Int J Game Theory (2013) 42:305–324
DOI 10.1007/s00182-012-0361-7

“Procedural” values for cooperative games

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Accepted: 15 November 2012 / Published online: 9 December 2012

Abstract This paper introduces a new notion of a “procedural” value for cooperative TU games. A procedural value is determined by an underlying *procedure* of sharing marginal contributions to coalitions formed by players joining in random order. We consider procedures under which players can only share their marginal contributions with their predecessors in the ordering, and study the set of all resulting values. The most prominent procedural value is, of course, the Shapley value obtaining under the simplest procedure of every player just retaining his entire marginal contribution. But different sharing rules lead to other interesting values, including the “egalitarian solution” and the Nowak and Radzik “solidarity value”. All procedural values are efficient, symmetric and linear. Moreover, it is shown that these properties together with two very natural monotonicity postulates characterize the class of procedural values. Some possible modifications and generalizations are also discussed. In particular, it is shown that dropping one of monotonicity axioms is equivalent to allowing for sharing marginal contributions with both predecessors and successors in the ordering.

Keywords Cooperative game · Procedure · Value · Efficiency · Weak monotonicity · Coalitional monotonicity · Extended procedure.

1 Introduction

A significant amount of research has been conducted on classes of efficient values for cooperative transferable utility games which obtain by various methods of distributing the coalition dividends (Harsányi 1959) among players. These values can be viewed as variations on the Shapley value obtained by equal division of every coalition’s dividend

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among its members. The notions of selector values (Derks et al. 2000), weighted Shapley values (Kalai and Samet 1987) or random order values (Weber 1988) are now well-established in the cooperative game theory.

In this paper I define and study another class of values for cooperative TU games. Though it might well be claimed that these values also deserve the name of "random order values", I use the term "procedural values" instead. These values are also close relatives of the Shapley value. The generalization, as well as the difference from the aforementioned classes, lies in the fact that, while preserving the assumption of equal probabilities of all possible orders in which the grand coalition is formed, I admit the possibility of redistribution—but of *marginal contributions of players* instead of dividends of coalitions. The method of redistribution will be called procedure. Every procedure is determined by a family of real coefficients and, on the other hand, it determines a value on the space of TU games. A reasonable interpretation is that a player who wishes to join a coalition that has already formed is obliged to pay an "entrance fee" to the players who already are in the coalition, and that this fee constitutes a specified fraction of the newcoming player's marginal contribution.

This section introduces the notions used in the paper. In Sect. 2 I present two equivalent definitions of a procedure and a couple of simple examples, together with the values resulting from the presented procedures. Section 3 is devoted to properties of procedural values; it also contains the main theorem on their axiomatization. The final section reviews some variations and generalizations.

Most of the notions and the notation used are standard. We shall be dealing with n-person cooperative games, n being a fixed but arbitrary positive integer; usually the case $n \geq 3$ is of interest. A cooperative game (transferable utility game) is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is the set of players and v is the characteristic function—any real function defined on the set of all coalitions, i.e. all subsets of N, satisfying $v(\emptyset) = 0$. Its values are called worths of coalitions. Since the set of players will usually be fixed in this paper, I shall identify every cooperative game with its characteristic function. The set of all n-person cooperative games will be denoted by $\mathcal{G}_n$; this set forms a $(2^n - 1)$-dimensional vector space.

The players will be denoted with small letters $(i, j, k, \ldots)$, and the coalitions with capitals $(S, T, U, \ldots)$. For a coalition denoted by a capital letter, the corresponding lowercase letter will denote the cardinality of the coalition, e.g. $t = \#T$. For brevity, I shall usually omit braces when enumerating players in a coalition, thus obtaining for instance ijk instead of $\{i, j, k\}$ and $T \cup j$ instead of $T \cup \{j\}$.

In a game (N, v), two players $i, j \in N$ are interchangeable if, for every coalition $T \subseteq N \setminus \{i, j\}$, $v(T \cup i) = v(T \cup j)$, and player $k \in N$ is a null player if, for every coalition $T \subset N$, $v(T \cup k) = v(T)$.

Π is the set of all permutations of the set N. For any player $j \in N$ and any permutation $\pi \in \Pi$, let us denote:

$$H_{\pi,j} = \pi^{-1}(\{1, 2, \ldots, \pi(j)\}),$$
$$N_{\pi,j} = \pi^{-1}(\{\pi(j), \pi(j) + 1, \ldots, n\})$$

—the sets, respectively, of all predecessors of j and all successors of j in the ordering π (in both cases including player j).

For any real number z, z^+ and z^- will denote its positive and negative part: $z^+ = \max(z, 0)$ and $z^- = \min(z, 0)$.

A value is any function ψ assigning to every game $(N, v) \in \mathcal{G}_n$ a vector $\psi(v) = (\psi_1(v), \psi_2(v), \ldots, \psi_n(v))$; $\psi(v)$ is the value of the game (N, v), and its components are individual values of the players. If $\sum_{i=1}^{n} \psi_i(v) = v(N)$ for every game $(N, v) \in \mathcal{G}_n$, then the value ψ is efficient. Efficient values are rules of dividing among all players the quantity $v(N)$, interpreted as the "gain" that all players together can achieve if they all agree to cooperate.

Many (though not all) commonly accepted values depend somehow on the marginal contributions of players to coalitions. The marginal contribution of player j to coalition T (containing j) is the difference $v(T) - v(T \setminus j)$. In particular, for a given ordering π of N we shall denote by $m_{j,\pi}(v)$ the marginal contribution of j to the set of his predecessors in the ordering π:

$$m_{j,\pi}(v) = v(H_{\pi,j}) - v(H_{\pi,j} \setminus j).$$

The historally first and undoubtedly most prominent value for cooperative games, the Shapley value (Shapley 1953) is defined as the vector of expected marginal contributions of players under the probability measure given by uniform distribution on the set Π:

$$\phi_i(v) = \mathbf{E}_\Pi\, m_{i,\pi}(v) = \sum_{\pi \in \Pi} \frac{v(H_{\pi,i}) - v(H_{\pi,i} \setminus i)}{n!}.$$

Various generalizations of the Shapley value are obtained when other probability distributions on the set of permutations are allowed; this leads to the class of random order values (see Weber 1988). The generalization in this paper goes in different direction. I follow Shapley in assuming that all permutations of players are equally probable, but, unlike Shapley and Weber, I do not require players to retain their marginal contributions for themselves. Instead, I assume some specific ways of redistribution of marginal contributions of successive players entering the coalitions between themselves and the players already present in the coalition.

2 Basic notions and examples. Equivalent representations

In this section the notions of procedure and procedural value will be introduced and demonstrated at simple examples.

The scenario of obtaining and computing the procedural value consists of the following steps:

1. The players arrive in a random order π; all orders (permutations of the set N) are equally probable.
2. Every arriving player, k, brings his marginal contribution, $m_{k,\pi}(v)$, to the coalition of his predecessors.
3. This marginal contribution is divided among all players in $H_{\pi,k}$ according to some fixed procedure.

