Separating equilibrium in quasi-linear signaling games

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Accepted: 28 April 2019 / Published online: 8 May 2019
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Abstract

Using a network approach we provide a characterization of a separating equilibrium for standard signaling games where the sender's payoff function is quasi-linear. Given a strategy of the sender, we construct a network where the node set and the length between two nodes are the set of the sender's type and the difference of signaling costs, respectively. Construction of a separating equilibrium is then equivalent to constructing the length between two nodes in the network under the condition that the response of the receiver is a node potential. When the set of the sender's type is a real interval, shortest path lengths are antisymmetric and a node potential is unique up to a constant. A strategy of the sender in a separating equilibrium is characterized by some differential equation with a unique solution. Our results can be readily applied to a broad range of economic situations, such as for example the standard job market signaling model of Spence, a model not captured by earlier papers.

Keywords Signaling game · Separating equilibrium · Node potential

JEL Classification C72 · D82

1 Introduction

A signaling game is a two-player game of incomplete information in which one player has private information—a type—and the other is affected by the information. The informed player, the sender, can send a signal, contingent on type, and the uninformed

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party, the receiver, takes an action, which may be conditional on the observed signal. In equilibrium, the informed player optimally chooses a signal, knowing that the uninformed player will take an optimal action conditional on his inferences from the signal, and inferences are correct. If the sender chooses to send different signals for different types, the uninformed party can unambiguously determine the informed player's type. In this case the equilibrium is called *separating*. Among many equilibria, separating equilibria are of central interest because informational asymmetry is resolved.

Contribution In the model described above, we focus on those type separating strategies of the sender that, combined with the induced optimal strategy of the receiver, form a (separating) equilibrium. Such strategies are called SE strategies. Earlier results on the characterization of SE strategies in signaling games, despite their seeming generality, fail to capture many simple and natural extensions of the classic job market model by Spence. We achieve full coverage of job market models with quasi-linear utilities. That is, settings in which the utility function of the sender is calculated as the payment a by the receiver, minus the cost $c(s, t)$ of signaling s when of type t (in formula, $U_S = a - c(s, t)$). The advantage of quasi-linearity is that it separates the cost of signaling from the response of the receiver, thereby making the cost independent of the response of the receiver. This advantage translates itself into two improvements upon previous literature, which we describe below.

First, we give a complete characterization of SE strategies that uses only four fairly mild conditions, the most prominent one being that the cost function should satisfy decreasing differences, an algebraic version of the single crossing property. Under these conditions many of the results on SE strategies can already be established. For example the—intuitive—result that any SE strategy is strictly increasing with type follows immediately. Notably, we do not need differentiability (or even continuity) of the utility function of the sender, so that these results apply in a wide variety of settings.

Second, when type space is a halfopen interval, we only need a further three relatively natural and straightforward conditions on model primitives to characterize existence and uniqueness of the SE strategy. The requirements on the signaling cost function are that it is continuous, and that it has a continuously differentiable and bounded partial derivative with respect to the signal.¹

Approach The technique we use to establish our results is in itself worth a short discussion. We use the—essentially algebraic—theory of node potentials on directed graphs to derive our results. This technique has recently been applied with much success in the, related, context of mechanism design.² See for example Heydenreich et al. (2009), Carbajal and Ely (2013), Chung and Olsewski (2007) and Kos and

¹ In contrast, Mailath (1987) and Mailath and von Thadden (2013) require the utility function of the sender to be at least twice continuously differentiable, and additionally need to impose several technical assumptions regarding signs both first and second order derivatives. In particular, the standard job market model by Spence where the utility of the sender equals the payment by the receiver minus the reciprocal of the type falls within our requirements, while it is excluded by the conditions imposed by Mailath (1987) and Mailath and von Thadden (2013).

² Note that signaling games are in general not an object of study in mechanism design. In mechanism design, reporting types is in essence free of cost. In a signaling game, a signal may have different costs for different types, which is an added complication that is typically not addressed in mechanism design.

Messner (2012) for applications of the theory of networks to mechanism design without differentiability.

The algebraic nature of the theory allows us to discard conditions on the first and second order partial derivatives of the utility function of sender, the conditions that drive most of the earlier results. Our proof technique highlights that differentiability is in itself not the driving force of these results. Instead, these results are shown to rely only on the implied algebraic structure of marginal gains in type pooling versus marginal costs of signaling captured by the decreasing differences condition.

Related literature Mailath (1987) studies a model in which the utility function of the receiver is C^2 , utility is strictly monotonic in types and actions, and strictly quasi-concave in signals, plus a technical condition on the first and second order derivatives of the utility function for the sender, called boundedness. Type space is assumed to be a compact interval, the signal space is assumed to be equal to the type space³ and the action space is the entire set of reals. Under the assumption of single crossing it is shown that an one-to-one strategy is a separating equilibrium strategy precisely when the strategy is strictly increasing and it solves a characterizing differential equation. Existence and uniqueness of a separating equilibrium strategy is then shown under the additional assumption that the derivative of the utility function of the sender with respect to signals satisfies a boundedness condition.

Mailath and von Thadden (2013) study a model in which the utility function of the sender and the best response function of the receiver under complete information are both C^2 . Further, under complete information the receiver needs to have a unique best response, and the induced valuation function of the sender needs to satisfy a technical condition on both first and second order derivatives of the valuation function. Thus, the conditions on the strategy of the receiver are only implicitly formulated, in terms of the induced payoff function for the sender. They show that if a separating equilibrium strategy of the receiver is differentiable at a certain point, it is characterized by a differential equation at that point. They also show that if (1) a strategy is separating, (2) it solves the characterizing differential equation, and (3) it satisfies a version of the single-crossing property, then that strategy is incentive-compatible. They apply their results in several examples.

Quinzii and Rochet (1985) study a model where both types and signals are higher dimensional. They show for the case of linear, separable costs, that a separating equilibrium exists under assumptions of smoothness of the productivity function, plus convexity of the surplus function.

Hoppe et al. (2009) study a matching market between a set of men and a set of women. Each man and each woman has private information regarding their type. The value of a match depends on both the type of the man and the woman in question.⁴ Signals are costly, but do not depend on type of either men or women, and are linearly deducted from utility. In this model, Hoppe et al. (2009) show that there exists a single symmetric, strictly monotonic, and continuously differentiable equilibrium. They also characterize this equilibrium by means of a differential equation. However, the analysis

³ By the revelation principle this is without loss of generality.

⁴ In fact it is defined as the product of both types. So, if a man is of type y and a woman of type x , and they are matched, then this generates utility xy , both for him and her. Thus, total utility generated by this match is $2xy$, equally divided over both partners.

does not rule out the existence of non-symmetric, non-monotonic, or non-differentiable equilibria, and uniqueness is not guaranteed.

