal for 3 if 1 plays D and 2 plays R. It is worthwhile to point out that this feature is in no way related to L and W being weakly dominated strategies (it is possible to extend the game by adding strategies for 1 such that this is not the case without affecting the minimal curb block).[9]

By contrast, this game has two Nash blocks; trivially, the just-described curb block, and also $T = \{U, D\} \times \{L, R\} \times \{W\}$. Therefore, the set of NBE coincides with the equilibrium component containing the SPE. To see why T is a Nash block, consider the block game in which 3 only considers W. This corresponds to the two-player extensive form obtained from the extensive form in Fig. 1 where the outcome is $(3, 3)$ if 1 plays U. The Nash equilibria of this block game has 1 playing U with probability 1. Thus, E is never optimal against any equilibrium in this component. The reason why U has to be included for 1 in the Nash block—hence why T is the unique minimal Nash block—is that (U, L, W) is a Nash equilibrium in the (one-player) block game where 1 plays U, 2 chooses between L and R, and 3 plays W. As is easily seen, this is not an equilibrium of the full game.

It can be shown that the SPE constitutes the unique minimal coarsely (and finely) tenable block. Note that this implies that tenable blocks sometimes fail to include the support of all Nash equilibria generating the same outcome in the corresponding extensive form. As will be shown by example below, this introduces a non-robustness to small payoff perturbations and to the addition of strictly dominated strategies which can all be avoided using the Nash block concept.

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[9] To see this, add a decision node for 1 after 1 plays U and 3 plays W, and after 1 plays D and 2 plays R. Let these decision nodes give 1 a choice between the respective end-node in the full game and an outcome that is very bad for all players. The modified game is still a generic perfect information extensive-form game. In the corresponding normal-form game, neither L nor W are weakly dominated.

3 Notation and definitions

A finite normal-form game (a *game*, for short) is a triple $G = \langle N, S, u \rangle$, where $N = \{1, 2, \ldots, n\}$ is the finite set of players, $S = \times_{i \in N} S_i$ the finite and nonempty set of pure-strategy profiles, and $u : S \to \mathbb{R}^n$ the payoff functions, with $u_i(s)$ representing the payoff to player i under strategy profile s.

Let player i's mixed-strategy set be denoted by $\Delta_i(S_i)$, the unit simplex in $\mathbb{R}^{m_i}$, where m_i is the number of elements in S_i. Let $m = \prod_{i \in N} m_i$. The mixed-strategy space, $\square[S] \subset \mathbb{R}^m$, is the Cartesian product of the simplices $\Delta_i(S_i)$. Each payoff function's domain is extended in the usual way from S to $\square[S]$ by

$$u_i(x) = \sum_{s \in S} \left[\prod_{j \in N} x_j(s_j) \right] \cdot u_i(s).$$

Let $u_i(x_{-i}, s_i)$ denote the payoff that player i gets from playing $s_i \in S_i$ when all other players play according to $x_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(S_j)$. The set of *pure best replies* of player i against x is given by $\beta_i(x) = \arg \max_{s_i \in S_i} u_i(x_{-i}, s_i)$ with $\beta(x) = \times_{i \in N} \beta_i(x)$.

A strategy profile x is a *Nash equilibrium* if $x_i(s_i) > 0$ implies $s_i \in \beta_i(x)$. The nonempty set of Nash equilibria of a game is denoted $\square[S]^{NE}$. This set is semi-algebraic and consists of finitely many disjoint, closed and path-connected sets, so-called *equilibrium components* (Jiang 1963). A Nash equilibrium is *strict* if it is the unique best reply against itself. Strict equilibria evidently constitute singleton equilibrium components.

A pure strategy $s_i \in S_i$ is *weakly dominated* if there exists a mixed-strategy $y_i \in \Delta(S_i)$ such that $u_i(x_{-i}, y_i) \geq u_i(x_{-i}, s_i)$ for all $x \in \square[S]$, with strict inequality for some $x \in \square[S]$. It is *strictly dominated* if there exists a $y_i \in \Delta(S_i)$ such that $u_i(x_{-i}, y_i) > u_i(x_{-i}, s_i)$ for all $x \in \square[S]$. For any $\varepsilon > 0$, a strategy profile x such that $x_i(s_i) > 0$ for all $s_i \in S_i$ and $i \in N$ is *ε-proper* if

$$u_i(x_{-i}, s_i) < u_i(x_{-i}, t_i) \quad \Rightarrow \quad x_i(s_i) \leq \varepsilon \cdot x_i(t_i).$$

A *proper equilibrium* (Myerson 1978) is any limit of some sequence of ε-proper strategy profiles as $\varepsilon \to 0$. The set of such Nash equilibria is nonempty in every game.

For any game G, a *block* is any set $T = \times_{i \in N} T_i$ such that $\emptyset \neq T_i \subseteq S_i \; \forall i \in N$. The associated *block game* is defined by $G^T = \langle N, T, u^T \rangle$ where u^T denotes the restriction of u to T. The mixed-strategy space of any block game is embedded in the strategy space of the *full game* G by identifying $\square[T]$ with $\{x \in \square[S] : x_i(s_i) = 0 \; \forall s_i \notin T_i, \; \forall i \in N\}$. A strategy profile x has support in a block T if $x \in \square[T]$. The set of Nash equilibria of G^T is denoted $\square[T]^{NE}$.

The following block concept is due to Basu and Weibull (1991): A block T is *curb* if $\beta(x) \subseteq T$ for all $x \in \square[T]$. Every game admits at least one minimal curb block.

4 Nash blocks

The block property to be introduced here requires that each player's set of strategies contains all best replies to any Nash equilibrium of the associated block game.

Definition 1 A block T is a *Nash block* if $\beta(x) \subseteq T$ for all $x \in \Box[T]^{NE}$.

Using this definition, the game G^T where T is a Nash block is said to be a *Nash-block game*. The set of Nash equilibria of any Nash-block game coincides with the set of Nash equilibria of the full game with support in the block.[10] Thus, the concept defines a selection from the set of Nash equilibria of the full game.