4. In this way, for every permutation π the whole quantity $v(N)$ is distributed among all players.
5. The procedural value of a player is the expected value (over all orders of arrival) of his part of $v(N)$.

In this paper I confine my attention to simplest possible procedures, under which the distribution of marginal contributions depends neither on players' names nor on their worths in the game. These procedures, thus, divide every marginal contribution in proportions determined solely by the order in which the players form coalitions. Moreover, every marginal contribution is retained within the coalition of the contributor and his predecessors—nothing is left for players who arrive later (some possible variations on relaxing these requirements will be briefly discussed in the last section).

Definition 1a A procedure s on $\mathcal{G}_n$ is a family of nonnegative coefficients $((s_{k,j})_{j=1}^{k})_{k=1}^{n}$ such that $(\forall k)\ \sum_{j=1}^{k} s_{k,j} = 1$.
The coefficient $s_{k,j}$ describes the share of player who is at place j in the order [that is, player $\pi^{-1}(j)$] in the marginal contribution of player $\pi^{-1}(k)$. Obviously, since the player $\pi^{-1}(1)$ has no predecessors, $s_{1,1} = 1$.

Definition 1b The procedural value ψ^s determined by the procedure s on $\mathcal{G}_n$ is defined by the formula

$$\psi_i^s(v) = \mathbf{E}_\pi \sum_{j \in N_{\pi,i}} s_{\pi(j),\pi(i)}\, m_{j,\pi}(v) = \sum_{\pi \in \Pi} \sum_{j \in N_{\pi,i}} \frac{s_{\pi(j),\pi(i)} m_{j,\pi}(v)}{n!}\,. \tag{1}$$

A few examples of procedures with very natural interpretation are presented below.

Procedure 1 $\forall k\ (s_{k,k} = 1$ and $\forall j < k\,,\ s_{k,j} = 0)$, that is, every player retains his entire marginal contribution to every coalition for himself. This procedure of course leads to the Shapley value: $\psi_i^s(v) = \phi_i(v)$.

Procedure 2a Every player gives his entire marginal contribution to his immediate predecessor:

$$s_{1,1} = 1; \quad \forall k > 1\,,\ s_{k,k-1} = 1\,.$$

Procedure 2b Under every ordering of players, all players pass their marginal contributions to the first player in the ordering:

$$\forall k \geq 1\,, \quad s_{k,1} = 1\,.$$

Procedure 2c Every marginal contribution to the formed coalition is divided in equal parts among all players in the coalition, excluding the newcomer (= contributor):

$$s_{1,1} = 1; \quad (\forall k > 1\ \ \forall j < k)\ \ s_{k,j} = \frac{1}{k-1}\,.$$

Procedure 3 All marginal contributions are divided in equal parts between the contributing player and his immediate predecessor:

$$s_{1,1} = 1; \quad \forall k > 1\,,\ s_{k,k} = s_{k,k-1} = \frac{1}{2}\,.$$

Procedure 4 Every marginal contribution is divided in equal parts among all players in the coalition that has just formed, including the contributor:

$$(\forall k \geq 1 \ \ \forall j \leq k) \ \ s_{k,j} = \frac{1}{k}\,.$$

The example below demonstrates an application of procedures 2a, 2c and 4 to a three-person game and the resulting values of the game.

Example 1 In the three-person game with the following characteristing function v:

$v(1) = 1\,,\ v(2) = 2\,,\ v(3) = 3\,,$
$v(12) = 5\,,\ v(13) = 7\,,\ v(23) = 8\,,$
$v(123) = 12\,,$

in the ordering given by $\pi(i) \equiv i$ (players arriving in the order of their numbering) procedure 2a divides $v(N) = 12$ in the following way: 5 to player 1 , 7 to player 2 and 0 to player 3. Procedure 2c under the same ordering leads to division $\left(1 + 4 + \frac{7}{2}\,,\ \frac{7}{2}\,,\ 0\right)$, and procedure 4—to division $\left(1 + \frac{4}{2} + \frac{7}{3}\,,\ \frac{4}{2} + \frac{7}{3}\,,\ \frac{7}{3}\right)$.

Averaging over all six orderings, procedures 2a and 2c yield the same division (4, 4, 4), i.e., the quantity $v(N)$ is shared in equal parts among all players. It is also quite easy to check that the equal division will result from applying procedure 2b in any game. Thus, all three procedures determine the same procedural value—the egalitarian value e, defined by $e(N, v) = \left(\frac{v(N)}{n}, \ldots, \frac{v(N)}{n}\right)$. This is a particular instance of theorem 1 below.

It follows that procedure 3 leads to the division being the arithmetic mean of the equal distribution and the Shapley value. On the other hand, the division resulting from applying procedure 4 to the game v is $\left(\frac{127}{36}\,,\ \frac{142}{36}\,,\ \frac{163}{36}\right)$. More generally, the value determined by procedure 4 is the *solidarity value* σ defined by Nowak and Radzik (1994) as

$$\sigma_i(v) = \sum_{T \ni i} \frac{(t-1)!(n-t)!}{n!} A^0(T)\,,$$

where $A^0(T)$ is the average marginal contribution to coalition T:

$$A^0(T) = \frac{\sum_{k \in T}(v(T) - v(T \setminus k))}{t}\,.$$

We have thus obtained as a by-product a new simple and plausible probabilistic interpretation of the solidarity value: a player's value is his expected gain when every marginal contribution is shared in equal parts among all players in the coalition.

Theorem 1 Equivalent representations: *If* $s = ((s_{k,j})_{j=1}^{k})_{k=1}^{n}$ *and* $t = ((t_{k,j})_{j=1}^{k})_{k=1}^{n}$ *are two procedures such that, for all* k, $s_{k,k} = t_{k,k}$, *then* $\psi^s = \psi^t$.

Proof Given a procedure s, a game $v \in \mathcal{G}_n$ and $i, j \in N$, denote

$$q_{j,i} = \sum_{\pi\,:\,\pi(j)\geq\pi(i)} \frac{1}{n!} s_{\pi(j),\pi(i)}\, m_{j,\pi}(v)\,.$$