Organization Section 2 presents the model of signaling games and several notions of graph theory. In Sect. 3 we fully characterize SE strategies in terms of node potentials for an induced length function. In Sect. 4 we derive conditions to guarantee existence of a unique SE strategy when type space is a halfopen interval. We show that the unique SE strategy can be computed by solving an associated differential equation. We illustrate the power of our techniques in four applications, among those the job market model and an example of a model without differentiability. All omitted proofs are in the “Appendix”.

2 The model

We consider the following standard signaling game. There are two risk neutral players, a sender and a receiver. The type space and signal space of the sender are denoted by T and S , respectively. We assume that the type space T is a subset of $\mathbb{R}_+$ and that the signal space S equals $\mathbb{R}_+$. Further, T is assumed to have a smallest element $\underline{t}$. The action space of the receiver is $A = \mathbb{R}_+$.

The game proceeds as follows. Nature decides the sender's type $t \in T$, and the result is communicated to the sender only. With knowledge of his type t , the sender chooses a signal $s \in S$, incurring a cost $c(s, t)$ depending on both type and signal. Upon observing the sender's signal s , the receiver chooses an action $a \in A$. The sender receives a payoff $U_S(a, s, t) = a - c(s, t)$ and the receiver receives a payoff $U_R(a, t)$. Then the game ends.

Note that the payoff of the receiver does not depend on the signal s . This is called the “pure signaling case” in Quinzii and Rochet (1985).

Assumptions We adopt the following four structural assumptions.

- [1] The cost function $c(s, t)$ is strictly increasing in s and for each $s > 0$ strictly decreasing in t . Further, for all t , $c(0, t) = 0$.
- [2] The cost function $c(s, t)$ satisfies decreasing differences. That is, for all $s, s' \in S$ and $t, t' \in T$ with $s > s'$ and $t > t'$ it holds that

$$c(s, t') - c(s', t') > c(s, t) - c(s', t).$$

- [3] For every $t \in T$ there exists an $\alpha(t) \in A$ with

$$U_R(\alpha(t), t) > U_R(a, t) \quad \text{for all } a \in A \setminus \{\alpha(t)\}.$$

In other words, for every $t \in T$, $\alpha(t) := \arg \max_{a \in A} U_R(a, t)$ is unique.

- [4] The resulting function α is strictly increasing in t , and $\alpha(\underline{t}) = 0$.

A (pure) strategy of the sender is a mapping $\sigma: T \rightarrow S$. A (pure) strategy of the receiver is a function $\gamma: S \rightarrow A$. Since in this paper we only focus on pure strategies, we omit the prefix “pure” from now on.

Definition A strategy pair (σ, γ) is a separating equilibrium (SE) if the strategy σ is one-to-one and, moreover, for all $t \in T$ we have

- [1] $U_S(\gamma(\sigma(t)), \sigma(t), t) \geq U_S(\gamma(s), s, t)$ for all $s \in S$
- [2] $U_R(\gamma(\sigma(t)), t) \geq U_R(a, t)$ for all $a \in A$.

Condition [1] states that the strategy σ is a best response for the sender given the strategy γ of the receiver. Condition [2] states that the strategy γ is a best response for the receiver given the strategy σ of the sender. Thus, in fact we use the concept of separating Bayesian Nash Equilibrium (BNE) to analyze sender-receiver games.⁵

If a strategy σ of the sender is part of a separating equilibrium it is called a separating equilibrium strategy (SE strategy). The aim of this paper is to characterize the set of SE strategies.⁶

We end this section with two short observations. In any SE, each type t gets paid $\alpha(t)$, and the lowest type sends the lowest signal possible.

Lemma 1 *Let (σ, γ) be an SE. Then $(\gamma \circ \sigma)(t) = \alpha(t)$ for all $t \in T$. Further, $\sigma(\underline{t}) = 0$.*

Proof Let (σ, γ) be an SE. By assumption [3] and part [2] of the definition of SE, $(\gamma \circ \sigma)(t) = \alpha(t)$ for all $t \in T$.

We argue that $\sigma(\underline{t}) = 0$. Using assumption [4], $(\gamma \circ \sigma)(\underline{t}) = \alpha(\underline{t}) = 0$. So, since type $\underline{t}$ receives the lowest possible payment, it is clear that type $\underline{t}$ sends the lowest cost signal. Hence, since $c(s, \underline{t})$ is strictly increasing in s , $\sigma(\underline{t}) = 0$. $\square$

3 Graph representation of SE strategies

We provide a graph theoretic interpretation of SE strategies. We recall a few basic terms from graph theory. A directed graph or digraph is a pair $G = (V, E)$ where V is an arbitrary set and E is a collection of pairs (u, v) of elements $u, v \in V$ with $u \neq v$. The elements of V are called nodes or vertices, and the elements of E are called arcs. The complete digraph is the digraph $G = (V, E)$ where E is the set of all possible ordered pairs of distinct elements of V .

3.1 The characterization

Let $G = (V, E)$ be a complete digraph. An arc length on G is a function $l: E \rightarrow \mathbb{R}$. A function $p: V \rightarrow \mathbb{R}$ is called a node potential for arc length l on $G = (V, E)$ if for all arcs $(u, v) \in E$

$$p(v) - p(u) \leq l(u, v).$$

Let σ be a strategy of the sender. Let $H = (T, Q)$ be the complete digraph on the type space T , so $Q = \{(u, v) \in T \times T \mid u \neq v\}$. Define the arc length l_σ on H by, for all $t, t' \in T$,

⁵ Since the BNE we focus on is separating, beliefs are uniquely determined, and obsolete for checking optimality of responses. Therefore we decided not to model beliefs at all in this paper.

⁶ Note that the analogous task for the strategies of the receiver is a triviality.

$$l_{\sigma}(t, t') = c(\sigma(t'), t) - c(\sigma(t), t)$$

Theorem 2 A strategy σ of the sender is an SE strategy if and only if σ is one-to-one, $\sigma(\underline{t}) = 0$, and α is a node potential for l_{σ} on H .

Proof We show both implications separately.

A. Let (σ, γ) be an SE. Since σ is separating, it is one-to-one. Further, by Lemma 1, $\sigma(\underline{t}) = 0$. Also by Lemma 1, $(\gamma \circ \sigma)(t) = \alpha(t)$ for all t . So, since σ is an equilibrium,

$$\alpha(t) - c(\sigma(t), t) = (\gamma \circ \sigma)(t) - c(\sigma(t), t) \geq \gamma(s) - c(s, t)$$

for any s . Substitution of $s = \sigma(t')$ and rewriting then yields

$$\alpha(t') - \alpha(t) \leq c(\sigma(t'), t) - c(\sigma(t), t) = l_{\sigma}(t, t')$$

for all $t, t' \in T$. Hence, α is a node potential for l_{σ} on H .

B. Conversely, since σ is one-to-one, for every $s \in S$ there is at most one $t \in T$ with $\sigma(t) = s$. So, we can define the strategy γ of the receiver by

$$\gamma(s) = \begin{cases} \alpha(t) & \text{if } t \in T \text{ is such that } \sigma(t) = s \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to show that (σ, γ) is a Nash equilibrium. First, for all $a \in A$

$$U_R(\gamma(\sigma(t)), t) = U_R(\alpha(t), t) \geq U_R(a, t)$$

by definition of γ and α . So, γ is a best response of the receiver to σ .