A Nash block T is *minimal* if no Nash block is a proper subset of T. Since the set of pure-strategy profiles, S, is finite and trivially a Nash block, at least one minimal Nash block always exists. By considering the set of Nash equilibria with support in such blocks, one obtains a point-valued solution concept, a refinement of Nash equilibrium. Every game has at least one such equilibrium.

Definition 2 A *Nash-block settled equilibrium*, or NBE, is any Nash equilibrium with support in some minimal Nash block.

I now derive a few properties of Nash blocks and relate the concept to curb blocks. It is easy to see that every curb block is a Nash block and that the converse does not hold, as in Game 1 below. Even so, the Nash block concept inherits many stability properties from the more demanding concept.

First, as noted by Basu and Weibull (1991), the curb concept can be interpreted as a generalization of strict Nash equilibrium. To see this, note that a strategy profile is a strict Nash equilibrium if and only if it constitutes a singleton curb block. This property is also satisfied by the Nash block concept.

Observation 1 *A pure strategy profile is a strict Nash equilibrium if and only if it constitutes a singleton Nash block.*

A second related observation is that, even if a player attaches a small probability to the possibility that other players will not play a NBE, all her best replies are still in the block. Hence, the Nash block property is robust against perturbations of mixed strategies.[11]

Proposition 1 *If T is a Nash block, then there exists an open set U such that $\Box[T]^{NE} \subset U$ and $\beta(x) \subseteq T$ for all $x \in U$.*

Proof Given a Nash block T, for any $x \in \Box[T]^{NE}$, $i \in N$, and $t_i \in \beta_i(x) \subseteq T_i$, we have $u_i(x_{-i}, t_i) > u_i(x_{-i}, s_i)$ for all $s_i \notin T_i$. Fix $x \in \Box[T]^{NE}$. Since each payoff function u_i is continuous and each S_i finite, there exists a neighborhood $U_{i,x,t} \subset \mathbb{R}^m$ including x such that $u_i(y_{-i}, t_i) > u_i(y_{-i}, s_i)$ for all $y \in U_{i,x,t} \cap \Box[S]$ and $s_i \notin T_i$. For all $x \in \Box[T]^{NE}$ and $i \in N$, let the finite intersection of such sets define $U_{i,x} =$

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[10] As shown by Myerson and Weibull (2015), this is also a property of every coarsely tenable block.

[11] Ritzberger and Weibull (1995) show that this is also the case for any strategy profile in $\Box[T]$ if T is a curb block.

$\cap_{t_i \in \beta_i(x)} U_{i,x,t}$, implying that $u_i(y_{-i}, t_i) > u_i(y_{-i}, s_i)$ for all $y \in U_{i,x} \cap \Box[S]$, $t_i \in \beta(x)$, and $s_i \notin T_i$. Define $U = \cup_{x \in \Box[T]^{NE}} \cap_{i \in N} U_{x,i}$ which defines a neighborhood of $\Box[T]^{NE}$. By construction, $\beta(x) \subseteq T$ for all $x \in U \cap \Box[S]$.   □

Third, Nash blocks 'respect' Nash equilibrium components; each such component is either disjoint from or contained in $\Box[T]$ of any Nash block T. As shown by Ritzberger and Weibull (1995), curb blocks also have this property.

Proposition 2 *If T is a Nash block and ζ is a Nash equilibrium component, then either $\zeta \subseteq \Box[T]^{NE}$ or $\zeta \cap \Box[T]^{NE} = \emptyset$.*

Proof The Nash equilibrium correspondence (assigning to each game a set of Nash equilibria) is semi-algebraic. This implies that any Nash equilibrium component is closed and path-connected. For any block T with $X = \Box[T]^{NE}$, let $x \in \zeta \cap X$ and $y \in \zeta \backslash X$ for a Nash equilibrium component ζ in G. By path-connectedness, there exists a continuous function $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. As $\zeta \cap X$ is a closed set, by continuity there exists a $t \in [0, 1)$ and an $\bar{\varepsilon}' \in (0, 1)$ such that $\gamma(t) \in \zeta \cap X$ and $\gamma(t + \varepsilon) \in \zeta \backslash X$ for any $\varepsilon \in (0, \bar{\varepsilon}')$. By Proposition 1, there exists an $\bar{\varepsilon} \in (0, 1)$ such that $\beta(\gamma(t + \varepsilon)) \subseteq \beta(\gamma(t))$ for any $\varepsilon \in (0, \bar{\varepsilon})$. Moreover, by assumption $\gamma(t)$ is a Nash equilibrium for all $t \in [0, 1]$, and for any $\varepsilon \in (0, \min\{\bar{\varepsilon}, \bar{\varepsilon}'\})$ there exists a $s_i \notin T_i$ for at least one $i \in N$ with $\gamma(t + \varepsilon)(s_i) > 0$. As $s_i \in \beta_i(\gamma(t))$, I conclude that T is not a Nash block.   □

The next example provides a generic normal-form game in which there is a minimal Nash block that is not contained in any minimal curb block.

Example 1 Consider the game

$$
\begin{array}{cccc}
 & L & C & R \\
U & 4,1 & 1,4 & 0,0 \\
Game\ 1:\ M & 3,3 & 2,2 & 0,0 \\
B & 5,0 & -3,0 & 1,1
\end{array}
$$

This game has three Nash equilibria given by $x = (\frac{1}{4}U + \frac{3}{4}M, \frac{1}{2}L + \frac{1}{2}C)$, $y = (B, R)$, and $z = (\frac{1}{14}U + \frac{3}{14}M + \frac{5}{7}B, \frac{1}{5}L + \frac{1}{5}C + \frac{3}{5}R)$. It has two minimal Nash blocks; $T = \{B\} \times \{R\}$ and $T' = \{U, M\} \times \{L, C\}$. Thus, both y and x are NBE while the completely mixed Nash equilibrium z is not. The unique minimal curb block is T.

According to the curb concept, T' is unstable since if 1 would assign high probability to the event that 2 will choose L, then 1's unique optimal strategy, B, is outside the block. Moreover, it does not suffice to simply add B to T', since the optimal strategy for 2 against B is R. By contrast, T' is a Nash block since all strategies that are optimal against the unique equilibrium x of the block game $G^{T'}$ are in T'.

This example shows that Nash blocks, like tenable blocks, evaluate stability in population play near Nash equilibrium components (of the block game), not in population play far away from block game Nash equilibria.

