



Non-emptiness of the alpha-core: sufficient and necessary conditions

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Abstract

We give sufficient and necessary conditions for the non-emptiness of the alpha-core in the setting of strategic games with non-ordered and discontinuous preferences. In order to prove our results, we can avoid the use of Scarf's Theorem for *NTU*-games, by suitably appealing to the Ky Fan minimax inequality. Examples clarify our conditions and allow the comparison of our results with the previous ones.

Keywords Games with non-ordered preferences · Discontinuous games · Non-emptiness of the alpha-core

JEL Classification C60 · C71

1 Introduction

A well known cooperative solution concept for strategic form games is the α -core by Aumann (1961). A strategy profile belongs to the α -core if no coalitions of players have profitable joint deviations against to all strategies of the opponents. Scarf (1971) proved the non-emptiness of the α -core when the players are endowed with continuous and quasi-concave utility functions and the sets of strategies are finite-dimensional. Kajii (1992) extended Scarf's result to the setting of games with continuous and non-ordered preferences: each player is characterized by a mapping defined on the set of strategy profiles, and it is assumed that all such mappings are open-graph and convex-valued.

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In the light of some recent advances on the existence of Nash equilibria in discontinuous games, new conditions for the non-emptiness of the α -core in games with discontinuous payoff functions have been provided by Uyanik (2015). In the mentioned paper, the author introduces a coalitional version of the *continuous-security* by Borelli and Meneghel (2013), and other conditions so that the non-emptiness of the α -core is obtained through the non-emptiness of the core of a suitable *NTU*-game. Here, Scarf's Theorem (Scarf 1967) plays a crucial role.

In the present paper, we give necessary and sufficient conditions for the non-emptiness of the α -core in games with discontinuous and non-ordered preferences. In doing this, we do not use Scarf's Theorem (Scarf 1967), but we adapt to our setting the approach to the Nash equilibrium existence problem considered by Scalzo (2019a). More precisely, given a game with non-ordered preferences, we define a real-valued function so that the solutions to the corresponding Ky Fan minimax inequality (Fan 1972) coincide with the α -core elements of the game. Hence, we apply the general results on the existence of solutions to the Ky Fan minimax inequality given by Scalzo (2013, 2019a) and identify sufficient (and necessary) conditions on the game for the non-emptiness of the α -core. The conditions we obtain are adaptations to our setting of those given by Scalzo (2019a, b) for the existence of Nash equilibria in discontinuous games.

The investigation on the non-emptiness of the α -core in games with discontinuous and non-ordered preferences can be motivated, for example, by the analysis of situations like the following one. Consider a family of clubs $\{C_i : i \in N\}$, each of which being represented by an individual $i \in N$, called administrator of C_i . Suppose that $i \neq j$ implies $C_i \neq C_j$. Assume that clubs get benefits from a common asset and the gain they obtain depends on the investments of all clubs. So, every club ranks profile of investments $x = (x_i)_{i \in N}$ by means of a rule which aggregates the preferences of the members (for example, unanimity rule, majority rule, etc.). Even if each member of a club has a utility function on the set of profiles of investments, the aggregation rule may give rise to a non necessarily ordered preference (for instance, the majority rule induces a non transitive preference). Moreover, the aggregate preference of a club C_i may not satisfy the continuity properties considered by the previous literature (open-graph preferences). This is the case, for example, if i is interpreted as the administrator of a condominium C_i . By the role, she/he is naturally led to identify amounts even when they are different but differ by less than a threshold level $\lambda > 0$. That is: x and y are indifferent for C_i if $\left| \sum_{j \in N} x_j - \sum_{j \in N} y_j \right| < \lambda$; and C_i prefers y on x if $\sum_{j \in N} y_j \geq \sum_{j \in N} x_j + \lambda$ (see also Campbell and Walker 1990). This preference cannot be represented by utility functions and the mapping P_i defined by $P_i(x) = \{y : C_i \text{ prefers } y \text{ on } x\}$ is not open-graph.¹ Now, remaining in the condominium interpretation, it is plausible that all C_i are located in a common district, reason why cooperation among administrators is beneficial. Consequently, the α -core solution concept plays obviously a role, whereas the results from the previous literature do not apply.

