Catch games: the impact of modeling decisions

János Flesch¹ · Dries Vermeulen¹ · Anna Zseleva²

Accepted: 29 September 2018 / Published online: 19 October 2018
© The Author(s) 2018

Abstract

We examine the guarantee levels of the players in a type of zero sum games. We show how these levels depend on the sigma algebras that are being employed on the players' action spaces. We further argue that guarantee levels may therefore also depend on set theoretic considerations. Additionally, we calculate the guarantee levels for finitely additive strategies. The solutions of catch games essentially differ among these setups. We find optimal strategies for almost all cases.

Keywords Infinite games · Two-person zero sum games · Countably additive strategies · Sigma algebras · Set theory · Finitely additive strategies

1 Introduction

This paper is a study of catch games, a class of two-person zero sum games with infinite action spaces. We use this class of games to highlight the effects of modeling decisions on the guarantee levels of the players, and hence on the existence of the value of the game. Catch games and similar games are used for example to compute optimal strategies in games of pursuit, or to study tactics of evasion. Besides that, modeling

We would like to thank Miklós Pintér and William Sudderth for their helpful comments, and the Dutch institute NWO (Grant no. 040.11.495) for providing the financial support for William's visit to Maastricht. We also thank an associate editor and two anonymous reviewers for their useful suggestions. Anna Zseleva: Support from the Basic Research Program of the National Research University Higher School of Economics is gratefully acknowledged.

✉ Dries Vermeulen
d.vermeulen@maastrichtuniversity.nl

János Flesch
j.flesch@maastrichtuniversity.nl

Anna Zseleva
zseleva.anna@gmail.com

¹ Maastricht University, Maastricht, The Netherlands

² Higher School of Economics, St. Petersburg, Russia

questions similar to the ones in catch games can arise in any economic model where the players' action spaces are assumed to be infinite.

Catch games A catch game is a game played by two players. Player 1 chooses an action of a given infinite set A of actions. An action $a \in A$ is interpreted as a location for player 1 to hide. Player 2 simultaneously selects a finite set $b \subseteq A$ of possible locations. The interpretation is that player 2 checks finitely many possible locations for the presence of player 1. The collection of actions of player 2 is the finite subsets of A which we denote by B . Player 1 wins the game if his choice $a \in A$ of location is not an element of the finite set $b \in B$ of locations that player 2 selects to check. Thus, player 1 wins if $a \notin b$. Otherwise, when $a \in b$, player 1 gets caught, and player 2 wins the game. Notice that the game is entirely specified by the choice of the action set A of player 1.

Players are assumed to use mixed strategies. In order to model this assumption, each of the sets A and B is endowed with a sigma algebra, $\mathcal{F}(A)$ and $\mathcal{F}(B)$, respectively. Player 1 selects an element of the set of probability measures on $\mathcal{F}(A)$, and player 2 selects an element of the set of probability measures on $\mathcal{F}(B)$. Here in the main model description, we assume that these probability measures are countably additive, but later we will also consider the case when probabilities are only finitely additive. The result of these choices is evaluated by means of computation of expected payoffs, which is identical to the probability of success, with respect to outer measure.¹ The main questions in our paper revolve around the possible set of mixed strategies and the corresponding guarantee levels, which are calculated through the expected payoffs.

Optimistic versus pessimistic evaluation In this context, we consider two possible attitudes for each player. An optimistic player 1 maximizes the probability that $a \notin b$, i.e. that player 1 avoids player 2. A pessimistic player 1 minimizes the probability that $a \in b$, i.e. that player 2 catches player 1. Similarly, an optimistic player 2 maximizes the probability that $a \in b$, i.e. that player 2 catches player 1. A pessimistic player 2 minimizes the probability that $a \notin b$, i.e. that player 1 avoids player 2.

Since we compute outer measures, these two attitudes of a player are not identical. It may happen that in a catch game an optimistic player 1 thinks he can successfully, with probability 1, avoid player 2, while in the same catch game a pessimistic player 1 thinks that, with probability 1, player 2 is able to catch player 1.

The central results.² Our first central result, Theorem 1, states that player 2, both in the optimistic and the pessimistic case, thinks that player 1 will win the game. Our central result for player 1, Theorem 2, considers three exhaustive and mutually exclusive cases. For two out of three cases we have conclusive answers, and in one case we only have a partial result.

Case [1]. In this case $\mathcal{F}(A)$ admits only trivial probability measures.³ Player 1, both in the optimistic and in the pessimistic case, thinks that player 2 will win the game. Consequently, the game does not have a value.

¹ In measure theory outer approximations are often preferred to inner approximations. For example, recall the construction of the Lebesgue measure on the real line.

² We elaborate on the precise formal conditions in Sects. 3 and 4.

³ A probability measure is called trivial if there is at least one singleton set with non-zero probability.

Case [2]. In this case, there exists a non-trivial probability measure on $\mathcal{F}(A)$ that does not have any atoms.⁴ Player 1, both in the optimistic and in the pessimistic case, thinks that player 1 will win the game. Consequently, the value of the game is 1.

Case [3]. In this case there exist non-trivial probability measures on $\mathcal{F}(A)$, but each such measure has an atom. The optimistic player 1 thinks that he will win the game, so that the value of the game is 1.

On the remaining situation, case [3], and player 1 being pessimistic, we only have a partial result, Theorem 3. We have an example of sigma algebras $\mathcal{F}(A)$ and $\mathcal{F}(B)$ that fall within the context of case [3] in which a pessimistic player 1 thinks that player 2 will win the game. Thus, here a pessimistic player 1 disagrees with the optimistic player 1.

Among the three cases and the two different attitudes not only the guarantee levels of player 1 vary, but also his sets of optimal strategies are quite different.

The consequences The modeling decisions we study affect the collection of mixed strategies available to the players, and hence have influence on which case is realized. The decisions we vary are as follows.

- [1] The choice of sigma algebra on which mixed strategies are defined.
- [2] Optimistic versus pessimistic players.
- [3] The set theoretic setting in which we execute our analysis.

Each of these decisions determines in which of the three above cases we end up. Thus, the guarantee levels change when we change our modeling decision. Apart from these modeling decisions, which take place in the setting of countably additive strategies, we also analyze the effect on the guarantee levels of the players when the players use finitely additive strategies instead of countably additive strategies. Thus, the fourth modeling decision we vary is as follows.

- [4] Countably additive strategies versus finitely additive strategies.

We first briefly discuss the possible effects of each modeling decision, under the assumption that players use countably additive strategies.

- [1] Different sigma algebras may allow different types of strategies. For example, in the partial result in case [3], the only non-trivial probability measures that the sigma algebra $\mathcal{F}(A)$ allows are atomic. In this case, a pessimistic player 1 thinks player 2 wins the game. However, in the case where $A = [0, 1]$, we can extend $\mathcal{F}(A)$ to the Borel sigma algebra $\mathcal{B}(A)$. Since the Lebesgue measure is an atomless probability measure on $\mathcal{B}(A)$, that would put us in case [2], where a pessimistic player 1 thinks he wins the game.
- [2] The attitude (pessimistic or optimistic) of player 1 affects the guarantee level of player 1, and thus the existence of the value, in case [3].
- [3] The set theoretic setting may also affect which case we end up in. For each of the three cases, we have a collection of set theoretic axioms that force the game to be in that specific case. We will provide an example where the action set $A = [0, 1]$

⁴ That is, under this probability measure all singletons have probability 0, and moreover for every measurable set with non-zero probability, there is a proper subset with a smaller non-zero probability.

and the sigma algebra $\mathcal{F}(A)$ is its power set. This is a rather natural setting and even so, player 1's guarantee levels can be quite different, 0 or 1, depending on the assumed set theoretic axioms (see Sect. 5).

- [4] Finally, we study the case where players use finitely additive probabilities as mixed strategies, under the assumption that both have the power set as sigma algebra. Both the guarantee level of a pessimistic player 1 and the guarantee level of an optimistic player 2 are equal to zero. However, the guarantee levels of an optimistic player 1 and a pessimistic player 2 are both equal to one. Thus, when one player is optimistic, and the other is pessimistic, the value of the game exists. Evidently, these results are in stark contrast to the countably additive case discussed earlier (see Sect. 6).

Related literature Instances of catch games have previously been considered—mainly as illustrative examples—in other papers as well, see Capraro and Scarsini (2013), Pivato (2014) and Flesch et al. (2017). A related line of literature is on hide-and-seek games, which often have a dynamic flavor where the seeker has to develop a search strategy to detect the hider. Applications of hide-and-seek games can be found in for example Crawford and Iriberri (2007) and Rubinstein et al. (1996). Other classes of related games are for example search games (Isaacs 1965), Stackelberg security games (Kiekintveld et al. 2009) and patrolling games (Abaffy et al. 2014).

Interplay between set theory and game theory Our paper is not the first paper where game theory and set theory interact. Set theory plays an important role in several game theoretic papers. For example, Prikry and Sudderth (2016) derive properties of the guarantee levels in certain zero sum games from descriptive set theory. Zame (2007) also investigated the relationship between economic theory and set theory, when he showed that the existence of an ethical preference relation is independent of a certain collection of set theoretic axioms. An ethical preference relation is a formalisation of equal regard of all individuals in an intergenerational model. This model can also be interpreted as a repeated game. Peters and Vermeulen (2012) show that the set of bargaining solutions satisfying a certain collection of properties differs when assuming the Axiom of Determinacy compared to assuming the Axiom of Choice.

Conversely, there is a line of research where game theory is used for set theoretic purposes. For example, the determinacy of Gale-Stewart games (Gale and Stewart 1953) has implications in set theory, see Kechris (1995), p. 138–139 and Martin (1975). Similarly, topological games such as Banach-Mazur games, Oxtoby (1980), p. 27, also have applications in set theory.

Finite additivity It is not immediately clear how probabilities on an infinite space should be defined. The usual approach is to use countably additive probability measures, as we do in most of the paper. A respectable, but lesser used alternative is to use finitely additive probabilities. Many contributions to decision theory, including de Finetti (1975), Savage (1972), Dubins and Savage (2014) have vehemently argued in favor of finite additivity. More on the history and use of this approach can be found in Bingham (2010).

For the effects of finite additivity on the value of two-person zero sum games, see Maitra and Sudderth (1993), Maitra and Sudderth (1998), Yanovskaya (1970),

Kindler (1983) and Schervish and Seidenfeld (1996). For similar studies on non-zero sum games, see Marinacci (1997), Harris et al. (2005) and Capraro and Scarsini (2013). **Organization of the paper** In Sect. 2 we discuss some preliminaries. In Sect. 3 we define catch games and the solution concepts that we study. In Sect. 4, we present our main results. The proofs are given in “Appendix A”. In Sect. 5 we show how set theoretic assumptions affect our results, and provide some examples to illustrate the different cases in our main theorem. In Sect. 6 we report our results for the finitely additive setup, and compare those to the countably additive setup. In Sect. 7 we briefly argue that our requirements on the sigma algebras $\mathcal{F}(A)$ and $\mathcal{F}(B)$ are fairly mild.