The quantity $q_{j,i}$ is the expected value of player i's share in player j's marginal contribution when procedure s is applied to game v. By (1),

$$\psi_i^s(v) = \sum_{j=1}^{n} q_{j,i} = q_{i,i} + \sum_{j\neq i} q_{j,i}$$

and so to prove the theorem it suffices to show that the quantities $q_{j,i}$ depend only on those coefficients $s_{k,l}$ for which $l = k$. For $i = j$ this is obvious. For $i \neq j$ let us partition the sum in the definition of $q_{j,i}$ into sums of terms over all possible coalitions T containing players i, j :

$$q_{j,i} = \sum_{T\supseteq\{i,j\}} \sum_{\pi\,:\,H_{\pi,j}=T} \frac{1}{n!} s_{t,\pi(i)}\, m_{j,\pi}(v)$$

and group the permutations in this sum according to the place (k) assigned to player i:

$$\begin{aligned} q_{j,i} &= \sum_{T\supseteq\{i,j\}} \sum_{k=1}^{t-1} \sum_{\pi\,:\,H_{\pi,j}=T,\pi(i)=k} \frac{1}{n!}\, s_{t,k}(v(T) - v(T\setminus j)) \\ &= \sum_{T\supseteq\{i,j\}} (v(T) - v(T\setminus j)) \sum_{k=1}^{t-1} \frac{(t-2)!(n-t)!}{n!} s_{t,k} \\ &= \sum_{T\supseteq\{i,j\}} \frac{(t-2)!(n-t)!}{n!}\, (v(T) - v(T\setminus j))(1 - s_{t,t})\,. \end{aligned}$$

Thus, we have shown that $q_{j,i}$—and therefore also $\psi_i^s(v)$—do not depend on $s_{k,l}$, $l < k$. □

It is of some interest to observe that for a fixed coalition T the term under the last sum does not depend on i: the expected shares of all players from the set $T \setminus j$ in the marginal contribution of player j to T are the same. It follows from the fact that equal probabilities of all orderings imply that every player in $T \setminus j$ has the same probability (both conditional and unconditional) of being assigned to any of places 1, 2, ... , $t - 1$, and thus of obtaining any of the shares in player j's marginal contribution.

Theorem 1 means that the value determined by procedure s on $\mathcal{G}_n$ depends only on the family of coefficients $(s_{k,k})_{k=1}^{n}$, that is, on the shares of players in their <u>own</u> marginal contributions to coalitions. In other words, changing the coefficients $(s_{k,j})_{j<k}$ leads just to equivalent representations of the same procedural value. Therefore, in what follows we shall use the abbreviated notation $s = (s_1, s_2, \ldots, s_n)$ for any procedure $((s_{k,j})_{j=1}^{k})_{k=1}^{n}$ on $\mathcal{G}_n$, with $s_{j,j} = s_j$ for $j = 1, 2, \ldots, n$.

Moreover, formula (1) for procedural values can be re-stated in a simpler form

$$\psi_i^s(v) = \sum_{\pi\in\Pi} \frac{1}{n!}\left(s_{\pi(i)} m_{i,\pi}(v) + \sum_{j\in N_{\pi,i}\setminus i} \frac{(1-s_{\pi(j)})m_{j,\pi}(v)}{\pi(j)-1}\right) \tag{2}$$

or equivalently

$$\psi_i^s(v) = \sum_{\pi\in\Pi} \frac{s_{\pi(i)} m_{i,\pi}(v)}{n!} + \sum_{\pi:\pi(i)=1} \sum_{j\neq i} \frac{(1-s_{\pi(j)})m_{j,\pi}(v)}{n!}\,. \tag{3}$$

Equation (2) is obtained by distributing each of the quantities $(1-s_{\pi(j)})m_{j,\pi}$ equally among all players in the set $H_{\pi,j} \setminus j$, and Eq. (3) by transferring all these quantities to the "first" player, $\pi^{-1}(1)$. Both these equations are often useful for computing particular procedural values.

A result converse to theorem 1 is also true: a procedural value ψ^s on $\mathcal{G}_n$ uniquely determines the coefficients $(s_1, s_2, \ldots, s_n)$ of its underlying procedure.

Proposition 1 *If $s = (s_1, s_2, \ldots, s_n)$ and $t = (t_1, t_2, \ldots, t_n)$ are two different procedures on $\mathcal{G}_n$, then $\psi^s \neq \psi^t$.*

Proof Denote by k the smallest integer for which $s_k \neq t_k$ (by definition of procedure, $k \geq 2$) and consider the simple game w defined by

$$w(U) = \begin{cases} 0 & \text{if } u < k-1 \quad \text{or} \quad (u = k-1 \text{ and } 1 \in U)\,, \\ 1 & \text{if } u \geq k \qquad\ \ \text{or} \quad (u = k-1 \text{ and } 1 \notin U)\,. \end{cases}$$

In this game player 1 is a null player and all other players are interchangeable, their marginal contributions $m_{j,\pi}(w)$ $(j > 1)$ given by

$$m_{j,\pi}(w) = \begin{cases} 1 & \text{if } ((\pi(j) = k-1 < \pi(1)) \text{ or } (\pi(1) < \pi(j) = k)) \\ 0 & \text{otherwise}\,, \end{cases}$$

and so by formula (3) the values of the null player are

$$\psi_1^s(w) = \frac{1-s_k}{n}, \quad \psi_1^t(w) = \frac{1-t_k}{n} \neq \psi_1^s(w).$$

□

Corollary 1 *There is a one-to-one correspondence $\psi^s \leftrightarrow s$ between procedural values on $\mathcal{G}_n$ and n-vectors $s = (s_1, s_2, \ldots, s_n)$ satisfying $s_1 = 1$, $s_2, \ldots, s_n \in [0, 1]$, with the procedural value ψ^s defined by formula (2) or (3).*

Remark 1 Linearity of values with respect to procedures: If $t = (1, t_2, \ldots, t_n)$ and $u = (1, u_2, \ldots, u_n)$ are two procedures, then for every $\lambda \in [0, 1]$

$$s = \lambda t + (1 - \lambda)u$$

is a procedure, and the corresponding value ψ^s is given by

$$\psi^s = \lambda\psi^t + (1 - \lambda)\psi^u .$$

This implies that the set $\mathcal{P}(\mathcal{G}_n)$ of all procedural values on $\mathcal{G}_n$ is convex, with extreme points being procedures with $s_2, \ldots, s_n \in \{0, 1\}$. Since the Shapley value ϕ and the egalitarian value e are its extreme points, the set of all *egalitarian Shapley values*, defined by Joosten (1996) as convex combinations of ϕ and e and studied in detail by van den Brink et al. (2011) is a diagonal of $\mathcal{P}(\mathcal{G}_n)$. Also, as the solidarity value σ is procedural and all coefficients of its procedure belong to $]0, 1[$, it is an interior point of $\mathcal{P}(\mathcal{G}_n)$. Therefore, also all convex combinations of the Shapley value and the solidarity value, studied by Nowak and Radzik (1996), are procedural and, except for ϕ, belong to the interior of $\mathcal{P}(\mathcal{G}_n)$ (but not to egalitarian Shapley values).

3 Procedural values: properties and axiomatizations

The following observation follows directly from the definition of procedural values:

Proposition 2 *For every procedure s the value ψ^s has the following properties:*

- *efficiency,*
- *linearity: $\psi^s(v+w) = \psi^s(v) + \psi^s(w)$ and $\psi^s(c \cdot w) = c\psi^s(w)$ for every games $v, w \in \mathcal{G}_n$ and every constant $c \in R$,*
- *equal treatment property: if players i and j are interchangeable in the game v, then $\psi_i^s(v) = \psi_j^s(v)$.*

Actually, all procedural values have a property of *symmetry* (“anonymity”): for every permutation π of the set N and every game $v \in \mathcal{G}_n$ and for the game $\pi^* v$ defined by $\pi^* v(S) = v(\pi(S))\ \forall S \subset N$, the equality $\psi_i^s(\pi^* v) = \psi_{\pi(i)}^s(v)$ holds for each $i \in N$. In general, symmetry is stronger than equal treatment property; however, for linear and efficient values the two are equivalent (Malawski 2007, theorem 2).