¹ One can easily see that the symmetric part of the preference is not transitive. Moreover, $\bar{x} \in P_i(x)$ for \bar{x} given by $\bar{x}_j = x_j + \frac{\lambda}{|N|}$ for each $j \in N$. Now, it is clear that there are profiles of investments x' indifferent to \bar{x} in every open neighborhood of x . This implies that the graph of P_i is not open.

The paper is organized as follows. Section 2 recalls definitions and results on the Ky Fan minimax inequality. The setting of the paper is introduced in Sect. 3, where the previous literature on the non-emptiness of the α -core is also recalled. Our main results, as well as the comparison with the previous ones, are given in Sect. 4. Section 5 concludes the paper.

2 Ky Fan minimax inequality background

Let X be a non-empty subset of a topological vector space and Θ be a real-valued function defined on $X \times X$. The following problem

$$(KF) \begin{cases} \text{find } x^* \in X & \text{such that} \\ \Theta(x, x^*) \leq 0 & \forall x \in X \end{cases}$$

is called *Ky Fan minimax inequality* corresponding to the function Θ (Ky Fan 1972). An element x^* , if any exists satisfying condition (KF), is a *solution* to the inequality; let S_Θ denote the set of solutions. Below, we present some properties which play a role in the existence of solutions to (KF).

Definition 2.1 (Scalzo 2019a) The function Θ is said to be *slightly diagonally transfer continuous* if $\Theta(x, z) > 0$ implies that there exists an open neighborhood O_z of z and $x' \in X$ so that $\Theta(x', z') > 0$ for all $z' \in O_z \setminus S_\Theta$.

Definition 2.2 The function Θ is said to be *generalized lower transfer continuous* if $\Theta(x, z) > 0$ implies that there exists an open neighborhood O_z of z and an upper semicontinuous mapping $\xi : O_z \rightrightarrows X$, with non-empty, convex and compact values, so that $\Theta(x', z') > 0$ for all $z' \in O_z \setminus S_\Theta$ and all $x' \in \xi(z')$.

The property in Definition 2.1 generalizes the *diagonal transfer continuity* by Baye et al. (1993) because it allows a non-solution to (KF) to be approached by nets of solutions. In the same spirit, Definition 2.2 presents a generalization of the *generalized 0-quasi-transfer continuity* given in Definition 2 by Scalzo (2013). It is clear that every slightly diagonally transfer continuous function is generalized lower transfer continuous, but the converse does not hold: see the example below.

Example 2.1 Let u be the real-valued function defined on $X = [0, 1] \times [0, 1]$ as follows:

$$u(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 = x_2 > 0 \\ -1 & \text{if } x_1 = x_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\Theta(x, z) = u(x_1, z_2) - u(z)$ for all x and z belonging to X . We have $S_\Theta = \{x : x_1 = x_2 > 0\} \cup (\{0, 1\} \times \{0\})$. Note that the non solution $(0, 0)$ is the limit (in the usual topology) of sequences of solutions to (KF). The function Θ is generalized lower transfer continuous. In fact, let $\Theta(x, z) > 0$ for some x and z . If $z \neq (0, 0)$, the inequality is possible only if $z_1 \neq z_2$ and $x_1 = z_2 > 0$. It is easy to see that

the generalized lower transfer continuity is satisfied for an open neighborhood O_z of z where $z'_1 \neq z'_2$ for all $z' \in O_z$ and $\xi(z') = \{z'_2\} \times [0, 1]$ for each $z' \in O_z$. If $z = (0, 0)$ and O_z is an open neighborhood of z , since every $z' \in O_z \setminus S_\Theta$ is such that $z'_1 \neq z'_2$, by using the mapping ξ defined above, we recognize that Θ is generalized lower transfer continuous (since $(0, 0)$ is approached by sequences of solutions to the Ky Fan minimax inequality, the function Θ is not generalized 0-quasi-transfer continuous). Finally, in order to have $\Theta(x', z') > 0$ for all $z' \in O_z \setminus S_\Theta$, it is clear that x' must change as z' does. So, Θ is not slightly diagonally transfer continuous.