Second, suppose the sender is of type t . Take any $s \in S$. If there is $t' \in T$ with $\sigma(t') = s$. Then $\gamma(s) = \alpha(t')$. So,

$$U_S(\gamma(\sigma(t)), \sigma(t), t) = \alpha(t) - c(\sigma(t), t) \geq \alpha(t') - c(\sigma(t'), t) = U_S(\gamma(s), s, t).$$

Otherwise $\gamma(s) = 0$. Then, since $\alpha(\underline{t}) = 0$, $\sigma(\underline{t}) = 0$, and $c(s, t)$ is increasing in s , we get

$$\begin{aligned} U_S(\gamma(\sigma(t)), \sigma(t), t) &= \alpha(t) - c(\sigma(t), t) \geq \alpha(\underline{t}) - c(\sigma(\underline{t}), t) \\ &= 0 - c(0, t) \geq 0 - c(s, t) \\ &= U_S(\gamma(s), s, t). \end{aligned}$$

Hence, σ is a best response for the sender to γ . □

3.2 A few useful consequences

We need a few direct consequences of the basic characterization in the remainder of this paper. We discuss these here in this section. One direct consequence of the above characterization is the following observation.

Lemma 3 *Let σ be an SE strategy. Then σ is strictly increasing in t .*

Proof Take $t, t' \in T$ with $t' > t$. To show: $\sigma(t') > \sigma(t)$. Since σ is an SE strategy, by Theorem 2

$$\alpha(t') - \alpha(t) \leq c(\sigma(t'), t) - c(\sigma(t), t).$$

By assumption [4] this implies $c(\sigma(t'), t) > c(\sigma(t), t)$. Hence, since $c(s, t)$ is strictly increasing in s by assumption [1], it follows that $\sigma(t') > \sigma(t)$. $\square$

An arc length l on H is strictly monotone if $l(t, t') + l(t', t) > 0$ for all $t \neq t'$. The arc length l is decomposition monotone if for any $t, t', t'' \in T$ with $t < t' < t''$ we have $l(t, t'') > l(t, t') + l(t', t'')$ and $l(t'', t) > l(t'', t') + l(t', t)$.

Lemma 4 *Suppose σ is strictly increasing. Then arc length l_σ is strictly monotone and decomposition monotone on H .*

Proof In order to show that l_σ is strictly monotone, take t, t' with $t < t'$. Then, since $\sigma(t') > \sigma(t)$ and c has decreasing differences,

$$l_\sigma(t, t') + l_\sigma(t', t) = c(\sigma(t'), t) - c(\sigma(t), t) + c(\sigma(t), t') - c(\sigma(t'), t') > 0.$$

In order to prove that l_σ is decomposition monotone, take $t, t', t'' \in T$ with $t < t' < t''$. Then, because $\sigma(t'') > \sigma(t')$ and c has decreasing differences,

$$\begin{aligned} l_\sigma(t, t'') &= c(\sigma(t''), t) - c(\sigma(t), t) \\ &= c(\sigma(t''), t) - c(\sigma(t'), t) + c(\sigma(t'), t) - c(\sigma(t), t) \\ &> c(\sigma(t''), t') - c(\sigma(t'), t') + c(\sigma(t'), t) - c(\sigma(t), t) \\ &= l_\sigma(t', t'') + l_\sigma(t, t'). \end{aligned}$$

Similarly, $l_\sigma(t'', t) > l_\sigma(t', t) + l_\sigma(t'', t')$. $\square$

4 Characterization of SE when T is a continuum

If the type space T is a continuum, we can characterize an SE strategy as a solution of an integral equation. The environment is the same as before, except that the sender's type space is an interval $T = [\underline{t}, \bar{t}]$, where $\bar{t}$ may be equal to ∞ . Recall that both the set of signals S and the action space A are $\mathbb{R}_+$.

Assumptions We adopt the following three additional structural assumptions.

- [5] The function α is continuous.
- [6] The cost function $c(s, t)$ is continuous on $S \times T$.
- [7] For all t , the partial derivative $c_s(s, t)$ of $c(s, t)$ with respect to s exists, and $c_s(s, t) > 0$ is continuous and bounded on $S \times T$.

Lemma 5 *The function $c_s(s, t)$ is non-increasing in t .*

Proof By assumption [2], for $h > 0$ and $t \leq t'$,

$$c(s + h, t') - c(s, t') \leq c(s + h, t) - c(s, t).$$

By assumption [7] we can divide both sides by h and take limits for $h \rightarrow 0$, which yields the inequality $c_s(s, t') \leq c_s(s, t)$. $\square$

We need the following observation in this setting.

Lemma 6 *Suppose that σ is an SE strategy. Then, σ is strictly increasing and continuous.*

Proof Let (σ, γ) be an SE. Then σ is strictly increasing by Lemma 3. We show that σ is continuous. Take $t \in T$. Define $\sigma(t+) := \lim_{\varepsilon \downarrow 0} \sigma(t + \varepsilon)$. Since (σ, γ) is an SE, Lemma 1 implies that

$$c(\sigma(t + \varepsilon), t + \varepsilon) - c(\sigma(t), t + \varepsilon) \leq \alpha(t + \varepsilon) - \alpha(t).$$

for every $\varepsilon > 0$. Since $\alpha(t)$ and $c(s, t)$ are continuous by assumptions [5] and [6], it follows that

$$c(\sigma(t+), t) - c(\sigma(t), t) \leq 0.$$

So, $\sigma(t+) \leq \sigma(t)$ by assumption [1], which implies that σ is right-continuous. Similarly, σ is left-continuous. Hence, σ is continuous. $\square$

We show that there is a unique SE strategy $\sigma = (\gamma^*)^{-1} \circ \alpha$, under the condition that the image $\gamma^*(S)$ of the solution γ^* of an associated differential equation equals the image $\alpha(T)$.⁷

Write $B = \{\alpha(t) \mid t \in T\}$. By assumption [4], $\alpha^{-1}: B \rightarrow T$ exists. Take $(s, a) \in S \times B$. Define

$$H(s, a) := c_s(s, \alpha^{-1}(a)).$$

Note that $H(s, a) > 0$ by assumption [7]. A subset I of $\mathbb{R}_+$ is called an initial segment if I is either a closed interval $[0, b]$ with $0 < b < \infty$, or a halfopen interval $[0, b)$

⁷ The fact that the SE strategy can be computed by solving an associated differential equation is the continuous counterpart of the result in Theorem 2 that α is a node potential for l_σ . Rewriting the formula $\sigma = (\gamma^*)^{-1} \circ \alpha$ for the SE strategy to $\alpha = \gamma^* \circ \sigma$ shows the connection.

with $0 < b \leq \infty$. A function $f: I \rightarrow \mathbb{R}$ is called a partial solution on initial segment I if $f(0) = 0$, f is differentiable on I , and

$$f'(s) = H(s, f(s))$$

for all $s \in I$. A partial solution $f: I \rightarrow \mathbb{R}$ encompasses a partial solution $g: J \rightarrow \mathbb{R}$ if $J \subseteq I$ and $f(x) = g(x)$ for all $x \in J$. A solution is a partial solution that is not encompassed by any other partial solution.