Remark 2 It follows immediately from proposition 2 and Shapley’s (Shapley 1953) axiomatization of his value that the Shapley value is the unique procedural value with the null player property, i.e. satisfying $\psi_i(v) = 0$ whenever i is a null player in v.

Proposition 2, although trivial, is of some use because it enables us to apply some established properties of linear, symmetric and efficient values in an attempt to find an axiomatic characterization of the class of all procedural values. In particular, the following lemma will be very useful:

Lemma 1 *(Ruiz et al. 1998) Every linear and efficient value ψ having the equal treatment property is of the form:*

$$\psi_i(v) = \sum_{T \subseteq N, T \ni i} \frac{p_t v(T)}{t} - \sum_{T \subset N, T \not\ni i} \frac{p_t v(T)}{n-t}$$

for every v and i, where $p_1, \ldots, p_n \in R$, $p_n = 1$.

Thus, the value of each player is a weighted sum of worths of all coalitions with weights depending only on cardinalities of coalitions.[1] We shall call $p_1, \ldots p_n$ RVZ coefficients of the value ψ.

It can be expected that some kind of *monotonicity* of values might serve as an axiom characterizing the procedural values together with the already established properties. Let us recall the main monotonicity notions[2] of values of TU games.

Definition 2 The value ψ on $\mathcal{G}_n$ is

- *monotonic* (Young 1985) if for every pair of games $v, w \in \mathcal{G}_n$ such that for every coalition T we have $v(T \cup i) - v(T) \geq w(T \cup i) - w(T)$, the inequality $\psi_i(v) \geq \psi_i(w)$ holds;
- *weakly monotonic* (Weber 1988) if in every monotone game v (i.e., satisfying $S \supset T \Rightarrow v(S) \geq v(T)$) the individual values of all players are nonnegative;
- *coalitionally monotonic* if for every coalition T and every two games v, w such that ($v(T) > w(T)$ and $v(S) = w(S)$ for every $S \neq T$) we have $\psi_i(v) \geq \psi_i(w)$ for each $i \in T$;
- *locally monotonic* if the inequality $v(S \cup i) \geq v(S \cup j)$ for *all* coalitions S containing neither player i nor j implies that $\psi_i(v) \geq \psi_j(v)$.

A well-known theorem by Young (1985) states that the Shapley value is the only efficient and monotonic value with equal treatment property[3], and it is clear that ϕ is the unique monotonic procedural value. On the other hand, it is easy to prove that all procedural values are weakly monotonic (just observe that players' values are nonnegative linear combinations of marginal contributions, which themselves are necessarily nonnegative in a monotone game) and it will be shown below that they are also coalitionally and locally monotonic. Moreover, the main theorem states that weak monotonicity together with coalitional monotonicity do indeed characterize the class of procedural values.

For the proof of this characterization, a number of lemmata will be needed. First, we relate coefficients of a procedure to the RVZ coefficients of its procedural value

[1] Some similar interesting forms of the same class of values have been recently presented by Radzik and Driessen (2011) and Chameni Nembua (2012).

[2] There are some ambiguities in the literature concerning the terminology of monotonicity notions—for instance, Weber (1988) and Young (1985) use the term "monotonicity" to denote two different properties. In what follows I use "weak monotonicity" for monotonicity in the sense of Weber (even though some authors use it for various other properties, see e.g. van den Brink et al. (2011)) in order to distinguish between this property and more fundamental Young's monotonicity. Coalitional monotonicity appears under the same name in Young (1985) and Ruiz et al. (1998); local monotonicity is a well-known generalization of a property of power indices, also known as desirability.

[3] In fact, Young assumes symmetry, but it is clear from his proof that equal treatment is sufficient.

(which exist by proposition 2). Further, we specify upper and lower bounds for the RVZ coefficients when the value is weakly and / or coalitionally monotonic.

Lemma 2 *For the value $\psi^s \in \mathcal{P}(\mathcal{G}_n)$ determined by a procedure $s = (s_1, s_2, \dots s_n)$, the coefficients $p_1, \dots, p_n$ are*

$$p_n = 1, \quad p_t = \frac{s_{t+1}}{\binom{n}{t}} \; \textit{for} \quad t < n.$$

Proof By the definition of procedure, a procedural value of a player is always a linear combination of worths of coalitions in the game, and we only need to find the coefficients of this combination. To this end, let us apply formula (3) for computing $\psi_i^s(v)$. In this formula, the worth $v(N)$ of the grand coalition appears

(1) for all $((n-1)!)$ orderings in which $\pi(i) = n$, with coefficient $\frac{s_n}{n!,}$ and
(2) for all $((n-1)!)$ orderings in which $\pi(i) = 1$, with coefficient $\frac{1-s_n}{n!,}$

so $\dfrac{p_n}{n} = (n-1)!\dfrac{s_n}{n!} + (n-1)!\dfrac{1-s_n}{n!}$, and so $\quad p_n = 1$.

The worth $v(T)$ of any other coalition containing player i appears

(1) in all $((t-1)!(n-t)!)$ orderings in which $\pi(i) = t$ and $\pi(T) = \{1, 2, \dots, t\}$, with coefficient $\frac{s_t}{n!,}$ and
(2) in all $((t-1)!(n-t)!)$ permutations in which $\pi(i) = 1$ and $\pi(T) = \{1, 2, \dots, t\}$, with coefficient $\frac{1-s_t}{n!} - \frac{1-s_{t+1}}{n!}$.

Therefore, $\dfrac{p_t}{t} = (t-1)!(n-t)!\dfrac{s_{t+1}}{n!}$, so $\quad p_t = s_{t+1}\binom{n}{t}^{-1}$.