Although strictly weaker than curb and being defined in terms of properties of Nash equilibria, the concept has cutting power in a variety of important classes of

games for which it is well-known that established solution concepts admits, arguably, implausible equilibria. In the "Appendix", I show that the Nash block concept have cutting power in sender-receiver and signaling games in which established solutions concepts, including curb and tenable blocks, are unable to reject unintuitive equilibria.

5 Nash blocks and tenability

In this section, I analyze the Nash block concept's relationship with tenable strategy blocks. Myerson and Weibull (2015) theory embeds the full game in a so-called consideration-set game. Such a meta-game endows every player role with a large population of boundedly rational individuals. For a block to be a potential convention, it is required that no individual should be able to do better by choosing a strategy outside the block when most individuals use strategies in the block. A coarsely tenable block formalizes such a convention when the overall population play constitutes a Nash equilibrium.

Formally, a game G is given a large population of individuals for each player role $i \in N$. One individual from each population is from time to time randomly drawn to play the game in her player role. Every individual is boundedly rational as she only considers a subset of the strategies available to her. Such a set of strategies is called the individual's consideration set, or her type. The type space for each player role i is given by $\Theta_i = \mathcal{C}(S_i)$, where $\mathcal{C}(S_i)$ is the collection of all nonempty subsets $C_i \subseteq S_i$. Let μ_i define a probability distribution on $\mathcal{C}(S_i)$ where $\mu_i(C_i) \in [0, 1]$ is the probability that the individual drawn to play in role i is of the type $\theta_i = C_i \in \mathcal{C}(S_i)$. A vector $\mu = (\mu_1, \ldots, \mu_n) \in \times_{i \in N} \Delta(\mathcal{C}(S_i))$ is called a type distribution, and the draws from each population are statistically independent.

Each type distribution μ defines a game of incomplete information $G^\mu = \langle N, \times_{i \in N} F_i, u^\mu \rangle$, called a consideration-set game. A pure strategy for player $i \in N$ is given by a function $f_i : \mathcal{C}(S_i) \to S_i$ such that $f_i(C_i) \in C_i$ for all $C_i \in \mathcal{C}(S_i)$. The set of all such functions is denoted F_i, and the simplex of mixed strategies is denoted $\Delta(F_i)$ with generic element τ_i. A consideration-set game is connected to the full game as each mixed-strategy profile $\tau \in \square[F] = \times_{i \in N} \Delta(F_i)$ induces a corresponding mixed-strategy profile $\tau^\mu \in \square[S]$ in G. The conditional probability distribution over the strategies in S_i, induced by a strategy used by some type $\theta_i = C_i$, is denoted $\tau_{i|C_i}$. Hence, the probability that player i will use a pure strategy s_i, given a strategy induced by τ_i, is

$$\tau_i^\mu(s_i) = \sum_{C_i \in \mathcal{C}(S_i)} \mu_i(C_i) \cdot \tau_{i|C_i}(s_i).$$

The expected payoff to each player i is given by $u_i^\mu(\tau) = u_i(\tau^\mu)$. This defines the vector of expected payoff functions $u^\mu : \square[F] \to \mathbb{R}^n$ in G^μ. Every consideration-set game admits at least a Nash equilibrium, and it is straightforward to show that the projections of Nash equilibria of G^μ to G converges to Nash equilibria of G^T as $\mu_i(T) \to 1$ for all i.

A block T is *coarsely tenable* if there exists an $\varepsilon \in (0, 1)$ such that $T \cap \beta(\tau^\mu) \neq \emptyset$ for every type distribution μ with $\mu_i(T_i) > 1 - \varepsilon$ for all $i \in N$ and every Nash equilibrium τ of G^μ. A *coarsely settled equilibrium* is any Nash equilibrium that has support in some minimal coarsely tenable block. For any block T and any $\varepsilon \in (0, 1)$, a type distribution μ is $\varepsilon - proper$ on T if for every player $i \in N$

$$\begin{cases} (a) \ \mu_i(T_i) > 1 - \varepsilon, \\ (b) \ \mu_i(C_i) > 0 \ \ \forall \ C_i \in \mathcal{C}(S_i), \\ (c) \ T_i \neq C_i \subset D_i \in \mathcal{C}(S_i) \ \ \Rightarrow \ \ \mu_i(C_i) \leq \varepsilon \cdot \mu_i(D_i). \end{cases}$$

A block T is *finely tenable* if there exists an $\bar{\varepsilon} \in (0, 1)$ such that $T \cap \beta(\tau^\mu) \neq \emptyset$ for every type distribution μ that is $\bar{\varepsilon} - proper$ on T and every Nash equilibrium τ of G^μ. Myerson and Weibull (2015) show that every finely tenable block contains the support of a proper equilibrium. Therefore, a *finely settled equilibrium* is any proper equilibrium that has support in some finely tenable block. Finally, a *fully settled equilibrium* is any Nash equilibrium that is both coarsely and finely settled. Such an equilibrium exists in every game.

Throughout the examples given in this paper (excluding the "Appendix"), minimal coarsely and finely tenable blocks coincide.

The Nash block concept is closely related to coarse tenability. In fact, it is possible to characterize the concept within the consideration-set framework that defines tenability. This not only highlights the relationship between the concepts but also provides a behavioral micro foundation for Nash blocks.

Call a block *strict coarsely tenable* if, when almost all individuals only consider strategies within the block, they do *strictly* better than any individual using strategies outside the block (given that the overall population play constitutes a Nash equilibrium). Formally:

Definition 3 A block T is *strict coarsely tenable* if there exists an $\varepsilon \in (0, 1)$ such that $\beta(\tau^\mu) \subseteq T$ for every type distribution μ with $\mu_i(T_i) > 1 - \varepsilon$ for all $i \in N$ and every Nash equilibrium τ of G^μ.

Evidently, every strict coarsely tenable block is coarsely tenable. The below characterization then implies that every Nash block is coarsely tenable.