2 Preliminaries

For a set A , we write $\mathcal{P}(A)$ to denote the collection of all subsets of A . For a finite set A , $|A|$ denotes the number of elements of A . We write $\mathbb{N} = \{1, 2, \dots\}$.

Let S be a non-empty set and let $\mathcal{F}$ denote a sigma algebra on S . A function $p: \mathcal{F} \rightarrow [0, \infty)$ is called countably additive if for all collections $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{F}$ it holds that

$$p\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} p(E_i).$$

A measure on S is a countably additive function p such that $p(S) > 0$. A measure p is called a probability measure if $p(S) = 1$. We denote by $\Delta(S)$ the set of all probability measures on S . For $s \in S$, the Dirac-measure $\delta(s)$ on S is defined by, for all $A \in \mathcal{F}$,

$$\delta(s)(A) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{otherwise.} \end{cases}$$

Let $p \in \Delta(S)$. A set $A \in \mathcal{F}$ is called an atom of p if

- [1] $p(A) > 0$, and
- [2] there is no set $B \in \mathcal{F}$ such that $B \subset A$ with $0 < p(B) < p(A)$.

If a measure has no atoms, then it is called atomless. Suppose that $\{s\} \in \mathcal{F}$ for every $s \in S$. A measure p is called non-trivial if $p(\{s\}) = 0$ for every $s \in S$. Note that an atomless measure is non-trivial.

Consider two non-empty sets S and T . Let $\mathcal{F}(S)$ be a sigma algebra on S and let $\mathcal{F}(T)$ be a sigma algebra on T . A set $Z \subseteq S \times T$ is called a rectangle if there are sets $A \in \mathcal{F}(S)$ and $B \in \mathcal{F}(T)$ such that $Z = A \times B$. We write $Z_1 = A$ and $Z_2 = B$.

A collection $\{Z^i\}_{i=1}^{\infty}$ is a cover of $X \subseteq S \times T$ if

- [1] for each $i \in \mathbb{N}$, the set Z^i is a rectangle in $S \times T$, and
- [2] $X \subseteq \bigcup_{i=1}^{\infty} Z^i$.

Let $p_1 \in \Delta(S)$, $p_2 \in \Delta(T)$ and $X \subseteq S \times T$. The outer measure $\langle p_1, p_2 \rangle(X)$ of X with respect to (p_1, p_2) is defined by

$$\langle p_1, p_2 \rangle(X) = \inf \left\{ \sum_{i=1}^{\infty} p_1(Z_1^i) \cdot p_2(Z_2^i) : \{Z^i\}_{i=1}^{\infty} \text{ is a cover of } X \right\}.$$

Note that for a rectangle $Z \subseteq S \times T$ we have $\langle p_1, p_2 \rangle(Z) = p_1(Z_1) \cdot p_2(Z_2)$.

3 Catch games

This paper is a study of catch games. A catch game is defined as follows. Let A be an infinite set, and let B be the set consisting of all finite subsets of A . Consider the zero sum game in which player 1 chooses an action a from A , and simultaneously, player 2 chooses an action b from B . The payoff is 0 if $a \in b$, i.e. when player 2 catches player 1, and 1 if $a \notin b$. Player 2 pays the payoff to player 1. Thus, the winning set of player 1 in a catch game is

$$W^1 = \{(a, b) \in A \times B \mid a \notin b\}.$$

Similarly $W^2 = \{(a, b) \in A \times B \mid a \in b\}$ is the winning set for player 2.⁵

Sigma algebras We endow A and B with sigma algebras $\mathcal{F}(A)$ and $\mathcal{F}(B)$, respectively. For our results we need various minimum requirements on $\mathcal{F}(A)$ and $\mathcal{F}(B)$. Therefore we first briefly discuss the requirements we need. For the interested reader, in Sect. 7 we discuss our requirements on the sigma algebras $\mathcal{F}(A)$ and $\mathcal{F}(B)$ in more detail, and argue that our conditions are relatively mild, in the sense that they allow for many different sigma algebras.

Condition [A] For every $a \in A$, $\{a\} \in \mathcal{F}(A)$.

We explain the condition on $\mathcal{F}(B)$ we use in the paper. For any $m \in \mathbb{N}$, let B_m be the set of actions b for player 2 with $|b| = m$. Let I be a finite subset of A , and let $m \in \mathbb{N}$. We define

$$Q(I, m) = \{b \in B_m \mid I \text{ is a subset of } b\}.$$

Condition [B] For any choice of I and m , $Q(I, m) \in \mathcal{F}(B)$.

Condition [B] has the following basic, but useful, consequences.

- [B1] For every m , $B_m \in \mathcal{F}(B)$. To see this, take $I = \emptyset$.
- [B2] For every $a \in A$, $B(a) = \{b \in B \mid a \in b\} \in \mathcal{F}(B)$. To see this, take $I = \{a\}$ and any $m \in \mathbb{N}$. Then $Q(I, m) = \{b \in B_m \mid a \in b\}$. Hence, $B(a)$ is the union over all m of the sets $Q(I, m)$.
- [B3] For every $c \in B$, $\{c\} \in \mathcal{F}(B)$. To see this, take $I = c$ and $m = |c|$. Then, for an arbitrary $b \in B$, we have that $b \in Q(I, m)$ precisely when b is a set with $|c|$ elements that contains c . This implies that $b = c$, so that $Q(I, m) = \{c\}$. Hence, $\{c\} \in \mathcal{F}(B)$.

⁵ When A is finite, the whole set A is an action for player 2, and he can win by choosing this action. Therefore we restrict our attention to cases where A is infinite.

Conditions [A] and [B] are assumed throughout this paper. Therefore, we do not mention their validity explicitly. Notice that [B1] and [B2] together are equivalent to [B], as $Q(\phi, m) = B_m$, and

$$Q(I, m) = B_m \cap \left[\bigcap_{a \in I} B(a) \right].$$

We sometimes need a stronger condition on $\mathcal{F}(B)$. Let D be any subset of A , and let $k \in \mathbb{N} \cup \{0\}$. We define

$$Y(D, k) = \{b \in B \mid D \cap b \text{ has exactly } k \text{ elements} \}.$$

Condition [C] For any choice of $D \in \mathcal{F}(A)$, and any k , $Y(D, k) \in \mathcal{F}(B)$.

Notice that condition [C] implies condition [B]. To see this, first notice that $Y(A, m) = B_m$. Hence, for any finite I and $m \in \mathbb{N}$, $Q(I, m) = Y(I, |I|) \cap Y(A, m)$.

Guarantee levels We examine 4 different guarantee levels for a catch game. Let $\mathcal{F}(A)$ and $\mathcal{F}(B)$ be fixed. The set of mixed strategies for player 1 is $\Delta(A)$, and the set of mixed strategies for player 2 is $\Delta(B)$. Write $p = \langle p_1, p_2 \rangle$ for the outer measure defined by p_1 and p_2 . We define

$$v_1^o := \sup_{p_1 \in \Delta(A)} \inf_{p_2 \in \Delta(B)} p(W^1) \quad (1)$$

$$v_1^p := \sup_{p_1 \in \Delta(A)} \inf_{p_2 \in \Delta(B)} [1 - p(W^2)] \quad (2)$$

$$v_2^p := \inf_{p_2 \in \Delta(B)} \sup_{p_1 \in \Delta(A)} p(W^1) \quad (3)$$

$$v_2^o := \inf_{p_2 \in \Delta(B)} \sup_{p_1 \in \Delta(A)} [1 - p(W^2)]. \quad (4)$$

In the definition of these guarantee levels the superscript p stands for pessimistic and superscript o stands for optimistic.⁶ The subscript in the notation for these levels indicates which player is under consideration. Thus subscript 1 indicates guarantee levels for the payoff to player 1, and subscript 2 indicates guarantee levels for the payoff of player 2.

Remarks A few comments on these definitions are in order here.

[1] Notice that $W^1 \cup W^2 = A \times B$. So, it follows immediately from the definition of outer measure that

$$\langle p_1, p_2 \rangle(W^1) + \langle p_1, p_2 \rangle(W^2) \geq 1$$

for any $p_1 \in \Delta(A)$ and $p_2 \in \Delta(B)$. Hence, $v_1^o \geq v_1^p$ and $v_2^p \geq v_2^o$.

⁶ The guarantee level v_1^o is considered to be a more optimistic assessment by player 1 than v_1^p because $v_1^o \geq v_1^p$.

- [2] Let $\mathcal{F}(A \times B)$ denote the sigma algebra induced by the rectangles of the product space $A \times B$. Based on Carathéodory's extension theorem, any two probability measures $p_1 \in \Delta(A)$ and $p_2 \in \Delta(B)$ generate a unique probability measure p over elements of $\mathcal{F}(A \times B)$. Hence, if $W^1 \in \mathcal{F}(A \times B)$, then $p(W^1) + p(W^2) = 1$. In that case, $v_1^p = v_1^o$ and $v_2^p = v_2^o$.
- [3] When player 1 is optimistic and player 2 is pessimistic, the resulting game is zero sum, just like the original game (in pure strategies). Naturally, this also holds when player 1 is pessimistic and player 2 is optimistic. However, when both players are optimistic or both are pessimistic, the game in mixed strategies need not be zero sum anymore.

Optimal strategies A strategy $p_1 \in \Delta(A)$ is optimistic optimal for player 1 if

$$\langle p_1, p_2 \rangle (W^1) \geq v_1^o$$

for all $p_2 \in \Delta(B)$. Similarly, p_1 is pessimistic optimal for player 1 if

$$1 - \langle p_1, p_2 \rangle (W^2) \geq v_1^p$$

for all $p_2 \in \Delta(B)$. For player 2, a strategy $p_2 \in \Delta(B)$ is pessimistic optimal if for all strategies $p_1 \in \Delta(A)$ it holds that $\langle p_1, p_2 \rangle (W^1) \leq v_2^p$. Strategy p_2 is optimistic optimal for player 2 if for all strategies $p_1 \in \Delta(A)$ it holds that $1 - \langle p_1, p_2 \rangle (W^2) \leq v_2^o$.

4 The main results

In this section we present our main results. The proofs of the main results, Theorems 1, 2, and 3, can be found in Appendix A. As a reminder we note that throughout this section we assume that $\mathcal{F}(A)$ satisfies condition [A], and $\mathcal{F}(B)$ satisfies condition [B]. The first result states that the guarantee levels for player 2 are always equal to 1.

Theorem 1 *The guarantee levels for player 2 are $v_2^p = v_2^o = 1$. In both cases (optimistic and pessimistic), any strategy of player 2 is optimal.*

The analysis of the guarantee levels for player 1 requires more effort. We distinguish three cases, in which the levels for player 1 may differ. It is peculiar that the pessimistic and optimistic approach can make a difference only in one case.

