An issue based power index

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Abstract

An issue game is a combination of a monotonic simple game and an issue profile. An issue profile is a profile of linear orders on the player set, one for each issue within the set of issues: such a linear order is interpreted as the order in which the players will support the issue under consideration. A power index assigns to each player in an issue game a nonnegative number, where these numbers sum up to one. We consider a class of power indices, characterized by weight vectors on the set of issues. A power index in this class assigns to each player the weighted sum of the issues for which that player is pivotal. A player is pivotal for an issue if that player is a pivotal player in the coalition consisting of all players preceding that player in the linear order associated with that issue. We present several axiomatic characterizations of this class of power indices. The first characterization is based on two axioms: one says how power depends on the issues under consideration (Issue Dependence), and the other one concerns the consequences, for power, of splitting players into several new players (no advantageous splitting). The second characterization uses a stronger version of Issue Dependence, and an axiom about symmetric players (Invariance with respect to Symmetric Players). The third characterization is based on a variation on the transfer property for values of simple games (Equal Power Change), besides Invariance with respect to Symmetric Players and another version of Issue Dependence. Finally, we discuss how an issue profile may arise from preferences of players about issues.

Keywords Power index · Simple game · Issue profile

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1 Introduction

1.1 Background

Power indices for simple games measure the power of players in such a game, independently of the issues at stake or the positions of players regarding these issues. For instance, a power index applied to a weighted majority game associated with a political parliament, typically considers how often a political party is needed to form a majority, without taking the issue at stake (for instance, a new law) into account. One may well argue that this is how it should be (for instance, Braham and Holler 2005), but one may also argue that this is a drawback (e.g., Napel and Widgrén 2005).

For a relatively recent overview of power indices for simple games see Bertini et al. (2013). These power indices include the Shapley value (Shapley 1953), also called Shapley–Shubik index (Shapley and Shubik 1954), the Banzhaf value (Banzhaf 1965; Shenoy 1982; Nowak 1997) and the Banzhaf–Coleman index (Coleman 1971), the Holler index (Holler 1982), and many more. Most of these power indices, including the ones mentioned, are based on counting in some way or another the number of times a player is pivotal in the simple game.

There are, however, also many approaches which do take the issues at stake, and the preferences of players regarding these issues, into account. Notably, so-called spatial power indices are defined on simple games, enriched by adding positions of the players in policy space. For instance, Owen (1971) and later Owen and Shapley (1989) add to the simple game a vector of positions of players in two-dimensional Euclidean space, and use this to obtain a variant of the Shapley–Shubik index, called the Owen–Shapley spatial power index, which takes these positions into consideration. See Peters and Zarzuelo (2017) for an axiomatic characterization of this index. As will become clear below, the approach in the present paper is based on a similar idea as the Owen–Shapley spatial power index. On the topic of spatial power indices, see also Enelow and Hinich (1984), Enelow and Hinich (1990), Grofman et al. (1987), Straffin (1994), Felsenthal and Machover (1998), Felsenthal and Machover (2005), and Laruelle and Valenciano (2008). More recent contributions are Alonso-Meijide et al. (2011), Benati and Marzetti (2013), Martin et al. (2017), and Blockmans and Guerry (2015).

An alternative and less known definition of a power index is proposed by Hoede and Bakker (1982), based on so-called inclination vectors: the positions of players with respect to certain issues may influence each other, and a power index may depend on the strengths of these influences. See also Rusinowska and de Swart (2003).

A theory of power measurement within corporate and/or financial networks is proposed by Gambarelli and Owen (1994). See also Karos and Peters (2015) and Mercik and Stach (2018). For power measurement on graphs, see e.g. van den Brink (2002),

or Peters et al. (2016). Karos and Peters (2018) discuss power indices when the possibilities of players and coalitions are described by an effectivity function.

Still other approaches model the impact of preferences on power by means of a noncooperative voting game: see, for instance, Schmidchen and Steunenbergh (2014).

1.2 Our approach

Our approach is best illustrated by a simple example. Consider a parliament with four parties, called 1, 2, 3, and 4, which have numbers of seats 49, 17, 17, and 17, respectively. To pass any law a strict majority (at least 51 seats) is required.¹ The Shapley-Shubik and (normalized) Banzhaf values assign power distribution $\frac{1}{6}(3, 1, 1, 1)$, and the Holler index assigns $\frac{1}{9}(3, 2, 2, 2)$. These power distributions are completely independent of what is at stake. Now suppose that during the period that this particular composition of the parliament is in vigor, there are three main issues (say, new laws) under consideration, and suppose that the big party 1 is highly in favor of these issues. This implies that party 1 is practically powerless, since it always depends on some other party (which is less enthusiastic about these issues) in order to pass the corresponding law. Pursuing the example somewhat further, call the issues a , b , and c , and let the ‘order of enthusiasm’ for each of the issues be given by 1, 2, 3, 4 for a , 1, 3, 4, 2 for b , and 1, 4, 3, 2 for c . If we assume that for each issue a supporting coalition is formed according to the given order, we see that the pivotal party for a would be party 2, for b party 3, and for c party 4. Taking the number of times a party is pivotal as a measure for its power, the resulting power distribution is $\frac{1}{3}(0, 1, 1, 1)$. In a nutshell, this is the (main) issue based power index that we consider in this paper.²

As mentioned above, this approach is closely inspired by the construction of the Owen–Shapley spatial power index (Owen and Shapley 1989). In the spatial game model of Owen and Shapley, each issue is a point u on the unit circle in $\mathbb{R}^2$ and each player i has a position p_i in $\mathbb{R}^2$. Then player i is more enthusiastic about issue u than player j if $p_i \cdot u < p_j \cdot u$ —thus, the inner product is interpreted as linear (dis)utility. A given simple game then determines, for each issue, which player is pivotal: this is the first player who makes the coalition, formed in the order of enthusiasm, winning. As a result, the unit circle is partitioned into subsets with different pivotal players, and the relative size of the subset (arc(s)) for which a player is pivotal, is defined to be that player’s power according to the Owen–Shapley spatial power index. Here, assuming that all positions are different, the set of issues for which the pivotal player is not unique has measure zero and can be neglected. This is no longer the case if the number of issues is finite, as in the present paper: this is one of the reasons that we take the orderings of the players per issue, rather than their positions (i.e., their preferences over the issues) as primitives in the model. For more discussion on this choice the reader is referred to the concluding Sect. 6.