Proposition 3 *T is a Nash block if and only if it is strict coarsely tenable.*

Proof ($\Leftarrow$) Suppose T is strict coarsely tenable. The set of Nash equilibria of a consideration-set game with $\mu_i(T_i) = 1$ for all $i \in N$ then induces the set $\Box[T]^{NE}$ in G. As a strict coarsely tenable block contains all best replies to the induced strategy profiles from the just mentioned consideration-set game, T is a Nash block.

($\Rightarrow$) Suppose T is a Nash block. By Proposition 1, there exists a neighborhood $U \subseteq \Box[S]$ such that $\Box[T]^{NE} \subset U$ and $\beta(x) \subseteq T$ for all $x \in U$. For any $\varepsilon \in (0, 1)$, let X^ε denote the closed and nonempty set of strategy profiles induced by the set of Nash equilibria of a consideration-set game with $\mu_i(T_i) = 1 - \varepsilon$ for all $i \in N$. By continuity, there exists an $\bar{\varepsilon} \in (0, 1)$ such that $X^\varepsilon \subset U$ for all $\varepsilon \in (0, \bar{\varepsilon})$. Thus, there exists an $\bar{\varepsilon} \in (0, 1)$ such that $X^\varepsilon \subseteq U$ for all $\varepsilon \in (0, \bar{\varepsilon})$ completing the proof. □

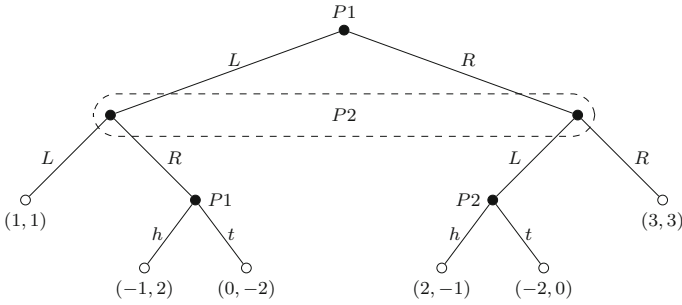**Corollary 1** *Every Nash block is coarsely tenable.*

Fig. 2 A coordination game with outside options in case of coordination failure

In generic normal-form games, it is straightforward to show that every coarsely tenable block is a Nash block. This follows because, in such games, all Nash equilibria are quasi-strict: for any quasi-strict Nash equilibrium x, if $s_i \in \beta_i(x)$ then $x_i(s_i) > 0$ (see, e.g., van Damme 1991).

Proposition 4 *Let G be a generic normal-form game. Then T is a Nash block if and only if it is coarsely tenable.*

As illustrated in Example 2, this result does not extend beyond generic normal-form games.

Example 2 Consider the generic extensive-form game given in Fig. 2.

It is an elaboration of a simple coordination game where one of the two pure Nash equilibria has been destabilized by giving one of the players an outside option if they fail to coordinate. The purely reduced normal-form representation of this extensive form is

$$
\begin{array}{cccc}
 & Lh & Lt & R \\
Lh & 1,1 & 1,1 & -1,2 \\
Game\ 2: Lt & 1,1 & 1,1 & 0,-2 \\
R & 2,-1 & -2,0 & 3,3
\end{array}
$$

This game has three Nash equilibrium components:

$$
\Theta = \{(pLh + (1-p)Lt, qLh + (1-q)Lt) : p, q \in [0, 3/4]\},
$$

the strict Nash equilibrium $x = (R, R)$, and the mixed Nash equilibrium $y = (\frac{1}{2}Lt + \frac{1}{2}R, \frac{1}{2}Lt + \frac{1}{2}R)$. The sole minimal Nash block is $T^1 = \{R\}^2$, the support of the NBE x. By contrast, the two minimal coarsely tenable blocks are T^1 and $T^2 = \{Lt\}^2$, thus, both x and $z = (Lt, Lt)$ are coarsely settled. The latter block is coarsely tenable as Lh is weakly dominated implying that it is redundant in any block including Lt. Note that if a strictly dominated strategy is added for 2 that gives higher payoff to Lh than Lt for 1 and is better than R for 2 if 1 plays Lt, then the unique minimal coarsely and finely tenable block coincides with the unique minimal Nash block T^1.

Since every coarsely tenable block contains the support of a proper equilibrium (Myerson and Weibull 2015), every game has a NBE that is proper. Using another machinery, it is possible to show that every Nash block also contains a set of Nash equilibria satisfying other demanding refinements in the literature. This is the topic of the upcoming section.

6 Index invariance

To provide robustness properties of sets of NBE, I here utilize results from the literature on index theory applied to Nash equilibrium components, as introduced by Ritzberger (1994)). In particular, it is shown that it suffices to analyze the associated Nash-block game to determine the index of a Nash equilibrium component with support in a Nash block.

I first provide an informal description of index theory for Nash equilibrium components.[12] To compare Nash equilibrium components of a given game, the theory adds a differential structure on the mixed-strategy space. Given this structure, it is possible to use concepts from differential topology to classify equilibrium components using an index. This index allows for inferring global properties—such as whether an equilibrium is proper—without computing perturbations of, e.g., mixed strategies.

More specifically, the differential structure on $\square[S]$ is induced by the system of equations obtained from the necessary Karush–Kuhn–Tucker conditions for a strategy to be a best reply against a given mixed-strategy profile. This system is then interpreted as a vector field.

In generic normal-form games, the determinant of the Jacobian of this vector field is non-zero, or regular, evaluated at any equilibrium. For such games, the index of a Nash equilibrium component (which, thus, is a singleton) is the sign of the determinant of -1 times the Jacobian evaluated at the equilibrium. Hence, it is either $+1$ or -1. To assign indices to non-regular components (that may be set-valued), the vector field is slightly perturbed so as to resemble the vector field of a generic game. The index is then defined as the sum of the indices of the 'Nash equilibria' of the perturbed vector field that are within an isolating neighborhood of the component.

The first result obtained using this machinery establishes that the index of a Nash equilibrium component with support in a Nash block is the same as the index of the same component in the associated Nash-block game. This result builds on Ritzberger (2002, Proposition 6.8), who showed that the index of an equilibrium component is invariant with respect to deletion of a strategy that is never a best reply against the component (see also McLennan 2016, Theorem 5).

