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ORIGINAL ARTICLE

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A new combined asymptotic-tolerance model of thermoelasticity problems for thin uniperiodic cylindrical shells

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Abstract The objects of consideration are thin linearly thermoelastic Kirchhoff–Love-type circular cylindrical shells having a periodically microheterogeneous structure in circumferential direction (*uniperiodic shells*). The aim of this contribution is to formulate and discuss *a new averaged mathematical model for the analysis of selected dynamic thermoelasticity problems for the shells under consideration*. This so-called *combined asymptotic-tolerance model* is derived by applying *the combined modelling* including *the consistent asymptotic and the tolerance non-asymptotic modelling techniques,* which are conjugated with themselves into *a new procedure*. The starting equations are the well-known governing equations of linear Kirchhoff–Love theory of thin elastic cylindrical shells combined with Duhamel–Neumann thermoelastic constitutive relations and coupled with the known linearized Fourier heat conduction equation. For the periodic shells, the starting equations have highly oscillating, non-continuous and periodic coefficients, whereas equations of the proposed model have constant coefficients dependent also on a cell size.

Keywords Thermoelasticity problems · Uniperiodic cylindrical shells · Combined asymptotic-tolerance modelling

1 Introduction

Thin linearly thermoelastic Kirchhoff–Love-type circular cylindrical shells with a periodically micro-inhomogeneous structure in the circumferential direction are objects of consideration. Shells of this kind are termed *biperiodic*. At the same time, the shells have constant structure in axial direction. By periodic inhomogeneity we shall mean periodically varying thickness and/or periodically varying inertial, elastic and thermal properties of the shell material. We restrict our considerations to those uniperiodic cylindrical shells,

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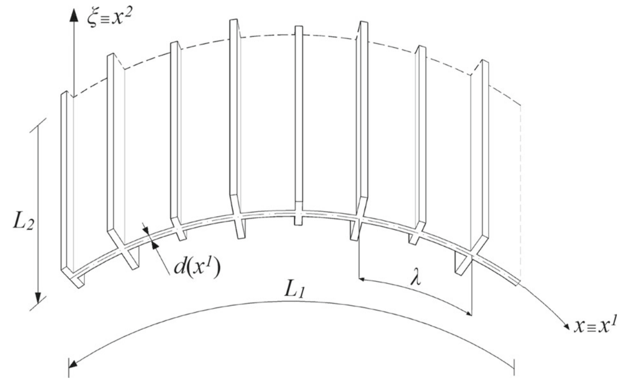

Fig. 1 Fragment of the shell reinforced by two families of uniperiodically spaced ribs

which are composed of a large number of identical elements. Moreover, every such element, called *a periodicity cell*, can be treated as a thin shell. Typical examples of such shells are presented in Figs. 1 (stiffened shell) and 2 (a shell composed of two kinds of periodically distributed materials).

Thermoelastic problems of periodic structures (shells, plates, beams) are described by partial differential equations with periodic, highly oscillating and discontinuous coefficients. Thus, these equations are too complicated to constitute the basis for investigations of the engineering problems. To obtain averaged equations with constant coefficients, many different approximate modelling methods for structures of this kind have been formulated. Periodic cylindrical shells (plates) are usually described using *homogenized models* derived by applying *asymptotic methods*. These asymptotic models represent certain equivalent structures with constant or slowly varying rigidities and averaged mass densities. Unfortunately, the asymptotic models neglecting *the effect of a periodicity cell size* on the overall shell behaviour (*the length-scale effect*). The mathematical foundations of this modelling technique can be found in Bensoussan et al. [1], Jikov et al. [2]. Applications of the asymptotic homogenization procedure to modelling of stationary and non-stationary phenomena for microheterogeneous shells (plates) are presented in a large number of contributions. From the extensive list on this subject we can mention paper by Lutoborski [3] and monographs by Lewiński and Telega [4], Andrianov et al. [5].

The length-scale effect can be taken into account using *the non-asymptotic tolerance averaging technique*. This technique is based on the concept of *the tolerance relations* related to the accuracy of the performed measurements and calculations. The mathematical foundations of this modelling technique can be found in Woźniak and Wierzbicki [6], Woźniak et al. [7,8] Ostrowski [9]. A certain extended version of the tolerance modelling technique has been proposed by Tomczyk and Woźniak in [10]. For periodic structures, *governing equations of the tolerance models have constant coefficients dependent also on a cell size*. Some applications of this averaging method to the modelling of mechanical and thermomechanical problems for various periodic structures are shown in many works. We can mention here monograph by Tomczyk [11] and papers by Tomczyk and Litawska [12–14], Tomczyk et al. [15–18], where the length-scale effect in mechanics of periodic cylindrical shells is investigated; papers by Baron [19], where dynamic problems of medium thickness periodic plates are studied and by Marczak and Jędrysiak [20], Marczak [21,22], where dynamics of periodic sandwich plates is analysed; papers by Jędrysiak [23–25], which deal with stability of thin periodic plates; papers by Łaciński and Woźniak [26], Rychlewska et al. [27], Ostrowski and Jędrysiak [28], Kubacka and Ostrowski [29], where problems of heat conduction in conductors with periodic structure are analysed. Let us also mention papers by Tomczyk and Gołąbczak [30], Tomczyk et al. [31,32], which deal with coupled thermoelasticity problems respectively for thin cylindrical shells with micro-periodic structure in circumferential direction (*uniperiodic shells*) and for thin cylindrical shells with micro-periodic structure in circumferential and axial directions (*biperiodic shells*). The extended list of references on this subject can be found in [6–9,11].

The tolerance averaging technique was also adopted to formulate mathematical models for analysis of various mechanical and thermomechanical problems for functionally graded solids, e.g. for heat conduction in longitudinally graded hollow cylinder by Ostrowski and Michalak [33,34], for thermoelasticity of transversally graded laminates by Pazera and Jędrysiak [35], Pazera et al. [36], for dynamics for functionally graded annular plates by Wirowski and Rabenda [37], for dynamics or stability of functionally graded thin cylindrical shells by Tomczyk and Szczerba [38–41].

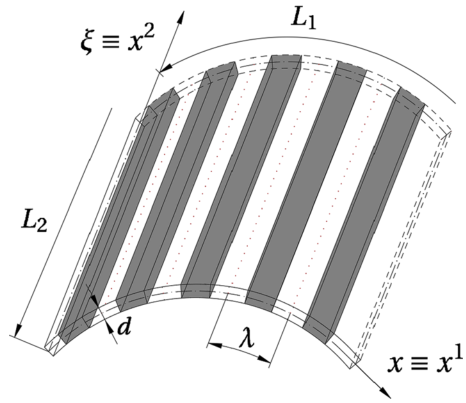

Fig. 2 Fragment of the shell composed of two materials periodically and densely distributed in circumferential direction

Let us note, that the comprehensive review of the literature on the existing theories dealing with modelling and analysis of functionally graded structures is presented by Sofiyev [42]. In this article, analytical solutions to various dynamic and stability problems for such structures, e.g. for functionally graded sandwich or layered conical shells, are discussed in detail.

The aim of this contribution is to formulate and discuss *a new averaged mathematical model for the analysis of selected dynamic thermoelasticity problems for the shells under consideration.* This so-called *combined asymptotic-tolerance model* is derived by applying *the combined modelling* [8,9] including *the consistent asymptotic and the tolerance non-asymptotic modelling techniques,* which are conjugated with themselves into *a new procedure*.