Definition 2.3 (Baye et al. 1993) The function Θ is said to be *diagonally transfer quasi-concave* if, for any $\{x^1, \dots, x^k\} \subset X$ there exists $\{z^1, \dots, z^k\} \subset X$, where $x^h \mapsto z^h$, such that, for each $z \in \text{co}\{z^1, \dots, z^k\}$ and $z = \sum_{j=1}^l \lambda_j z^{h_j}$ with $\lambda_j > 0$ for $j = 1, \dots, l$, one has $\Theta(x, z) \leq 0$ for some $x \in \{x^{h_1}, \dots, x^{h_l}\}$.

Definition 2.4 (Zhou and Chen 1988) The function Θ is said to be *0-diagonally quasi-concave* if, for any $\{x^1, \dots, x^k\} \subset X$ and any $z \in \text{co}\{x^1, \dots, x^k\}$ with $z = \sum_{j=1}^l \lambda_j x^{h_j}$ and $\lambda_j > 0$ for $j = 1, \dots, l$, there exists $x \in \{x^{h_1}, \dots, x^{h_l}\}$ so that $\Theta(x, z) \leq 0$.

Obviously, every 0-diagonally quasi-concave function is diagonally transfer quasi-concave, but the converse does not hold.

We have the following results on the existence of solutions to (KF).

Proposition 2.1 (Scalzo 2019a) *Assume that X is a convex and compact subset of a Hausdorff topological vector space and Θ be slightly diagonally transfer continuous. Then, S_Θ is non-empty if and only if Θ is diagonally transfer quasi-concave.*

Proposition 2.2 *Assume that X is a convex and compact subset of a locally convex Hausdorff topological vector space. If Θ is generalized lower transfer continuous and 0-diagonally quasi-concave, then S_Θ is non-empty.*

Proof By contradiction, assume $S_\Theta = \emptyset$. Because of the generalized lower transfer continuity and $S_\Theta = \emptyset$, one has that Θ is generalized 0-quasi-transfer continuous. So, Proposition 2 by Scalzo (2013) applies and we get $S_\Theta \neq \emptyset$, which is a contradiction.² □

Remark 2.1 In Sect. 4, we provide α -core non-emptiness results by using the Ky Fan minimax inequality. So, in the light of Proposition 4.2 and Remark 4.2, one can deduce that Proposition 2.2 cannot be obtained from Proposition 2.1, and viceversa.

3 Setting and preliminaries

Let N be a finite set of players and, for each $i \in N$, let X_i be a non-empty set. Each player i is endowed with a mapping $P_i : X = \prod_{j \in N} X_j \rightrightarrows X$. Given $x \in X$, $P_i(x)$

² **Proposition 2** (Scalzo 2013). *Assume that X is a convex and compact subset of a locally convex Hausdorff topological vector space. If Θ is generalized 0-quasi-transfer continuous and 0-diagonally quasi-concave, then S_Θ is non-empty and compact.*

represents the set of all strategy profiles that player i ranks better than x . For instance, if player i compares strategy profiles by means of a binary relation and \succ_i denotes its asymmetric part, we have $P_i(x) = \{z \in X : z \succ_i x\}$. In the same way, if u_i is a utility function of player i on X , $P_i(x) = \{z \in X : u_i(z) > u_i(x)\}$. However we point out that the mapping P_i does not necessarily come from an ordered preference. The list $G = \langle X_i, P_i \rangle_{i \in N}$ of data is named *strategic form game with non-ordered preferences* (game, briefly).