Theorem 7 *The initial value problem*

$$\gamma'(s) = H(s, \gamma(s)) \text{ with } \gamma(0) = 0,$$

has a unique solution $\gamma^*: I \rightarrow B$, where I is an initial segment of B . Moreover, γ^* is strictly increasing.

Proof The function H is continuous, strictly positive, and non-increasing in the second argument due to assumptions [2], [4], and [7]. So, the initial value problem

$$\gamma'(s) = H(s, \gamma(s)) \text{ with } \gamma(0) = 0$$

has a unique solution γ^* on an initial segment I by Theorem 17. Moreover, γ^* is continuously differentiable, and $(\gamma^*)'(s) = H(s, \gamma^*(s)) > 0$ for all s . So, γ^* is strictly increasing. $\square$

Using the solution to the above initial value problem we can now characterize under which conditions SE strategies exist.

Theorem 8 *Let $\gamma^*: I \rightarrow B$ be the unique solution to the above initial value problem. Then*

- [1] *if $\gamma^*(I) \neq B$, there does not exist an SE strategy.*
- [2] *if $\gamma^*(I) = B$, the strategy $\sigma = (\gamma^*)^{-1} \circ \alpha$ is the unique SE strategy.*

Proof A. Proof of [1]. Let σ be an SE strategy. By Lemma 6, σ is continuous and strictly increasing. So, $J := \sigma(T)$ is an initial segment of S . Define $\gamma(s) = (\alpha \circ \sigma^{-1})(s)$ for all $s \in J$.

We first argue that γ is a partial solution to the initial value problem. Note that $\gamma(0) = 0$ by Lemma 1 and assumption [4]. Further, by Lemma 15, it follows that

$$\alpha(t) = \int_t^t h(\sigma)(x) d\sigma(x),$$

where $h(\sigma)(x) = c_s(\sigma(x), x)$. The change of variable $y = \sigma(x)$ yields for $s = \sigma(t)$ that

$$\gamma(s) = (\alpha \circ \sigma^{-1})(s) = \int_{\sigma(t)}^{\sigma(t)} c_s(y, \sigma^{-1}(y)) dy = \int_0^s H(y, \gamma(y)) dy.$$

Since the function $y \mapsto H(y, \gamma(y))$ is continuous by Lemma 6 and assumptions [5] and [7], the Fundamental Theorem of Calculus states that γ is differentiable, and that for all $s \in J$

$$\gamma'(s) = H(s, \gamma(s)).$$

It follows that γ is a partial solution to the initial value problem.

Now suppose that $\gamma^*(I) \neq B$. Since γ is a partial solution, γ^* encompasses γ by Theorem 17. Take any $a \in B$ with $a \notin \gamma^*(I)$. Since $a \in B$, there is $t \in T$ with $\alpha(t) = a$. Then

$$a = \alpha(t) = (\alpha \circ \sigma^{-1})(\sigma(t)) = \gamma(\sigma(t)) = \gamma^*(\sigma(t)) \in \gamma^*(I).$$

This contradicts the assumption that $a \notin \gamma^*(I)$. This proves claim [1].

B. Proof of [2]. Suppose that $\gamma^*(I) = B$. Then, for every t there exists an s with $\gamma^*(s) = \alpha(t)$. So we can define $\sigma = (\gamma^*)^{-1} \circ \alpha$. We show that σ is an SE strategy.

Note that σ is strictly increasing and continuous by assumptions [4] and [5]. Moreover, $\sigma(\underline{t}) = (\gamma^*)^{-1}(\alpha(\underline{t})) = (\gamma^*)^{-1}(0) = 0$. So, by Theorem 2 it suffices to show that α is a node potential for l_σ . Take $t, t' \in T$. We show that

$$\alpha(t') - \alpha(t) \leq l_\sigma(t, t').$$

On the one hand, by definition of σ ,

$$\begin{aligned} \alpha(t') - \alpha(t) &= (\gamma^* \circ \sigma)(t') - (\gamma^* \circ \sigma)(t) = \int_{\sigma(t)}^{\sigma(t')} (\gamma^*)'(x) dx \\ &= \int_{\sigma(t)}^{\sigma(t')} H(x, \gamma^*(x)) dx = \int_{\sigma(t)}^{\sigma(t')} c_s(x, \alpha^{-1}(\gamma^*(x))) dx \\ &= \int_{\sigma(t)}^{\sigma(t')} c_s(x, \sigma^{-1}(x)) dx. \end{aligned}$$

On the other hand,

$$l_\sigma(t, t') = c(\sigma(t'), t) - c(\sigma(t), t) = \int_{\sigma(t)}^{\sigma(t')} c_s(x, t) dx.$$

Now note that, since σ is strictly increasing, for any $x \geq \sigma(t)$ we have $\sigma^{-1}(x) \geq t$. Therefore $c_s(x, \sigma^{-1}(x)) \leq c_s(x, t)$ for any $x \geq \sigma(t)$ by Lemma 5. Hence,

$$\alpha(t') - \alpha(t) = \int_{\sigma(t)}^{\sigma(t')} c_s(x, \sigma^{-1}(x)) dx \leq \int_{\sigma(t)}^{\sigma(t')} c_s(x, t) dx = l_\sigma(t, t').$$

This completes the proof. $\square$

Remark We do not know of any example where [1] of Theorem 8 does indeed occur. On the other hand, we are also not aware of any reason why [1] could not occur. So, existence of such examples remains a possibility in our results.

5 Applications

In this section we present a few applications and direct consequences of our results. We first discuss a simple example to illustrate the computational power of our results. The second application is the continuous-type version of the classic job market model by Spence (1973). Third, we illustrate the scope of our results in an example outside the context of differentiable utility. Fourth, we discuss the consequences of our results for mechanism design.

5.1 Application: the differentiable case

As a first application we consider the case where α is differentiable.

Corollary 9 *Suppose α is differentiable on T . Then, the unique SE strategy σ is also differentiable and its inverse $\tau = \sigma^{-1}$ solves the initial value problem*

$$\alpha'(\tau(s)) \cdot \tau'(s) = c_s(s, \tau(s)) \text{ with } \tau(0) = \underline{t}.$$

Proof From Theorem 8 it follows immediately that the SE strategy σ is differentiable as well. Further, by differentiating both sides in the integral expression for α with respect to s , we have the differential equation.