Theorem 2 *For player 1, we distinguish three exhaustive but mutually exclusive settings.*

- [1] *Suppose there exists no non-trivial measure on $\mathcal{F}(A)$. Then $v_1^p = v_1^o = 0$. In both cases (optimistic and pessimistic), any strategy of player 1 is optimal.*
- [2] *Suppose there exists an atomless measure on $\mathcal{F}(A)$. Assume condition [C]. Then $v_1^p = v_1^o = 1$. In both cases (optimistic and pessimistic), any atomless probability measure is optimal for player 1.*

- [3] Suppose there is a non-trivial measure on $\mathcal{F}(A)$, but each non-trivial measure has an atom. Then $v_1^o = 1$. Moreover, any non-trivial $\{0, 1\}$ -valued probability measure on A is optimistic optimal for player 1.

The following table summarizes the results for player 1's guarantee levels in Theorem 2:

		v_1^p	v_1^o
Case [1]	No non-trivial measure on $\mathcal{F}(A)$	0	0
Case [2]	Atomless measure on $\mathcal{F}(A)$	1	1
Case [3]	Non-trivial measure on $\mathcal{F}(A)$, and each has an atom	?	1

Thus, the guarantee levels of player 1 depend on what type of (probability) measures player 1's action space A admits. For each case, we present an example. In the first two examples we do not specify the sigma algebra $\mathcal{F}(B)$, we only require that the conditions from Sect. 3 are satisfied. In Example 3 we need an additional requirement on $\mathcal{F}(B)$.

Example 1 Take $A = \mathbb{N}$ and $\mathcal{F}(A) = \mathcal{P}(\mathbb{N})$. Since $A = \mathbb{N}$ is countable, $\mathcal{F}(A)$ does not admit a non-trivial measure. So, this example belongs to case [1]. Hence, $v_1^p = v_1^o = 0$.

Example 2 Take $A = [0, 1]$. Let $\mathcal{F}(A) = \mathcal{M}$ be the collection of Lebesgue measurable subsets of $[0, 1]$. Then, the Lebesgue measure is an atomless measure on $\mathcal{M}$. So, this example belongs to case [2]. Hence, $v_1^p = v_1^o = 1$.

Example 3 Take $A = [0, 1]$. Let $\mathcal{F}(A)$ consist of all subsets of $A = [0, 1]$ that are at most countable, together with their complements. In Lemma 9 it is shown that this example falls under case [3]. Hence, $v_1^o = 1$, and the unique non-trivial probability measure on $\mathcal{F}(A)$ is the only optimistic optimal strategy for player 1.

Remark The value of a game exists when the guarantee levels for the two players coincide. In finite games the value always exists, but for infinite games this is not necessarily so. Since in catch games the guarantee level for player 2 is always 1, the value exists when player 1's guarantee level is also 1. So the value exists in a catch game in case [2], and in case [3] when player 1 is optimistic.

In general, the guarantee level of v_1^p for a pessimistic player 1 remains an open problem in case [3]. We briefly discuss a partial result. We assume that A is uncountable.⁷ The sigma algebra $\mathcal{F}(A)$ on A that consists of all subsets of A that are at most countable, together with their complements is called the minimal sigma algebra on A . Note that the minimal sigma algebra on A satisfies condition [A].

For the pessimistic case, we have the following partial result.

Theorem 3 Let A be uncountable. Let $\mathcal{F}(A)$ be the minimal sigma algebra on A . Let $\mathcal{F}(B)$ be the smallest sigma algebra that satisfies condition [B]. Then $v_1^p = 0$. Moreover, any strategy is pessimistic optimal for player 1.

⁷ Note that for countable sets A we are automatically in case [1].

We conclude that in case [3] the optimistic and pessimistic guarantee levels for player 1 may differ.

We conjecture that $v_1^p = 0$ is valid in general in case [3]. In fact, determining v_1^p is much more challenging than that of v_1^o . When we prove in case [3] that $v_1^o = 1$, we focus on player 1 and construct a very specific strategy q_1 that works against any strategy of player 2: $\langle q_1, p_2 \rangle (W^1) = 1$ for every $p_2 \in \Delta(B)$. However, assuming that our conjecture is correct and $v_1^p = 0$, for every strategy $p_1 \in \Delta(A)$ of player 1, we would now have to construct a strategy $p_2 \in \Delta(B)$ for player 2 such that $\langle p_1, p_2 \rangle (W^2) = 1$, or at least very close to 1. Since the action set of player 2 is quite different from that of player 1, we cannot mimic the idea of the construction of q_1 . We could also consider generalizing the proof method of our partial result in Theorem 3. However, given an arbitrary strategy of player 1 it is difficult to construct a strategy for player 2 such that he catches player 1 with a high probability. For this reason we could only prove this result under specific conditions.

5 Consequences from set theory

In this section we discuss the consequences of set theory on player 1's guarantee levels in our main theorem. In Theorem 2 we distinguished three exhaustive but mutually exclusive cases depending on what type of (probability) measures player 1's action set A admits. Our previous examples showed that different choices for $\mathcal{F}(A)$ may change the case we end up in.

In this section we fix the choice of sigma algebra for player 1 to $\mathcal{F}(A) = \mathcal{P}(A)$. Thus, the case distinctions now boil down to the question of existence of certain types of probability measures defined on the entire power set of A . For each case, we present well-known axioms from set theory that force our model to be in that specific case. Thus in this section under cases [A], [B] and [C] we mention set theoretic axioms that force the framework into cases [1], [2] and [3] respectively. The definitions of the various axioms can be found in Appendix C. In Sect. 5.4 we provide an example to illustrate our results in this section.

5.1 Case [A]

In case [A] we assume the Zermelo-Fraenkel axioms ZF, plus the axiom of constructability $V = L$. We show that under these conditions, $\mathcal{P}(A)$ belongs to case [1] for any set A . It can be shown that, if ZF is consistent,⁸ then ZF plus " $V = L$ " is consistent.

Theorem 4 *Assume the axiom of constructability $V = L$. Then there exists no non-trivial measure on $\mathcal{P}(A)$. Hence*

$$v_1^p = v_1^o = 0.$$

⁸ Intuitively speaking, consistency of an axiom system means that it is without contradiction.

Proof Let A be any infinite set. We show that there exists no non-trivial measure on $\mathcal{P}(A)$. Suppose the opposite. Take the smallest cardinal κ on which a non-trivial measure μ exists. Thus, $\kappa \leq \text{card}(A)$.

Suppose first that $\kappa > 2^{\aleph_0}$. By corollary 10.7 in Jech (2006), κ is a real-valued measurable cardinal. Hence, by corollary 10.10 in Jech (2006), κ is simply a measurable cardinal. This is however impossible in view of Scott's theorem (Jech 2006, Theorem 17.1), which states that under the assumption of the axiom of constructability $V = L$, there is no measurable cardinal.

So, $\kappa \leq 2^{\aleph_0}$. Then, the measure μ can be extended to a non-trivial measure $\bar{\mu}$ on $2^{\aleph_0}$. This is however impossible. Based on Gödel's theorem (Jech 2006, Theorem 13.20) under the assumption of the axiom of constructability $V = L$, the generalized continuum hypothesis holds: $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α . Banach and Kuratowski showed that under the assumption of the continuum hypothesis $2^{\aleph_0} = \aleph_1$, there exists no non-trivial measure on the continuum $2^{\aleph_0}$, i.e. on any set having the same cardinality as $[0, 1]$ (Jech 2006, pp. 133 and 138).

We conclude that there exists no non-trivial measure on $\mathcal{P}(A)$. Thus, we are in case [1] of Theorem 2. Hence, $v_1^p = v_1^o = 0$. $\square$

Remark We briefly mention that $\mathcal{F}(A) = \mathcal{P}(A)$ falls under case [1] when A is countable, and when A is a subset of $\mathbb{R}$ or $\mathbb{R}^n$, provided that the continuum hypothesis CH is valid.

5.2 Case [B]

In case [B] we assume the Zermelo-Fraenkel axioms ZF plus the axiom of determinacy AD. We show that, under these conditions $\mathcal{P}(A)$ belongs to case [2] for any set A whose cardinality exceeds, or equals, the cardinality of $\mathbb{R}$.⁹ Woodin showed that the consistency of ZF+AD is equivalent to the consistency of ZF plus the statement that infinitely many Woodin cardinals exist.

Theorem 5 *Assume the axiom of determinacy AD. Let A be such that the cardinality of A is at least the cardinality of $[0, 1]$. Then there exists an atomless measure on $\mathcal{P}(A)$. Hence*

$$v_1^p = v_1^o = 1.$$

Proof Let A be any infinite set with a cardinality at least the cardinality $[0, 1]$. By the axiom of determinacy, $\mathcal{P}([0, 1]) = \mathcal{M}$, that is the power set of $[0, 1]$ is equal to the collection of Lebesgue measurable subsets of $[0, 1]$ (see Mycielski and Świerczkowski 1964; Jech 2006, Theorem 33.3(i)). So, the Lebesgue measure is an atomless probability measure on $\mathcal{P}([0, 1])$. Since the cardinality of A is at least the cardinality of $[0, 1]$, we can extend this measure to a measure on $\mathcal{P}(A)$. $\square$

⁹ The cardinality of X is at least the cardinality of Y if there exists a one-to-one map from Y to X .

5.3 Case [C]

In case [C] we assume ZFC (ZF plus AC), existence of a measurable cardinal MC, and the continuum hypothesis CH. In this case we show that $\mathcal{P}(A)$ belongs to case [3], whenever the cardinality of A is larger or equal to the cardinality of a measurable cardinal. In ZF plus MC it can be shown that ZF is consistent. It was shown by Lévy and Solovay (1967) that if ZFM (ZF plus MC) is consistent, then ZFM plus CH is consistent.

Theorem 6 *Assume ZFC and CH. Also assume MC, so there exists a measurable cardinal α . Let A be such that $\text{card}(A) \geq \alpha$. Then there exists a non-trivial measure on $\mathcal{P}(A)$, and each such measure has an atom. Hence*

$$v_1^o = 1.$$

Proof We show that $\mathcal{P}(A)$ belongs to case [3]. Since α is a measurable cardinal, and $\text{card}(A) \geq \alpha$, it follows that there is a non-trivial probability measure on $\mathcal{P}(A)$. Let μ be such a non-trivial probability measure on $\mathcal{P}(A)$. We show that μ has an atom.

Suppose that μ does not have an atom. Lemma 10.9 of Jech (2006) states that, if there is an atomless non-trivial σ -additive measure on A , then there exists a non-trivial countably additive measure on some cardinal $\kappa \leq 2^{\aleph_0}$. So, there exists a non-trivial measure on some $\kappa \leq 2^{\aleph_0}$. Since by CH we have that $2^{\aleph_0} = \aleph_1 = \omega_1$, this implies that there exists a non-trivial measure on ω_1 . This however contradicts Lemma 10.13 in Jech (2006). This concludes the proof. $\square$

5.4 An example

This section can be considered to be a simpler, illustrative version of the preceding discussion. We explain how the various set theoretic settings affect the analysis for the specific case $A = [0, 1]$ and $\mathcal{F}(A) = \mathcal{P}(A)$.