The starting equations are the well known governing equations of linear Kirchhoff–Love theory of thin elastic cylindrical shells combined with Duhamel–Neumann thermoelastic constitutive relations and coupled with the known linearized Fourier heat conduction equation, in which the heat sources are neglected. For the micro-periodic shells under consideration, the starting equations mentioned above have highly oscillating, non-continuous and periodic coefficients. Contrary to the starting equations, governing equations of the averaged model proposed here have constant coefficients. Moreover, some of them *depend also on a characteristic cell length dimension*. Hence, this model makes it possible to study the effect of a microstructure size on the thermoelastic shell behaviour (the length-scale effect). This effect plays an important role in many special dynamic thermoelasticity problems in micro-periodic structures.

The combined modelling will be realized in two steps. The variational approach to the asymptotic and tolerance modelling of microheterogeneous media will be applied, cf. [8,9]. The first step is based on *the consistent asymptotic averaging of integral functional* describing thermoelastic behaviour of the shells and then on using *the extended stationary action principle* [8,9]. *Asymptotic (macroscopic) model* obtained in this step has *constant coefficients, but independent of the period length*. The second step is based on *the tolerance averaging of integral functional* describing thermoelastic behaviour of the shells and then on using *the extended stationary action principle*. It is worth mentioning that the asymptotic or tolerance model equations cannot be derived from the principle of stationary action in its classical form, because heat conduction equation contains the odd derivatives of unknown functions with respect to time argument. *The tolerance microscopic model* obtained in the second step of the combined modelling *has constant coefficients*. Moreover, *some of these coefficients depend on a cell size*. Asymptotic and tolerance models are coupled with each other under assumption that in the framework of the macroscopic model the solutions to the problem under consideration are known.

Note that *a new mathematical asymptotic-tolerance model of selected dynamic thermoelasticity problems for thin cylindrical shells with two-directional periodic microstructure in directions tangent to the shell mid-surface* (*biperiodic shells*) has been proposed by Tomczyk et al. [32]. However, this model does not make it possible to analyse thermoelasticity problems of *uniperiodic shells* being objects of considerations here. In the tolerance approach applied in the combined asymptotic-tolerance modelling, uniperiodic shells are not special cases of biperiodic ones. The tolerance (microscopic) model for uniperiodic shells obtained in the second step of the combined modelling and that of biperiodic shells have to be led out independently. It follows from the fact that *the modelling physical reliability conditions* for uniperiodic shells are hold only in one periodicity direction, whereas for biperiodic shells these conditions are hold in two periodicity directions tangent to the shell midsurface. It means that *the modelling physical reliability conditions* for uniperiodic shells are less restrictive than pertinent conditions for biperiodic shells. Similarities and differences between the combined model for uniperiodic shells proposed here and the corresponding combined model for biperiodic shells presented in [32] will be discussed. It will be shown that tolerance part of the combined model for uniperiodic shells is more complicated than tolerance part of the combined model for biperiodic shells. It will be shown that microscopic equations for uniperiodic shells contain a lot of length-scale terms, which do not have counterparts in the microscopic equations for biperiodic shells.

As examples, two special length-scale problems will be analysed. The first of them refers to the derivation of formula for the frequency of the cell-dependent transversal free micro-vibrations. The second one deals with investigations of the effect of a cell size on the shape of initial distributions of temperature micro-fluctuations caused by a micro-periodic structure of the shells under consideration.

Note, that the combined asymptotic-tolerance model can also be derived by applying the *orthogonalization approach* to the asymptotic and tolerance modelling of microheterogeneous media. The orthogonalization method is based on the asymptotic/tolerance averaging of the partial differential equations describing thermoelasticity behaviour of the micro-periodic shells under consideration and then on using *the residual orthogonality conditions* [6,9,10].

The periodic shells being objects of consideration in this contribution are widely applied in civil engineering, most often as roof or bridge girders. They are also widely used as housings of reactors and tanks. Periodic shells having small length dimensions are elements of air-planes, ships and machines.

2 Formulation of the problem: starting equations

We assume that x^1 and x^2 are coordinates parametrizing the shell midsurface M in circumferential and axial directions, respectively. We denote $x \equiv x^1 \in \Omega \equiv (0, L_1)$ and $\xi \equiv x^2 \in \Xi \equiv (0, L_2)$, where L_1, L_2 are length dimensions of M, cf. Figs. 1 and 2. Let $O\,\bar{x}^1\bar{x}^2\bar{x}^3$ stand for a Cartesian orthogonal coordinate system in the physical space E^3 and denote $\bar{\mathbf{x}} \equiv (\bar{x}^1, \bar{x}^2, \bar{x}^3)$. Let us introduce the orthonormal parametric representation of the underformed cylindrical shell midsurface M by means of $M \equiv \left\{ \bar{\mathbf{x}} \in E^3 : \bar{\mathbf{x}} = \bar{\mathbf{r}}\left(x^1, x^2\right), \left(x^1, x^2\right) \in \Omega \times \Xi \right\}$, where $\bar{\mathbf{r}}(\cdot)$ is the smooth invertible function such that $\partial\,\bar{\mathbf{r}}/\partial x^1 \cdot \partial\,\bar{\mathbf{r}}/\partial x^2 = 0, \partial\,\bar{\mathbf{r}}/\partial x^1 \cdot \partial\,\bar{\mathbf{r}}/\partial x^1 = 1, \partial\,\bar{\mathbf{r}}/\partial x^2 \cdot \partial\,\bar{\mathbf{r}}/\partial x^2 = 1$. Note, that derivative $\partial\,\bar{\mathbf{r}}/\partial x^\alpha, \alpha = 1, 2$, should be understood as differentiation of each component of $\bar{\mathbf{r}} \in E^3$, i.e. $\partial\,\bar{\mathbf{r}}/\partial x^\alpha = [\partial \bar{r}^1/\partial x^\alpha, \partial \bar{r}^2/\partial x^\alpha, \partial \bar{r}^3/\partial x^\alpha]$.

Let $d(x)$, r stand for the shell thickness and the midsurface curvature radius, respectively.

Throughout the paper, indices $\alpha, \beta, \ldots$ run over 1, 2 and are related to midsurface parameters x^1, x^2, summation convention holds. Partial differentiation related to x^α is represented by ∂_α, where $\partial_\alpha = \partial/\partial x^\alpha$. Moreover, it is denoted $\partial_{\alpha\ldots\delta} \equiv \partial_\alpha \ldots \partial_\delta$. Differentiation with respect to time coordinate $t \in \mathrm{I} = [t_0, t_1]$ is represented by the overdot.

Let $a_{\alpha\beta}$ and $a^{\alpha\beta}$ stand for the covariant and contravariant midsurface first metric tensors, respectively. Under orthonormal parametrization introduced on M, $a_{\alpha\beta}$ and $a^{\alpha\beta}$ are unit tensors. Denote by $b_{\alpha\beta}$ the covariant midsurface second metric tensor. Under orthonormal parametrization introduced on M, components of tensor $b_{\alpha\beta}$ are: $b_{22} = b_{12} = b_{21} = 0, b_{11} = -r^{-1}$.