Given a game G , we call *coalition* every non-empty subset of players; \mathcal{N} denotes the set of coalitions of G . For each $S \in \mathcal{N}$, let $X_S = \prod_{j \in S} X_j$ and $X_{-S} = \prod_{j \notin S} X_j$; a strategy profile $x \in X$ will be also denoted by $x = (x_S, x_{-S})$.

In this paper, we are interested in the cooperative solution concept for games proposed by Aumann (1961) and recalled in the following definition:

Definition 3.1 Let G be a game. The strategy profile $z \in X$ is said to be α -blocked if there exists a coalition S and $x_S \in X_S$ such that $\{x_S\} \times X_{-S} \subseteq \bigcap_{i \in S} P_i(z)$. The set of all non α -blocked strategy profiles is called α -core of G and is denoted by ${}^\alpha\mathcal{C}(G)$.

The first result on the non-emptiness of the α -core was given by Scarf (1971) for games whose players are endowed with continuous, quasi-concave utilities, and the strategy sets are finite-dimensional. Later, Kajii (1992) extended the result by Scarf (1971) to the games where the players' preferences are non-ordered and the sets of strategies are included in normed vector spaces. In this result, the mappings of players are assumed to be open graph and convex-valued. More recently, in the framework of finite-dimensional spaces of strategies, Uyanik (2015) obtained the non-emptiness of the α -core in normal form games with quasi-concave payoff functions: inspired by Borelli and Meneghel (2013), the author considered a condition on payoffs which allows discontinuities.

Let us note that Scarf's Theorem (Scarf 1967) on the non-emptiness of the core of NTU-games plays a significant role in all the above mentioned papers. The approach of the present paper is different and we do not need Scarf's Theorem for our existence result.

4 Non-emptiness of the alpha-core

In this section, we give sufficient and necessary conditions for the non-emptiness of the α -core in the framework of discontinuous games. We deal with games with non-ordered preferences. To do this, we consider a suitable real-valued function Θ such that the solution set to the Ky Fan minimax inequality corresponding to Θ coincides with the α -core.

Let $G = \langle X_i, P_i \rangle_{i \in N}$ be a game with non-ordered preferences. Define the function Θ on $X \times X$ as follows:

$$\Theta(x, z) = \sum_{S \in \mathcal{N}} \min_{y_{-S} \in X_{-S}} \min_{i \in S} \widehat{P}_i((x_S, y_{-S}), z), \tag{4.1}$$

where $\widehat{P}_i((x_S, y_{-S}), z) = 1$ if $(x_S, y_{-S}) \in P_i(z)$ and $\widehat{P}_i((x_S, y_{-S}), z) = 0$ otherwise.

We have: $x^* \in {}^\alpha\mathcal{C}(G)$ if and only if $\Theta(x, x^*) = 0$ for all $x \in X$. Indeed, if $\Theta(x, x^*) = 0$ for all $x \in X$, we have $\min_{y_{-S} \in X_{-S}} \min_{i \in S} \widehat{P}_i((x_S, y_{-S}), x^*) = 0$ for each coalition S . This implies that, for every strategy profile x and every coalition S , there exists $y_{-S} \in X_{-S}$ and $i \in S$ such that $(x_S, y_{-S}) \notin P_i(x^*)$. In other words: no coalition can use a joint deviation to α -block x^* . So, x^* is an α -core element. Similarly, one gets that x^* is a solution to the Ky Fan minimax inequality corresponding to the function Θ if $x^* \in {}^\alpha\mathcal{C}(G)$.

Since $\Theta \geq 0$, the α -core of G coincides with the solution set to the Ky Fan minimax inequality corresponding to Θ .

The following definitions introduce to our framework some *deviation properties* used in recent Nash equilibrium existence results (see Remark 4.1).