Example 4 Take $A = [0, 1]$ and $\mathcal{F}(A) = \mathcal{P}(A) = \mathcal{P}([0, 1])$.

- [A] Assume ZF and the axiom of constructibility $V = L$. Then, based on Theorem 4, $\mathcal{P}(A)$ belongs to case [1]. So, $v_1^p = v_1^o = 0$. Note that, in this setting, also AC is valid, so that $\mathcal{P}(A) \neq \mathcal{M}$.
- [B] Assume ZF and AD. Then, it follows from Mycielski and Świerczkowski (1964) that $\mathcal{P}([0, 1]) = \mathcal{M}$. So, we are in fact in the situation of Example 2. It follows that $v_1^p = v_1^o = 1$.
- [C] Assume ZFC and CH. Suppose that there exists a non-trivial measure on $\mathcal{P}([0, 1])$. Then there exists a non-trivial measure on $2^{\aleph_0}$. Banach and Kuratowski showed that this implies that $2^{\aleph_0} > \aleph_1$. This violates CH. Hence, under these assumptions, $\mathcal{P}(A)$ does not belong to case [3], but to case [1].

This ends our discussion of the example. $\square$

6 Finitely additive strategies

In this section we will compare the setup with countable additivity with that of finite additivity. Finite additivity is a reasonable alternative to countable additivity, as de Finetti (1975), Savage (1972) and Dubins and Savage (2014) argued. For an extensive comparison between these two approaches, see Bingham (2010).

We show that the guarantee levels defined using finitely additive strategies (called charges) differ from those specified in Theorems 1 and 2, where the player used countably additive strategies. We restrict our analysis to the case where the sigma algebra equals the power set of the action space.

Let S be a non-empty set. A probability charge on S is a function $\mu: \mathcal{P}(S) \rightarrow [0, 1]$ such that $\mu(S) = 1$, and for all disjoint sets $E, F \in \mathcal{P}(S)$ it holds that $\mu(E \cup F) = \mu(E) + \mu(F)$. We denote by $\mathcal{C}(S)$ the set of all probability charges on S . If S is infinite and we assume the axiom of choice, the set of (countably-additive) probability measures $\Delta(S)$ is a strict subset of $\mathcal{C}(S)$.

We can define an outer measure for probability charges very similarly to how it was carried out in the Preliminaries. However, in this case we would use only finite covers instead of countable ones. Furthermore, the guarantee levels can be defined similarly to Sect. 3, but now the set of strategies for player 1 is $\mathcal{C}(A)$, and the set of strategies for player 2 is $\mathcal{C}(B)$. We denote the 4 guarantee levels defined through probability charges by w_1^p, w_1^o, w_2^p and w_2^o . The following result follows from Theorems 4.1 and 5.1 in Flesch et al. (2017).

Theorem 7 *Assume the axiom of choice. The guarantee levels in finitely-additive strategies are*

$$w_1^p = w_2^o = 0 \quad \text{and} \quad w_1^o = w_2^p = 1.$$

Both players have pessimistic and optimistic optimal strategies.

So optimal strategies always exist, independent of assuming finitely or countably additive strategies.

However, the guarantee levels differ drastically between the two different models. The finitely additive guarantee levels are 0 when the definition of a guarantee level considers 1 minus the outer measure of W^2 (Eqs. (2) and (4)), and the guarantee levels are 1 when the definition considers the outer measure of W^1 (Eqs. (1) and (3)). Thus the finitely additive guarantee levels of a player depend on whether we take the pessimistic or the optimistic approach.

Also note that, when the players agree on how to compute expected payoffs—that is, when one player is pessimistic and the other player is optimistic—in the finitely additive case, the players agree on the value of the game. This is in stark contrast to for example, case [1] of the countably additive setup, where both players think that their opponent will win, even when they agree on how to compute expected payoffs.

7 Remarks on our requirements for the sigma algebras

In this section, we briefly examine how demanding conditions [A], [B], and [C] are. We argue that conditions [A] and [B] are relatively mild, while condition [C] is more demanding.

We start with condition [A] on $\mathcal{F}(A)$. The condition requires that all singleton sets are in $\mathcal{F}(A)$. This is a relatively mild condition. It is for example satisfied by any sigma algebra that includes a Hausdorff topology. Thus, any Borel sigma algebra induced by such a topology, as well as the familiar sigma algebra $\mathcal{M}$ on $\mathbb{R}$ satisfy this condition. One of the consequences of this condition - from a game theoretic perspective - is that it allows for pure strategies.

We will now argue that condition [B] on $\mathcal{F}(B)$ is mild. If a sigma algebra $\mathcal{F}(B)$ satisfies condition [B], then a larger sigma algebra also satisfies it. That is why in Lemma 8 we look at the smallest sigma algebra satisfying [B]. We show that this sigma algebra is not particularly rich in the following sense. It certainly contains a certain type of sets, the so-called layers. Moreover, any other set in this smallest sigma algebra splits every layer into an at most countable set and its complement. This means that it does not contain any sets that split a layer into two uncountable sets. So our goal is to comment on the richness of the smallest sigma algebra satisfying our condition [B], and moreover to compare our conditions [B] and [C]. To formalise these we need a bit more terminology.

A subset L of B is called a layer if there is a finite subset F of A such that $L = \{F \cup \{a\} \mid a \in A \setminus F\}$.

Remarks Note that every finite set F defines a (unique) layer. Moreover, if L is a layer generated by F , then $L \subseteq B_m$ with $m = |F| + 1$. Also note that B_1 is a layer (for $F = \emptyset$), and that condition [B] implies that every layer is an element of $\mathcal{F}(B)$.

A set $C \subseteq B$ is *minimalistic* if for every layer L , either $L \cap C$ is at most countable, or $L \setminus C$ is at most countable.

Condition [M] For every $C \in \mathcal{F}(B)$, C is minimalistic.

We have the following result. The proof can be found in Appendix A.

Lemma 8 Let $\mathcal{F}(B)$ be the smallest sigma algebra that satisfies condition [B]. Then $\mathcal{F}(B)$ satisfies condition [M].

Thus, no element of the smallest sigma algebra that satisfies condition [B] will split a layer into two uncountable sets. In this sense, $\mathcal{F}(B)$ only contains a limited collection of subsets of B .

Remark Suppose that $\mathcal{F}(A)$ is not minimal. Then any sigma algebra $\mathcal{F}(B)$ that satisfies condition [C] automatically violates condition [M]. To see this, since $\mathcal{F}(A)$ is not minimal, $\mathcal{F}(A)$ contains an uncountable set D whose complement is also uncountable.¹⁰ Then $Y(D, 1) \in \mathcal{F}(B)$ by [C]. However, $L = B_1$ is a layer—for $F = \emptyset$ —while both $Y(D, 1) \cap L$ and $L \setminus Y(D, 1)$ are uncountable. Hence, $Y(D, 1)$ is not minimalistic.

¹⁰ This statement is not entirely innocuous. Showing the existence of such a set D for any given uncountable set A requires Zorn's Lemma, hence AC. In our case it follows though from the assumption that $\mathcal{F}(A)$ is not minimal.

Hence, when $\mathcal{F}(A)$ is not minimal, the smallest sigma algebra that satisfies condition [B] does not satisfy condition [C]. The smallest sigma algebra that satisfies condition [B] is a strict subset of any sigma algebra that satisfies condition [C]. Nevertheless, given $\mathcal{F}(A)$, also condition [C] still leaves some flexibility for the choice of $\mathcal{F}(B)$. For a set $D \subseteq A$, write

$$D^* = \{\{a\} \mid a \in D\}.$$

Let $\mathcal{F}(B)$ be the smallest sigma algebra that satisfies condition [C]. Then

$$\{E \subseteq B_1 \mid E \in \mathcal{F}(B)\} = \{D^* \mid D \in \mathcal{F}(A)\}.$$

Thus, the restriction of $\mathcal{F}(B)$ to B_1 is “identical” to $\mathcal{F}(A)$.

8 Concluding remarks

We examined a class of zero sum games, catch games, in which the guarantee levels are quite varied. The countably additive levels depend on many details of the model, including the cardinality of the action set, the sigma algebra defined on the action set, and on set theoretic axioms.

At first sight, when A is infinite, as player 2 can only choose finite subsets of A , it would seem that player 2 has an impossible task. We showed that this speculation is just partly justified, and the situation is more complex than first intuition might suggest.

We distinguished three cases. In case [1] player 1 only has trivial strategies. This makes it easier to catch him, which is reflected in his guarantee levels being 0, signifying that player 1 expects to be caught. Paradoxically, player 2 disagrees, and thinks player 1 will win the game.

In case [2] player 1 has atomless measures, which is more favorable when trying not to be caught. The guarantee levels for player 1 are 1 in this case. So, both players expect player 1 to win the game.

Case [3] is an intermediate case. The optimistic guarantee level for player 1 is again 1. We also established that $v_1^p = 0$ for a specific type of sigma algebra. In general, it remains an open problem to determine v_1^p for this case.

The situation changes again when we consider finitely additive strategies. In this case the guarantee levels of the players are driven by the extension (pessimistic or optimistic) we take. When player 1 is optimistic, and player 2 is pessimistic, the value of the game is 1. So, both players expect player 1 to win the game. When player 1 is pessimistic, and player 2 is optimistic, both players expect player 2 to win the game.

Thus, we should be cautious with claims about the value of a zero sum game. As we argued, the value, its size, and even its existence, may depend on many details in our choices regarding which model to select.

In economic theory there is an abundance of models with infinite action spaces. Modeling the mixed strategies of players in this case is not trivial, since similar questions can be raised to the ones presented in this paper. In our examples the exact model

of mixed strategies significantly changed the prediction, both in terms of the value of the game and in terms of optimal strategies.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Appendix: The proofs of Theorems 1 and 2

In this section we provide the proofs of our main theorems. Throughout this section, we assume that $\mathcal{F}(A)$ satisfies condition [A]. So, $\{a\} \in \mathcal{F}(A)$ for every $a \in A$. Also, unless explicitly stated otherwise, $\mathcal{F}(B)$ satisfies condition [B]. Thus, $Q(I, m) \in \mathcal{F}(B)$ for any choice of I and m .

A. The guarantee levels for player 2: Proof of Theorem 1

We show that

$$v_2^p = v_2^o = 1.$$

Since $v_2^o \leq v_2^p$, it is sufficient to prove that $v_2^o = 1$. Take an arbitrary $k \in \mathbb{N}$. We prove that $v_2^o \geq 1 - \frac{1}{k}$. For this purpose, take a $p_2 \in \Delta(B)$. We construct a $p_1 \in \Delta(A)$ such that

$$1 - \langle p_1, p_2 \rangle (W^2) \geq 1 - \frac{1}{k}.$$

We need to show that $\langle p_1, p_2 \rangle (W^2) \leq \frac{1}{k}$. It is sufficient to find a cover of W^2 (as defined in the Preliminaries) with a probability less or equal than $\frac{1}{k}$. We need to find collections of rectangles in $\mathcal{F}(A) \times \mathcal{F}(B)$ that cover W^2 .