The basic cell Δ and an arbitrary cell $\Delta(x)$ with the centre at point $x \in \Omega_\Delta$ are defined by means of: $\Delta \equiv [-\lambda/2,\ \lambda/2]$, $\Delta(x) \equiv x + \Delta$, $\Omega_\Delta \equiv \{x \in \Omega : \Delta(x) \subset \Omega\}$, where $\lambda \equiv \lambda_1$ is a cell length dimension in $x \equiv x^1$-direction, cf. Figs. 1 and 2. Period λ, called *the microstructure length parameter*, satisfies conditions: $\lambda / \sup_{x\in\Omega} d(x) \gg 1, \lambda/r \ll 1$ and $\lambda/\mathrm{L}_1 << 1$.

It is assumed that the cell Δ has a symmetry axis for $z = 0$, where $z \equiv z^1 \in [-\lambda/2,\ \lambda/2]$. It is also assumed that inside the cell the geometrical, elastic, inertial and thermal properties of the shell are described by even functions of argument z.

Denote by $u_\alpha = u_\alpha(x, \xi, t)$, $w = w(x, \xi, t)$, $(x, \xi) \in \Omega \times \Xi$, $t \in \mathrm{I}$, the shell displacements in directions tangent and normal to M, respectively. Elastic properties of the shell are described by shell stiffness tensors $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $x \in \Omega$. Let $\mu(x)$ stand for a shell mass density per midsurface unit area. In the thermoelasticity problems discussed in this contribution, the external forces tangent and normal to M will be neglected.

Denote by $\theta(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, the temperature field treated as the temperature increment from a certain constant reference temperature T_0 (by reference temperature we shall mean the zero stress temperature). It is assumed that $\theta/T_0 \ll 1$. Let $\bar{d}^{\alpha\beta}(x)$, $x \in \Omega$, stand for the membrane thermal stiffness tensor (tensor of thermoelastic moduli: $\bar{d}^{\alpha\beta} = D^{\alpha\beta\gamma\delta}\alpha_{\gamma\delta}$, where $\alpha_{\gamma\delta}$ are coefficients of thermal expansion). Denote by $K^{\alpha\beta}(x)$ and $c(x)$, $x \in \Omega$, the tensor of heat conductivity and the specific heat, respectively. The heat sources will be neglected. For biperiodic shells, $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $\mu(x)$, $\bar{d}^{\alpha\beta}(x)$, $K^{\alpha\beta}(x)$, $c(x)$ are periodic, highly oscillating and non-continuous functions with respect to argument $x \in \Omega$.

It is assumed that the temperature along the shell thickness is constant. From this restriction it follows that only the coupling between temperature field θ and membrane stresses occurs (this coupling is described by tensor $\bar{d}^{\alpha\beta}(x)$), while the coupling of temperature and bending stresses is absent.

The starting equations are the well-known governing equations of linear Kirchhoff–Love theory of thin elastic cylindrical shells combined with Duhamel–Neumann thermoelastic constitutive relations and coupled with the known linearized Fourier heat conduction equation, in which the heat sources are neglected [43–47]. Thus, the starting equations consist of:

(a) the Duhamel–Neumann stress–strain-temperature relations

$$\begin{aligned} &n^{\alpha\beta}(x,\xi,t) = D^{\alpha\beta\gamma\delta}(x)\varepsilon_{\gamma\delta}(x,\xi,t) - \bar{d}^{\alpha\beta}(x)\theta(x,\xi,t),\\ &m^{\alpha\beta}(x,\xi,t) = B^{\alpha\beta\gamma\delta}(x)\kappa_{\gamma\delta}(x,\xi,t), \quad (x,\xi,t) \in \Omega\times\Xi\times \mathrm{I}, \end{aligned} \tag{1}$$

where

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\partial_\beta u_\alpha + \partial_\alpha u_\beta) - b_{\alpha\beta}w, \quad \kappa_{\alpha\beta} = -\partial_{\alpha\beta}w, \tag{2}$$

b) the dynamic equilibrium equations

$$\partial_\beta n^{\alpha\beta} - \mu a^{\alpha\beta}\ddot{u}_\beta = 0, \quad \partial_{\alpha\beta}m^{\alpha\beta} + b_{\alpha\beta}n^{\alpha\beta} - \mu\ddot{w} = 0, \tag{3}$$

which after combining with (1) and (2) are expressed in displacement fields u_α, w and temperature field θ

$$\begin{aligned} &\partial_\beta(D^{\alpha\beta\gamma\delta}\partial_\delta u_\gamma) + r^{-1}\partial_\beta(D^{\alpha\beta 11}w) - \partial_\beta(\bar{d}^{\alpha\beta}\theta) - \mu a^{\alpha\beta}\ddot{u}_\beta = 0,\\ &r^{-1}D^{\alpha\beta 11}\partial_\beta u_\alpha + \partial_{\alpha\beta}(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w) - r^{-1}\bar{d}^{11}\theta + r^{-2}D^{1111}w + \mu\ddot{w} = 0, \end{aligned} \tag{4}$$

c) the linearized heat conduction equation based on the Fourier law coupled with (4)

$$\partial_\alpha(K^{\alpha\beta}\partial_\beta\theta) - c\dot{\theta} = T_0(\bar{d}^{\alpha\beta}\partial_\alpha\dot{u}_\beta + r^{-1}\bar{d}^{11}\dot{w}). \tag{5}$$

We recall that $b_{\alpha\beta}$ in (2), (3) is the second metric tensor of the shell midsurface; under orthonormal parametrization introduced on M, components of tensor $b_{\alpha\beta}$ are: $b_{22} = b_{12} = b_{21} = 0$, $b_{11} = -r^{-1}$.

Equations (4) and (5) describe selected dynamic thermoelasticity problems for the periodically micro-heterogeneous shells under consideration. For these shells, coefficients of Eqs. (4), (5) are periodic, highly oscillating and non-continuous functions with respect to argument x, $x \in \Omega$. That is why, in the most cases it is impossible to obtain the exact analytical solutions to initial/boundary value problems for Eqs. (4), (5).

In order to replace Eqs. (4), (5) by averaged equations with constant coefficients dependent also on the cell size, a certain modelling technique proposed by Woźniak [8] will be applied. However, *this so-called asymptotic-tolerance modelling technique* will not be used directly to Eqs. (4), (5), but to the integral functional determined by Lagrange function describing the thermoelastic behaviour of the shells under consideration. *The appropriate form of this function will be implied by the well-known thermoelasticity equations* (4), (5). The variational formulation of the thermoelasticity problem under consideration is based on *the extended principle of stationary action*, cf. [8]. The principle of stationary action in its classical form can not be applied because heat conduction equation (5) involves the odd derivatives of unknown functions $\theta = \theta(x,\xi,t)$, $u_\alpha = u_\alpha(x,\xi,t)$, $w = w(x,\xi,t)$, $(x,\xi,t) \in \Omega\times\Xi\times \mathrm{I}$, with respect to argument t.