Definition 4.1 A game G is said to satisfy the *coalitional deviation property* if, for all $z \notin {}^\alpha\mathcal{C}(G)$, there exists an open neighborhood O_z of z and $x' \in X$ such that, for any $z' \in O_z \setminus {}^\alpha\mathcal{C}(G)$, there exists a coalition S for which $\{x'_S\} \times X_{-S} \subseteq P_i(z')$ for every $i \in S$.

Definition 4.2 A game G is said to satisfy the *generalized coalitional deviation property* if, for all $z \notin {}^\alpha\mathcal{C}(G)$, there exists an open neighborhood O_z of z and an upper semicontinuous mapping $\xi_z : O_z \rightrightarrows X$, with non-empty, convex and compact values, such that, for each $z' \in O_z \setminus {}^\alpha\mathcal{C}(G)$ and each $x' \in \xi_z(z')$, there exists a coalition S for which $\{x'_S\} \times X_{-S} \subseteq P_i(z')$ for every $i \in S$.

Remark 4.1 The definitions above are given in the spirit of the *slight single deviation property* and the *generalized single deviation property* introduced by Scalzo (2019a, b) in the framework of the Nash equilibrium existence problem, where new existence results have been obtained. The properties are generalizations of the *single deviation property* given by Nassah and Tian (2008) and Reny (2009). As it is showed in the following, the coalitional versions of the mentioned properties allow to obtain α -core non-emptiness results in the setting of discontinuous games.

The class of games which satisfy the coalitional deviation property strictly includes the class of games where the mappings P_i are open graph, as the following Proposition 4.1 and Example 4.1 show.

Proposition 4.1 Let $G = \langle X_i, P_i \rangle_{i \in N}$ be a game. If P_i is open graph for all $i \in N$, then G satisfies the coalitional deviation property.

Proof Suppose that $z \notin {}^\alpha\mathcal{C}(G)$. So, for some coalition S and $x_S \in X_S$, we get $\{x_S\} \times X_{-S} \subseteq P_i(z)$ for all $i \in S$, that is: $\{z\} \times (\{x_S\} \times X_{-S}) \subseteq Gr(P_i)$.³ For every $i \in S$, since P_i is open graph, there exists an open neighborhood O_z^i of z and an open set $A^i \subseteq X$, which includes $\{x_S\} \times X_{-S}$, such that $O_z^i \times A^i \subseteq Gr(P_i)$. This implies that $\{z'\} \times (\{x_S\} \times X_{-S}) \subseteq Gr(P_i)$ for all $z' \in O_z^i$. Now, set $O_z = \bigcap_{i \in S} O_z^i$ and let x' be a point in X such that $x'_S = x_S$. We get $\{x'_S\} \times X_{-S} \subseteq P_i(z')$ for all $i \in S$ and all $z' \in O_z$, which proves that the coalition deviation property is satisfied (in the present situation, the coalition which α -blocks $z' \in O_z$ does not depend on z'). \square

³ $Gr(P_i)$ denotes the graph of the mapping P_i , that is: $Gr(P_i) = \{(u, v) \in X \times X : v \in P_i(u)\}$.

Example 4.1 Consider the normal form game with discontinuous and non quasi-concave payoff functions $G = \langle X_i, u_i \rangle_{i=1,2}$ (borrowed from Example 2 by Scalzo 2019a) such that: $X_1 = X_2 = [0, 1]$ and $u(x) = (u_1(x), u_2(x))$ is defined as below

$$u(x) = \begin{cases} (0, 1) & \text{if } x_1 < x_2 \text{ and } x \notin B \\ (1, 0) & \text{if } x_1 > x_2 \text{ and } x \notin B \\ (0, 0) & \text{if } x_1 = x_2 = 0 \\ (1, 1) & \text{if } x \in B, \end{cases}$$

where $B = \{x : x_1 = x_2 > 0\} \cup (\{0\} \times]0, 1]) \cup (]0, 1] \times \{0\})$. Define $P_i(x) = \{z : u_i(z) > u_i(x)\}$. It is easy to check that ${}^\alpha C(G) = B$. Moreover, in order to see that the coalitional deviation property holds true, it is sufficient to set $x' = (1, 1)$ for every $z \notin {}^\alpha C(G)$. Finally, it is obvious that P_i is not an open-graph mapping.