¹ The associated simple game is a weighted majority game. In this case, it is also an apex game with player 1 as the apex player.

² Of course, it may well be possible that the biggest party 1 has had an important say in setting the agenda. Here, however, we assume that the agenda—set of issues—is given.

Instead of spatial games, in this paper we consider issue games. An issue game consists of a monotonic simple game for some player set N , and for each issue within the set of issues M , a linear order over the set of players N . This linear order expresses the order of enthusiasm or support of the players for the issue under consideration. Such a collection of linear orders is called an issue profile. In the example above, the simple game is the four-player weighted majority game, and the issue profile is the set of three linear orders over the four players, associated with the issues a , b , and c . A power index assigns to each player in an issue game a nonnegative number, where these numbers sum up to one. The power indices studied in this paper are characterized by weight vectors (nonnegative vectors with coordinates summing up to one) for the set of issues, such that the power assigned to each player in an issue game is equal to the weighted sum of the issues for which this player is pivotal. (In the example above, all three issues have equal weight $\frac{1}{3}$). The main part of the paper is devoted to presenting several axiomatic characterizations of this class of power indices.

The first result in the paper is Theorem 3.1, in which this class is characterized by two axioms: Issue Dependence and No Advantageous Splitting. Issue Dependence requires that in a situation where each player is pivotal for at most one issue, the power of a pivotal player depends (only) on the issue at stake. This axiom will imply that we can attach fixed weights to issues. No Advantageous Splitting says that no player, by splitting up into several players, can change the power distribution. For instance, in the above example, if party 2 would split up into two parties of sizes, say, 8 and 9 seats, its total power should stay the same, and also the powers of players 1, 3, and 4 should not change. The axiom is similar in spirit (and equal in name) to an axiom for bankruptcy problem rules in De Frutos (1999), where it is considered as a kind of non-manipulability. It is also closely related to the 2-efficiency axiom in Nowak (1997), used in a characterization of the Banzhaf index. The proof of this result is quite transparent: given an issue game, if a player is pivotal for more than one issue, we split this player up in new players such that each of the new players is pivotal for exactly one issue; and then apply Issue Dependence. In order for this to work, we assume that the set of players may vary, as a finite subset of the universal set of players identified with $\mathbb{N}$. The set of issues is regarded as fixed.

In the second characterization (Theorem 4.1) we consider the axiom of Invariance with respect to Symmetric Players: given any issue game and two players i and i' who are symmetric in the associated simple game, if i is pivotal for an issue and we change the linear order of that issue so that i' becomes pivotal, then the power assigned to every other player should not change. Additionally, we strengthen Issue Dependence to Strong Issue Dependence: if, in two issue games, the partitions of the set of issues, obtained by grouping together those issues which have the same pivotal player, coincide, then the power assigned to the players should only depend on the issues for which they are pivotal. These two conditions again characterize the same class of power indices. The proof of Theorem 4.1 is in some sense dual to the proof of Theorem 3.1: we start with an issue game in which the simple game is a unanimity game for the grand coalition (in particular, all players are symmetric), and by using Invariance with respect to Symmetric Players, change this to an issue game with any arbitrary partition of the set of issues, by grouping issues together; then, we apply Strong Issue Dependence.

In our third characterization (Theorem 5.3) we use the axiom Equal Power Change, which concerns the simple game rather than the issue profile. This axiom is a variation on the transfer property of Dubey (1975) and makes that the power index is uniquely determined by its values on issue games in which the simple games are unanimity games. Adding this axiom, allows for a considerable weakening of Strong Issue Dependence, namely to Symmetric Player Issue Dependence. The two axioms, plus Invariance with respect to Symmetric Players, again characterize the same class of power indices.

The organization of the paper is as follows. After preliminaries in Sect. 2 the three characterizations are presented in Sects. 3, 4, and 5. Section 6 concludes with further discussion.

2 Preliminaries

We identify the universe of potential *players* with $\mathbb{N}$. A *monotonic simple game* is a pair (N, v) , where $N \subseteq \mathbb{N}$ is a nonempty finite set of players and $v: 2^N \rightarrow \{0, 1\}$ satisfies $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$. Throughout we only consider simple games that are monotonic, and just refer to these as simple games. A *coalition* is a subset of N . A coalition S is *minimal winning* if $v(S) = 1$ and $v(S') = 0$ for every $S' \subsetneq S$. Player $i \in N$ is a *null player* in the simple game (N, v) if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N$. Two players i and i' in a simple game (N, v) are *symmetric* if $v(S \cup \{i\}) = v(S \cup \{i'\})$ for every $S \subseteq N \setminus \{i, i'\}$. A *unanimity game* is a simple game (N, u_T) for some $\emptyset \neq T \subseteq N$, defined by $u_T(S) = 1$ if and only if $T \subseteq S$, for every $S \subseteq N$.