We assume that the thermoelastic problems for the thin shells considered here are described by the following action functional

$$A(u_\alpha, w) = \int_0^{L_1}\int_0^{L_2}\int_{t_0}^{t_1} L(x,\xi,t,\partial_\beta u_\alpha,\dot{u}_\alpha,\partial_{\alpha\beta}w, w,\dot{w}, p^{\alpha\beta},\hat{r})dt\,d\xi\,dx, \tag{6}$$

where Lagrangian L is defined by

$$
\begin{aligned}
L = & -\frac{1}{2}(D^{\alpha\beta\gamma\delta}\partial_\beta u_\alpha \partial_\delta u_\gamma + 2r^{-1}D^{\alpha\beta 11} w \partial_\beta u_\alpha + r^{-2}D^{1111}ww \\
& + B^{\alpha\beta\gamma\delta}\partial_{\alpha\beta} w \partial_{\gamma\delta} w - K^{\alpha\beta}\partial_\alpha\theta\partial_\beta\theta - \mu a^{\alpha\beta}\dot{u}_\alpha\dot{u}_\beta - \mu\dot{w}^2) \\
& + p^{\alpha\beta}\partial_\beta u_\alpha + \frac{1}{r}p^{11}w + \hat{r}\theta,
\end{aligned} \tag{7}
$$

and where functions $p^{\alpha\beta}(x,\xi,t), \hat{r}(x,\xi,t)$, $(x,\xi,t) \in \Omega \times \Xi \times \mathrm{I}$, are determined by independent equations

$$
\begin{aligned}
p^{\alpha\beta} &= \bar{d}^{\alpha\beta}\theta, \\
\hat{r} &= c\dot{\theta} + T_0(\bar{d}^{\alpha\beta}\partial_\alpha\dot{u}_\beta + r^{-1}\bar{d}^{11}\dot{w}).
\end{aligned} \tag{8}
$$

Equation (8) are called *the constitutive equations for functions* $p^{\alpha\beta}(x,\xi,t), \hat{r}(x,\xi,t)$, $(x,\xi,t) \in \Omega \times \Xi \times \mathrm{I}$, cf. [8]. It has to be emphasized that functions $p^{\alpha\beta}, \hat{r}$ are not arguments of Lagrangian (7); they play the role of non-variational parameters. Due to the non-continuous and highly oscillating form of functions describing elastic, inertial and thermal properties of the microheterogeneous shells under consideration, i.e. due to the non-continuous and highly oscillating coefficients $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $\mu(x)$, $\bar{d}^{\alpha\beta}(x)$, $K^{\alpha\beta}(x)$, $c(x)$, $x \in \Omega$, occurring in (7), (8), functions L, $p^{\alpha\beta}, \hat{r}$ are also non-continuous and highly oscillating with respect to x, $x \in \Omega$.

Under assumption that $\partial L/\partial(\partial_\beta u_\alpha)$, $\partial L/\partial(\partial_{\alpha\beta} w)$ and $\partial L/\partial(\partial_\beta\theta)$ are continuous, from *the extended principle of stationary action* applied to $A(u_\alpha, w)$, we obtain the following system of Euler–Lagrange equations

$$
\begin{aligned}
&\partial_\beta \frac{\partial L}{\partial(\partial_\beta u_\alpha)} + \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{u}_\alpha} = 0, \\
&-\partial_{\alpha\beta}\frac{\partial L}{\partial(\partial_{\alpha\beta} w)} - \frac{\partial L}{\partial w} + \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{w}} = 0, \\
&\partial_\beta \frac{\partial L}{\partial(\partial_\beta \theta)} - \frac{\partial L}{\partial \theta} = 0.
\end{aligned} \tag{9}
$$

Combining (9) with (7) and (8) we arrive finally at the explicit form of *the fundamental equations of the thermoelasticity shell theory under consideration*. These equations coincide with the well-known equations (4), (5).

The passage from action functional (6) to Euler–Lagrange equations (9), in which $p^{\alpha\beta}, \hat{r}$ are given by constitutive equations (8) represents *the extended principle of stationary action* or *the principle of stationary action extended by constitutive equations*.

Applying *the combined asymptotic-tolerance modelling technique* to action functional (6) determined by lagrangian (7), we will derive an averaged *combined asymptotic-tolerance model* describing thermoelastic phenomena in the uniperiodic shells under consideration. *Governing equations of this model have constant coefficients*. Moreover, *some of these coefficients depend on a cell size*. *The combined model* will be formulated by using *the consistent asymptotic modelling procedure coupled with the tolerance non-asymptotic modelling technique*. *The combined asymptotic-tolerance modelling technique* is proposed by Woźniak et al. [8] and discussed in detail by Ostrowski in the book [9].

To make this paper self-consisted, in the subsequent section we shall outline the main concepts and the fundamental assumptions of *the tolerance modelling procedure* and of *the consistent asymptotic approach*, which in the general form are given in monographs [8,9].

3 Concepts and assumptions of the tolerance and asymptotic modelling techniques

Following the monographs by Woźniak et al. [8] and Ostrowski [9], we outline below the basic concepts and assumptions of the tolerance and consistent asymptotic modelling procedures.

3.1 Main concepts of the tolerance averaging procedure

The fundamental concepts of the tolerance modelling procedure under consideration are those of *two tolerance relations between points and real numbers determined by tolerance parameters, slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and the averaging operation*.

Below, the mentioned above concepts and assumptions will be specified with respect to one-dimensional region $\Omega = (0, L_1)$ (region of midsurface parameters) defined in this paper.

3.1.1 Tolerance between points

Let λ be a positive real number. Points x, y belonging to $\Omega = (0, L_1) \subset E$ are said to be in tolerance determined by λ, if and only if the distance between points x, y does not exceed λ, i.e. $\|x - y\|_E \leq \lambda$, where $\|\cdot\|$ is the Euclidean norm in E.

3.1.2 Tolerance between real numbers

Let $\tilde{\delta}$ be a positive real number. Real numbers $\mu, \ \nu$ are said to be in tolerance determined by $\tilde{\delta}$, if and only if $|\mu - \nu| \leq \tilde{\delta}$.

The above relations are denoted by: $x \overset{\lambda}{\approx} y$, $\mu \overset{\tilde{\delta}}{\approx} \nu$. Positive parameters λ, $\tilde{\delta}$ are called *tolerance parameters.*

3.1.3 Slowly-varying functions

Let F be a function defined in $\bar{\Omega} = [0, L_1] \subset E$, which is differentiable in $\bar{\Omega}$ together with its derivatives up to the Rth order. It can be observed that function F is said to be differentiable in closed set $\bar{\Omega}$; however, we do not specify how derivatives are defined on its fringe $\partial\bar{\Omega}$, because differentiation may look differently for any particular problem. Nonnegative integer R is assumed to be specified in every problem under consideration. Note, that function F can also depend on arguments $\xi \in \Xi$ and $t \in \mathrm{I}$ as parameters. Denote by $\partial_1^k F(\cdot)$, $k = 1, \ldots, R$, the *k-th* derivative in $\bar{\Omega}$. Let $\delta \equiv (\lambda, \delta_0, \delta_1, \ldots, \delta_R)$ be the set of tolerance parameters. The first of them represents the distances between points in $\bar{\Omega}$. The second one is related to the upper limit of the norm in appropriate space between the values of function $F(\cdot)$ in the points x, y belonging to $\bar{\Omega}$ such that $\|x - y\|_E \leq \lambda$. Each tolerance parameter δ_k, $k = 1, \ldots, R$, refers to the upper limit of the norm in appropriate space between the values of derivative $\partial_1^k F(\cdot)$ in the points x, y belonging to $\bar{\Omega}$ such that $\|x - y\|_E \leq \lambda$. A function $F(\cdot)$ is said to be *slowly-varying of the R-th kind* with respect to cell Δ and tolerance parameters δ, $F \in SV_\delta^R(\Omega, \Delta)$, if and only if the following conditions are fulfilled

$$(\forall (x, y) \in \Omega^2)[(x \overset{\lambda}{\approx} y) \Rightarrow |F(x) - F(y)| \leq \delta_0 \ \ and \ \ \left|\partial_1^k F(x) - \partial_1^k F(y)\right| \leq \delta_k, \quad k = 1, 2, \ldots, R], \tag{10}$$

$$(\forall x \in \Omega) \left[\lambda \left|\partial_1^k F(x)\right| \overset{\delta_k}{\approx} 0, \quad k = 1, 2, \ldots, R\right]. \tag{11}$$