Proposition 5 *If a Nash equilibrium component ζ has support in a Nash block T, then the index of ζ in G is also the index of ζ in the Nash-block game G^T.*

Proof Let ζ be a Nash equilibrium component in G. The proof of Proposition 6.8 in Ritzberger (2002, pp. 327–328) implies that if $s_i \notin \beta_i(x)$ for $x \in \zeta$, then the index of ζ is the same in G as in the block game $G^{S'} = \langle N, S', u \rangle$ for $S' = S_{-i} \times (S_i \backslash \{s_i\})$.

---

[12] See Ritzberger (2002) for a textbook treatment.

As S is a finite set, any Nash-block game can be obtained by reducing the strategy space of G by removing a finite number of strategies that are not best replies to the set of Nash equilibria with support in the corresponding block. Thus, the index of this equilibrium components stays the same. □

The above result implies that global properties of any equilibrium component consisting of NBE can be analyzed by restricting attention to the associated Nash-block game. Moreover, due to the Poincaré–Hopf theorem, it is possible to say something about the robustness properties of sets of Nash equilibria with support in Nash blocks. This theorem has the remarkable implication that the index sum across all equilibrium components in any game is $+1$. This, in turn, implies that the index sum across the equilibrium components with support in the same Nash block is $+1$.[13]

Corollary 2 *If T is a Nash block, then the index sum across all Nash equilibrium components with support in T is $+1$.*

Proof By the Poincaré–Hopf theorem, the index sum across all equilibrium components in a game G is $+1$. Moreover, every block game is a finite normal-form game independent of the structure of the full game, thus, the same holds true for all such games. An application of Proposition 5 completes the proof. □

Corollary 2 implies that every game has a NBE that is contained in an essential component (Jiang 1963) and an M-stable (strategically stable in the sense of Mertens 1989, 1991) set. Roughly speaking, a Nash equilibrium component is essential if a nearby component exists in all games with nearby payoffs. A minimal connected set of Nash equilibria is strategically stable if it is robust against arbitrary small perturbations of mixed strategies and satisfies an additional technical condition.

Corollary 3 *Every Nash block contains the support of an essential component and an M-stable set.*

The corollary follows from Ritzberger (1994,Theorem 4) and Demichelis and Ritzberger (2003, Theorem 2), who showed that a component with non-zero index contains the claimed sets. As the index sum across all components with support in a Nash block is $+1$, there is at least a component with non-zero index.

Another observation pertains to the cutting power of minimal Nash blocks.

Observation 2 *If a game admits more than one minimal Nash block, then there exists at least one Nash equilibrium component that does not have support in any of them.*

This observation a straightforward implication of the Poincaré–Hopf theorem together with Corollary 2 since the index sum across all components in n Nash blocks is $+n$. The last observation was pointed out by one of the referees and implies an aversion to completely mixed Nash equilibria.

---

[13] Two other result for *generic* finite normal-form games also hold for the set of equilibria of any generic Nash-block game: (1) the number of Nash equilibrium components is finite and odd (Harsanyi 1973) and (2) if the game has $m \geq 1$ pure Nash equilibria, then it has at least $m - 1$ mixed Nash equilibria (Gül et al. 1993).

Observation 3 *A completely mixed Nash equilibrium is a NBE if and only if there is only one Nash block constituting the entire set of pure-strategy profiles.*

Thus, whenever the minimal Nash block concept has any cutting power it rejects completely mixed Nash equilibria.

As illustrated in Example 3 below, the above index properties are not inherited by coarsely (and thus finely) tenable blocks.[14] In fact, there exist coarsely tenable blocks consisting of a single pure Nash equilibrium that lives in a component with index 0. Clearly, this equilibrium has index $+1$ in the (trivial) block game it generates.

Example 3 Consider the game

$$
\begin{array}{c c c c}
 & L & C & R \\
\textit{Game } 3 : U & 0, 0 & 2, 0 & 3, 3 \\
D & 2, 2 & 0, 2 & 1, -1
\end{array}
$$

It has two Nash equilibrium components; the strict Nash equilibrium $x = (U, R)$, and the non-convex component Γ consisting of the union of the connected components

$$
A = \left\{ \left(\frac{1}{2}U + \frac{1}{2}D, \frac{1}{2}L + p\frac{1}{2}C + (1 - p)\frac{1}{2}R \right) : p \in [0, 1] \right\}
$$

$$
B = \{ (D, pL + (1 - p)C) : p \in [1/2, 1] \}
$$

$$
C = \left\{ \left(qU + (1 - q)D, \frac{1}{2}L + \frac{1}{2}C \right) : q \in [0, 1/2] \right\}.
$$

Since x is strict, its index is $+1$. This implies that Γ has index zero.

The sole minimal Nash block is $T = \{U\} \times \{R\}$, implying that x is an NBE. However, there exists another minimal coarsely tenable block $T' = \{D\} \times \{L\}$. The latter is coarsely tenable since L and C are payoff equivalent for 2, implying that both are never included in the same minimal coarsely tenable block. Note that the unique strategy profile with support in T' belongs to the component with index 0. As x is the unique Nash equilibrium of any game in which 2's payoff from (D, C) is increased by any $\varepsilon > 0$, the component with index 0 is not essential.[15]

7 Dominated strategies

A perhaps surprising property is that a minimal Nash block may not survive the deletion of strictly dominated strategies. Furthermore, such strategies may even be *included* in minimal Nash blocks. This failure of invariance is also inherited by minimal coarsely and finely tenable blocks.[16] I here provide the intuition behind this property which

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[14] Of course, since every curb block is a Nash block, the former satisfies all of them.

[15] An open question is whether every coarsely tenable block includes the support of an M-stable set.

[16] In contrast, Basu and Weibull (1991) have shown that minimal curb blocks only contain strategies that survive the iterated removal of strictly dominated strategies.

is illustrated in Example 4 below. Thereafter, I discuss the Nash block concept's invariance under the addition of strictly dominated strategies and its implications.

It is useful to begin with a couple of simple observations. First, Myerson and Weibull (2015, p. 954) observe that a block containing all strategies that are not weakly dominated is coarsely tenable. The same observation holds for Nash blocks if weak is replaced by strict dominance. Second, if a Nash block includes a strategy that is strictly dominated in its block game, it is not minimal. This follows from the simple observation that a strictly dominated strategy is never a best reply to any strategy profile. Thus, it can be excluded from the block without loss.