The *set of issues* is a nonempty finite set M ; we usually write $M = \{1, \dots, |M|\}$. Throughout, the set of issues is fixed and therefore suppressed from notation. An *issue game* is a triple (N, v, Q) with (N, v) a simple game and Q a mapping from M to the set of linear orders on N .³ The mapping Q is an *issue profile*; for $j \in M$, Q^j denotes the linear order on N assigned by Q to issue j . By $\mathcal{G}$ we denote the set of all issue games. An *issue based power index* or simply *power index* is a map φ on $\mathcal{G}$ such that for every $G = (N, v, Q) \in \mathcal{G}$, $\varphi_i(G) \geq 0$ for all $i \in N$ and $\sum_{i \in N} \varphi_i(G) = 1$. Hence, a power index is *individually rational* and *efficient* by definition.⁴

In an issue game $G = (N, v, Q)$, player $i \in N$ is *pivotal* for issue $j \in M$ if $v(S \cup \{i\}) - v(S) = 1$, where $S = \{k \in N: k Q^j i\}$. In this case we also write $i = p_G(j)$, that is, $p_G(j)$ denotes the player who is pivotal for issue j in the issue game G . We use the notation $M_G^i = \{j \in M: i = p_G(j)\}$ for the set of issues for which player i is pivotal. Further, $P(G)$ denotes the set of pivotal players in G , i.e., $P(G) = \{i \in N: M_G^i \neq \emptyset\}$. Clearly, $|P(G)| \leq |M|$. Also, if player $i \in N$ is a null player in the simple game (N, v) , then $i \notin P(G)$.

³ A linear order R on a set of players N is, as usual, an irreflexive, complete, transitive and asymmetric binary relation on N . Instead of $(i, j) \in R$ we write $i R j$.

⁴ We include these conditions in the definition since they will be imposed throughout. We also repeat that the set of players may vary, but the set of issues is fixed.

A *weight vector* is a vector $w \in \mathbb{R}^M$ satisfying $w_j \geq 0$ for all $j \in M$, and $\sum_{j \in M} w_j = 1$. The power index Φ^w is defined by

$$\Phi_i^w(G) = \sum_{j \in M_G^i} w_j$$

for every $G = (N, v, Q) \in \mathcal{G}$. Hence, player i 's power according to Φ^w in the issue game G is simply the issue-weighted number of times that player i is pivotal. For the special case that all issues always have equal weights, i.e., $w_j = \frac{1}{|M|}$ for every $j \in M$, we write Φ .

3 Issue dependence and no advantageous splitting

Let φ be a power index. In this section we consider two axioms for φ , and show that these characterize the family of power indices Φ^w .

The first axiom says that in games where each player is pivotal for at most one issue, the power of each pivotal player depends only on the issue for which that player is pivotal. Observe that in such a situation every pivotal player is pivotal for exactly one issue, and then it is reasonable to assume that the power of each pivotal player depends exclusively on the issue for which that player is pivotal.

Issue Dependence (ID) For all $G = (N, v, Q) \in \mathcal{G}$ and $\widehat{G} = (\widehat{N}, \widehat{v}, \widehat{Q}) \in \mathcal{G}$ such that $|P(G)| = |P(\widehat{G})| = |M|$, and all $j \in M$, $\varphi_{p_G(j)}(G) = \varphi_{p_{\widehat{G}}(j)}(\widehat{G})$.

The second axiom says that players should not be able to increase (or decrease) their power by splitting up into more than one separate players. Think of a (pivotal) political party which could split up into smaller, still pivotal, parties: this should not influence the distribution of power within a parliament. More precisely, suppose that the game $\widehat{G}$ arises from the game G by (a) replacing player i , pivotal in G , by player set I , disjoint with the player set N in G , (b) changing the simple game only in the sense that in $\widehat{G}$ every coalition of players containing at least one player from I has the same worth as the original coalition with player i , and (c) replacing in the linear order associated with any issue j , player i by the player set I in any arbitrary order, without changing the rest of the order for issue j . The axiom then requires that the original power of i in G is equal to the total power of the players of I in the new situation $\widehat{G}$, whereas all the other players just keep their original power. Formally:

No Advantageous Splitting (NAS) For every $G = (N, v, Q) \in \mathcal{G}$, every $i \in P(G)$, every player set I with $I \cap N = \emptyset$, and every $\widehat{G} = (I \cup N \setminus \{i\}, \widehat{v}, \widehat{Q})$ such that

- $\widehat{v}(S) = v(S)$ for all $S \subseteq N \setminus \{i\}$ and $\widehat{v}(S) = v(\{i\} \cup (S \setminus I))$ for every $S \subseteq N \cup I$ with $S \cap I \neq \emptyset$, and
- for every $j \in M$, $\widehat{Q}^j$ satisfies $k \widehat{Q}^j k'$ if and only if $k Q^j k'$, for all $k, k' \in N \setminus \{i\}$; and $k \widehat{Q}^j i'$ if and only if $k Q^j i$, for all $k \in N \setminus \{i\}$ and $i' \in I$,

we have

$$\varphi_i(G) = \sum_{i' \in I} \varphi_{i'}(\widehat{G}) \text{ and } \varphi_\ell(G) = \varphi_\ell(\widehat{G}) \text{ for all } \ell \in N \setminus \{i\}.$$

As already mentioned in the Introduction, an axiom with the same name and spirit occurs in De Frutos (1999) in the context of bankruptcy problems. There, combined with an analogous condition of ‘no advantageous merging’, it is interpreted as a non-manipulability condition. It is also closely related to the ‘2-efficiency’ condition in Nowak (1997), which is used in a characterization of the normalized Banzhaf value without additivity.

We now show that these two axioms characterize the family of power indices Φ^w .

Theorem 3.1 *Let φ be a power index. Then φ satisfies ID and NAS, if and only if there is a weight vector w such that $\varphi = \Phi^w$.*

Proof We leave verification of the if-part of the theorem to the reader. For the only-if part, assume that φ satisfies ID and NAS.