From condition (10) it follows that *the slowly-varying function* can be treated (together with its derivatives up to the *R-th* order) as constant on an arbitrary cell, for sufficiently small tolerance parameter δ_R. Condition (11) states that *the products of the absolute values of derivatives of slowly-varying functions and microstructure length parameter* λ *are negligibly small.*

It is worth to known that tolerance parameter λ in every problem under consideration is known *a priori* as a characteristic cell length dimension, whereas values of tolerance parameters $\delta_0, \delta_1, \ldots, \delta_R$ can be determined only *a posteriori*, i.e. after obtaining unique solution to the considered initial-boundary value problem.

3.1.4 Tolerance-periodic functions

An essentially bounded and weakly differentiable function φ defined in $\bar{\Omega} = [0, L_1] \subset E$, which can also depend on $\xi \in \bar{\Xi}$ and time coordinate t as parameters, is called *tolerance-periodic of the R-th kind* in reference to cell Δ and tolerance parameters $\delta \equiv (\lambda, \delta_0)$, if for every $x \in \Omega_\Delta$ there exist Δ-periodic function $\tilde{\varphi}(\cdot)$ defined in E such that $\varphi\left|_{\Omega_x \cap Dom\, \varphi_x}\right.$ and $\tilde{\varphi}\left|_{\Omega_x}\right.$ are indiscernible in tolerance determined by $\delta \equiv (\lambda, \delta_0)$, where $\Omega_x \equiv \Omega \cap \cup_{z \in \Delta(x)} \Delta(z)$, $x \in \bar{\Omega}$, is a cluster of 2 cells having common sides. Function $\tilde{\varphi}$ is a Δ-*periodic approximation* of φ in $\Delta(x)$. For function $\varphi(\cdot)$ being tolerance-periodic together with its derivatives up to the *R-th* order, we shall write $\varphi \in TP_\delta^R(\Omega, \Delta)$, $\delta \equiv (\lambda, \delta_0, \delta_1, \ldots, \delta_R)$. It should be noted that for periodic structures being objects of considerations in this paper, function $\tilde{\varphi}$ has the same analytical form in every cell $\Delta(x)$ with a centre at $x \in \Omega_\Delta$. Hence, $\tilde{\varphi} = \tilde{\varphi}(z)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$, is independent of x. In the general case, i.e. for tolerance-periodic structures (i.e. structures, which in small neighbourhoods of $\Delta(x)$ can be approximately regarded as periodic), $\tilde{\varphi}$ depends on x and hence we have $\tilde{\varphi} = \tilde{\varphi}(x, z)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$.

3.1.5 Fluctuation shape functions

Let h be a continuous, λ-periodic function defined in $\bar{\Omega} = [0, L_1]$, which has continuous derivatives $\partial_1^k h, \quad k = 1, \ldots, R-1$, and either continuous or piecewise continuous bounded derivative $\partial_1^R h$. Function h will be called *the fluctuation shape function of the R-th kind*, $h \in FS^R(\Omega, \Delta)$, if it satisfies conditions: $h \in O(\lambda^R), \quad \partial_1^k h \in O(\lambda^{R-k}), \quad k = 1, 2, \ldots, R, \quad \int_{\Delta(x)} \mu(z)\, h(z) dz = 0, \quad \forall x \in \Omega_\Delta$, where $\mu(\cdot)$ is a certain positive-valued λ-periodic function defined in $\bar{\Omega}$.

Note, that in the tolerance and asymptotic modelling procedures applied here, these λ-dependent fluctuation shape functions describe the expected forms of kinematic or thermal fluctuations caused by the highly oscillating character of the shell micro-structure. They are assumed to be known in every special problem. Due to the micro-periodic structure of the cylindrical shells under consideration, these functions are strongly oscillating in $x \in \Omega$.

3.1.6 Averaging operation

Let f be a function defined in $\bar{\Omega} = [0, L_1]$, which is integrable and bounded in every cell $\Delta(x)$, $x \in \Omega_\Delta$. *The averaging operation of* $f(\cdot)$ is defined by

$$< f > (x) \equiv \frac{1}{|\Delta|} \int_{\Delta(x)} f(z) dz, \quad x \in \Omega_\Delta. \tag{12}$$

where $|\Delta| = \lambda$. It can be observed that if f is Δ-periodic, then $< f >$ is constant.

3.2 Modelling assumptions of the tolerance averaging procedure

The tolerance modelling under consideration is based on two assumptions. The first of them is termed *the tolerance averaging approximation*. The second one is called *the micro–macro decomposition*.

3.2.1 Tolerance averaging approximation

For an integrable periodic function f defined in $\bar{\Omega} \equiv [0, L_1]$ and for *slowly-varying function* $F \in SV_\delta^R(\Omega, \Delta)$ and *fluctuation shape function* $h \in FS^R(\Omega, \Delta)$, the following tolerance relations, called *the tolerance averaging approximation*, hold for every $x \in \Omega_\Delta$

$$\begin{aligned} &< f \partial_1^k F > = < f > \partial_1^k F + O(\delta_k), \\ &< f\, \partial_1^k (hF) > = < f\, \partial_1^k h > F + O(\delta_k), \\ &k = 0, 1 \ldots, R, \quad \partial_1^0 F \equiv F, \quad \partial_1^0 (hF) \equiv hF, \quad \partial_1^0 h \equiv h. \end{aligned} \tag{13}$$

In the course of modelling, terms $O(\delta_k)$ in (13), i.e. terms that are much smaller than the tolerance parameter δ_k, are neglected.

Approximations (13) follow directly from conditions (10), (11) satisfied by the slowly-varying functions and from conditions: $h \in O(\lambda^R), \quad \partial^k h \in O(\lambda^{R-k}), \quad k = 1, 2, \ldots, R$, which hold for the fluctuation shape functions.

In the problem discussed in this contribution, R is equal either 1 or 2.

Let us observe that the slowly-varying functions can be regarded as invariant under averaging.

3.2.2 Micro–macro decomposition assumption

The second fundamental assumption, called *the micro–macro decomposition*, states that the displacement and temperature fields occurring in the starting lagrangian under consideration can be decomposed into *macroscopic and microscopic parts.* The macroscopic part is represented by *unknown averaged displacements and averaged temperature* being slowly-varying functions in periodicity direction. The microscopic part is described by *the known* strongly oscillating periodic *thermal fluctuation shape functions* multiplied by *unknown temperature* *fluctuation amplitudes, and by the known* strongly oscillating periodic *kinematic fluctuation shape functions* multiplied by *unknown displacement fluctuation amplitudes*. Fluctuation amplitudes for temperature and for displacements are slowly-varying functions in x.