For $m \in \mathbb{N}$, write $C_m = B_1 \cup \dots \cup B_m$. So C_m contains elements of B which have at most m elements of the action set A . By condition [B1], $C_m \in \mathcal{F}(B)$ for each m . Choose $m \in \mathbb{N}$ such that

$$p_2(C_m) \geq 1 - \frac{1}{2k}.$$

Since A is infinite, we can choose a sequence $a_1, \dots, a_{2km}$ of $2km$ different elements of A . Since by condition [A] singleton sets are elements of $\mathcal{F}(A)$, we can define

$$p_1 = \frac{1}{2km} \cdot \delta(a_1) + \dots + \frac{1}{2km} \cdot \delta(a_{2km}),$$

where $\delta(a)$ denotes the Dirac measure on A . We show that $\langle p_1, p_2 \rangle (W^2) \leq \frac{1}{k}$.

Let $A^* = \{a_1, \dots, a_{2km}\}$. For each $I \subseteq A^*$ with $|I| \leq m$, we define $R(I)$ in $A \times C_m$ as follows. Define $R(I)_1 = I \cup (A \setminus A^*)$, and $R(I)_2 = \{b \in C_m \mid b \cap A^* = I\}$. Define

$$R(I) = R(I)_1 \times R(I)_2.$$

The set $R(I)$ will be used as part of the cover of $W^2 = \{(a, b) \in A \times B \mid a \in B\}$.

By condition [B], since

$$R(I)_2 = C_m \cap \left[\left(\bigcup_{a \in I} B(a) \right) \setminus \left(\bigcup_{a \in A^* \setminus I} B(a) \right) \right],$$

we have that $R(I)_2 \in \mathcal{F}(B)$. Also, since singleton sets are in $\mathcal{F}(A)$, it follows that $R(I)_1 \in \mathcal{F}(A)$. So, $R(I)$ is a rectangle. Let $\mathcal{R}$ denote the set of all such sets $R(I)$ for each choice of I . Clearly, $\mathcal{R}$ is finite. In the proof of the following claim we show that $\mathcal{R}$ is a cover of $W^2 \cap (A \times C_m)$. We created the individual sets $R(I)$ in such a way to help with the cover.

Claim The collection $\mathcal{R} \cup \{A \times (B \setminus C_m)\}$ is a cover of W^2 .

Proof of claim Take an $(a, b) \in W^2 \cap (A \times C_m)$. Then $a \in b$. Take $I = b \cap A^*$. Note that, since $b \in C_m$, the set I has at most m elements. Consider the rectangle $R(I) \in \mathcal{R}$. We argue that $(a, b) \in R(I)$. Clearly,

$$b \in \{b \in C_m \mid b \cap A^* = I\}$$

by definition of I . We argue that $a \in I \cup (A \setminus A^*)$. Suppose that $a \notin I$. Then, since $a \in b$, $a \notin A^*$. So, $a \in A \setminus A^*$. End proof of claim. $\square$

We continue to prove that $\langle p_1, p_2 \rangle (W^2) \leq \frac{1}{k}$. By the choice of m , we have

$$\langle p_1, p_2 \rangle (A \times (B \setminus C_m)) = p_1(A) \cdot p_2(B \setminus C_m) \leq \frac{1}{2k}.$$

Next, take $R(I) \in \mathcal{R}$. Define $k(I) = p_2(R(I)_2)$. Then

$$\begin{aligned} \langle p_1, p_2 \rangle (R(I)) &= p_1(R(I)_1) \cdot p_2(R(I)_2) = p_1(I) \cdot k(I) \leq m \cdot \frac{1}{2km} \\ &\cdot k(I) = \frac{1}{2k} \cdot k(I). \end{aligned}$$

Also, let $\mathcal{I}$ be the collection of subsets I of A^* with $|I| \leq m$. Then the collection $\{R(I)_2\}_{I \in \mathcal{I}}$ constitutes a finite partition of C_m . Indeed, $b \in R(I)_2$ precisely when $I = b \cap A^*$. Hence, using the above claim, and the fact that

$$\sum_{I \in \mathcal{I}} k(I) = p_2(C_m) \leq 1,$$

we have that

$$\langle p_1, p_2 \rangle (W^2) \leq \langle p_1, p_2 \rangle (A \times (B \setminus C_m)) + \sum_{I \in \mathcal{I}} \langle p_1, p_2 \rangle (R(I))$$

$$\begin{aligned} &\leq \frac{1}{2k} + \sum_{I \in \mathcal{I}} \frac{1}{2k} \cdot k(I) \\ &\leq \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \end{aligned}$$

Since the guarantee levels for player 2 are 1, any probability measure $p_2 \in \Delta(B)$ is automatically pessimistic and optimistic optimal for player 2. $\square$

B. The guarantee levels for player 1: Proof of Case [1] in Theorem 2

In this section, we assume that there exists no non-trivial measure on $\mathcal{F}(A)$. We also only need condition [B3] on $\mathcal{F}(B)$ instead of [B].

Since there exists no non-trivial measure on $\mathcal{F}(A)$, by Lemma 10, every strategy of player 1 has a countable support. Take a $p_1 \in \Delta(A)$ and let C be the countable support for p_1 . Let $\varepsilon > 0$. Take a finite subset $D \subseteq C$ such that $p_1(D) \geq 1 - \varepsilon$. By condition [B3], $\{D\} \in \mathcal{F}(B)$. So, $\delta(D) \in \Delta(B)$, and $\langle p_1, \delta(D) \rangle (W^1) \leq \varepsilon$. Hence

$$\inf_{p_2 \in \Delta(B)} \langle p_1, p_2 \rangle (W^1) = 0.$$

Therefore $v_1^0 = 0$, which implies that $v_1^p = 0$ as well. Since these levels for player 1 are 0, it follows that all probability measures $p_1 \in \Delta(A)$ are pessimistic and optimistic optimal for player 1. $\square$

C. The guarantee levels for player 1: Proof of Case [2] in Theorem 2

In this section, assume condition [C]. We also assume that there is an atomless measure on $\mathcal{F}(A)$.

Let p_1 be any atomless probability measure on $\mathcal{F}(A)$. It is sufficient to prove that $v_1^p = 1$. Thus, it is sufficient to prove that $\langle p_1, p_2 \rangle (W^2) = 0$ for every $p_2 \in \Delta(B)$. Take any $p_2 \in \Delta(B)$ and $k \in \mathbb{N}$. We prove that

$$\langle p_1, p_2 \rangle (W^2) \leq \frac{1}{k}.$$

For every $m \in \mathbb{N}$, by applying km times Theorem 11 in Appendix B there exists a partition $\mathcal{A} = \{A_m^1, \dots, A_m^{km}\}$ of A such that $p_1(A_m^i) = \frac{1}{km}$ for each $i = 1, \dots, km$. We explain the first step. By Theorem 11 the probability space $(A, \mathcal{F}(A), p_1)$ has a set $A_m^1 \in \mathcal{F}(A)$ such that $p_1(A_m^1) = \frac{1}{km}$. Decreasing the space by A_m^1 and applying Theorem 11 $km - 1$ more times leads to the partition $\mathcal{A} = \{A_m^1, \dots, A_m^{km}\}$.

For a vector $g = (g(1), \dots, g(km))$ whose coordinates are non-negative integers, write

$$X(g, m) = \bigcup_{\ell: g(\ell) > 0} A_m^\ell,$$

$$Y(g, m) = \{b \in B_m \mid b \cap A_m^\ell \text{ has } g(\ell) \text{ elements for every } \ell = 1, \dots, km\},$$

and

$$Z(g, m) = X(g, m) \times Y(g, m).$$

For fixed m , let $\mathcal{Z}(m)$ denote the collection of all such sets $Z(g, m)$, and let $\mathcal{Z}$ denote the collection of all sets $Z(g, m)$. For fixed m , the partition $\{A_m^1, \dots, A_m^{km}\}$ is finite, so there are finitely different $X(g, m)$. For fixed m , the set $Y(g, m)$ is either empty or not empty. The latter is only possible if $\sum_{i=1}^{km} g(i) = m$. Since for all i the integer $g(i) \geq 0$, there are only finitely many different $Y(g, m)$. Thus, $\mathcal{Z}(m)$ is finite, and $\mathcal{Z}$ is countable. We note the following facts.

- [1] $\mathcal{Z}$ is a cover of W^2 . We first show that each $Z(g, m)$ is a rectangle. Since $X(g, m)$ is a union of sets A_m^ℓ , it follows that $X(g, m) \in \mathcal{F}(A)$. Also, since $Y(g, m)$ is the intersection of B_m with sets of the form $Y(A_m^\ell, g(\ell))$, it follows from condition [C] that $Y(g, m) \in \mathcal{F}(B)$. Next, take any $(a, b) \in W^2$. Define $m = |b|$. Define g by $g(\ell) = |b \cap A_m^\ell|$. Then $b \in Y(g, m)$. Further, take ℓ with $a \in A_m^\ell$. Since $(a, b) \in W^2$, we know that $a \in b$. So $g(\ell) > 0$. Thus, $a \in X(g, m)$. It follows that $(a, b) \in Z(g, m)$. Hence, the sets $Z(g, m)$ cover W^2 .
- [2] For any non-empty set $Z(g, m) \in \mathcal{Z}$ we have

$$p_1(X(g, m)) = \sum_{\ell | g(\ell) > 0} p_1(A_m^\ell) \leq m \cdot \frac{1}{km} = \frac{1}{k}.$$

- [3] For fixed m , the sets $Y(g, m)$ constitute a partition of B_m . To see this, note that for $b \in B_m$, $b \in Y(g, m)$ precisely when each $g(\ell)$ is the number of elements of $b \cap A_m^\ell$.

Using the properties above, we compute that

$$\begin{aligned} \langle p_1, p_2 \rangle(W^2) &\leq \sum_{Z(g, m) \in \mathcal{Z}} \langle p_1, p_2 \rangle(Z(g, m)) \\ &= \sum_m \sum_{Z(g, m) \in \mathcal{Z}(m)} p_1(X(g, m)) \cdot p_2(Y(g, m)) \\ &\leq \sum_m \frac{1}{k} \cdot \sum_{Z(g, m) \in \mathcal{Z}(m)} p_2(Y(g, m)) \\ &= \sum_m \frac{1}{k} \cdot p_2(B_m) = \frac{1}{k}. \end{aligned}$$

It also follows that p_1 is pessimistic, and hence optimistic, optimal for player 1. $\square$

D. The guarantee levels for player 1: Proof of Case [3] in Theorem 2

In this section we assume that there is a non-trivial measure on $\mathcal{F}(A)$, but each such measure has an atom.