Example Take $\alpha(t) = t^2$ and $c(s, t) = \frac{s}{t+1}$. It is straightforward to check that this model satisfies all conditions [1] till [7]. Thus, the above corollary applies, and

$$2 \cdot \tau(s) \cdot \tau'(s) = \frac{1}{\tau(s) + 1}.$$

This can be rewritten to the differential equation

$$2 \cdot \tau(s) \cdot (\tau(s) + 1) \cdot \tau'(s) = 1$$

which yields $\frac{2}{3}\tau(s)^3 + \tau(s)^2 = s$. It immediately follows that $\sigma(t) = \frac{2}{3}t^3 + t^2$.

5.2 Application: job market signaling

The job market model of Spence (1973) is one of the earlier seminal papers in signaling games. Consider the following version of the job market model. The ability of the worker is characterized by an attribute $t \in T = [\underline{t}, \infty)$ ⁸ with $\underline{t} \geq 0$. After having

⁸ In the original Spence model, $T = \{1, 2\}$.

observed his ability, the worker has to choose a level of education $s \in S = [0, \infty)$. The employer then observes the level s of education chosen by the worker, and based on that information chooses a wage $a \in A = [0, \infty)$.

We assume that the employer has a unique optimal wage schedule $\alpha: T \rightarrow A$ that is increasing in the attribute t , and $\alpha(\underline{t}) = 0$. We also assume that the cost of signaling for the worker is a separable function

$$c(s, t) = h(s) \cdot g(t),$$

where h and g are continuous functions from S and T respectively to $\mathbb{R}_+$, $h(0) = 0$, h is strictly increasing, and g is strictly decreasing. Thus, the utility to the worker is given by

$$U_S(a, s, t) = a - h(s) \cdot g(t).$$

This model, a straightforward and natural version of the classic model by Spence, satisfies all conditions we specified in the previous sections.

We analyze the following instance of the job market signaling game. The type space is $T = [1, \infty)$, so that $\underline{t} = 1$, and the signal space is $S = \mathbb{R}_+$. The best action of the receiver when the sender is of type $t \in T$ is $\alpha(t) = t$, and the cost for the sender when he is of type $t \in T$ and sends signal $s \in S$ is $c(s, t) = \frac{s}{t}$.

Since this example is an instance of our model, our results apply. By Corollary 9, the inverse σ of the solution τ of the differential equation

$$\tau'(s) = \frac{1}{\tau(s)} \quad \text{with } \tau(0) = 1$$

is the unique SE strategy. It is straightforward to check that $\tau(s) = \sqrt{2s + 1}$. Hence, $\sigma(t) = \frac{1}{2}(t^2 - 1)$ is the unique SE strategy of this job signaling model.

In fact, the same analysis applies to the situation where α is strictly increasing and continuously differentiable, and $c(s, t) = s \cdot g(t)$, where g is any strictly positive, continuous, and strictly decreasing function on $\mathbb{R}_+$. Then by the above corollary

$$\frac{\alpha'(\tau(s))}{g(\tau(s))} \cdot \tau'(s) = 1.$$

Integrating both sides with respect to s and using the substitution $z = \tau(s)$ and $dz = \tau'(s)ds$ for the left-hand side yields

$$\sigma(t) = (s) = \int_0^t \frac{\alpha'(z)}{g(z)} dz.$$

This integral immediately renders the unique (continuously differentiable) separating equilibrium strategy.

5.3 Application: an example without SE strategies

Consider the setting in the previous paragraph. Take $S = T = [0, \infty)$, $\alpha(t) = t$, $h(s) = s$, and

$$g(t) = \begin{cases} 1-t & \text{if } t \in [0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Note that in this example $c_s(s, t) = g(t)$, so that requirement [7] is not satisfied.

The formula for the SE strategy yields

$$\sigma(t) = \int_0^t \frac{\alpha'(z)}{g(z)} dz = \int_0^t \frac{1}{1-z} dz = \ln \left(\frac{1}{1-t} \right).$$

This SE strategy is only defined for values of t in the interval $[0, 1)$. Since the range of signals that are used by types in $[0, 1)$ is equal to $S = [0, \infty)$, it is clear that types $t \geq 1$ do not have a signal available that would separate them from lower types. Hence, when T strictly includes the interval $[0, 1)$ —for example when $T = [0, \infty)$ —the resulting model does not admit an SE strategy.

5.4 Application: the non-differentiable case

Consider the job market model where $T = A = S = \mathbb{R}_+$, $g(t) = \frac{1}{t+1}$, $h(s) = s$, and

$$\alpha(t) = \begin{cases} t & \text{if } 0 \leq t \leq 3 \\ \frac{1}{3}t + 2 & \text{if } 3 \leq t. \end{cases}$$

Note that α is continuous, but not differentiable. Nevertheless, our results apply. We first compute $H(s, a)$. Note that $c(s, t) = s \cdot g(t)$. So, $c_s(s, t) = g(t)$, and therefore $H(s, a) = c_s(s, \alpha^{-1}(a)) = g(\alpha^{-1}(a))$. It is now easy to compute that

$$\alpha^{-1}(a) = \begin{cases} a & \text{if } 0 \leq a \leq 3 \\ 3a - 6 & \text{if } 3 \leq a \end{cases} \quad \text{and} \quad H(s, a) = \begin{cases} \frac{1}{a+1} & \text{if } 0 \leq a \leq 3 \\ \frac{1}{3a-5} & \text{if } 3 \leq a. \end{cases}$$

Thus, Theorem 8 states that a separating equilibrium strategy σ is determined by $\sigma = \gamma^{-1} \circ \alpha$ where γ solves the initial value problem that $f(0) = 0$ and

$$f'(s) = (H(s, f(s))) = \begin{cases} \frac{1}{f(s)+1} & \text{if } 0 \leq f(s) \leq 3 \\ \frac{1}{3f(s)-5} & \text{if } 3 \leq f(s). \end{cases}$$

The (unique) solution to the differential equation above is

$$\gamma(s) = \begin{cases} \sqrt{s+1} - 1 & \text{if } 0 \leq s \leq 15 \\ \frac{1}{3} \cdot (\sqrt{6s-74} + 5) & \text{if } 15 \leq s. \end{cases}$$

We get that

$$\gamma^{-1}(a) = \begin{cases} (a+1)^2 - 1 & \text{if } 0 \leq a \leq 3 \\ \frac{1}{6} \cdot ((3a-5)^2 + 74) & \text{if } 3 \leq a. \end{cases}$$

Hence, since $\sigma = \gamma^{-1} \circ \alpha$, we find that

$$\sigma(t) = \begin{cases} (t+1)^2 - 1 & \text{if } 0 \leq t \leq 3 \\ \frac{1}{6} \cdot ((t+1)^2 + 74) & \text{if } 3 \leq t. \end{cases}$$

Finally, note that the SE strategy σ is indeed continuous, and that, although γ is differentiable, the SE strategy σ is not.

5.5 Application: incentives for truthful reporting

Our results also have implications for mechanism design. Note that T is a subset of S . Also note in the job market signaling application that merely the payment scheme $\alpha(t) = t$ does not induce the truthful report $\sigma(t) = t$, but $\sigma(t) = \frac{1}{2}(t^2 - 1)$ instead.