Finally, the worth $v(U)$ of any coalition not including player i appears with coefficient $\frac{-s_{u+1}}{n!}$ for all $(u!(n-u-1)!)$ permutations in which $\pi(i) = u+1$ and $\pi(U) = \{1, 2, \dots, u\}$, so $\dfrac{-p_u}{n-u} = -u!(n-u-1)!\dfrac{s_{u+1}}{n!}$, and again $p_u = s_{u+1}\binom{n}{u}^{-1}$. □

Corollary 2 *Every efficient linear value with equal treatment property on $\mathcal{G}_n$ with coefficients $p_1, \dots p_n$ satisfying*

$$p_n = 1, \quad 0 \le p_t \le \frac{1}{\binom{n}{t}} \; \textit{for} \quad t = 1, 2, \dots n-1 \tag{4}$$

is procedural, and the coefficients of its procedure are

$$s_1 = 1, \quad s_k = p_{k-1} \cdot \binom{n}{k-1} \; \textit{for} \quad k = 2, 3, \dots n\,. \tag{5}$$

Remark 3 Out-of-range coefficients:
It should be noted that in the proofs of theorem 1, propositions 1 and 2 and lemma 2 non-negativity of coefficients $s_{k,j}$ is not used, i.e., it is only required that $\sum_{j=1}^{k} s_{k,j} = 1$ for

each k. Thus, if definition 1a is modified by removing the assumption of $s_{k,j} \in [0, 1]$, all these results will still hold for such "generalized procedures" and "generalized procedural values". A counterpart to corollary 2 will then be true with inequality constraints in (4) dropped and "procedural" replaced by "generalized procedural", and it will imply the following theorem:
A value on $\mathcal{G}_n$ satisfies efficiency, linearity and equal treatment property if and only if it is "generalized procedural". Moreover, the coefficients of its generalized procedure are given by formula (5).

Lemma 3 *A linear efficient value having the equal treatment property is coalitionally monotonic if and only if, for every $t < n$, $p_t \geq 0$.*

Proof Straightforward from the definition of coalitional monotonicity. □

Lemma 4 *Let $c \in R$ and let $t < n$ be a positive integer. If (N, v) is a game such that $v(T) = c$ for every t-person coalition T, then for every player i*

$$\sum_{S:S \ni i\,,\, s=t} \frac{v(S)}{s} - \sum_{S:S \not\ni i\,,\, s=t} \frac{v(S)}{n-s} = 0.$$

Proof Obvious: there are $\binom{n-1}{t-1}$ t-person coalitions to which i belongs and $\binom{n-1}{t}$ t-person coalitions to which i does not belong. □

Lemma 5 *If a linear efficient value on $\mathcal{G}_n$ with the equal treatment property is weakly monotonic, then for every $t = 1, 2, \ldots n-1$ its RVZ coefficients satisfy*

$$(a)\ \binom{n}{t} p_t \leq 1, \quad (b)\ \forall u = 1, 2, \ldots t\ , \ \sum_{s=u}^{t} \binom{n}{s} p_s \geq -1\,.$$

Proof Both inequalities are proved by computing the value of player 1 in an appropriate monotone simple game and using the fact that it must be non-negative because of weak monotonicity of the value. For any fixed but arbitrary $t = 1, 2, \ldots n-1$ and $u = 1, 2, \ldots t$, consider the games w and y with the characteristic functions defined by:

$$w(S) = \begin{cases} 0 & \text{if } s < t \text{ or } (s = t \text{ and } 1 \in S)\,, \\ 1 & \text{if } s > t \text{ or } (s = t \text{ and } 1 \notin S), \end{cases}$$

$$y(S) = \begin{cases} 0 & \text{if } s < u \text{ or } (u \leq s \leq t \text{ and } 1 \notin S)\,, \\ 1 & \text{if } s > t \text{ or } (u \leq s \leq t \text{ and } 1 \in S)\,. \end{cases}$$

When ψ is an efficient linear value with equal treatment property, by lemma 1

$$\psi_1(w) = \sum_{S \subseteq N\,,\, S \ni 1} \frac{p_s w(S)}{s} - \sum_{S \subset N\,,\, S \not\ni 1} \frac{p_s w(S)}{n-s}$$

and similarly for $\psi_1(y)$. Further, since for every coalition S with more than t players $w(S) = y(S) = 1$, both games satisfy the assumption of lemma 4 for every s such that $t < s < n$ (and y also for every $s < u$). Thus, the above sum reduces to

$$\psi_1(w) = \frac{1}{n} + \sum_{S:s\le t\,,\,S\ni 1} \frac{p_s w(S)}{s} - \sum_{S:s\le t\,,\,S\not\ni 1} \frac{p_s w(S)}{n-s}\,,$$

$$\psi_1(y) = \frac{1}{n} + \sum_{S:u\le s\le t\,,\,S\ni 1} \frac{p_s y(S)}{s} - \sum_{S:u\le s\le t\,,\,S\not\ni 1} \frac{p_s y(S)}{n-s}\,.$$

Substituting the worths of coalitions gives

$$\psi_1(w) = \frac{1}{n} + 0 - \binom{n-1}{t} \frac{p_t}{n-t} = \frac{1}{n} \cdot \left[1 - \binom{n}{t} p_t\right],$$

$$\psi_1(y) = \frac{1}{n} + \sum_{s=u}^{t} \binom{n-1}{s-1} \frac{p_s}{s} - 0 = \frac{1}{n} \cdot \left[1 + \sum_{s=u}^{t} \binom{n}{s} p_s\right].$$

Since w and y are monotone games, both expressions must be non-negative if ψ is a weakly monotonic value. The inequality $\psi_1(w) \ge 0$ gives the upper bound for p_t in (a), and $\psi_1(y) \ge 0$ is equivalent to (b). □

Theorem 2 *A value on $\mathcal{G}_n$ is procedural if and only if it satisfies efficiency, linearity, equal treatment property, weak monotonicity and coalitional monotonicity.*

Proof It has been noticed earlier that all procedural values are efficient, linear, symmetric and weakly monotonic, and it follows immediately from the first three properties and lemmata 2 and 3 that they are also coalitionally monotonic. To prove the converse, observe that the RVZ coefficients for any value satisfying the assumptions of the theorem must

- exist, because of efficiency, linearity and equal treatment property,
- be nonnegative, because of coalitional monotonicity,
- satisfy $p_n = 1$ by lemma 1, and
- satisfy $p_t \le \binom{n}{t}^{-1}$ for $t < n$ because of weak monotonicity [lemma 5 (a)],

and so they are exactly such as in corollary 2 and determine the coefficients of a procedure by formula (5). □

All procedural values are also locally monotonic, and in fact local monotonicity can be used instead of coalitional monotonicity in the axiomatization of the class of procedural values. This follows immediately from the following

Lemma 6 *(a) Every coalitionally monotonic value with equal treatment property is locally monotonic.*
(b) Every linear, efficient and locally monotonic value is coalitionally monotonic and has equal treatment property.

Proof (a) We need to show that a player who is "at least as strong" as another player has at least the same value. To this end, consider two players i and j such that $v(S \cup i) \geq v(S \cup j)$ for every coalition S to which neither i nor j belongs, and the game $v_{j \to i}$ defined as follows:

$$v_{j \to i}(T) = \begin{cases} v(T) & \text{when } j \notin T, \\ v(T \cup i \setminus j) & \text{when } i \notin T\,,\; j \in T\,, \\ v(T \setminus j) + v(T) - v(T \setminus i) & \text{when } i, j \in T\,. \end{cases}$$

In this new game player i's marginal contributions are the same as in v but player j's marginal contributions are changed so that for every S such that $i, j \notin S$ we have

$$v_{j \to i}(S \cup j) - v_{j \to i}(S) = v_{j \to i}(S \cup i) - v_{j \to i}(S)\,.$$

This is, i and j are interchangeable in this game, and so if the value ψ has the equal treatment property, then $\psi_i(v_{j \to i}) = \psi_j(v_{j \to i})$.