Let p_1 be a non-trivial probability measure on $\mathcal{F}(A)$. By our assumption, it has an atom, say T . Let q_1 be the probability measure on $\mathcal{F}(A)$ defined by

$$q_1(E) = \frac{p_1(E \cap T)}{p_1(T)}$$

for every $E \in \mathcal{F}(A)$. Note that q_1 is non-trivial and $\{0, 1\}$ -valued.

We prove that $v_1^o = 1$. In particular, we show that $\langle q_1, p_2 \rangle (W^1) = 1$ for every $p_2 \in \Delta(B)$. Take any $p_2 \in \Delta(B)$. Let $\{Z^i\}_{i \in \mathbb{N}}$ be a cover of W^1 . We prove that

$$\sum_{i \in \mathbb{N}} \langle q_1, p_2 \rangle (Z^i) = 1.$$

The proof is in three parts.

A. Let

$$E = \{i \in \mathbb{N} \mid q_1(Z_1^i) = 1\} \quad \text{and} \quad P = \bigcap_{i \in E} Z_1^i \cap \left(\bigcap_{i \notin E} (A \setminus Z_1^i) \right).$$

We argue that $q_1(P) = 1$. First note that $q_1(A \setminus Z_1^i) = 0$ for all $i \in E$. However, since q_1 is a $\{0, 1\}$ -valued measure, we also have $q_1(Z_1^i) = 0$ for all $i \notin E$. It follows that $q_1(A \setminus P) = 0$, so that $q_1(P) = 1$.

B. Write

$$Q = \bigcup_{i \in E} Z_2^i.$$

We argue that $Q = B$. Suppose there is a fixed $c \in B$ with $c \notin Q$. We derive a contradiction.

Take any $a \in P$. Take $i \in E$. Then, since $c \notin Q$, also $c \notin Z_2^i$. It follows that $(a, c) \notin Z^i$. Take $i \notin E$. Then, since $a \in P$, $a \notin Z_1^i$. So, also in this case, $(a, c) \notin Z^i$. We conclude that $(a, c) \notin Z^i$ for every $i \in \mathbb{N}$. This implies that $(a, c) \notin W^1$. It follows that $a \in c$. So, $P \subseteq c$.

However, as q_1 is non-trivial and $q_1(P) = 1$, the set P is uncountable, hence infinite. This contradicts the fact that c is a finite subset of A .

C. Therefore, we obtain

$$\sum_{i \in \mathbb{N}} \langle q_1, p_2 \rangle (Z^i) = \sum_{i \in \mathbb{N}} q_1(Z_1^i) \cdot p_2(Z_2^i) = \sum_{i \in E} p_2(Z_2^i) \geq p_2(Q) = p_2(B) = 1.$$

This completes the proof that $v_1^o = 1$. It also implies that q_1 is an optimistic optimal strategy for player 1. $\square$

E. The proofs for the partial result in Theorem 3

Let $\mathcal{F}(A)$ be the minimal sigma algebra on A . Define q_1 on $\mathcal{F}(A)$ by, for every $E \in \mathcal{F}(A)$,

$$q_1(E) = \begin{cases} 0 & \text{if } E \text{ is at most countable} \\ 1 & \text{otherwise.} \end{cases}$$

Notice that q_1 is a non-trivial probability measure for which each uncountable $E \in \mathcal{F}(A)$ is an atom. We start with the following observation.

Lemma 9 *The probability measure q_1 is the unique non-trivial probability measure on $\mathcal{F}(A)$. Hence, $\mathcal{F}(A)$ falls in case [3].*

Proof We already observed that q_1 is a non-trivial measure on $\mathcal{F}(A)$. Let p_1 be any non-trivial probability measure on $\mathcal{F}(A)$. Then $p_1(E) = 0 = q_1(E)$ for every $E \in \mathcal{F}(A)$ that is at most countable. Since $\mathcal{F}(A)$ is minimal, it follows that $p_1 = q_1$. $\square$

Proof of Lemma 8 The proof is in four parts.

- A. Take any non-empty set $Q(I, m)$. We show that $Q(I, m)$ is minimalistic. For this purpose, let L be the layer corresponding to a finite subset F of A . We need to show that $L \cap Q(I, m)$ is at most countable, or $L \setminus Q(I, m)$ is at most countable. Take an arbitrary action $b \in L \cap Q(I, m)$ for player 2. Then there is an action $a \in A \setminus F$ for player 1 with $b = F \cup \{a\}$, $|F| = m - 1$, and $I \subseteq b$. We distinguish two cases. Assume first that $a \in I$. Then $L \cap Q(I, m) = \{b\}$. So, $L \cap Q(I, m)$ is at most countable. Assume now that $a \notin I$. Then $I \subseteq F$, so that $L \subseteq Q(I, m)$. In that case $L \setminus Q(I, m)$ is empty, hence at most countable.
- B. Let $C_1, C_2, \dots$ be a sequence of minimalistic subsets of B . Write $C = \bigcup_n C_n$. We argue that C is also minimalistic. Let L be any layer. If for each n it holds that $L \cap C_n$ is at most countable, then also $L \cap C$ is at most countable. If on the other hand there is an n such that $L \setminus C_n$ is at most countable, then clearly also $L \setminus C$ is at most countable.
- C. Let C be a minimalistic subset of B . Then also $B \setminus C$ is minimalistic, as for every layer L , $L \cap (B \setminus C) = L \setminus C$ and $L \setminus (B \setminus C) = L \cap C$.
- D. Our claim now follows from the observation that the smallest sigma algebra that satisfies condition [B] can be constructed by building the countable Borel hierarchy starting with the collection of sets $Q(I, m)$. $\square$

Proof of Theorem 3 Let $p_1 \in \Delta(A)$ be a strategy of player 1. We have to show that

$$\inf_{p_2 \in \Delta(B)} [1 - \langle p_1, p_2 \rangle (W^2)] = 0, \quad \text{or, equivalently,} \quad \sup_{p_2 \in \Delta(B)} \langle p_1, p_2 \rangle (W^2) = 1.$$

$\square$

By Lemmas 9 and 10 there are a countable set $I \subseteq A$ and coefficients $d \geq 0$ and $c_a \geq 0$ for $a \in I$ such that

$$p_1 = d \cdot q_1 + (1 - d) \cdot \sum_{a \in I} c_a \cdot \delta(a),$$

where q_1 is the unique non-trivial probability measure on $\mathcal{F}(A)$. Take any $\varepsilon > 0$. Take a finite set $F \subseteq I$ such that $\sum_{a \notin F} c_a < \varepsilon$. If $d = 0$, then for $p_2 = \delta(F)$ it holds that $\langle p_1, p_2 \rangle (W^2) \geq 1 - \varepsilon$.

We assume that $d > 0$. Let L be the layer associated with F , that is $L = \{F \cup \{a\} \mid a \in A \setminus F\}$. For a set $C \in \mathcal{F}(B)$, define

$$C_* = \{a \in A \mid F \cup \{a\} \in L \cap C\}.$$

Note that $C_* = \{a \in A \setminus F \mid F \cup \{a\} \in C\}$.

Intuitively speaking, the goal is to show that for this strategy p_1 of player 1 there is a strategy p_2 of player 2 such that player 2 can catch player 1 with high probability (outer measure of at least $1 - \varepsilon$). To achieve this it is favourable for player 2 to mimic the strategy of player 1. We will describe shortly how this can be done. Roughly speaking, the strategy p_1 of player 1 has a non-trivial probability part and a trivial part, that is a part made out of Dirac-measures. There might be countably many Dirac-measures, but we only consider finitely many of them (the set F), by excluding the part with smaller than ε weight. (We do this, because the action set of player 2 contains only finite subsets of A , but not countably infinite ones.) This finite set F of actions of player 1 is a single action for player 2. Starting with an action C of player 2, an action $a \notin F$ of player 1 is in C_* if the action $F \cup \{a\}$ of player 2 remains in C . This property will be helpful when we construct p_2 in such a way to mimic p_1 . So far we considered the trivial part of p_1 , now we take into account the non-trivial part. Therefore we define the strategy p_2 for player 2 as $p_2(C) = q_1(C_*)$ for every $C \in \mathcal{F}(B)$ where q_1 is the unique non-trivial probability measure on the action set of player 1.

The structure of the proof is the following. For every action set C of player 2 we constructed a useful corresponding action set C_* of player 1. In part A we show that due to the special structure of $\mathcal{F}(A)$ and $\mathcal{F}(B)$, the set C_* is measurable. In part B we define a strategy p_2 for player 2 where $p_2(C) = q_1(C_*)$ for every $C \in \mathcal{F}(B)$. In part C we show that $\langle p_1, p_2 \rangle (W^2) \geq 1 - \varepsilon$.

- A.** We show that $C_* \in \mathcal{F}(A)$. Since, by Lemma 8, $\mathcal{F}(B)$ satisfies condition [M], either $L \cap C$ is at most countable, or $L \setminus C$ is at most countable. If $L \cap C$ is at most countable, it follows that C_* is at most countable. In this case, by condition [A], $C_* \in \mathcal{F}(A)$. If $L \setminus C$ is at most countable, the set

$$D = \{a \in A \mid F \cup \{a\} \in L \setminus C\}$$

is at most countable. So, also in this case $C_* = A \setminus (D \cup F)$ is an element of $\mathcal{F}(A)$.

- B.** Thus, we can define p_2 by, for every $C \in \mathcal{F}(B)$, $p_2(C) = q_1(C_*)$. Then $p_2(B) = q_1(B_*) = q_1(A \setminus F) = 1$. Further, if $C_1, C_2, \dots$ is a sequence of

mutually disjoint sets in $\mathcal{F}(B)$, then $C_{1*}, C_{2*}, \dots$ are mutually disjoint sets in $\mathcal{F}(A)$. So,

$$\sum_k p_2(C_k) = \sum_k q_1(C_{k*}) = q_1 \left[\bigcup_k C_{k*} \right] = q_1 \left[\left(\bigcup_k C_k \right)_* \right] = p_2 \left(\bigcup_k C_k \right).$$

Hence, p_2 is a probability measure on $\mathcal{F}(B)$.

C. We show that $\langle p_1, p_2 \rangle (W^2) \geq 1 - \varepsilon$. Notice that

$$U = \{(a, F \cup \{a\}) \mid a \in A \setminus F\} \quad \text{and} \quad V = \{(c, F \cup \{a\}) \mid c \in F \text{ and } a \in A \setminus F\}$$

are subsets of W^2 . Thus, it suffices to show that $\langle p_1, p_2 \rangle (U \cup V) \geq 1 - \varepsilon$.

C1. We first argue that

$$\langle p_1, p_2 \rangle (U \cup V) = \langle p_1, p_2 \rangle (U) + \langle p_1, p_2 \rangle (V).$$

Clearly

$$\langle p_1, p_2 \rangle (U \cup V) \leq \langle p_1, p_2 \rangle (U) + \langle p_1, p_2 \rangle (V).$$

We show the reverse inequality. Notice that $V = F \times L$. Let $\{Z^i\}_{i \in \mathbb{N}}$ be a cover of $U \cup V$. Let $Z^i = Z_1^i \times Z_2^i$ be a rectangle in the cover. Define $G_1^i = Z_1^i \cap F$, $H_1^i = Z_1^i \setminus F$, and $H_2^i = Z_2^i \cap L$.