If we want $\sigma(t) = t$ for all $t \in T$, then the SE condition that $\sigma = (\gamma^*)^{-1} \circ \alpha$ yields $\gamma^*(t) = \alpha(t)$ for all $t \in T$. Hence, by Theorem 8,

Theorem 10 *The model allows truthful reporting in equilibrium precisely when α solves*

$$\alpha'(t) = H(t, \alpha(t)) = c_s(t, t).$$

If we consider the signaling cost function $c(s, t) = \frac{s}{t+1}$, then $c_s(t, t) = \frac{1}{t+1}$, so that the differential equation $\alpha'(t) = \frac{1}{t+1}$ yields $\alpha(t) = \log(t+1)$. Thus, in order to induce truthful reporting, the planner needs to apply the specific reward function $\alpha(t) = \log(t+1)$.

6 Discussion

In the context of one-dimensional signaling games, we characterized the set of separating equilibrium strategies, using techniques known from network theory and mechanism design with non-differentiable utility. In case type space is a closed halfline, we identified conditions under which there exists a unique SE strategy, and showed how to compute the SE strategy by means of a differential equation. We applied these results in several examples, including a generalization of the Spence job market model, and an example with non-differentiable primitives.

Finally we briefly compare the class of models we consider in this paper to Mailath (1987). Consider the class of job market games, where the cost of signaling is separable, so that

$$U_S(a, s, t) = a - h(s) \cdot g(t).$$

Condition [4] of Mailath (1987) states that for every t the equation $h'(s) \cdot g(t) = 0$ should have a unique solution in s . This only holds when $g(t) \neq 0$ and $h'(s) = 0$ has a unique solution. Thus, none of the models where the cost of signaling is linear in s , so that $c(s, t) = s \cdot g(t)$, are covered by the analysis in Mailath (1987).⁹ Also condition [1] of Mailath (1987), stating that

$$U(t, \hat{t}, s) = \alpha(\hat{t}) - h(s) \cdot g(t)$$

is C^2 , does not necessarily hold. When α is continuous, but not C^2 , then clearly U is not C^2 . Thus, for example the application in Sect. 5.3 is not covered by Mailath (1987).

Acknowledgements We gratefully acknowledge the helpful and very detailed comments of three anonymous referees and the associate editor.

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A Appendix: Proof of Theorem 8

This section is devoted to the proof of Theorem 8. Let σ be a continuous and strictly increasing strategy for the sender. Then the function $h(\sigma): T \rightarrow \mathbb{R}$ defined by $h(\sigma)(t) = c_s(\sigma(t), t)$ is continuous in t by assumption [7]. Hence, the Riemann-Stieltjes integral $\int_r^t h(\sigma)(x) d\sigma(x)$ exists for all $r, t \in T$ with $r \neq t$.

Consider a complete digraph $G = (V, E)$ on a finite set V of nodes. A path is a vector $\pi = (v_0, \dots, v_k)$ with $(v_{i-1}, v_i) \in E$ for all $i = 1, \dots, k$.¹⁰ For two nodes $u, v \in V$, a path from u to v is a path $(v_0, \dots, v_k)$ with $v_0 = u$ and $v_k = v$. For u, v with $u < v$, a partition from u to v is a path $(v_0, \dots, v_k)$ from u to v with $v_{i-1} < v_i$ for all $i = 1, \dots, k$. A similar definition applies to the case where $u > v$. For $r, t \in T$, the set of all paths from r to t is denoted by $\Pi(r, t)$, and the set of all partitions from r to t is denoted by $P(r, t)$. Let l be an arc length on G . The length of a path $\pi = (v_0, \dots, v_k)$, denoted by $\text{length}(l)(\pi)$, is defined as

$$\text{length}(l)(\pi) = \sum_{i=0}^{k-1} l(v_i, v_{i+1}).$$

The distance from r to t with respect to the the arc length l_σ is defined by

$$\text{dist}(\sigma)(r, t) = \inf_{\pi \in \Pi(r, t)} \text{length}(l_\sigma)(\pi).$$

⁹ For the same reason the more general results in Mailath and von Thadden (2013) do not apply here, since [4] and [5] are equivalent to assumption 2 in that paper.

¹⁰ Note that, due to the definition of a complete digraph, the requirement $(v_i, v_{i+1}) \in E$ is equivalent to $v_i \neq v_{i+1}$.

Lemma 11 Suppose that σ is strictly increasing and c has decreasing differences. Then for any $r, t \in T$ with $r \neq t$,

$$\text{dist}(\sigma)(r, t) = \inf_{\pi \in P(r, t)} \text{length}(l_\sigma)(\pi).$$

Proof We prove the statement in case $r < t$. The proof for the case where $r > t$ runs along similar lines. Since by definition

$$\text{dist}(\sigma)(r, t) = \inf_{\pi \in \Pi(r, t)} \text{length}(l_\sigma)(\pi).$$

it is obvious that

$$\text{dist}(\sigma)(r, t) \leq \text{length}(l_\sigma)(\pi)$$

for any partition π from r to t . To show the reverse inequality, by definition of $\text{dist}(\sigma)(r, t)$ there exists a path $\pi = (t_0, \dots, t_n)$ in H from r to t such that

$$\text{length}(l_\sigma)(\pi) < \text{dist}(\sigma)(r, t) + \varepsilon.$$

Now suppose there is a k with $t_k < t_{k+1}$ and $t_{k+1} > t_{k+2}$. It follows from monotonicity and decomposition monotonicity that $\text{length}(\rho) \leq \text{length}(\pi)$ for the path

$$\rho = (t_0, \dots, t_k, t_{k+2}, \dots, t_n).$$

Iteration now yields a partition π^* from r to t with $\text{length}(l_\sigma)(\pi^*) \leq \text{length}(l_\sigma)(\pi)$. $\square$

For $\pi \in P(r, t)$, define $L(\pi, h, \sigma)$ by

$$L(\pi, h, \sigma) = \sum_{i=0}^{k-1} n_i \cdot [\sigma(t_{i+1}) - \sigma(t_i)]$$

where $n_i = \min\{c_s(\sigma(x), t_i) \mid t_i \leq x \leq t_{i+1}\}$. Replacement of the minimum by a maximum yields $U(\pi, h, \sigma)$.