First, we fix an issue game $\widehat{G} = (\widehat{N}, \widehat{v}, \widehat{Q})$ such that $P(\widehat{G}) = \widehat{N}$ and $|P(\widehat{G})| = |M|$. We define the vector $w \in \mathbb{R}^M$ by $w_j = \varphi_{P(\widehat{G})(j)}(\widehat{G})$ for every $j \in M$. Then w is a weight vector since, in particular, $|P(\widehat{G})| = |\widehat{N}| = |M|$.

Now let $G = (N, v, Q) \in \mathcal{G}$ be an arbitrary issue game. Without loss of generality let $P(G) = \{1, \dots, s\}$, where $s \leq |M|$. Note that $M_G^1, \dots, M_G^s$ is a partition of M . Since w is a weight vector, it is sufficient to prove that $\varphi_\ell(G) = \sum_{j \in M_G^\ell} w_j$ for every $\ell \in P(G)$.

We start with constructing an issue game G_1 in which player 1 is replaced by a collection of new players I_1 , as many as the number of issues for which player 1 is pivotal, i.e., $|I_1| = |M_G^1|$. Without loss of generality, write $M_G^1 = \{1, \dots, \ell\}$ and $I_1 = \{1_1, \dots, 1_\ell\}$, such that $I_1 \cap N = \emptyset$. Define the simple game $(I_1 \cup N \setminus \{1\}, v_1)$ by $v_1(S) = v(S)$ whenever $S \subseteq N \setminus \{1\}$, and $v_1(S) = v(\{1\} \cup S \setminus I_1)$ otherwise. Define the issue profile Q_1 as follows. For $j \in M \setminus M_G^1$ let Q_1^j be such that $k Q_1^j k'$ if and only if $k Q^j k'$, for all $k, k' \in N \setminus \{1\}$; and $k Q_1^j 1'$ if and only if $k Q^j 1$, for all $k \in N \setminus \{1\}$ and $1' \in I_1$. For $j = 1, \dots, \ell$ let Q_1^j be such that $k Q_1^j k'$ if and only if $k Q^j k'$, for all $k, k' \in N \setminus \{1\}$; $k Q_1^j 1'$ if and only if $k Q^j 1$, for all $k \in N \setminus \{1\}$ and $1' \in I_1$; and $1_j Q_1^j 1_{j'}$ for all $j' \in \{1, \dots, \ell\} \setminus \{j\}$.

In words, in the new profile Q_1 , everywhere player 1 is replaced by the player set I_1 such that for the issues where player 1 was pivotal, each time a different player from I_1 is pivotal. In particular, in the new issue game $G_1 = (I_1 \cup N \setminus \{1\}, v_1, Q_1)$ every player in I_1 is pivotal exactly once. By NAS we have

$$\varphi_1(G) = \sum_{i \in I_1} \varphi_i(G_1), \text{ and } \varphi_i(G) = \varphi_i(G_1) \text{ for all } i \in N \setminus \{1\}.$$

Next, we apply a similar construction to G_1 by replacing player 2 by player set I_2 with $|I_2| = |M_G^2|$ and with $I_2 \cap (N \cup I_1) = \emptyset$, resulting in a new issue game G_2 for which by NAS we have

$$\varphi_2(G) = \varphi_2(G_1) = \sum_{i \in I_2} \varphi_i(G_2) \text{ and } \varphi_i(G_1) = \varphi_i(G_2) \text{ for all } i \in I_1 \cup N \setminus \{1, 2\}.$$

Repeating this construction for players $3, \dots, s$, we end up with an issue game G_s with player set $I_1 \cup \dots \cup I_s \cup (N \setminus P(G))$ for which $|P(G_s)| = |I_1 \cup \dots \cup I_s| = |M|$, and where by repeated application of NAS we have

$$\varphi_\ell(G) = \varphi_\ell(G_{\ell-1}) = \sum_{i \in I_\ell} \varphi_i(G_\ell) = \sum_{i \in I_\ell} \varphi_i(G_s) \text{ for every } \ell = 1, \dots, s. \quad (1)$$

By ID, comparing G_s and $\widehat{G}$, we obtain

$$\varphi_i(G_s) = w_j \text{ for } j \in M \text{ such that } p_{G_s}(j) = i, \text{ for every } i \in I_1 \cup \dots \cup I_s. \quad (2)$$

By (1) and (2) we have

$$\varphi_\ell(G) = \sum_{i \in I_\ell} \varphi_i(G_s) = \sum_{j \in M_\ell^\ell} w_j \text{ for every } \ell = 1, \dots, s,$$

which is what we had to prove. $\square$

Observe that in the above proof we need the assumption of an infinite universe of potential players: if not, then the proof would not work if all potential players are already involved in the issue game, since then we cannot add new players.

The axioms in Theorem 3.1 are independent, as the following examples show. Verification of the claims in these examples is left to the reader.

Example 3.2 (i) For every $G = (N, v, Q) \in \mathcal{G}$ and every $i \in N$ define

$$\varphi_i^1(G) = \begin{cases} 0 & \text{if } i \notin P(G) \\ \frac{1}{|P(G)|} & \text{if } i \in P(G). \end{cases}$$

Then φ^1 is a power index satisfying ID, but not NAS.

(ii) We define φ^2 as follows. Let $M = \{1, 2\}$, $w = (1, 0)$, and $\bar{w} = (0, 1)$. We define φ^2 by

$$\varphi^2(N, v, Q) = \begin{cases} \Phi^w(N, v, Q) & \text{if } (N, v) \text{ has no null players} \\ \Phi^{\bar{w}}(N, v, Q) & \text{otherwise.} \end{cases}$$

Then φ^2 is a power index satisfying NAS, but not ID. $\triangleleft$

In case there is no reason to distinguish between issues in terms of importance, the following condition is plausible.