Notice that $F \in \mathcal{F}(A)$, and $L = Q(F, |F| + 1) \in \mathcal{F}(B)$. So, $G_1^i \in \mathcal{F}(A)$, $H_1^i \in \mathcal{F}(A)$, and $H_2^i \in \mathcal{F}(B)$.

Moreover, $G_1^i \times H_2^i$ and $H_1^i \times H_2^i$ are disjoint subsets of Z^i . Also, if $(a, F \cup \{a\}) \in U$, then $(a, F \cup \{a\}) \in H_1^i \times H_2^i$. If $(a, F \cup \{a\}) \in V$, then $(a, F \cup \{a\}) \in G_1^i \times H_2^i$.

C2. We show that $\langle p_1, p_2 \rangle (V) = (1 - d) \cdot (1 - \varepsilon)$. We already observed that $V = F \times L$. Moreover, $F \in \mathcal{F}(A)$, and $L = Q(F, |F| + 1) \in \mathcal{F}(B)$. So,

$$\begin{aligned} \langle p_1, p_2 \rangle (V) &= p_1(F) \cdot p_2(L) = p_1(F) \cdot q_1(A \setminus F) \\ &= (1 - d) \cdot \left[\sum_{a \in F} c_a \right] \cdot 1 \geq (1 - d) \cdot (1 - \varepsilon). \end{aligned}$$

C3. It remains to show that $\langle p_1, p_2 \rangle (U) \geq d$. Let $\{Z^i\}_{i \in \mathbb{N}}$ be a cover of $U = \{(a, F \cup \{a\}) \mid a \in A \setminus F\}$. Let $Z^i = Z_1^i \times Z_2^i$ be a rectangle in the cover. Define

$$Y_1^i = \{a \in A \setminus F \mid (a, F \cup \{a\}) \in Z^i\} \quad \text{and} \quad Y_2^i = \{F \cup \{a\} \mid a \in Y_1^i\}.$$

Clearly, Y_1^i is a subset of Z_1^i and Y_2^i is a subset of Z_2^i .

Claim It holds that $Y_2^i \in \mathcal{F}(B)$ and $Y_1^i \in \mathcal{F}(A)$.

We first show that $Y_2^i \in \mathcal{F}(B)$. Write $J_2^i = \{F \cup \{a\} \mid a \in Z_1^i\}$. Then $Y_2^i = J_2^i \setminus \{F\}$. Thus, by condition [B3], it suffices to show that $J_2^i \in \mathcal{F}(B)$. We do this in two steps.

Step 1. If J_2^i is countable. Then $J_2^i \in \mathcal{F}(B)$ by condition [B3], plus the observation that $\mathcal{F}(B)$ is a sigma algebra.

Step 2. If J_2^i is not countable. Then Z_1^i is not countable. So, since $Z_1^i \in \mathcal{F}(A)$ and $\mathcal{F}(A)$ is the minimal sigma algebra on A , this implies that $A \setminus Z_1^i$ is countable. So, $D = \{F \cup \{a\} \mid a \notin Z_1^i\}$ is countable, and hence an element of $\mathcal{F}(B)$. Then either $J_2^i = L \setminus D$ or $J_2^i = (L \setminus D) \cup \{F\}$. In either case, since $\mathcal{F}(B)$ is a sigma algebra, and $L = Q(F, |F| + 1)$ is an element of $\mathcal{F}(B)$, J_2^i is an element of $\mathcal{F}(B)$. Now note that

$$Y_{2*}^i = \{a \in A \mid F \cup \{a\} \in Y_2^i\} = \{a \in A \setminus F \mid a \in Y_1^i\} = Y_1^i.$$

It follows that $Y_1^i = Y_{2*}^i \in \mathcal{F}(A)$.

C4. Define $Y^i = Y_1^i \times Y_2^i$. Clearly the sets Y^i cover U . Thus, it suffices to show that

$$\sum_{i \in \mathbb{N}} p_1(Y_1^i) \times p_2(Y_2^i) \geq d.$$

For a set $Y^i = Y_1^i \times Y_2^i$,

$$< p_1, p_2 > (Y^i) = p_1(Y_1^i) \cdot p_2(Y_2^i) \geq d \cdot q_1(Y_1^i) \cdot q_1(Y_{2*}^i) = d \cdot q_1(Y_1^i) \cdot q_1(Y_1^i).$$

Further, since the sets Y^i cover U , the sets Y_1^i cover $A \setminus F$. Then there is a k with $q_1(Y_1^k) > 0$. Then however, since A is an atom of q_1 , this implies that $q_1(Y_1^k) = 1$.

Finally, since $v_1^p = 0$, all probability measure in $\Delta(A)$ are pessimistic optimal for player 1. This completes the proof. $\square$

B Toolbox

In this section we briefly discuss two well-known theorems needed in our arguments.

Lemma 10 *Let $p \in \Delta(S)$. Then there are*

- [1] *a countable set I*
- [2] *coefficients $c \geq 0$ and $c_i \geq 0$ for all $i \in I$, with $c + \sum_{i \in I} c_i = 1$, and*
- [3] *a non-trivial probability measure q on S and Dirac measures $\delta(s_i)$ on $s_i \in S$ for every $i \in I$*

such that $p = c \cdot q + \sum_{i \in I} c_i \cdot \delta_{s_i}$.

Proof Let $I = \{s \in S \mid p(\{s\}) > 0\}$. Note that I is countable. Define $c_i = p(\{i\})$, for all $i \in I$, and $c = 1 - \sum_{i \in I} c_i$. If $c > 0$, then $q(T) = p(T \setminus I)/c$, for all $T \subseteq S$. $\square$

The following theorem has been proved by Sierpiński (1922).

Theorem 11 *Let $(S, \mathcal{F}, p)$ be an atomless probability measure space. For any $t \in (0, 1)$ there exists a set $U \in \mathcal{F}$ for which $p(U) = t$.*

Proof The proof is in two parts.

- A.** Let $V \in \mathcal{F}$ with $p(V) > 0$ be given. Take c and d such that $2c \leq d < p(V)$. We show that there is a set $W \in \mathcal{F}$ such that $W \subseteq V$ and $c \leq p(W) \leq d$.
If $c = 0$, we take $W = \phi$. Suppose $c > 0$. Let $\mathcal{V}$ be the collection of sets $C \in \mathcal{F}$ with $C \subseteq V$. Define

$$s = \sup\{p(C) : p(C) < c, C \in \mathcal{V}\}.$$

Note that $s \leq c$. Since S does not have atoms, $s > 0$. Take a sequence $(C_k)_{k \in \mathbb{N}}$ in $\mathcal{V}$ where $p(C_k) \rightarrow s$. Let $D_m = \bigcup_{k=1}^m C_k$. Suppose there is an m with $p(D_m) \geq c$. Let n be the first such m . Then $p(D_n) \leq 2c \leq d$. So, we can take $W = D_n$.

Suppose alternatively that $p(D_m) < c$ for all m . Define $D = \bigcup_{n \in \mathbb{N}} D_n$. It follows from monotone convergence that $p(D) = s \leq c$.

Suppose that $p(D) = c$. Then we can take $W = D$.

Suppose alternatively that $p(D) < c$. We will derive a contradiction. Since $0 < c - p(D)$, there is an $n \in \mathbb{N}$ such that $\frac{p(V \setminus D)}{2^n} < c - p(D)$. Since p is atomless, the set $V \setminus D$ is not an atom. Then there is a set $F \subseteq V \setminus D$ with $0 < p(F) < p(V \setminus D)$. If $p(F) \leq \frac{p(V \setminus D)}{2}$, then let $E_1 = F$, otherwise let $E_1 = V \setminus (D \cup F)$. By further splitting E_1 we can create a sequence of sets $\{E_2, \dots, E_n\}$ such that $E_m \subseteq V \setminus D$ and $0 < p(E_m) \leq \frac{p(V \setminus D)}{2^m}$ for any $m \in \{1, \dots, n\}$. Thus $0 < p(E_n) \leq \frac{p(V \setminus D)}{2^n} < c - p(D)$. It then follows that $s < p(E_n \cup D) < c$. This contradicts the definition of s .

- B.** Now take any $t \in (0, 1)$. We show that there exists a set $U \in \mathcal{F}$ for which $p(U) = t$. We will take a sequence of sets $(U_n)_{n \in \mathbb{N}}$ in such a way that their probabilities get closer and closer to t . The first set's probability will have at most $t/2$ -distance from t , the next set's probability's distance from t will be less than half the previous distance and so on. Until the probability of a set is not exactly t , we use the procedure from part A to construct the next set.

Since S is not an atom, there is a $U_1 \in \mathcal{F}$ such that $\frac{t}{2} \leq p(U_1) \leq t$.

Suppose U_n is defined with $p(U_n) \leq t$ and $t - p(U_n) \leq \frac{t - p(U_1)}{2^{n-1}}$. If $p(U_n) = t$, we take $U = U_n$. Otherwise, take $c = \frac{t - p(U_n)}{2}$ and $d = t - p(U_n)$. By the previous argument applied to $V = S \setminus U_n$, there is a $W \in \mathcal{F}$ and $W \subseteq V$ such that $\frac{t - p(U_n)}{2} \leq p(W) \leq t - p(U_n)$. Let $U_{n+1} = U_n \cup W$. Then $p(U_{n+1}) = p(U_n) + p(W) \leq t$ and $t - p(U_{n+1}) = t - p(U_n) - p(W) \leq \frac{t - p(U_n)}{2} \leq \frac{t - p(U_1)}{2^n}$.

If the iterative procedure does not terminate, let $U = \bigcup_{n \in \mathbb{N}} U_n$. Then by monotone convergence, $p(U) = t$. $\square$

C Set Theory

In this section we briefly define and discuss the set theoretic axioms that feature in this paper.

Zermelo Fraenkel The Zermelo Fraenkel axiom system is the universally accepted axiom system on which modern mathematics is built. We do not venture into all its separate axioms, but only observe that the ZF axiom system is conservative, in the sense that it does not include controversial axioms, and it is therefore widely accepted. There are statements that cannot be proven in ZF, but they can be useful in certain areas of mathematics. For example, in functional analysis many relevant theorems make use of the Axiom of Choice. Instead of the Axiom of Choice, assuming the Axiom of Determinacy can help avoid certain counter-intuitive theorems, for example the Banach-Tarski paradox. We assume the ZF axiom system, and the following axioms are possible additions to it.

Axiom of Choice Let I be any non-empty set, and suppose that for every $i \in I$, V_i is a non-empty set. Write

$$V = \bigcup_{i \in I} V_i.$$

A function $f: I \rightarrow V$ with $f(i) \in V_i$ for every $i \in I$ is called a choice function. The Axiom of Choice asserts the following.