However, a strictly dominated strategy can be undominated in a Nash-block game if the strategy that dominates it is not in the block.[17] This implies that a strictly dominated strategy can, if included in such a block, stop a strategy profile from being a Nash equilibrium of the block game. Without this strategy, the block is not a Nash block as the block game includes a Nash equilibrium that is not an equilibrium in the full game (since, by assumption, the inclusion of the dominated strategy eliminates it).

Note that this feature is consistent with the interpretation of Nash blocks as potential conventions: If the game is played by large populations of individuals, then they may be oblivious of strategies outside the conventional block, including strategies that strictly dominate strategies included in the current block game.

By definition, every strictly dominated strategy must have at least one (possibly mixed) strategy that dominates it. And although the support of this strategy could replace the strategy it dominates in the block, the resulting block may still not be minimal. To see this, consider a minimal Nash block that contains a strictly dominated strategy. Replace this strategy by the support of the strategy that dominates it. This block is *not* minimal if there exist a subblock including the just-added strategies that constitutes a minimal Nash block.

The above reasoning is illustrated in the upcoming example. In this example, the addition of a strictly dominated strategy allows for a new minimal Nash block and increases the set of NBE. The game is based on an elaboration of a simple coordination game, introduced by Myerson and Weibull (2015), in which the 'miscoordination end-nodes' (in its extensive form) are replaced by zero-sum subgames.

Example 4 Consider two versions of the following game

		Lh'	Lt'	Rh'	Rt'	B'	C'	D'
	Lh	1, 1	1, 1	$-2, 2$	2, -2	0, 0	0, 3	$-1, -1$
	Lt	1, 1	1, 1	2, -2	$-2, 2$	0, 0	$-1, -6$	$-1, -1$
Game 4 :	Rh	2, -2	$-2, 2$	1, 1	1, 1	5, 5	$-4, -4$	$-3, 3$
	Rt	$-2, 2$	2, -2	1, 1	1, 1	5, 5	$-4, -4$	$-3, 3$
	A	3, 0	$-6, -1$	$-4, -4$	$-4, -4$	7, 7	4, 4	$-8, 2$

one version where D' is available for 2 and one where it is deleted. This game is the reduced normal form of an extension of the simultaneous-move extensive-form

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[17] It might be the case that a strictly dominated strategy is dominated by a mixed-strategy profile. Then, the same observation holds true if any of the strategies in the support of the mixed strategy profile is missing from the block.

representation of the simple coordination game presented in the introduction. Here, each of (L, R) and (R, L) has been replaced by a zero-sum subgame with value zero and the strategies A, B', C' and D' are added. Note that D' is strictly dominated by B'.

This game has five Nash equilibrium components including:

$$\Theta = \{(pLh + (1 - p)Lt, qLh' + (1 - q)Lt') : p, q \in [1/4, 3/4]\},$$

and $\{x\} = (A, B')$. Of course, the set of Nash equilibria does not depend on whether D' is included in 2's strategy set. By contrast, as will be seen this is not true for the set of settled equilibria.

In the version of the game where D' is available for 2, it is possible to show that there exist two minimal Nash blocks, $T' = \{A\} \times \{B'\}$ and $T = \{Lh, Lt, Rh, Rt\} \times \{Lh', Lt', Rh, Rt, D'\}$. Thus, the set of NBE is given by $\{x\}$ and Θ. Notice that D' is undominated in the block game G^T. In both versions of this game, the set of Nash, coarsely tenable and finely tenable blocks coincide.

Consider now the version of Game 4 where D' is deleted. In this game, T' is the unique minimal Nash block, and $\{x\}$ is the set of NBE. To see this, consider the block $T^* = T_1 \times (T_2 \backslash D')$, that is, T excluding D' for 2. It is not a Nash block as the set of Nash equilibria of the corresponding block game includes a Nash equilibrium component where B' is the unique optimal strategy for 2.[18] However, adding B' to T^* does *not* generate a minimal Nash block as the resulting block properly contains the Nash block T'.

It is easy to show that Nash blocks are invariant under the addition of strictly dominated strategies. That is, the introduction of such a strategy can never make a Nash block seize to exist. This property extends to the addition of strategies that are never best replies to any Nash equilibrium of a Nash-block game.

Observation 4 *Let T be a minimal Nash block in G. Then T is also a minimal Nash block in any game G' obtained from G by the addition of strategies that are never best replies against the set of NBE with support in T.*

The above observation implies that every game has a minimal Nash block, and therefore a set of NBE, that is invariant against the addition and deletion of strictly dominated strategies. The set of such NBE is easy to identify: reduce the strategy space of a game by iteratively deleting strictly dominated strategies and then applying the Nash block concept. The reduced game contains all the minimal Nash blocks (which exist in the full game) that are invariant against the addition and removal of dominated strategies.

It is interesting to note that most established Nash equilibrium refinements are not invariant to the addition of strictly dominated strategies. This failure of invariance has been defended by Kohlberg and Mertens (1986) when discussing strategic stability on the grounds that strategic stability "depends on the whole given situation.

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[18] The set of Nash equilibria of G^{T^*} includes the component $\Omega = \{(pRh' + (1 - p)Rt', qRh + (1 - q)Rt) : p, q \in [1/4, 3/4]\}$, where $x \in \Omega$ implies $\beta_2(x) = \{B'\}$ in G. The same component also exists in the block games in which either A is added for 1 or C' is added for 2, or both.

So, when some implausible alternatives are deleted, the analysis has already taken their unlikeliness into account. However, adding possibilities that were physically not present previously cannot and should not have been anticipated" (Kohlberg and Mertens 1986, p.1017).

In the setting analyzed in this paper, where Nash equilibrium is interpreted as the outcome of a dynamic process and individuals tend to ignore strategies that are unconventional (and therefore never use them), it might often be hard to exactly pin down what is meant by "the whole given situation." A more pragmatic approach is to concede, in agreement with McLennan (2016), that "[e]conomic modeling requires strategic simplification. A model necessarily specifies only a few features of the world. The social scientist hopes that the selected features are the critical ones..." (McLennan 2016, p.26-27). Confining the analysis of a game to one of its Nash-block games can be interpreted as such a 'strategic simplification.' If such an approach is taken, it is desirable that 'inessential' strategies do not affect the criterion used to predict potential outcomes. In this view, the above observation is an important robustness requirement that shows that the concept is *not*, in the words of McLennan (2016, p.27), "excessively sensitive to minor details of model specification."