Micro–macro decomposition introduced in the thermoelastic problems discussed in this contribution is presented in Sect. 4.2.

3.3 Basic concepts and assumptions of the consistent asymptotic modelling procedure

3.3.1 Basic concepts

The basic concepts of the consistent asymptotic procedure [8,9] are those of *the fluctuation shape function and the averaging operation.* These notions have been explained in Sect. 3.1. *In the consistent asymptotic modelling there are no concepts of the tolerance-periodic and slowly-varying functions*. Also, for periodic structures, the tolerance parameters play here no role anymore.

3.3.2 The consistent asymptotic decomposition assumption

The consistent asymptotic decomposition is the basic assumption imposed on the starting Lagrangian under consideration. It states that the displacement fields and temperature field occurring in the Lagrangian must be replaced by families of fields depending on parameter $\varepsilon \in (0, 1]$ and defined in an arbitrary cell. These families of displacements and temperature are decomposed into averaged part independent of ε and highly oscillating part depending on ε.

Consistent asymptotic decomposition introduced in the thermoelastic problems discussed in this contribution is presented in Sect. 4.1.

4 Combined asymptotic-tolerance modelling

The combined modelling includes both the consistent asymptotic and the tolerance non-asymptotic modelling techniques, which are merged into a single new procedure. The variational approach to the asymptotic and tolerance modelling of microheterogeneous media will be applied. This variational approach is proposed by Woźniak et al. [8] and discussed in detail by Ostrowski in the book [9]. *The combined modelling is realized in two steps*. In the first step, applying *the consistent asymptotic averaging technique to starting lagrangin* (7) describing thermoelastic behaviour of the shells under consideration and *independently to invariable parameters* (8), and then using *the extended stationary action principle* [8,9], we obtain *the consistent asymptotic model equations.* Coefficients of the asymptotic model equations are constant, but independent of a characteristic cell length dimension. Hence the model obtained in the first step of the combined modelling is referred to as *the macroscopic model.* Assuming that in the framework of the macroscopic model the solutions to the considered problem are known, we can pass to the second step. The second step of the combined modelling is realized by means of *the tolerance* (*non-asymptotic*) *modelling procedure*. This step is based on *the tolerance averaging of starting lagrangian* (7) and independently on *the tolerance averaging of non-variational parameters* (8). Then, applying *the extended stationary action principle* to the averaged action functional determined by averaged Lagrange function, we arrive to *the tolerance model equations superimposed on the solutions obtained in the first step of the combined modelling.* Coefficients of the tolerance model equations are constant. Moreover, some of these coefficients depend on a cell size. For this reason, this model is referred to as *the superimposed microscopic model*. Asymptotic (macroscopic) and tolerance (microscopic) models are conjugated with themselves under assumption that in the framework of the macroscopic model the solutions to the problem under consideration are known. It will be shown that the combined model proposed here makes it possible to separate the macroscopic description of certain thermoelasticity problems from their microscopic description. This is an important advantage of the combined model proposed here. We recall that the asymptotic or tolerance model equations cannot be derived from the principle of stationary action in its classical form, because heat conduction Eq. (5) involves the odd derivatives of unknown functions $\theta = \theta(x, \xi, t)$, $u_\alpha = u_\alpha(x, \xi, t)$, $w = w(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, with respect to argument t.

4.1 Step 1. Consistent asymptotic modelling

Let us start with *the consistent asymptotic averaging of lagrangian* (7) and independently with *the consistent asymptotic averaging of constitutive equations* (8) *for functions* $p^{\alpha\beta}(x,\xi,t)$, $\hat{r}(x,\xi,t)$ being the non-variational parameters of Lagrange function (7).

In order to do it, we shall restrict considerations to displacement fields $u_\alpha = u_\alpha(z,\xi,t)$, $w = w(z,\xi,t)$ and temperature field $\theta(z,\xi,t)$ defined in $\Delta(x) \times \Xi \times \mathrm{I}$, $z \in \Delta(x)$, $x \in \Omega_\Delta$, $(\xi,t) \in \Xi \times I$. Then, we replace $u_\alpha(z,\xi,t)$, $w(z,\xi,t)$ and $\theta(z,\xi,t)$ by families of displacements $u_{\varepsilon\alpha}(z,\xi,t) \equiv u_\alpha(z/\varepsilon,\xi,t)$, $w_\varepsilon(z,\xi,t) \equiv w(z/\varepsilon,\xi,t)$ and family of temperature field $\theta_\varepsilon(z,\xi,t) \equiv \theta(z/\varepsilon,\xi,t)$, respectively, where $0<\varepsilon\le 1$, $z\in\Delta_\varepsilon(x)$,, $\Delta_\varepsilon \equiv (-\varepsilon\lambda_1/2, \varepsilon\lambda_1/2)$ (*scaled cell*), $\Delta_\varepsilon(x) \equiv x+\Delta_\varepsilon$, $x\in\Omega_{\Delta_\varepsilon}$ (*scaled cell with a centre at* $x \in \Omega_{\Delta_\varepsilon}$).

We introduce *the consistent asymptotic decomposition* of displacement and temperature families $u_{\varepsilon\alpha}(z,\xi,t)$, $w_\varepsilon(z,\xi,t)$, $\theta_\varepsilon(z,\xi,t)$, $(z,\xi,t)\in\Delta_\varepsilon\times\Xi\times\mathrm{I}$, in the area of every ε-scaled cell

$$\begin{aligned}
&u_{\varepsilon\alpha}(z,\xi,t) \equiv u_\alpha(z/\varepsilon,\xi t) = u^0_\alpha(z,\xi,t) + \varepsilon h_\varepsilon(z) U_\alpha(z,\xi,t),\\
&w_\varepsilon(z,\xi,t) \equiv w(z/\varepsilon,t) = w^0(z,\xi,t) + \varepsilon^2 g_\varepsilon(z) W(z,\xi,t),\\
&\theta_\varepsilon(z,\xi,t) \equiv \theta(z/\varepsilon,t) = \theta^0(z,\xi,t) + \varepsilon q_\varepsilon(z)\Theta(z,\xi,t).
\end{aligned} \tag{14}$$

Functions u^0_α, $w^0$and U_α, W are termed *macrodisplacements* and *displacement fluctuation amplitudes*, respectively. Functions θ^0, Θ are called *macrotemperature* and *temperature fluctuation amplitude*, respectively. Unknowns u^0_α, U_α, θ^0, Θ are assumed to be continuous and bounded in $\bar{\Omega}$ together with their first derivatives. Unknowns w^0, W are assumed to be continuous and bounded in $\bar{\Omega}$ together with their derivatives up to the second order. Moreover, all unknowns mentioned above are independent of ε. We recall that they are not referred to as the slowly-varying functions introduced in the tolerance averaging.