AC: For every choice of sets I and V_i , a choice function $f: I \rightarrow V$ exists.

Axiom of Determinacy Let A be a subset of $\mathbb{N}^{\mathbb{N}}$. This set A defines a game $G(A)$ in the following way. There are two players. The players take turns to choose elements of $\mathbb{N}$. The game has perfect information, so that players observe each other's choice. The game is then played as follows.

Player 1 chooses n_1 ,
 player 2 chooses n_2 ,
 player 1 chooses n_3 ,
 player 2 chooses n_4 ,
 and so on. This way a sequence

$$s = (n_1, n_2, n_3, n_4, \dots) \in \mathbb{N}^{\mathbb{N}}$$

is constructed. Player 1 wins the game $G(A)$ if $s \in A$, otherwise player 2 wins. The game $G(A)$ is called determined if either player 1 or player 2 has a winning strategy. The Axiom of Determinacy states the following.

AD: For every set $A \subseteq \mathbb{N}^{\mathbb{N}}$, the resulting game $G(A)$ is determined.

The Axiom of Determinacy implies the continuum hypothesis.

Continuum Hypothesis We say that two sets X and Y have the same cardinality if there exists a one-to-one and onto $f: X \rightarrow Y$.

CH: Let X be an infinite subset of the unit interval $[0, 1]$. Then either X has the cardinality of $\mathbb{N}$, or X has the cardinality of $\mathbb{R}$.

Generalized Continuum Hypothesis Let A be a non-empty set. A binary relation on A is a subset of $A \times A$. Let $\leq$ be a binary relation on A . When $x \leq y$ and not

$y \preceq x$ we write $x \prec y$. Let Y be a subset of A . An element $y \in Y$ is called the smallest element of Y when $y \prec z$ for all $z \in Y$ with $z \neq y$. The binary relation $\preceq$ is called a well-ordering when $\preceq$ satisfies

- [1] (*reflexivity*) for all $x \in X$ we have $x \preceq x$,
- [2] (*transitivity*) for all $x, y, z \in X$ we have $x \preceq z$ whenever both $x \preceq y$ and $y \preceq z$,
- [3] (*totality*) for all $x, y \in X$ with $x \neq y$ we have $x \prec y$ or $y \prec x$, and
- [4] (*well order*) every non-empty subset of A has a smallest element.

A set A is called transitive when for every $a \in A$ it holds that $a \subseteq A$. A set A is called an ordinal number when A is transitive and moreover A is well-ordered by set inclusion. For example,

$$A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$$

is an ordinal number. The set $a = \{\emptyset, \{\emptyset\}\}$ is both an element and a subset of A .

It can be shown that, for any two ordinal numbers α and β , either $\alpha \in \beta$ or $\beta \in \alpha$. Conversely, each element of an ordinal number is an ordinal number itself.

When two sets X and Y have the same cardinality, we write $X \sim Y$. If there is a one-to-one map $f: X \rightarrow Y$, we write $X \leq Y$. The Theorem of Cantor-Bernstein-Schroeder shows that $X \leq Y$ and $Y \leq X$ imply $X \sim Y$. We write $X < Y$ when $X \leq Y$ and not $X \sim Y$.

An ordinal number κ is called a cardinal number if there does not exist an ordinal number $\alpha \in \kappa$ with $\alpha \sim \kappa$. For every cardinal number κ , there exists a smallest cardinal number μ with $\kappa < \mu$. We write $\mu = \kappa^+$.

Now assume that the Axiom of Choice is true. Then for every set X there is a cardinal number κ such that $X \sim \kappa$. We define $\text{Card}(X) = \kappa$. The generalized continuum hypothesis states the following.

GCH: For every cardinal number κ , $\text{Card}(\mathcal{P}(\kappa)) = \kappa^+$.

Measurable Cardinals Let κ be a cardinal number. A binary map on κ is a map $\mu: \mathcal{P}(\kappa) \rightarrow \{0, 1\}$. A binary map on κ is called κ -additive if for every cardinal number $\lambda \in \kappa$ and for every mutually disjoint collection $(A_\xi)_{\xi \in \lambda}$ of sets $A_\xi \subseteq \kappa$ it holds that

$$\mu \left(\bigcup_{\xi \in \lambda} A_\xi \right) = \sum_{\xi \in \lambda} \mu(A_\xi).$$

A cardinal number κ is called measurable if there exists a binary κ -additive binary map on κ with $\mu(\kappa) = 1$ and $\mu(\{\lambda\}) = 0$ for all $\lambda \in \kappa$.

MC: There exists a measurable cardinal.

Constructable sets and $V = L$ The statement $V = L$ is called the Axiom of Constructability. Its formulation and its associated constructions are due to Gödel. Let A be a set. An expression ϕ is called a formula for A if

- [1] ϕ has one free variable x
- [2] the only predicate symbols in ϕ are “ $\in$ ” and “ $=$ ”
- [3] the individual-constants in ϕ are names of elements of A
- [4] the quantifiers are restricted to A . So, it only uses $\forall y \in A$ and $\exists y \in A$.

Let A be a set, and let B be a subset of A . We say that B is a definable subset of A when there is a formula ϕ for A such that

$$B = \{x \in A \mid \phi(x)\}.$$

We define for every set A

$$\text{Def}(A) = \{B \subseteq A \mid B \text{ is a definable set of } A\}.$$

Define $L(\phi) = \{\phi\}$. Using transfinite induction, for every ordinal α we define

$$L(\alpha) = \bigcup_{\beta \in \alpha} \text{Def}(L(\beta))$$

We say that a set B is constructable (by abuse of notation denoted by $B \in L$) if there is an ordinal α with $B \in L(\alpha)$. The class L is the collection of all constructable sets. $V = L$: Every set is constructable.

References

- Abaffy M, Brázdil T, Řehák V, Bošanský B, Kučera A, Krčál J (2014) Solving adversarial patrolling games with bounded error: (extended abstract). In: Proceedings of the 2014 international conference on Autonomous agents and multi-agent systems (AAMAS '14), International Foundation for Autonomous Agents and Multiagent Systems, Richland, SC, pp 1617–1618
- Bingham NH (2010) Finite additivity versus countable additivity. *Electron J Hist Prob Stat* 6(1)
- Blackwell D (1942) Idempotent Markoff chains. *Ann Math Sec Ser* 43(3):560–567
- Capraro V, Scarsini M (2013) Existence of equilibria in countable games: an algebraic approach. *Game Econ Behav* 79(C):163–180
- Crawford VP, Iriberri M (2007) Fatal attraction: salience, Naïveté, and sophistication in experimental “Hide-and-Seek” games. *Am Econ Rev* 97(5):1731–1750
- Dubins LE, Savage LJ (2014) Edited and updated by Sudderth, WD and Gilat D: How to gamble if you must: inequalities for stochastic processes. Dover Publications, New York
- de Finetti B (1975) The theory of probability, vol 2. Wiley, Chichester
- Flesch J, Vermeulen D, Zseleva A (2017) Zero-sum games with charges. *Game Econ Behav* 102:666–686
- Gale D, Stewart FM (1953) Infinite games with perfect information, contributions to the theory of games, vol. 2. *Annals of Mathematics Studies*, no. 28, Princeton University Press, Princeton, p. 245–266
- Harris CJ, Stinchcombe MB, Zame WR (2005) Nearly compact and continuous normal form games: characterizations and equilibrium existence. *Games Econ Behav* 50:208–224
- Heath D, Sudderth W (1972) On a theorem of de Finetti, Oddsmaking and game theory. *Ann Math Stat* 43:2072–2077
- Isaacs R (1965) *Differential games*. Wiley, New York
- Jech T (2006) *Set theory, the third millennium edition, revised and expanded*. Springer, New York
- Kechris A (1995) *Classical descriptive set theory*. Springer, New York
- Kiekintveld C, Jain J, Tsai J, Pita J, Ordóñez F, Tambe M (2009) Computing optimal randomized re-source allocations for massive security games. In: Proceedings of the 8th international joint conference on autonomous agents and multi-agent systems (AAMAS), Budapest, Hungary, p. 689–696
- Kindler J (1983) A general solution concept for two-person zero sum games. *J Optim Theory Appl* 40:105–119
- Maitra A, Sudderth W (1993) Finitely additive and measurable stochastic games. *Int J Game Theory* 22:201–223
- Maitra A, Sudderth W (1998) Finitely additive stochastic games with Borel measurable payoffs. *Int J Game Theory* 27:257–267

- Marinacci M (1997) Finitely additive and epsilon nash equilibria. *Int J Game Theory* 26(3):315–333
- Martin DA (1975) Borel determinacy. *Ann Math Sec Ser* 102(2):363–371
- Mycielski J, Świerczkowski S (1964) On the Lebesgue measurability and the axiom of determinateness. *Fund Math* 54(1):67–71
- Lévy A, Solovay RM (1967) Measurable cardinals and the continuum hypothesis. *Isr J Math* 5:234–248
- Oxtoby JC (1980) *Measure and category: a survey of the analogies between topological and measure spaces*, 2nd edn. Springer, New York
- Pivato M (2014) Additive representation of separable preferences over infinite products. *Theory Decis* 77(1):31–83
- Prikry K, Sudderth WD (2016) Measurability of the value of a parametrized game. *Int J Game Theory* 45:675–683
- Rao KPSB, Rao B (1983) *Theory of charges: a study of finitely additive measures*. Academic Press, New York
- Rubinstein A, Tversky A, Heller D (1996) naive strategies in competitive games. In: Guth W et al (eds), *Understanding strategic interaction-essays in honor*, Springer, New York, p. 394–402
- Peters H, Vermeulen D (2012) WPO, COV and IIA bargaining solutions for non-convex bargaining problems. *Int J Game Theory* 41(4):851–884
- Savage LJ (1972) *The foundations of statistics*. Dover Publications, New York
- Schervish MJ, Seidenfeld T (1996) A fair minimax theorem for two-person (zero-sum) games involving finitely additive strategies. In: Berry DA, Chaloner KM, Geweke JK (eds) *Bayesian analysis in statistics and econometrics*. Wiley, New York, pp 557–568
- Sierpiński W (1922) Sur les fonctions d'ensemble additives et continues. *Fund Math* 3:240–246
- Solovay RM (1971) Real-valued measurable cardinals. *Axiomatic set theory*. In: *Proceedings of Symposium Pure Mathematics, Vol. XIII, Part I*, University of California, Los Angeles, CA, 1967), pp 397–428. American Mathematics Society, Providence, RI
- Sudderth W (2015) Finitely additive dynamic programming, forthcoming in *mathematics of operations research*
- Yanovskaya EB (1970) The solution of infinite zero-sum two-person games with finitely additive strategies. *Theory Probab Appl* 15(1):153–158
- Zame WR (2007) Can intergenerational equity be operationalized? *Theor Econ* 2:187–202