Note that the invariance under the addition of strictly dominated strategies depends crucially on the requirement that *all* best replies to the Nash equilibria of the Nash-block game are included in the block. Therefore, coarse and fine tenability do *not* have this robustness property.

8 Related literature

In this section, I present other related ideas and discuss their relationship with Nash blocks. A concept that can be reformulated as a block property was introduced by Kalai and Samet (1984). In any game in which no player has any payoff-equivalent strategies among her undominated strategies, the set of mixed-strategy profiles $\Box[T]$ is an *absorbing retract* if there exists an open set U containing $\Box[T]$ such that $\beta(x) \cap T \neq \emptyset$ for all $x \in U$. A *persistent retract* is a minimal absorbing retract. A *persistent equilibrium* is any Nash equilibrium with support in a persistent retract. It is easy to show that if T is a curb block, then $\Box[T]$ is an absorbing retract, and if $\Box[T]$ is an absorbing retract, then T is coarsely tenable. Another closely related idea is so-called prep sets, as introduced by Voorneveld (2004)). A prep set is a block T such that $\beta(x) \cap \Box[T] \neq \emptyset$ for all $x \in \Box[T]$. If $\Box[T]$ is an absorbing retract then T is a prep set. However, not all prep sets are coarsely tenable and vice versa.

Neither prep sets nor absorbing retracts are implied by or imply that the associated block is a Nash block. In Game 1 above, the two concepts coincide with the minimal coarsely tenable block, thus is a subset of the minimal Nash block. In Game 1, there exists a Nash block that is neither in a subblock of a prep set nor in a subblock that spans an absorbing retract. Game 2 provides an example of a reduced normal-form game, obtained from a generic extensive form, where there exists an absorbing retract that is not spanned by a subblock of a minimal Nash block. Moreover, in Game 3, there is a singleton coarsely tenable block that spans an absorbing retract which is not a Nash block. Finally, in the version of Game 4 when the strictly dominated strategy

is added, there is a Nash block that is neither a prep set nor spans an absorbing retract. In the version of this game where the dominated strategy is removed, all the concepts considered so far in this paper coincide.

Voorneveld (2004) and Voorneveld (2005) shows that curb blocks, prep sets and absorbing retracts coincide in generic normal-form games (T is, e.g., a prep set if and only if $\Box[T]$ is an absorbing retract). As seen in Game 1, although coarsely tenable and Nash blocks coincide in generic normal-form games, they are generically distinct from curb blocks.

Balkenborg et al. (2013)) and Balkenborg et al. (2015)) analyze refinements of the best-reply correspondence that are upper hemi-continuous, closed- and convex-valued. They use these refined correspondences to provide weaker variations of curb blocks and prep sets. However, their correspondences coincide with the usual best-reply correspondence on generic normal-form games. Thus, these concepts generically differ from Nash blocks.

Another related idea is p-best response sets by Tercieux (2006a) (see also Tercieux 2006b) for an analysis of a weaker requirement). A block is a p-best response set if it contains all best replies to all beliefs putting at least probability p on the block. However, in his paper, beliefs are not constrained to treat other players' strategy choices as statistically independent. In two-player games this generalization is vacuous, implying that any curb block is a p-best response set for some $p < 1$ (Ritzberger and Weibull 1995).

Finally, the Nash block concept is related to concepts based on evolutionary stability arguments. For such concepts, a Nash equilibrium is seen as an outcome trial and error where more successful individual behavior tends to be more prevalent. Maynard Smith and Price (1973) notion of evolutionary stability is defined for symmetric games and features a single population of individuals that are uniform random matched with each other. The incumbents constitute a population of individuals utilizing a given mixed strategy. Such a strategy is *evolutionarily stable* (ESS) if it does better on average than any small population share of unconventional individuals when the whole population is made up of mostly conventional and a small share of unconventional individuals.

Although ESS is consistent with strong notions of rationality (see, e.g., (van Damme 1991, Theorem 9.3.4)), such strategies fail to exist in many games. The same is true for its extension to asymmetric games and for related set-valued notions (see, e.g., Swinkels 1992). In Wikman (2020), I show that tenable blocks, and thus Nash blocks, can be given evolutionary interpretations in which they are seen as necessary robustness conditions when almost all play a conventional strategy and only a few individuals play the new unconventional strategy. I also provided formal relationships between tenable blocks and several notions of evolutionary stability.

9 Discussion

I have here developed a block concept that captures candidates for potential conventions in a setting in which individuals are repeatedly and randomly drawn from large populations to play a game against each other, as in Nash's mass-action interpretation of his equilibrium concept. While not explored here, the concept captures some

notion of dynamic stability when explicitly modeled. As shown by Demichelis and Ritzberger (2003), a necessary condition for an equilibrium component to be stable with respect to any 'natural' dynamic process, in the sense that the individuals adjust their strategies toward those that generate higher payoffs, is that its index agrees with its Euler characteristic. Without going into detail on what the Euler characteristic is, in generic normal-form games, and in normal forms of generic two-player extensive-form games, this condition is fulfilled for any component that corresponds to a unique Nash equilibrium component of a Nash-block game.

It would be interesting to further explore connections between Nash blocks and explicit models of population dynamics. For example, Balkenborg et al. (2013) show that a sufficient condition for the set of mixed strategies with support in a block to be asymptotically stable under the best-reply dynamics is that the block is curb (see, e.g., Young 1993) for an analysis of stochastic dynamics). In such dynamic population models, the robustness properties of Nash blocks suggest that sets of NBE with support in the same minimal Nash block could be good predictors.

Appendix

This appendix includes additional examples highlighting the Nash block concept's cutting power in sender-receiver and signaling games.

Sender-receiver games

Refinements based on rationalistic interpretations of Nash equilibrium are known to have little 'bite' in sender-receiver games. Instead, block concepts such as curb and settled equilibria have shown to effectively discard implausible equilibria in such games (Blume 1994, 1998; Myerson and Weibull 2015).