The following example proves that the property introduced in Definition 4.2 strictly generalizes that given in Definition 4.1

Example 4.2 Let $G = \langle X_i, P_i \rangle_{i=1,2}$ be the game where $X_1 = X_2 = [0, 1]$ and

$$P_i(x) = \begin{cases} \{x_{-i}\} \times [0, 1] & \text{if } x_i > x_{-i} \\ \emptyset & \text{otherwise.} \end{cases}$$

One can see that ${}^\alpha C(G) = \{x : x_1 = x_2\}$. Now, if $z \notin {}^\alpha C(G)$, for some $i \in \{1, 2\}$, we have $z_i > z_{-i}$. So, there exists an open neighborhood O_z such that $z'_i > z'_{-i}$ for all $z' \in O_z$ and, by setting $\xi_z(z') = \{z'_{-i}\} \times [0, 1]$ for all $z' \in O_z$, one recognizes that the generalized coalitional deviation property holds true. Moreover, only player i can α -block every $z' \in O_z$, and the strategy he uses depends on z' . Hence, the game does not satisfy the coalitional deviation property.

The definitions below give coalitional versions of some quasi-concavity like properties employed to solve the Nash equilibrium existence problem in ordinal and normal form games. We refer to the *transfer uniform quasi-concavity* by Nassah and Tian (2016) and Scalzo (2019a) and the *C-condition* by Scalzo (2019b). Given a subset A of a vector space, with $\text{sco}A$ we denote the set of all strict convex combinations of all elements of A .

Definition 4.3 A game G is said to be *coalitional transfer quasi-concave* if, for each $\{x^1, \dots, x^k\} \subset X$ there exists $\{z^1, \dots, z^k\} \subset X$, where $x^h \mapsto z^h$, such that, for every $z \in \text{sco}\{z^{h_1}, \dots, z^{h_l}\}$, with $\{z^{h_1}, \dots, z^{h_l}\} \subseteq \{z^1, \dots, z^k\}$, for at least one $x \in \{x^{h_1}, \dots, x^{h_l}\}$ we have that no coalition can α -block z by using x , that is: for all $S \in \mathcal{N}$, there is $w_{-S} \in X_{-S}$ and $i \in S$ so that $(x_S, w_{-S}) \notin P_i(z)$.

Definition 4.4 A game G is said to be *coalitional quasi-concave* if, for each $\{x^1, \dots, x^k\} \subset X$ and each $z \in \text{sco}\{x^1, \dots, x^k\}$, there exists $x \in \{x^1, \dots, x^k\}$ so that no coalition can α -block z by using x .

It is clear that coalitional quasi-concavity implies coalitional transfer quasi-concavity. The coalitional transfer quasi-concavity is a necessary condition for the non-emptiness

of the α -core. Indeed, if $x^* \in {}^\alpha\mathcal{C}(G)$, to see that the property is satisfied, it is sufficient to set $\{z^1, \dots, z^k\} = \{x^*\}$ for every subset of strategy profiles $\{x^1, \dots, x^k\}$. However, the coalitional transfer quasi-concavity is not enough to guarantee the non-emptiness of the α -core, as the following example shows.

Example 4.3 Consider the 2-player game with mappings P_1 and P_2 defined on $[0, 1] \times [0, 1]$ as follows:

$$P_1(x) = \begin{cases}]x_1, 1] \times [0, 1] & \text{if } x_1 \in [0, 1[\\]0, 1[\times [0, 1] & \text{if } x_1 = 1 \end{cases} \quad P_2(x) \equiv \emptyset.$$

It is easy to see that ${}^\alpha\mathcal{C}(G) = \emptyset$. Moreover, the game is coalitional quasi-concave.