Equal Treatment of Issues (ETI) For all $G = (N, v, Q) \in \mathcal{G}$ such that $|P(G)| = |M|$, $\varphi_i(G) = \varphi_k(G)$ for all $i, k \in P(G)$.

It is easy to verify that ETI implies ID. The following result is a straightforward consequence of Theorem 3.1.

Corollary 3.3 *Let φ be a power index. Then φ satisfies ETI and NAS, if and only if $\varphi = \Phi$.*

Clearly, the conditions in Corollary 3.3 are again independent. In particular, for independence of ETI, any $\Phi^w \neq \Phi$ can be considered.

4 Strong issue dependence and invariance with respect to symmetric players

In this section we consider the following condition for a power index φ .

Invariance with respect to Symmetric Players (ISP) For all $G = (N, v, Q)$, $\widehat{G} = (N, v, \widehat{Q}) \in \mathcal{G}$ such that there are $j \in M$ and symmetric players i, i' in (N, v) with $\widehat{Q}^\ell = Q^\ell$ for all $\ell \in M \setminus \{j\}$, $p_G(j) = i$ and $p_{\widehat{G}}(j) = i'$, we have $\varphi_k(G) = \varphi_k(\widehat{G})$ for every $k \neq i, i'$.

This condition says that if we replace, for one issue, the pivotal player by another player, where these two players are symmetric in the simple game, then this does not change the power of any of the players not involved in this switch. Of course, this also implies that, with notations as in the definition,

$$\varphi_i(G) + \varphi_{i'}(G) = \varphi_i(\widehat{G}) + \varphi_{i'}(\widehat{G}) \quad (3)$$

due to efficiency of a power index. We will show that NAS in Theorem 3.1 can be replaced by ISP if, additionally, we strengthen Issue Dependence (ID) to the following property.

Strong Issue Dependence (SID) For all $G = (N, v, Q)$, $\widehat{G} = (\widehat{N}, \widehat{v}, \widehat{Q}) \in \mathcal{G}$, if the partitions $(M_G^i)_{i \in P(G)}$ and $(M_{\widehat{G}}^i)_{i \in P(\widehat{G})}$ are equal, then $\varphi_{p_G(j)}(G) = \varphi_{p_{\widehat{G}}(j)}(\widehat{G})$ for every $j \in M$.

Any issue game induces a partition of its set of issues: the issues in each element of this partition share the same pivotal player. SID says that if in two issue games these partitions coincide, then the power of the player who is pivotal for the issues in an element of the partition in one issue game, should be equal to the power of the player pivotal for the issues in the same partition element, in the other issue game. Clearly, for the case where the two partitions in this definition are equal to the partition of the set of issues into singletons, this condition reduces to ID. Hence, SID is indeed stronger than ID. We need this strengthening since, without the NAS property, ID has no implications for issue games where the number of pivotal players is lower than the number of issues.

The announced theorem is as follows.

Theorem 4.1 *Let φ be a power index. Then φ satisfies SID and ISP, if and only if there is a weight vector w such that $\varphi = \Phi^w$.*

Proof It is straightforward to verify that each power index Φ^w satisfies SID and ISP.

For the only-if direction, suppose φ satisfies these two conditions. As in the proof of Theorem 3.1, we fix an issue game $\widehat{G} = (\widehat{N}, u_{\widehat{N}}, \widehat{Q})$ such that $P(\widehat{G}) = \widehat{N}$ and $|P(\widehat{G})| = |M|$. (We now take as simple game the unanimity game $(\widehat{N}, u_{\widehat{N}})$, since in this simple game all players are symmetric.) We define the vector $w \in \mathbb{R}^M$ by $w_j = \varphi_{p_{\widehat{G}}(j)}(\widehat{G})$ for every $j \in M$. Then w is a weight vector since, in particular, $|P(\widehat{G})| = |\widehat{N}| = |M|$.

Now let $G = (N, v, Q) \in \mathcal{G}$ be an arbitrary issue game. Without loss of generality let $P(G) = \{1, \dots, s\}$, where $s \leq |M|$. Then $M_G^1, \dots, M_G^s$ is a partition of M . Since w is a weight vector, it is sufficient to prove that $\varphi_\ell(G) = \sum_{j \in M_G^\ell} w_j$ for every $\ell \in P(G)$.

Consider, first, the set M_G^1 , take some $j_1 \in M_G^1$, and let $k_1 \in P(\widehat{G})$ with $k_1 = p_{\widehat{G}}(j_1)$. Define $\widehat{G}_1 = (\widehat{N}, u_{\widehat{N}}, \widehat{Q}_1) \in \mathcal{G}$ by letting $\widehat{Q}_1^j = \widehat{Q}^j$ for all $j \in M \setminus M_G^1$, and with $M_{\widehat{G}_1}^{k_1} = M_G^1$. (Hence, in $\widehat{G}_1$ player k_1 is pivotal for all issues in M_G^1 , i.e., all issues for which player 1 is pivotal in G .) By repeated application of ISP, noting that in $(\widehat{N}, u_{\widehat{N}})$ all players are symmetric, it follows that $\varphi_i(\widehat{G}_1) = \varphi_i(\widehat{G})$ for all $i \in P(\widehat{G}_1) \setminus \{k_1\}$. Hence, $\varphi_{k_1}(\widehat{G}_1) = \sum_{j \in M_G^1} w_j$, and for every $i \in P(\widehat{G}_1) \setminus \{k_1\}$ we have $\varphi_i(\widehat{G}_1) = w_j$ for $i = p_{\widehat{G}_1}(j)$.