Consider first the simple sender-receiver game analyzed by Balkenborg et al. (2015)). In this game, a sender receives one of two signals $\{\omega_1, \omega_2\}$ and sends a message $m \in \{\alpha, \beta\}$ to the receiver who then implements an action $a \in \{A, B\}$. They both get 1 if they match the state of the world with the action (A if ω_1 and B if ω_2)

and -1 if they fail. The different states occur with equal probability.

	A, A	A, B	B, A	B, B
α, α	$0, 0$	$0, 0$	$0, 0$	$0, 0$
α, β	$0, 0$	$1, 1$	$-1, -1$	$0, 0$
β, α	$0, 0$	$-1, -1$	$1, 1$	$0, 0$
β, β	$0, 0$	$0, 0$	$0, 0$	$0, 0$

This game has infinitely many Nash equilibria that all are strategically stable in the sense of Kohlberg and Mertens (1986). By contrast, minimal curb, tenable and Nash blocks are $\{\alpha, \beta\} \times \{A, B\}$ and $\{\beta, \alpha\} \times \{B, A\}$ which constitutes the efficient coordinate outcomes.

However, as is shown by the example below, the curb concept loses much of its cutting power when the sender's and the receiver's interests are misaligned. By contrast, the Nash block equilibrium concept discards all but the efficient equilibria.

Consider now a modified sender-receiver game in which there are two equally likely states of nature $\omega = A$, and $\omega = B$. Player 1, the sender, observes the state of the world and sends one of two messages a or b, to player 2. The receiver, after having received the sender's message, selects one of two actions α or β. However, before this interaction both players can choose to implement an action. That is, before the state of the world is known to the sender, player 1 can choose between playing T or B, and player 2 can chose between playing L or R. The players' payoffs are given by the subgame $\mathcal{A}$ if action $\alpha(\beta)$ is taken in state $A(B)$, and subgame $\mathcal{SD}$ otherwise.

		L	R			L	R
Subgame $\mathcal{A}$:	T	$4, 0$	$0, 4$	Subgame $\mathcal{SD}$:	T	$-10, 6$	$6, -10$
	B	$0, 4$	$4, 0$		B	$6, -10$	$-10, 6$

The (purely) reduced normal form of the entire game is given by

	$L : \alpha\alpha/\beta\beta$	$R : \alpha\alpha/\beta\beta$	$L : \alpha\beta$	$R : \alpha\beta$	$L : \beta\alpha$	$R : \beta\alpha$
$T : aa/bb$	$-3, 3$	$3, -3$	$-3, 3$	$3, -3$	$-3, 3$	$3, -3$
$B : aa/bb$	$3, -3$	$-3, 3$	$3, -3$	$-3, 3$	$3, -3$	$-3, 3$
$T : ab$	$-3, 3$	$3, -3$	$4, 0$	$0, 4$	$-10, 6$	$6, -10$
$B : ab$	$3, -3$	$-3, 3$	$0, 4$	$4, 0$	$6, -10$	$-10, 6$
$T : ba$	$-3, 3$	$3, -3$	$-10, 6$	$6, -10$	$4, 0$	$0, 4$
$B : ba$	$3, -3$	$-3, 3$	$6, -10$	$-10, 6$	$0, 4$	$4, 0$

In this game, there exists a plethora of Nash equilibria giving zero payoff to both players. These equilibria involve player 1 playing $T : ab$ and $T : ba$, and $B : ab$ and $B : ba$ with equal probability respectively, while player 2 plays $L : \alpha\beta$ and $L : \beta\alpha$, and $R : \alpha\beta$ and $R : \beta\alpha$ with equal probability respectively. The two efficient equilibria involve both players selecting each action with equal probability. In that case the players have incentives to coordinate on the $\mathcal{A}$ subgame. The two equilibria, giving

both players a payoff of two, are given by $x_1 = (0.5[T : ab] + 0.5[B : ab], 0.5[L : \alpha\beta] + 0.5[R : \alpha\beta])$ and $x_2 = (0.5[T : ba] + 0.5[B : ba], 0.5[L : \beta\alpha] + 0.5[R : \beta\alpha])$.

The support of x_1 and x_2 are the only minimal Nash blocks in this game, hence they are both NBE. Here, the unique curb block is the whole strategy space, and all equilibria are strategically stable in the sense of Kohlberg and Mertens (1986). Minimal coarse tenability and minimal Nash block make the same predictions in the above game. By contrast, $\{T : aa/bb, B : aa/bb\} \times \{L : \alpha\alpha/\beta\beta, R : \alpha\alpha/\beta\beta\}$ is also a minimal finely tenable block. Arguably, the fine tenability concept is too permissive in this game.

Signaling: the beer-quiche game

The following is a version of the beer-quiche game (Kohlberg and Mertens 1986, Figure 14), with scaled payoffs (see also Cho and Kreps 1987).

	$Fight, Fight$	$Fight, \neg Fight$	$\neg Fight, Fight$	$\neg Fight, \neg Fight$
$Beer, Beer$	$-21, -8$	$-21, -8$	$-1, 0$	$-1, 0$
$Beer, Quiche$	$-2, -8$	$-18, -9$	$-2, 1$	$0, 0$
$Quiche, Beer$	$-30, -8$	$-12, 1$	$-28, -9$	$-10, 0$
$Quiche, Quiche$	$-29, -8$	$-9, 0$	$-29, -8$	$-9, 0$

In this game, minimal curb, coarse tenability and Nash block agree and contain the support of the so-called "intuitive equilibrium" in which 1 sends the strong signal (Beer, Beer) and 2 fights with at least probability 1/2 given a weak signal (Quiche) but not otherwise. The block is given by

$$T = \{(Beer, Beer), (Beer, Quiche)\} \times \{(\neg Fight, Fight), (\neg Fight, \neg Fight)\}.$$

By contrast fine tenability also permits the "unintuitive component" in which 1 sends the weak signal and 2 fights with at least probability 1/2 given a strong signal but not otherwise. The second minimal finely tenable block is given by

$$T = \{Quiche, Quiche\} \times \{Fight, \neg Fight\}.$$

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