Fluctuation shape functions for displacements $h_\varepsilon(z) \equiv h(z/\varepsilon)$, $h_\varepsilon \in FS^1(\Omega,\Delta_\varepsilon)$, $g_\varepsilon(z)\equiv g(z/\varepsilon)$, $g_\varepsilon \in FS^2(\Omega,\Delta_\varepsilon)$, and fluctuation shape function for temperature $q_\varepsilon(z)\equiv q(z/\varepsilon)$, $q_\varepsilon \in FS^1(\Omega,\Delta_\varepsilon)$, in (14) are highly oscillating and Δ_ε-periodic. They have to be known in every problem under consideration. They depend on $\varepsilon\lambda$ as a parameter and have to satisfy conditions: $h_\varepsilon \in O(\varepsilon\lambda)$, $\varepsilon\lambda\partial_1 h_\varepsilon \in O(\varepsilon\lambda)$, $g_\varepsilon \in O((\varepsilon\lambda)^2)$, $\varepsilon\lambda\partial_1 g_\varepsilon \in O((\varepsilon\lambda)^2)$, $(\varepsilon\lambda)^2\partial_{11} g_\varepsilon \in O((\varepsilon\lambda)^2)$, $q_\varepsilon \in O(\varepsilon\lambda)$, $\varepsilon\lambda\partial_1 q_\varepsilon \in O(\varepsilon\lambda)$, $<\mu h_\varepsilon>=<\mu g_\varepsilon>=<cq_\varepsilon>=0$. It has to be emphasized that $\partial_1 h_\varepsilon(z) \equiv \frac{1}{\varepsilon}\bar{\partial}_1 h(z/\varepsilon)$, $\partial_1 g_\varepsilon(z) \equiv \frac{1}{\varepsilon}\bar{\partial}_1 g(z/\varepsilon)$, $\partial_{11} g_\varepsilon(z) \equiv \frac{1}{\varepsilon^2}\bar{\partial}_{11} g(z/\varepsilon)$, $\partial_1 q_\varepsilon(z) \equiv \frac{1}{\varepsilon}\bar{\partial}_1 q(z/\varepsilon)$, where differential operator $\bar{\partial}_1$ means differentiation over z/ε.

Because of Lagrangian L defined by (7) is highly oscillating with respect to x and essentially bounded in its domain, then there exists Lagrangian $\tilde{L}(z,\xi,t,\partial_\beta u_\alpha,\dot{u}_\alpha,\partial_{\alpha\beta}w,w,\dot{w},p^{\alpha\beta},\hat{r})$ being the periodic approximation of Lagrangian L in $\Delta(x)$, $z\in\Delta(x)$, $x\in\Omega_\Delta$. Let $\tilde{L}_\varepsilon$ be a family of functions given by

$$\begin{aligned}
\tilde{L}_\varepsilon &= \tilde{L}(z/\varepsilon,\xi,t,\partial_\beta u_{\varepsilon\alpha},\dot{u}_{\varepsilon\alpha},\partial_{\alpha\beta}w_\varepsilon,w_\varepsilon,\dot{w}_\varepsilon,\partial_\beta\theta_\varepsilon,\theta_\varepsilon,p^{\alpha\beta}_\varepsilon,\hat{r}_\varepsilon)\\
&= -\frac{1}{2}[D^{\alpha\beta\gamma\delta}\partial_\beta u_{\varepsilon\alpha}\partial_\delta u_{\varepsilon\gamma} + 2r^{-1}D^{\alpha\beta 11}w_\varepsilon\partial_\beta u_{\varepsilon\alpha}\\
&\quad + r^{-2}D^{1111}w_\varepsilon w_\varepsilon + B^{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w_\varepsilon\partial_{\gamma\delta}w_\varepsilon\\
&\quad - K^{\alpha\beta}\partial_\alpha\theta_\varepsilon\partial_\beta\theta_\varepsilon - \mu a^{\alpha\beta}\dot{u}_{\varepsilon\alpha}\dot{u}_{\varepsilon\beta} - \mu(\dot{w}_\varepsilon)^2]\\
&\quad + p^{\alpha\beta}_\varepsilon\partial_\beta u^\alpha_\varepsilon + r^{-1}p^{11}_\varepsilon w_\varepsilon + \hat{r}_\varepsilon\theta_\varepsilon,
\end{aligned} \tag{15}$$

where $p^{\alpha\beta}_\varepsilon$, $\hat{r}_\varepsilon$ play the role of invariational parameters and are given by independent equations

$$\begin{aligned}
&p^{\alpha\beta}_\varepsilon = \bar{d}^{\alpha\beta}\theta_\varepsilon,\\
&\hat{r}_\varepsilon = c\dot{\theta}_\varepsilon + T_0(\bar{d}^{\alpha\beta}\partial_\alpha\dot{u}_{\varepsilon\beta} + r^{-1}\bar{d}^{11}\dot{w}_\varepsilon).
\end{aligned} \tag{16}$$

We substitute the right-hand sides of (14) into (15) and independently into (16). Then, we take into account that under limit passage $\varepsilon\to 0$, terms depending on ε can be neglected and every continuous and bounded function of argument $z\in\Delta_\varepsilon(x)$, tends to function of argument $x\in\bar{\Omega}$. Moreover, if $\varepsilon\to 0$ then, by means of a property of the mean value, cf. Jikov et al. [2], the obtained result tends weakly to function L_0 being *the averaged form of starting Lagrangian* (7) *under consistent asymptotic decomposition* (14). Introducing the extra approximation $1+\lambda/r\approx 1$ and assuming that the fluctuation shape functions for displacements and for temperature are either even or odd functions with respect to argument $z\in\Delta$, this result has the form

$$
\begin{aligned}
&L_0(\partial_\beta u^0_\alpha, U_\alpha, \dot{u}^0_\alpha, \partial_{\alpha\beta} w^0, w^0, W, \dot{w}^0, \partial_\beta\theta^0, \theta^0, \Theta, < p^{\alpha\beta} >, < p^{\alpha 1}\partial_1 h >, < \hat{r} >)\\
&\quad = -\frac{1}{2}[< D^{\alpha\beta\gamma\delta} > \partial_\beta u^0_\alpha \partial_\delta u^0_\gamma + 2 < D^{\alpha\beta\gamma 1}\partial_1 h > \partial_\beta u^0_\alpha U_\gamma\\
&\qquad + < \partial_1 h\, D^{\alpha 1\gamma 1}\partial_1 h > U_\gamma\, U_\alpha + 2r^{-1}(< D^{\alpha\beta 11} > \partial_\beta u^0_\alpha w^0\\
&\qquad + < D^{\alpha 111}\partial_1 h > w^0 U_\alpha) + r^{-2} < D^{1111} > (w^0)^2\\
&\qquad + < B^{\alpha\beta\gamma\delta} > \partial_{\alpha\beta} w^0 \partial_{\gamma\delta} w^0 + 2 < B^{\alpha\beta 11}\partial_{11} g > \partial_{\alpha\beta} w^0 W\\
&\qquad + < \partial_{11} g\, B^{1111}\partial_{11} g > (W)^2\\
&\qquad - < K^{\alpha\beta} > \partial_\alpha\theta^0\partial_\beta\theta^0 - 2 < K^{1\beta}\partial_1 q > \partial_\beta\theta^0\Theta\\
&\qquad - < \partial_1 q\, K^{11}\partial_1 q > (\Theta)^2 - < \mu > a^{\alpha\beta}\dot{u}^0_\alpha\dot{u}^0_\beta - < \mu > (\dot{w}^0)^2]\\
&\qquad + < p^{\alpha\beta} > \partial_\beta u^0_\alpha + < p^{\alpha 1}\partial_1 h > U_\alpha + r^{-1} < p^{11} > w^0 + < \hat{r} > \theta^0,
\end{aligned}
\tag{17}
$$

where averaged constitutive equations for functions $< p^{\alpha\beta} >, < \hat{r} >$ are given by

$$
\begin{aligned}
&< p^{\alpha\beta} > = < \bar{d}^{\alpha\beta} > \theta^0, \quad < p^{\alpha 1}\partial_1 h > = < \bar{d}^{\alpha 1}\partial_1 h > \theta^0,\\
&< \hat{r} > = < c > \dot{\theta}^0 + T_0[< \bar{d}^{\alpha\beta} > \partial_\alpha \dot{u}^0_\beta + < \bar{d}^{\alpha 1}\partial_1 h > \dot{U}_\beta + r^{-1} < \bar{d}^{11} > \dot{w}^0].
\end{aligned}
\tag{18}
$$