Next, consider the set M_G^2 , take some $j_2 \in M_G^2$, and let $k_2 \in P(\widehat{G})$ such that $k_2 = p_{\widehat{G}}(j_2)$. Define $\widehat{G}_2 = (\widehat{N}, u_{\widehat{N}}, \widehat{Q}_2) \in \mathcal{G}$ by letting $\widehat{Q}_2^j = \widehat{Q}^j$ for all $j \in M \setminus M_G^2$, and with $M_{\widehat{G}_2}^{k_2} = M_G^2$. (Hence, in $\widehat{G}_2$ player k_2 is pivotal for all issues in M_G^2 .) Again, by repeated application of ISP it follows that $\varphi_i(\widehat{G}_2) = \varphi_i(\widehat{G}_1)$ for all $i \in P(\widehat{G}_2) \setminus \{k_2\}$. Hence, $\varphi_{k_2}(\widehat{G}_2) = \sum_{j \in M_G^2} w_j$, $\varphi_{k_1}(\widehat{G}_2) = \varphi_{k_1}(\widehat{G}_1) = \sum_{j \in M_G^1} w_j$, and for every $i \in P(\widehat{G}_2) \setminus \{k_1, k_2\}$ we have $\varphi_i(\widehat{G}_2) = w_j$ for $i = p_{\widehat{G}_2}(j)$.

By repeating this construction we obtain, after s steps in total, an issue game $\widehat{G}_s = (\widehat{N}, u_{\widehat{N}}, \widehat{Q}_s)$ with $P(\widehat{G}_s) = \{k_1, \dots, k_s\}$ and such that $M_{\widehat{G}_s}^{k_\ell} = M_G^\ell$, and $\varphi_{k_\ell}(\widehat{G}_s) = \sum_{j \in M_G^\ell} w_j$, for every $\ell = 1, \dots, s$. By SID applied to $\widehat{G}_s$ and G , we obtain $\varphi_\ell(G) = \varphi_{k_\ell}(\widehat{G}_s) = \sum_{j \in M_G^\ell} w_j$ for every $\ell = 1, \dots, s$. This completes the proof. $\square$

Note that the proof of Theorem 4.1 parallels that of Theorem 3.1, but in a converse way. In the proof of Theorem 3.1 we use the NAS property to split a player who is pivotal for a subset of issues, into a number of new players such that each new player is pivotal for exactly one issue. In the proof of Theorem 4.1 we use the ISP property to merge several different players, pivotal for different issues, into one player pivotal for the set of those issues.

We can also use the same power indices as in Example 3.2 to show independence of the axioms.

Example 4.2 (i) Power index φ^1 in Example 3.2(i) satisfies SID but not ISP.

(ii) Power index φ^2 in Example 3.2(ii) satisfies ISP but not SID. $\triangleleft$

5 Equal power change

For our last characterization, we consider a condition which is closely related to the familiar condition of additivity for solutions of cooperative games (Shapley 1953), and more specifically to the transfer property for solutions of simple games (Dubey 1975; Dubey et al. 2005). Compared to the other axioms, this condition (which is also used in Peters and Zarzuelo (2017)) is concerned with the simple game in an issue game, rather than the issue profile.

Equal Power Change (EPC) For all issue games (N, v, Q) , (N, v', Q) , (N, w, Q) , and (N, w', Q) , if $v(S) - v'(S) = w(S) - w'(S) \geq 0$ for all $S \subseteq N$, then

$$\varphi_i(N, v, Q) - \varphi_i(N, v', Q) = \varphi_i(N, w, Q) - \varphi_i(N, w', Q) \text{ for every } i \in N.$$

In words, given a fixed issue profile, if v arises from v' and w arises from w' by adding the same winning coalitions, then for every player the change in power in both transitions should be equal.

We first show that every power index Φ^w has this property.

Lemma 5.1 *For every weight vector w , Φ^w satisfies EPC.*

Proof Let N, Q, v, v', w, w' as in the definition of EPC. Let $j \in M, i \in N$, and let $Q^j = \{i_1, \dots, i_s, \dots, i_t, \dots, i_n\}$, where i_s is a pivotal player for issue j according to (N, v) , and i_t is a pivotal player for issue j according to (N, v') . Since $v \geq v'$, we have $s \leq t$. We consider the following cases.