Moreover, notice that for every coalition T containing player j we have $v_{j \to i}(T) \geq v(T)$ (by the definition of $v_{j \to i}$ and the fact that $v(S \cup i) \geq v(S \cup j)$ for evry S , $i, j \notin S$).

Further, the original game v obtains from $v_{j \to i}$ by sucessive replacements of worths of all coalitions T containing player j, $v_{j \to i}(T)$, by their initial worths, $v(T)$. Since for these coalitions $v_{j \to i}(T) \geq v(T)$, every such replacement (worths of all other coalitions remaining unchanged) cannot increase the value of player j for a coalitionally monotonic value ψ, and so $\psi_j(v) \leq \psi_j(v_{j \to i})$. Therefore,

$$\psi_j(v) \leq \psi_j(v_{j \to i}) = \psi_i(v_{j \to i}) = \psi_i(v)\,.$$

(b) Equal treatment property follows trivially from local monotonicity. To prove coalitional monotonicity, take a value ψ satisfying the assumptions and two games v, w differing only on one coalition T, and assume without loss of generality that $v(T) > w(T)$. Then $v = w + \mu\chi_T$, where $\mu = v(T) - w(T)$ and χ_T is the game defined by

$$\chi_T(S) = \begin{cases} 1 & \text{for } S = T, \\ 0 & \text{for } S \neq T. \end{cases}$$

In this game all marginal contributions of any player $i \in T$ are nonnegative and all marginal contributions of any player $j \notin T$ are nonpositive, so by local monotonicity $\psi_i(\chi_T) \geq \psi_j(\chi_T)$. Together with equal treatment and efficiency, this implies that $\psi_i(\chi_T) \geq 0$ and $\psi_j(\chi_T) \leq 0$. Therefore, since $\mu > 0$,

$$\psi_i(v) = \psi_i(w) + \mu \cdot \psi_i(\chi_T) \geq \psi_i(w)$$

which gives the coalitional monotonicity. □

An immediate corollary of theorem 2 and lemma 6 is another characterization of procedural values using local monotonicity (and disposing of symmetry-type assumption):

Theorem 3 *A value on $\mathcal{G}_n$ is procedural if and only if it satisfies efficiency, linearity, weak monotonicity and local monotonicity.*

I conclude this section with a number of examples demonstrating that the conditions for procedural values in theorems 2 and 3 are independent.

Constructing examples of values fulfilling all conditions except weak or coalitional monotonicity is trivial – by lemmata 3 and 5 it suffices to define values by their RVZ coefficients so that, for some $t < n$, $p_t > \binom{n}{t}^{-1}$ or $p_t < 0$. Examples 2, 2' and 3 present some such values which otherwise are of interest.

Example 2 The *equal surplus value* defined by Driessen and Funaki (1991)

$$c_i(v) = v(i) + \frac{1}{n}\left(v(N) - \sum_{j=1}^{n} v(j)\right)$$

is linear, symmetric, efficient and coalitionally monotonic, but—except for two-person games, for which c is equal to the Shapley value—it is not weakly monotonic. To see this, it suffices to compute its RVZ coefficients which are

$$p_1 = 1 - \frac{1}{n}, \quad p_2 = \ldots = p_{n-1} = 0, \quad p_n = 1$$

and notice that for $n > 2$ we have $p_1 > \binom{n}{1}^{-1}$. Alternatively, take $z_T \in \mathcal{G}_n$—the game of incidence with coalition $T \subset N$,

$$z_T(S) = \begin{cases} 1 & \text{when } S \cap T \neq \emptyset\,, \\ 0 & \text{when } S \cap T = \emptyset\,. \end{cases}$$

and check that the value of any player $j \notin T$ in this (monotone) game is $c_j(z_T) = \frac{1-t}{n}$, so for $2 \leq t < n$ it is negative. Thus, c is not weakly monotonic and hence not procedural.

Example 2' The *least square prenucleolus* introduced by Ruiz et al. (1996) as the solution of a particular minimization problem and characterized by the formula

$$\lambda_i(v) = \frac{v(N)}{n} + \frac{1}{2^{n-2}} \sum_{S \ni i} v(S) - \frac{1}{n \cdot 2^{n-2}} \sum_{j=1}^{n} \sum_{S \ni j} v(S)$$

also satisfies all conditions in theorems 2 and 3 except weak monotonicity. A straightforward calculation of its RVZ coefficients demonstrates that $p_t = \dfrac{t(n-t)}{n \cdot 2^{n-2}}$ and so—except for two- and three-person games, where λ is equal to the Shapley value—some of them exceed $\binom{n}{t}^{-1}$, which cannot be the case for a weakly monotonic value (lemma 5).

Example 3 The value $\tilde{\phi}$ assigning to each player the arithmetic mean of Shapley values of all *other* players:

$$\tilde{\phi}_i(v) = \frac{1}{n-1}\sum_{j\neq i}\phi_j(v) = \frac{v(N)-\phi_i(v)}{n-1}$$

is obviously linear, symmetric, efficient and weakly monotonic, but neither coalitionally nor locally monotonic and hence not procedural.

In view of these examples, it would be interesting to find reasonable interpretation of classes of values on $\mathcal{G}_n$ which contain $\mathcal{P}(\mathcal{G}_n)$ but are assumed only to satisfy one of the monotonicity postulates (weak or coalitional) instead of both. In case of weak monotonicity theorem 4 in the next section provides an answer.

Example 4 The *Banzhaf value* β

$$\beta_i(v) = \frac{1}{2^{n-1}}\sum_{T\subset N}(v(T\cup i) - v(T))$$

satisfies all conditions in theorems 2 and 3 except efficiency.

Example 5 The value k dividing $v(N)$ among players proportionally to their numbers,

$$k_i(v) = \frac{2i}{n(n+1)}\cdot v(N)$$

satisfies all conditions in theorem 2 except equal treatment property, and all conditions in theorem 3 except local monotonicity.

Example 6 The value ζ dividing $v(N)$ equally among all players with greatest "standalone" worths when $v(N)$ is nonnegative and among all players with least "standalone" worths when $v(N)$ is negative, and assigning zero to all other players:

$$\zeta_i(v) = \begin{cases} \frac{v(N)^+}{y} & \text{if } i\in Y\setminus Z\,, \\ \frac{v(N)^-}{z} & \text{if } i\in Z\setminus Y\,, \\ 0 & \text{if } i\notin Y\cup Z \end{cases}$$

where $Y = \text{Argmax}_{j\in N}v(j)\,,\quad Z = \text{Argmin}_{j\in N}v(j)$ (in case $Y\cap Z\neq\emptyset$, which is equivalent to $v(1) = v(2) = \cdots = v(n)$, define $\zeta(v) = e(v)$), satisfies all assumptions of theorems 2 and 3 except linearity. Efficiency, equal treatment and weak monotonicity of ζ are obvious, and it is clear that ζ is also locally monotonic. Moreover, the individual value $\zeta_i(v)$ of (any) player i (i) by definition, does not depend on $v(T)$ for any $T\ni i$ such that $\{i\}\neq T\neq N$, and an elementary analysis of all possible cases of signs of $v(N)$ and orderings of standalone worths shows that for every player i, $\zeta_i(v)$ (ii) is a non-decreasing function of $v(i)$ and (iii) is a non-decreasing function of $v(N)$. Therefore, ζ is also coalitionally monotonic.