Lemma 12 Suppose that σ is continuous and strictly increasing. Then, for any partition π from r to t ,

$$L(\pi, h, \sigma) \leq \text{length}(l_\sigma)(\pi) \leq U(\pi, h, \sigma).$$

Proof We construct the proof for the case where $r < t$. Again, the remaining case where $r > t$ runs along similar lines. Take a partition $\pi = (t_0, \dots, t_k)$ from r to t . Take i fixed. By assumption [6], the map $y \mapsto c_s(y, t_i)$ is continuous on the interval

$[\sigma(t_i), \sigma(t_{i+1})]$. Write $m_i = \min\{c_s(y, t_i) \mid \sigma(t_i) \leq y \leq \sigma(t_{i+1})\}$. Now, since σ is strictly increasing,

$$\begin{aligned} \text{length}(l_\sigma)(\pi) &= \sum_{i=0}^{k-1} l_\sigma(t_i, t_{i+1}) \\ &= \sum_{i=0}^{k-1} [c(\sigma(t_{i+1}), t_i) - c(\sigma(t_i), t_i)] \\ &\geq \sum_{i=0}^{k-1} m_i \cdot [\sigma(t_{i+1}) - \sigma(t_i)]. \end{aligned}$$

On the other hand,

$$L(\pi, h, \sigma) = \sum_{i=0}^{k-1} n_i \cdot [\sigma(t_{i+1}) - \sigma(t_i)]$$

where $n_i = \min\{c_s(\sigma(x), t_i) \mid t_i \leq x \leq t_{i+1}\}$. Since σ is continuous and strictly increasing, it follows that $m_i = n_i$ for all i . Hence, $\text{length}(l_\sigma)(\pi) \geq L(\pi, h, \sigma)$. Similarly, $\text{length}(l_\sigma)(\pi) \leq U(\pi, h, \sigma)$. $\square$

For two partitions $\pi = (t_1, \dots, t_k)$ and $\rho = (r_1, \dots, r_m)$, we write $\pi \vee \rho$ for the (unique) partition $(u_1, \dots, u_p)$ with

$$\{u_1, \dots, u_p\} = \{r_1, \dots, r_m\} \cup \{t_1, \dots, t_k\}.$$

Lemma 13 *Let σ be strictly increasing and continuous. Suppose that c has decreasing differences. Then*

$$\sup_{\pi \in P(r, t)} L(\pi, h, \sigma) \leq \text{dist}(\sigma)(r, t).$$

Proof By Lemma 4, l_σ is decomposition monotone. So, $\text{length}(l_\sigma)(\pi \vee \rho) \leq \text{length}(l_\sigma)(\pi)$ for any $\pi, \rho \in P(r, t)$. Since also $L(\pi, h, \sigma) \leq L(\pi \vee \rho, h, \sigma)$ for any $\pi, \rho \in P(r, t)$, we get the result by Lemma 12. $\square$

Lemma 14 *Let σ be strictly increasing and continuous. Suppose that c has decreasing differences. Then for all $r, t \in T$ with $r \neq t$,*

$$\int_r^t h(\sigma)(x) d\sigma(x) = \text{dist}(\sigma)(r, t).$$

Proof By Lemmas 11 and 12,

$$\int_r^t h(\sigma)(x) d\sigma(x) = \inf_{\pi \in P(r, t)} U(\pi, h, \sigma) \geq \inf_{\pi \in P(r, t)} \text{length}(l_\sigma)(\pi) = \text{dist}(\sigma)(r, t).$$

On the other hand, by Lemma 13,

$$\int_r^t h(\sigma)(x) d\sigma(x) = \sup_{\pi \in P(r,t)} L(\pi, h, \sigma) \leq \text{dist}(\sigma)(r, t).$$

This completes the proof. $\square$

Let f and g be two real-valued functions. Suppose that g is non-decreasing. By $\int_r^t f(x) dg(x)$ we denote the Riemann-Stieltjes integral of f with respect to g on $[r, t]$. It is known that $\int_r^t f(x) dg(x)$ exists whenever f is continuous.¹¹

Lemma 15 *Let σ be an SE strategy. Suppose that c has decreasing differences. Then for all $t \in T$,*

$$\alpha(t) = \int_{\underline{t}}^t h(\sigma)(x) d\sigma(x).$$

Proof Take $r, t \in T$ with $r \neq t$. We show that

$$\alpha(t) - \alpha(r) = \int_r^t h(\sigma)(x) d\sigma(x).$$

The claim then follows by taking $r = \underline{t}$ and using the fact that $\alpha(\underline{t}) = 0$. Take r, t with $r \neq t$. Let $\pi = (t_0, \dots, t_k)$ be a partition from r to t . Then

$$\alpha(t) - \alpha(r) = \sum_{i=0}^{k-1} (\alpha(t_{i+1}) - \alpha(t_i)) \leq \sum_{i=0}^{k-1} l_\sigma(t_i, t_{i+1}) = \text{length}(l_\sigma)(\pi).$$

Hence, by Lemmas 11 and 14

$$\alpha(t) - \alpha(r) \leq \inf_{\pi \in P(r,t)} \text{length}(l_\sigma)(\pi) = \int_r^t h(\sigma)(x) d\sigma(x).$$

This also implies that

$$\alpha(r) - \alpha(t) \leq \int_t^r h(\sigma)(x) d\sigma(x) = - \int_r^t h(\sigma)(x) d\sigma(x)$$

so that

$$\int_r^t h(\sigma)(x) d\sigma(x) \leq \alpha(t) - \alpha(r).$$

This completes the proof. $\square$

¹¹ See Rudin (1976), p.125.

The proof of the following fact is loosely based on a similar proof by Peano.

Theorem 16 *Let $V = \mathbb{R}_+ \times \mathbb{R}_+$. Suppose $H: V \rightarrow \mathbb{R}_{++}$ is continuous and bounded. Then there exists at least one solution of the initial value problem*

$$f'(s) = H(s, f(s)) \text{ with } f(0) = 0.$$

Moreover, any solution is continuously differentiable.

Proof For $n, k \in \mathbb{N}$ define $s_{n,k} = k \cdot 2^{-n}$. Now take $n \in \mathbb{N}$ fixed for the moment. We set $x_{n,0} = 0$, and for $k \in \mathbb{N}$

$$x_{n,k+1} = x_{n,k} + 2^{-n} \cdot H(s_{n,k}, x_{n,k}).$$

Define $f_n: \mathbb{R}_+ \rightarrow \mathbb{R}$ by, for $s \in [s_{n,k}, s_{n,k+1})$,

$$f_n(s) = x_{n,k} + 2^n \cdot (s - s_k) \cdot (x_{n,k+1} - x_{n,k}).$$

Let $M > 0$ be such that $|H(s, a)| \leq M$ for all $(s, a) \in V$. First we show for all $s, s' \in \mathbb{R}_+$ that

$$|f_n(s') - f_n(s)| \leq M \cdot |s' - s|.$$

Take k, m such that $s \in [s_{n,k}, s_{n,k+1})$ and $s' \in [s_{n,m}, s_{n,m+1})$. Then $f_n(s)$ is a linear interpolation between $x_{n,k}$ and $x_{n,k+1}$, and $f_n(s')$ is a linear interpolation between $x_{n,m}$ and $x_{n,m+1}$. Therefore it suffices to prove that

$$|x_{n,k+i} - x_{n,k}| \leq M \cdot |s_{n,k+i} - s_{n,k}|.$$

This however immediately follows from the observation that

$$\begin{aligned} |x_{n,k+1} - x_{n,k}| &= |2^{-n} \cdot H(s_{n,k}, x_{n,k})| \\ &\leq 2^{-n} \cdot M \\ &= M \cdot |s_{n,k+1} - s_{n,k}|. \end{aligned}$$

Now write

$$D = \{k \cdot 2^{-n} \mid k, n \in \mathbb{N}\}.$$

Obviously D is a countable set. Let $d_1, d_2, \dots$ be an enumeration of D . Consider $d_1 \in D$. Then for all n we have $|f_n(d_1)| \leq M \cdot d_1$. So, there is a subsequence $(f_n^1)_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ for which $(f_n^1(d_1))_{n=1}^\infty$ is convergent. Iteratively for $k = 1, 2, \dots$ we can take a subsequence $(f_n^k)_{n=1}^\infty$ of $(f_n^{k-1})_{n=1}^\infty$ for which $(f_n^k(d_l))_{n=1}^\infty$ is convergent for all $l = 1, \dots, k$. Then for the sequence $(f_k^k)_{k=1}^\infty$ we have that $(f_k^k(d))_{k=1}^\infty$ is convergent for all $d \in D$.