1. If $i = i_s$ and $s < t$, we have $v(i_1, \dots, i_s) = 1$,⁵ $v(i_1, \dots, i_{s-1}) = v'(i_1, \dots, i_s) = v'(i_1, \dots, i_{s-1}) = 0$. Since $v - v' = w - w' \geq 0$, we have $w(i_1, \dots, i_s) = 1$, $w(i_1, \dots, i_{s-1}) = w'(i_1, \dots, i_s) = w'(i_1, \dots, i_{s-1}) = 0$. This means player i is pivotal for j according to (N, w) but not pivotal for j according to (N, w') .
2. If $i = i_t$ and $s < t$, we have $v(i_1, \dots, i_t) = v(i_1, \dots, i_{t-1}) = v'(i_1, \dots, i_t) = 1$ and $v'(i_1, \dots, i_{t-1}) = 0$. Since $v - v' = w - w' \geq 0$, it holds that $w(i_1, \dots, i_t) = w(i_1, \dots, i_{t-1}) = w'(i_1, \dots, i_t) = 1$ and $w'(i_1, \dots, i_{t-1}) = 0$. This means player i is pivotal for j according to (N, w') but not pivotal for j according to (N, w) .
3. If $i \neq i_s$ and $i \neq i_t$ and $s < t$, we have the following three cases. We write $i = i_k$.
 - (i) If $k < s$, then, as in Case 1, $w(i_1, \dots, i_s) = 1$ and $w(i_1, \dots, i_{s-1}) = 0$. Hence, $w(i_1, \dots, i_k) = w'(i_1, \dots, i_k) = 0$ since $w(i_1, \dots, i_{s-1}) \geq w(i_1, \dots, i_k)$ and $w \geq w'$. Thus, i is not pivotal for j according to both (N, w) and (N, w') .
 - (ii) If $s < k < t$, then $v(i_1, \dots, i_k) = v(i_1, \dots, i_{k-1}) = 1$ and $v'(i_1, \dots, i_k) = v'(i_1, \dots, i_{k-1}) = 0$. Hence, $w(i_1, \dots, i_k) = w(i_1, \dots, i_{k-1}) = 1$ and $w'(i_1, \dots, i_k) = w'(i_1, \dots, i_{k-1}) = 0$. Thus, i is not pivotal for j according to both (N, w) and (N, w') .
 - (iii) If $k > t$, then $v(i_1, \dots, i_k) = v(i_1, \dots, i_{k-1}) = 1$ and $v'(i_1, \dots, i_k) = v'(i_1, \dots, i_{k-1}) = 1$. Therefore, $w(i_1, \dots, i_k) = w'(i_1, \dots, i_k) \geq w(i_1, \dots, i_{k-1}) = w'(i_1, \dots, i_{k-1})$ since $v - v' = w - w' \geq 0$. Thus, either player i is pivotal for j according to both (N, w) and (N, w') , or player i is not pivotal for j according to both (N, w) and (N, w') .
4. If $i = i_s = i_t$, then $v(i_1, \dots, i_s) = v'(i_1, \dots, i_s) = 1$ and $v(i_1, \dots, i_{s-1}) = v'(i_1, \dots, i_{s-1}) = 0$. Since $v - v' = w - w' \geq 0$, we have $w(i_1, \dots, i_s) = w'(i_1, \dots, i_s) \geq w(i_1, \dots, i_{s-1}) = w'(i_1, \dots, i_{s-1})$. Thus, either player i is pivotal for j both in (N, w) and (N, w') , or player i is not pivotal for j according to both (N, w) and (N, w') .

⁵ We write $v(i_1, \dots, i_s)$ instead of $v(\{i_1, \dots, i_s\})$, etc.

5. If $i \neq i_s = i_t$, we have the following two cases.

- (i) $k < s$. This case is similar to Case 3(i).
- (ii) $k > t$. This case is similar to Case 3(iii).

By Cases 1–5, it follows that $\Phi^w(N, v, Q) - \Phi^w(N, v', Q) = \Phi^w(N, w, Q) - \Phi^w(N, w', Q)$ for every weight vector w . $\square$

The next lemma implies that, if φ satisfies EPC, then it is completely determined by its values on issue games based on unanimity games. The lemma follows from Lemma 2.3 in Einy (1987), see also Einy and Haimanko (2011).

Lemma 5.2 *Let φ be a power index satisfying EPC and let (N, v, Q) be an issue game. Let $T_1, \dots, T_k$ be the minimal winning coalitions in (N, v) . Then*

$$\varphi(N, v, Q) = \sum_{\emptyset \neq I \subseteq \{1, \dots, k\}} (-1)^{|I|+1} \varphi(N, u_{\cup_{m \in I} T_m}, Q^M). \quad (4)$$

Adding EPC to the conditions in Theorem 4.1 will allow us to weaken the SID property, by restricting it to players who are symmetric in the associated simple games, as follows.

Symmetric Player Issue Dependence (SPID) For all $G = (N, v, Q)$, $\widehat{G} = (\widehat{N}, \widehat{v}, \widehat{Q}) \in \mathcal{G}$, such that

- all players in $P(G)$ are symmetric in (N, v) and all players in $P(\widehat{G})$ are symmetric in $(\widehat{N}, \widehat{v})$, and
- the partitions $(M_G^i)_{i \in P(G)}$ and $(M_{\widehat{G}}^i)_{i \in P(\widehat{G})}$ are equal,

we have $\varphi_{P_G(j)}(G) = \varphi_{P_{\widehat{G}}(j)}(\widehat{G})$ for every $j \in M$.

Theorem 5.3 *Let φ be a power index. Then φ satisfies SPID, ISP, and EPC, if and only if there is a weight vector w such that $\varphi = \Phi^w$.*

Proof The if-direction follows from Theorem 4.1 and Lemma 5.1. For the proof of the only-if direction we copy the proof of Theorem 4.1, with now G in that proof of the form $G = (N, u_T, Q)$. The proof is then complete by applying Lemma 5.2. $\square$

The independence of the conditions in Theorem 5.3 is demonstrated by the power indices in the following example.

Example 5.4 (i) For each issue game $G = (N, v, Q)$ where (N, v) is a unanimity game, define

$$\varphi_i^3(G) = \begin{cases} 0 & \text{if } i \notin P(G) \\ \frac{1}{|P(G)|} & \text{if } i \in P(G) \end{cases}$$

for every $i \in N$. Hence, φ^3 coincides with φ^1 for unanimity games. For an arbitrary issue game G , define $\varphi^3(G)$ by (4). Then φ^3 is a power index satisfying SPID and EPC, but not ISP.

- (ii) We define φ^4 as follows. Let $M = \{1, 2\}$, $w = (1, 0)$, and $\bar{w} = (0, 1)$. Then, for every $G = (N, v, Q)$ with (N, v) a unanimity game,

$$\varphi^4(N, v, Q) = \begin{cases} \Phi^w(N, v, Q) & \text{if } (N, v) \text{ has no null players} \\ \Phi^{\bar{w}}(N, v, Q) & \text{otherwise.} \end{cases}$$

Hence, φ^4 coincides with φ^2 for unanimity games. For an arbitrary issue game G , define $\varphi^4(G)$ by (4). Then φ^4 is a power index satisfying ISP and EPC, but not SPID.