4 Inverse procedures and other generalizations

This paper has dealt only with simplest kind of procedures under which the proportion s_k of own marginal contribution retained by the contributing player depends only on his position in the ordering, k. The most obvious extension consists, of course, of admitting more general classes of procedures.

It has been observed (remark 3) that allowing for negative coefficients $s_{k,j}$ leads to all linear, efficient and symmetric values. This generalization, however, is of relatively little interest, since such "generalized procedures" have no natural interpretation as sharing rules. In this section I analyse in some detail the values resulting from changing the set of beneficiaries of the redistribution of marginal contributions, and briefly signal other possibilities.

4.1 Sharing with successors: self-dual values

One modest and simple variation on procedural values comprises sharing the marginal contributions with successors instead of predecessors in the order of arrival. Let us call sharing procedures of this kind inverse procedures, and the resulting values—inversely procedural values. An inverse procedure on $\mathcal{G}_n$ is a family of nonnegative coefficients

$$((z_{k,j})_{j=k}^n)_{k=1}^n \quad \text{such that } \forall k\,, \sum_{j=k}^n z_{k,j} = 1\,;$$

$z_{k,j}$ is the share of player who comes as j-th ($j \geq k$) in the marginal contribution of the player coming as k-th to the coalition $H_{\pi,k}$. Since one can easily repeat the proof of theorem 1 to show that an analogous theorem on equivalent representation holds also for these sharing systems, an inverse procedure z is uniquely determined by the family $\left(z_{k,k}\right)$. Let us then take $z_k := z_{k,k}$ and $z = (z_k)_{k=1}^n$. The inversely procedural value ${}^z\psi$ obtained from an inverse procedure z is defined by the equation

$${}^z\psi_i(v) = \sum_{\pi\in\Pi} \frac{z_{\pi(i)} m_{i,\pi}(v)}{n!} + \sum_{\pi:\pi(i)=n} \sum_{j\neq i} \frac{(1 - z_{\pi(j)}) m_{j,\pi}(v)}{n!}\,. \tag{6}$$

Proposition 3 *(a) Every inversely procedural value ${}^z\psi$ is linear, symmetric and efficient, and its RVZ coefficients are*

$$p_n = 1, \quad p_t = \frac{z_t}{\binom{n}{t}} \quad \textit{for} \quad t = 1, 2, \ldots n-1\,;$$

(b) ${}^{z}\psi(v) = \psi^{t}(v^{*})$, *where* $t_k = z_{n+1-k}$ *and* v^{*} *is the dual game of* v, *i.e.* $v^{*}(S) = v(N) - v(N \setminus S) \ \ \forall S$ *;*
(c) ${}^{z}\psi(v) = \psi^{s}(v)$, *where* $s_1 = 1$ *and* $s_k = z_{k-1}$ *for* $k = 2, 3, \ldots, n$.

Proof The proof of (a) repeats that of lemma 2: one has to count how many times and with what coefficients worths of coalitions occur in (inversely procedural) values of players. Part (b) is trivial, and (c) follows immediately from lemma 2 and (a). □

It follows that sharing with successors does not extend the class of obtained values: every inversely procedural value ${}^{z}\psi$ coincides with a procedural value derived from the procedure under which every player coming as j-th transfers to the predecessors the same fraction of his marginal contribution to $H_{\pi,j}$ which he had to leave to his successors when coming as $(j-1)$-th under the inverse procedure z.

One nice by-product of proposition 3 is a simple characterization of self-dual procedural values. A value ψ is self-dual if and only if, for every game v, $\psi(v) = \psi(v^{*})$.

Proposition 4 ψ^{s} *is a self-dual procedural value on* $\mathcal{G}_n$ *if and only if*

$$s_k = s_{n+2-k} \quad \textit{for every } k = 2, 3 \ldots n\,.$$

Proof By proposition 3 (b) and (c), self-duality of a procedural value ψ^{s} on $\mathcal{G}_n$ [where $s = (1, s_2, \ldots s_n)$] is equivalent to

$$\psi^{s}(v) = \psi^{s}(v^{*}) = {}^{z}\psi(v) = \psi^{t}(v)$$

for every $v \in \mathcal{G}_n$, where

$$z = (z_1, \ldots, z_{n-1}, 1)\ ,\ t = (1, t_2, \ldots t_n),$$
$$z_1 = s_n\ ,\ z_2 = s_{n-1}, \ldots, z_n = s_1\ ,\ t_1 = 1\ ,\ t_2 = z_1, \ldots, t_n = z_{n-1}$$

and so $t_2 = s_n\ ,\ \ldots, t_n = s_2$. Since $\psi^{t} = \psi^{s} \ \Rightarrow t = s$ (proposition 1), this completes the proof. □

This implies that, in particular, all self-dual procedural values on $\mathcal{G}_2$ and $\mathcal{G}_3$ are egalitarian Shapley values (on $\mathcal{G}_2$, moreover, all procedural values are self-dual). For games with more than three players there are self-dual procedural values which are not egalitarian Shapley—for instance, one such value is given by a procedure $s = (1, a, b, a)$ on $\mathcal{G}_4$, $a, b \in [0, 1]$, $a \neq b$.

4.2 Sharing with everybody. Extended procedures

One could also demand that both predecessors and successors of a contributing player have positive shares in his marginal contribution. A simplest sharing system of this kind was proposed as the "sock story" by Joosten[4] in which a fixed proportion $1-\beta$ of marginal contribution of each player in every ordering is put into a sock; afterwards,

[4] Joosten (personal communication).

this common pool is divided equally among all players. Clearly, the resulting value is the egalitarian Shapley value $(1-\beta)\cdot e+\beta\cdot\phi$ and as such it is also procedural with $s=(1,\beta,\ \dots,\beta)$ and inversely procedural.