It is interesting to note that, if one looks at the function Θ defined by means of (4.1), the properties introduced in Definitions 4.1, 4.2, 4.3 and 4.4 can be reformulate in terms of properties recalled in Sect. 2 for the (KF) inequality. In particular, we have the following proposition, whose proof is obvious.

Proposition 4.2 *Let G be a game and let Θ be defined by (4.1). One has:*

- (i) *G satisfies the coalitional deviation property if and only if Θ is slightly diagonally transfer continuous.*
- (ii) *G satisfies the generalized coalitional deviation property if and only if Θ is generalized lower transfer continuous.*
- (iii) *G is coalitional transfer quasi-concave if and only if Θ is diagonally transfer quasi-concave.*
- (iv) *G is coalitional quasi-concave if and only if Θ is 0-diagonally quasi-concave.*

Now, from Proposition 4.2, in the light of Propositions 2.1 and 2.2, we obtain the following results on the non-emptiness of the α -core in games with non-ordered preferences.

Theorem 4.1 *Assume that X_i is a convex and compact subset of a Hausdorff topological vector space, for each $i \in N$. Let $G = \langle X_i, P_i \rangle_{i \in N}$ be a game that satisfies the coalitional deviation property. Then, ${}^\alpha\mathcal{C}(G)$ is non-empty if and only if G is coalitional transfer quasi-concave.*

Theorem 4.2 *Assume that X_i is a convex and compact subset of a locally convex Hausdorff topological vector space for each $i \in N$. Let $G = \langle X_i, P_i \rangle_{i \in N}$ be a game that satisfies the generalized coalitional deviation property and is coalitional quasi-concave. Then ${}^\alpha\mathcal{C}(G)$ is non-empty.*

Remark 4.2 Theorem 4.1 presents a class of discontinuous games (games satisfying the coalitional deviation property) where the non-emptiness of the α -core is characterized by means of a quasi-concavity like property (the coalitional transfer quasi-concavity). In Theorem 4.2, the coalitional deviation property is replaced with a more general condition; at the same time, the coalitional transfer quasi-concavity is strengthened with the coalitional quasi-concavity. However, Theorems 4.1 and 4.2 are each other independent.

For instance, the game presented in Example 4.2, which satisfies the generalized coalitional deviation property but not the coalitional deviation property, is coalitional quasi-concave. To prove this, we proceed by contradiction and assume that there exists $z \in \text{sco}\{x^1, \dots, x^k\}$ such that: for each $x \in \{x^1, \dots, x^k\}$ there is a coalition S so that $\{x_S\} \times X_{-S} \subseteq P_i(z)$ for all $i \in S$. If $z_1 = z_2$, we have that $z \in {}^\alpha\mathcal{C}(G)$, and this is impossible. So, we have that $z_1 \neq z_2$. Suppose $z_1 > z_2$. Only coalition $\{1\}$ can α -block z , and we obtain $\{x_1\} \times [0, 1] \subseteq P_1(z) = \{z_2\} \times [0, 1]$ for all $x \in \{x^1, \dots, x^k\}$, which implies $x_1^h = z_2$ with $h = 1, \dots, k$. Since $z \in \text{sco}\{x^1, \dots, x^k\}$, we get the contradiction $z_1 = z_2$. Similarly if $z_1 < z_2$, and the coalitional quasi-concavity holds true. So, in this example, Theorem 4.2 applies but Theorem 4.1 not.

On the other hand, Theorem 4.1 gives a characterization of the non-emptiness of the α -core, while Theorem 4.2 provides only sufficient conditions.