Write $g_k = f_k^k$. Take any $s \geq 0$. We argue that $(g_k(s))_{k=1}^\infty$ is Cauchy. Take $\varepsilon > 0$. Since D is dense in $\mathbb{R}_+$, we can take a $d \in D$ with $|s - d| < \frac{\varepsilon}{3M}$. Further, since $(g_k(d))_{k=1}^\infty$ is convergent, we can take $K > 0$ such that

$$|g_k(d) - g_m(d)| < \frac{\varepsilon}{3}$$

whenever $k, m > K$. Then, for $k, m > K$,

$$\begin{aligned} |g_k(s) - g_m(s)| &= |g_k(s) - g_k(d) + g_k(d) - g_m(d) + g_m(d) - g_m(s)| \\ &\leq |g_k(s) - g_k(d)| + |g_k(d) - g_m(d)| + |g_m(d) - g_m(s)| \\ &< M \cdot |s - d| + \frac{\varepsilon}{3} + M \cdot |s - d| \\ &\leq M \cdot \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + M \cdot \frac{\varepsilon}{3M} = \varepsilon. \end{aligned}$$

Define $f(s) = \lim_{n \rightarrow \infty} g_n(s)$. We claim that f is continuous. Take any $s, s' \geq 0$. Since $(g_n)_{n=1}^\infty$ is a subsequence of $(f_n)_{n=1}^\infty$, we know from the argument above that

$$|g_n(s') - g_n(s)| \leq M \cdot |s' - s|$$

for all n . The claim now follows by taking limits for $n \rightarrow \infty$.

So, the function $s \mapsto H(s, f(s))$ is continuous. Then it is integrable on bounded intervals. Take any $s \geq 0$. We show that

$$f(s) = \int_0^s H(u, f(u)) du.$$

Take $n \in \mathbb{N}$ fixed. Let $p(n)$ be the natural number such that $g_n = f_n^n = f_{p(n)}$. Define

$$h_n(s) = H(s_{p(n),k}, x_{p(n),k}) \quad \text{if } s \in [s_{p(n),k}, s_{p(n),k+1}).$$

Then clearly $g_n(s) = \int_0^s h_n(u) dy$. We argue that $h_n(s) \rightarrow H(s, f(s))$ as $n \rightarrow \infty$. First note that

$$h_n(s_{p(n),k}) = H(s_{p(n),k}, x_{p(n),k}) = H(s_{p(n),k}, f_{p(n)}(s_{p(n),k})) = H(s_{p(n),k}, g_n(s_{p(n),k})).$$

Take $s \geq 0$ arbitrary. Choose $k(n)$ such that $s \in [s_{p(n),k(n)}, s_{p(n),k(n)+1})$. Then

$$h_n(s) = H(s_{p(n),k(n)}, x_{p(n),k(n)}) = H(s_{p(n),k(n)}, g_n(s_{p(n),k(n)})).$$

Since clearly $p(n) \rightarrow \infty$ as $n \rightarrow \infty$, we know that $s_{p(n),k(n)} \rightarrow s$ as $n \rightarrow \infty$. Therefore it suffices to show that

$$g_n(s_{p(n),k(n)}) \rightarrow f(s).$$

This however follows from the observations that $g_n(s) \rightarrow f(s)$ and $|g_n(s') - g_n(s)| \leq M \cdot |s' - s|$.

Hence, since $|h_n(s)| \leq M$ for all s and n , the theorem of bounded convergence yields

$$f(s) = \lim_{n \rightarrow \infty} g_n(s) = \lim_{n \rightarrow \infty} \int_0^s h_n(u) du = \int_0^s H(u, f(u)) du.$$

This completes the proof. $\square$

Now we are ready to show Theorem 17.

Theorem 17 *Let $B \subseteq A$ be an initial segment. Suppose that $H: S \times B \rightarrow \mathbb{R}_{++}$ is continuous, bounded, and non-increasing in the second coordinate. Then the initial value problem*

$$f'(s) = H(s, f(s)) \text{ with } f(0) = 0$$

has a unique solution f^ on initial segment $I \subseteq B$. Moreover, for any partial solution g on J it holds that $J \subseteq I$, and $f^*(s) = g(s)$ for all $s \in J$.*

Proof Using Theorem 16, we can show that there exists a solution f on I of the initial value problem.¹² Suppose that g is a partial solution on J . It suffices to show that $J \subseteq I$, and that $f(s) = g(s)$ for all $s \in I$.

Since I and J are initial segments, either $I \subseteq J$, or $J \subseteq I$. Define $K = I \cap J$. For $s \in K$, define $h(s) = f(s) - g(s)$. It suffices to show that $h(s) = 0$ for all $s \in K$. Suppose that $h(s) > 0$ for some $s \in K$. Since $h(0) = 0$, obviously $s > 0$. Define

$$s^* = \inf\{x \in K \mid 0 \leq x \leq s \text{ and } h(y) > 0 \text{ for all } y \in (x, s)\}.$$

Since h is continuous, $h(s) > 0$, and $h(0) = 0$, we know that $s^* < s$ and $h(s^*) = 0$. So, since h is continuously differentiable, by the mean value theorem there exists $\tau \in (s^*, s)$ with

$$h'(\tau) = \frac{h(s) - h(s^*)}{s - s^*} = \frac{h(s)}{s - s^*} > 0.$$

However, $\tau \in (s^*, s)$ implies that $h(\tau) > 0$ by definition of s^* . Then $f(\tau) > g(\tau)$. So, since H is non-increasing in the second argument, $f'(\tau) = H(\tau, f(\tau)) \leq H(\tau, g(\tau)) = g'(\tau)$. Hence, $h'(\tau) \leq 0$. Contradiction. $\square$

¹² In order to apply Theorem 16, we need to extend H from $S \times B$ to $S \times A$. If B is a closed interval, this is done by projection of the second coordinate onto B . If B is a half-open interval $B = [0, b)$, we choose $c \in B$, apply the previous construction to $[0, c]$, and then let c approach b .

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