When the proportion of marginal contribution put into the "sock" may depend on the place of the contributing player in the ordering, we obtain sharing rules defined by n-tuples of coefficients $u=(u_1,u_2,\dots u_n)$, $\forall i\ u_i\in[0,1]$; each player $\pi^{-1}(k)$ retains $u_k\cdot m_{k,\pi}(v)$ for himself and gives the rest to the common pool. The resulting "common pool values" are given by the formula

$$\psi_i^{(u)}(N,v)=\sum_{\pi\in\Pi}\left(\frac{u_{\pi(i)}m_{i,\pi}(v)}{n!}+\frac{1}{n}\sum_{j=1}^{n}\frac{(1-u_{\pi(j)})m_{j,\pi}(v)}{n!}\right)$$

and it is not difficult to check that they are again all procedural—the coefficients of the procedure s equivalent[5] to the common pool system u are

$$s_1=1,\quad s_k=\frac{n-(k-1)}{n}u_{k-1}+\frac{k-1}{n}u_k\quad\text{for}\quad k=2,3,\dots,n\,.$$

A more general way of sharing marginal contributions with all players is described by an extended procedure—a family of nonnegative coefficients

$$(q,r,s)=(q_k,r_k,s_k)_{k=1}^{n}$$

such that $q_1=r_n=0$ and, for each k , $q_k+r_k+s_k=1$. The underlying sharing rule prescribes dividing every marginal contribution of the player at k-th position in the ordering ($k=1,2,\dots,n$) in the following proportions: q_k jointly for all predecessors, r_k jointly for all successors and s_k for the contributing player. Thus, analogously to (3) and (6), the extended procedural value $\psi_1^{(q,r,s)}$ is given by the formula

$$\psi_i^{(q,r,s)}(v)=\sum_{\pi\in\Pi}\frac{s_{\pi(i)}m_{i,\pi}(v)}{n!}+\sum_{\pi:\pi(i)=1}\sum_{j\neq i}\frac{q_{\pi(j)}m_{j,\pi}(v)}{n!}+\sum_{\pi:\pi(i)=n}\sum_{j\neq i}\frac{r_{\pi(j)}m_{j,\pi}(v)}{n!}\,.$$

Obviously, extended procedural values are linear, symmetric and weakly monotonic. However, this generalization is real: not all extended procedural values are procedural. This is best seen at the example of the extended procedure on $\mathcal{G}_2$ under which $r_1=q_2=1$, i.e., the entire marginal contribution of the first player in the ordering is given to the second player and vice versa. Then, of course,

$$\psi_1^{(q,r,s)}(v)=\phi_2(v)\quad\text{and}\quad\psi_2^{(q,r,s)}(v)=\phi_1(v)$$

[5] However, many different systems may lead to the same procedural value: e.g., the egalitarian Shapley value on $\mathcal{G}_3$ with $\beta=\frac{1}{2}$ is given, among others, by "common pool" systems $\left(\frac{1}{4},1,\frac{1}{4}\right)$ and $\left(\frac{3}{4},0,\frac{3}{4}\right)$.

– that is, this extended procedural value is obtained by swapping the individual Shapley values between players, which clearly violates local and thus also coalitional monotonicity. Thus, $\psi = \tilde{\phi}$ on $\mathcal{G}_2$ (cf. example 3), and it is easily seen that also for $n > 2$ the value $\tilde{\phi}$ is extended procedural, with $r_i = \frac{n-i}{n-1}, q_i = \frac{i-1}{n-1}$ for $i = 1, 2, \ldots, n$.

However, it turns out that the class of all extended procedural values also has an axiomatic characterization analogous to that in theorem 2 for procedural values and using a natural monotonicity postulate. This is theorem 4 below, preceded by a lemma on the RVZ coefficients of extended procedural values.

Lemma 7 *For the value $\psi^{(q,r,s)}$ on $\mathcal{G}_n$ determined by an extended procedure $(q, r, s) = (q_1, r_1, s_1, q_2, r_2, s_2, \ldots q_n, r_n, s_n)$, the RVZ coefficients $p_1, \ldots, p_n$ are*

$$p_n = 1, \quad p_t = \frac{s_t + q_t - q_{t+1}}{\binom{n}{t}} = \frac{1 - r_t - q_{t+1}}{\binom{n}{t}} \quad \textit{for} \quad t < n.$$

The proof goes exactly the same way as that of lemma 2.

Theorem 4 *A value on $\mathcal{G}_n$ is extended procedural if and only if it satisfies efficiency, linearity, equal treatment property and weak monotonicity.*

Proof The “only if” part is obvious. To prove “if” we shall construct, for any value ψ satisfying all four conditions, a (not necessarily unique) extended procedure (q, r, s) such that $\psi = \psi^{(q,r,s)}$. To this end, let us take the RVZ coefficients $p_1, p_2, \ldots p_{n-1}$ of ψ and denote $\hat{p}_t = \binom{n}{t} p_t$ and

$$l_0 = 0, \quad l_{t+1} = (l_t + \hat{p}_{t+1})^- \quad \text{for} \quad t = 0, 1, \ldots n-1\,.$$

Further, define the family of coefficients $(q, r, s) = (q_t, r_t, s_t)_{t=1}^n$ as follows: $q_1 = r_n = 0$ and

$$r_t = \begin{cases} 1 + l_{t-1} & \text{if } l_t < 0\,, \\ 1 - \hat{p}_t & \text{if } l_t = 0\,, \end{cases}$$
$$q_{t+1} = -l_t,\, s_t = 1 - q_t - r_t$$

for $t = 1, 2, \ldots n-1$. Since ψ is weakly monotonic, it follows from lemma 5 that $-1 \leq \hat{p}_t \leq 1$ and $-1 \leq l_t \leq 0$ for each t. Therefore it is easy to check that all coefficients q_t, r_t and s_t are non-negative, and this implies that (q, r, s) is indeed an extended procedure. Moreover, for every $t = 1, 2, \ldots, n-1$, we have $\hat{p}_t = 1 - r_t - q_{t+1}$, which by lemma 7 means that $\psi = \psi^{(q,r,s)}$. □

In particular, an extended procedural value $\psi^{(q,r,s)}$ on $\mathcal{G}_n$ is procedural if and only if for each $t < n$, $r_t + q_{t+1} \leq 1$—i.e., if the extra condition of coalitional monotonicity is fulfilled – and then the coefficients of its procedure are

$$s = (1, 1 - q_2 - r_1, \ldots, 1 - q_n - r_{n-1})\,.$$

4.3 Other extensions

A further departure from simple procedures considered in this paper involves making the sharing proportions depend on worths of coalitions in the game whose value is determined. This significantly broadens the class of values obtained; it will then include, in particular, the equal surplus value, the Ju et al. (2004) "consensus value" and various non-linear values.

Another direction of generalization is abandoning equal treatment by assigning exogenous weights to players as in Kalai and Samet (1987). Such weights can then give rise to potentially interesting classes of "weighted procedural values", determining either the shares of individual players in portions of marginal contributions left for contributors' predecessors or the (unequal) probabilities of particular orderings of players.

Acknowledgments I am grateful to René van den Brink, Reinoud Joosten, Gerard van der Laan, Andrzej Nowak, Tadeusz Radzik, Andrzej Wieczorek and in particular to Elena Yanovskaya for many helpful discussions. My thanks are also to two anonymous referees and the associate editor of IJGT whose numerous insightful remarks led to a large improvement of this paper. Any remaining errors are, of course, my own.

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