4.1 Comparison with the previous literature

Examples 4.1 and 4.2 provide games where our results apply but not those by Scarf (1971) and Kajii (1992): the payoff functions in the first example are not continuous and the mappings from the second one are not open-graph.

Concerning the non-emptiness of the α -core in normal form games with discontinuous payoff functions, Theorem 1 by Uyanik (2015) assumes finite-dimensional spaces of strategies, quasi-concave payoffs and a condition called *coalitionally C-security*. This condition require that, if $z \notin {}^\alpha\mathcal{C}(G)$, for each coalition S there exist a numerical vector $(v_i)_{i \in S}$, $x_S \in X_S$ and an open neighborhood O_z of z such that:

- (a) $u_i(x_S, w_{-S}) \geq v_i$ for all $w_{-S} \in X_S$ and all $i \in S$, and
- (b) for each $z' \in O_z$ there is a coalition S so that $u_i(z') < v_i$ for all $i \in S$.

It is easy to see that the coalitionally C-security fails on the game by Example 4.1 because of the point $z = (0, 0)$.

In the setting of games with discontinuous non-ordered preferences, Martins-da-Rocha and Yannelis (2011) introduced the following condition, called α -continuity: the mapping $P_S : X \rightrightarrows X_S$, defined for each coalition S by

$$P_S(x) = \left\{ z_S \in X_S : \{z_S\} \times X_{-S} \subseteq \bigcap_{i \in S} P_i(x) \right\} \quad \forall x \in X,$$

has open lower sections. They obtained the non-emptiness of the α -core in games satisfying the α -continuity and such that $x \notin \text{co}P_i(x)$ for all $x \in X$ and all $i \in N$. This property can be characterized as follows:

Proposition 4.3 *Let $G = \langle X_i, P_i \rangle_{i \in N}$ be a game. Then, α -continuity holds true if and only if the following property is satisfied: a) for each $z \notin {}^\alpha\mathcal{C}(G)$, there exist an open neighborhood O_z of z , a coalition S , and $x'_S \in X_S$ such that $\{x'_S\} \times X_{-S} \subseteq \bigcap_{i \in S} P_i(z')$ for all $z' \in O_z$.*

Proof Suppose that α -continuity holds and $z \notin {}^\alpha\mathcal{C}(G)$. So, at least one coalition S α -blocks z by using some $x_S \in X_S$, that is: $\{x_S\} \times X_{-S} \subseteq \bigcap_{i \in S} P_i(z)$. Because

of α -continuity, we have $O_z \subseteq P_S^{-1}(x_S)$ for some open neighborhood O_z of z . Set $x'_S = x_S$, we get that the property *a*) is satisfied. Similarly one gets the viceversa. \square

Proposition 4.3 shows that the class of α -continuous games is included in the class of games which satisfy the coalitional deviation property. Moreover, the inclusion is strict because the α -continuity fails on the game introduced in Example 4.1; in fact, one has $P_{\{1,2\}}^{-1}(1, 1) = \{(0, 0)\}$.

5 Conclusions

This paper has provided new conditions which guarantee the non-emptiness of the α -core in the setting of games with discontinuous and non-ordered preferences and infinite-dimensional spaces of strategies. In particular, we have introduced sufficient and necessary conditions for the existence of α -core elements. Inspired by a recent literature on the existence of Nash equilibria in games with non-ordered preferences, our results have been obtained by means of the Ky Fan minimax inequality. More precisely, given a game G with non-ordered preferences, we have defined a real-valued functions Θ such that a strategy profile x^* belongs to the α -core of G if and only if $\Theta(x, x^*) \leq 0$ for all strategy profiles x . We have obtained two results. In the first one, we have proved that the α -core of a game satisfying the *coalitional deviation property* is non-empty if and only if the game is *coalitional transfer quasi-concave*. In the second one, we have considered a condition more general than the coalitional deviation property and we have obtained the non-emptiness of the α -core by using a strengthening of the coalitional transfer quasi-concavity.

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