- (iii) Define

$$\varphi^5(N, v, Q) = \frac{1}{2^k - 1} \sum_{\emptyset \neq I \subseteq \{1, \dots, k\}} \Phi(N, u_{\cup_{m \in I} T_m}, Q),$$

for every issue game (N, v, Q) , where $T_1, \dots, T_k$ are the minimal winning coalitions in (N, v) (cf. Lemma 5.2). Then φ^5 is a power index satisfying ISP and SPID, but not EPC. (This example is analogous to an example in Peters and Zarzuelo (2017)). $\triangleleft$

6 Further discussion

Naturally, there are further questions which could be explored.

The power indices in this paper are defined and characterized for a fixed set of issues; indeed, all axioms that we consider are formulated for this fixed set of issues. In fact, the results extend straightforwardly if different sets of issues are allowed, but in that case they imply no relation between the associated weight vectors. However, it would not be unnatural to assume that if an issue j is more important than an issue j' within a set of issues M , then that is still the case within some other set of issues M' containing both j and j' .

Also, simple games in an issue game are exclusively used to determine the pivotalness of a player for an issue; but one might, for instance, also want to take into account the number of times that a player, even if not pivotal, is in a winning coalition resulting from the issue profile—think of player 1 in the example in Sect. 1.2.

We conclude with establishing some relations with other power indices, and with a discussion on the relation with preferences of players over alternatives.

6.1 Relation with other power indices

By fixing specific issue profiles we obtain several well-known power indices. We consider three examples. Fix a player set N with $|N| = n$.

- (i) Let $M = \{1, \dots, n!\}$ and let Q be an issue profile such that $Q^j \neq Q^k$ for all $j, k \in M$ with $j \neq k$. Hence, every permutation of the player set N is assigned to some issue by Q . Then $\Phi(N, v, Q)$ is the Shapley value for every simple game (N, v) .

- (ii) For a simple game (N, v) , and for every $i \in N$, denote $M_i^v = \{S \subseteq N : v(S \cup \{i\}) - v(S) = 1\}$. Let M be a set of issues with $|M| = \sum_{i \in N} |M_i^v|$ such that the elements of M correspond to the pairs (i, S) with $S \in M_i^v$. Let Q be an issue profile such that for each $j = (i, S) \in M$ we have $k Q^j i Q^j k'$ for all $k \in S$ and $k' \in N \setminus (S \cup \{i\})$. Then $\Phi(N, v, Q)$ is the normalized Banzhaf value of (N, v) .
- (iii) Let (N, v) be a simple game, and let $\mathcal{W}$ be the set of minimal winning coalitions of (N, v) . For every $i \in N$ denote $W_i^v = \{S \in \mathcal{W} : i \in S\}$. Let M be a set of issues with $|M| = \sum_{i \in N} |W_i^v|$ such that the elements of M correspond to the pairs (i, S) with $S \in W_i^v$. Let Q be an issue profile such that for each $j = (i, S) \in M$ we have $k Q^j i Q^j k'$ for all $k \in S$ and $k' \in N \setminus (S \cup \{i\})$. Then $\Phi(N, v, Q)$ is the Holler index of (N, v) .

6.2 Player preferences

As noted in the Introduction, an alternative approach, closer to the approach in Owen and Shapley (1989), would be to take the profile of preferences of players over issues as a primitive in the model. The following example illustrates how this could work.

Example 6.1 Let $N = \{1, \dots, 4\}$. Suppose that there are three alternatives a, b , and c , and that the players have preferences on these alternatives as in the following table:

1	2	3	4
a	a	b	b
b	b	a	c
c	c	c	a

Hence, player 1 prefers a over b and b over c , etc. We may translate these preferences to an issue profile as follows. For alternative a , players 1 and 2 have a on top, player 3 has a at second position, and player 4 at the last position. This gives rise to two issues a_1 and a_2 with orderings respectively 1,2,3,4 and 2,1,3,4. Similarly, b gives rise to four issues and c results in six issues. The resulting issue profile is given in the following table:

a_1	a_2	b_1	b_2	b_3	b_4	c_1	c_2	c_3	c_4	c_5	c_6
1	2	3	3	4	4	4	4	4	4	4	4
2	1	4	4	3	3	1	1	2	2	3	3
3	3	1	2	1	2	2	3	1	3	1	2
4	4	2	1	2	1	3	2	3	1	2	1

Suppose in the simple game the minimal winning coalitions are those with three players. Then Φ would assign to the resulting game G the power distribution

$\frac{1}{12}(4, 4, 4, 0) = \frac{1}{18}(6, 6, 6, 0)$. With weight vector

$$w = \frac{1}{36}(6, 6, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2),$$

which results from assigning equal importance to the original alternatives a , b , and c , we obtain $\Phi^w(G) = \frac{1}{18}(5, 5, 8, 0)$. $\triangleleft$

As is the case in this example, if the number of alternatives is small relative to the number of players, then often two or more players rank the same alternative at the same position, resulting in many issues. This does not occur in the Owen–Shapley model where the number of alternatives is infinite, namely the unit circle, and thus the set of issues, obtained by the above procedure, coincides with the number of alternatives up to a set of measure zero.

While this potential abundance of issues, resulting from preferences of players over alternatives, is an important practical reason to take linear orders over players as a primitive in the model, it is certainly not the only reason. Deriving issue profiles as in Example 6.1 presupposes that these preferences are comparable, but this by itself is a strong assumption. This problem is avoided by taking issue profiles as a primitive. Moreover, in practice it seems often easier to determine an issue profile than to assess players' preferences over alternatives. For instance, concerning an issue on which a political parliament has to decide, it is usually not difficult to establish an order in which the parties are enthusiastic about and thus support the issue; on the other hand, it may be quite senseless to talk about preferences of parties over (possibly quite) different issues.

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