In the framework of consistent asymptotic procedure we introduce *the consistent asymptotic action functional*

$$
A^0_{hgq}(u^0_\alpha, U_\alpha, w^0, W, \theta^0, \Theta) = \int_0^{L_1}\int_0^{L_2}\int_{t_0}^{t_1} L_0 dt d\xi dx,
\tag{19}
$$

where L_0 is given by (17).

Under assumption that $\partial L_0/\partial(\partial_\beta u^0_\alpha)$, $\partial L_0/\partial(\partial_{\alpha\beta} w^0)$, $\partial L_0/\partial(\partial_\beta\theta^0)$ are continuous and recalling that expressions (18) are treated as non-variational parameters, from *the extended principle of stationary action* applied to (19) we obtain the following system of Euler–Lagrange equations for u^0_α, w^0, U_α, W, θ^0, Θ as the basic unknowns

$$
\begin{aligned}
&\partial_\beta \frac{\partial L_0}{\partial(\partial_\beta u^0_\alpha)} + \frac{\partial}{\partial t}\frac{\partial L_0}{\partial \dot{u}^0_\alpha} = 0,\\
&-\partial_{\alpha\beta}\frac{\partial L_0}{\partial(\partial_{\alpha\beta} w^0)} - \frac{\partial L_0}{\partial w^0} + \frac{\partial}{\partial t}\frac{\partial L_0}{\partial \dot{w}^0} = 0,\\
&\frac{\partial L_0}{\partial U_\alpha} = 0, \qquad \frac{\partial L_0}{\partial W} = 0,\\
&\partial_\beta\frac{\partial L_0}{\partial(\partial_\beta\theta^0)} - \frac{\partial L_0}{\partial\theta^0} = 0, \qquad \frac{\partial L_0}{\partial\Theta} = 0.
\end{aligned}
\tag{20}
$$

Combining (20) with (17) and (18) we arrive at the explicit form of *the consistent asymptotic model equations* for *macrodisplacements* $u^0_\alpha(x,\xi,t)$, $w^0(x,\xi,t)$, *displacement fluctuation amplitudes* $U_\alpha(x,\xi,t)$, $W(x,\xi,t)$, macrotemperature $\theta^0(x,\xi,t)$ and temperature fluctuation amplitude $\Theta(x,\xi,t)$, $(x,\xi,t)\in\Omega\times\Xi\times I$

$$
\begin{aligned}
&< D^{\alpha\beta\gamma\delta} > \partial_{\beta\delta} u^0_\gamma + r^{-1} < D^{\alpha\beta 11} > \partial_\beta w^0 + < D^{\alpha\beta\gamma 1}\partial_1 h > \partial_\beta U_\gamma\\
&\quad - < \bar{d}^{\alpha\beta} > \partial_\beta\theta^0 - < \mu > a^{\alpha\beta}\ddot{u}^0_\beta = 0,\\
&< B^{\alpha\beta\gamma\delta} > \partial_{\alpha\beta\gamma\delta} w^0 + < D^{\alpha\beta 11}\partial_{11} g > \partial_{\alpha\beta} W + r^{-1} < D^{11\gamma\delta} > \partial_\delta u^0_\gamma\\
&\quad + r^{-2} < D^{1111} > w^0 + r^{-1} < D^{111\delta}\partial_1 h > U_\delta - < \bar{d}^{11} > \theta^0) + < \mu > \ddot{w}^0 = 0,\\
&< \partial_1 h\, D^{1\beta\gamma 1}\partial_1 h > U_\gamma = - < \partial_1 h\, D^{1\beta\gamma\delta} > \partial_\delta u^0_\gamma - r^{-1} < \partial_1 h D^{1\beta 11} > w^0\\
&\quad + < \partial_1 h\, \bar{d}^{1\beta} > \theta^0,\\
&< \partial_{11} g\, B^{1111}\partial_{11} g > W = - < \partial_{11} g B^{11\gamma\delta} > \partial_{\gamma\delta} w^0,\\
&< K^{\alpha\beta} > \partial_{\alpha\beta}\theta^0 + < K^{1\beta}\partial_1 q > \partial_\beta\Theta - < c > \dot{\theta}^0\\
&= T_0[< \bar{d}^{\alpha\beta} > \partial_\alpha\dot{u}^0_\beta + < \bar{d}^{1\beta}\partial_1 h > \dot{U}_\beta + r^{-1} < \bar{d}^{11} > \dot{w}^0],\\
&< \partial_1 q\, K^{11}\partial_1 q > \Theta = - < K^{1\beta}\partial_1 q > \partial_\beta\theta^0.
\end{aligned}
\tag{21}
$$

Averages $< \cdot >$ occurring in (21) are constant and calculated by means of (12).

Equation (21) consist of partial differential equations for macrodisplacements $u^0_\alpha,\ w^0$ and macrotemperature θ^0 coupled with linear algebraic equations for kinematic fluctuation amplitudes $U_\alpha,\ W$ and thermal fluctuation amplitude Θ. After eliminating fluctuation amplitudes from the governing equations by means of

$$\begin{aligned}
&U_\gamma = -G^{-1}_{\gamma\eta}\ [< \partial_1 h D^{1\eta\mu\vartheta} >\ \partial_\vartheta u^0_\mu + r^{-1} < \partial_1 h D^{1\eta 11} > w^0 - < \partial_1 h\, \bar{d}^{1\eta} > \theta^0],\\
&W = -E^{-1} < \partial_{11} g B^{11\gamma\delta} > \partial_{\gamma\delta} w^0,\\
&\Theta = -C^{-1} < K^{1\beta}\partial_1 q > \partial_\beta \theta^0,
\end{aligned} \tag{22}$$

where $G_{\alpha\gamma} =< \partial_1 h\, D^{\alpha 1\gamma 1}\partial_1 h >,\ E =< \partial_{11} g\, B^{1111}\partial_{11} g >,\ C =< \partial_1 q\, K^{11}\partial_1 q >,\ G_{\alpha\gamma}G^{-1}_{\gamma\eta} = \delta_{\alpha\eta}$ ($\delta_{\alpha\eta}$ is an unit tensor), we arrive finally at *the asymptotic model equations expressed only in macrodisplacements* $u^0_\alpha,\ w^0$ and *macrotemperature* θ^0

$$\begin{aligned}
&D^{\alpha\beta\gamma\delta}_h \partial_{\beta\delta} u^0_\gamma + r^{-1} D^{\alpha\beta 11}_h \partial_\beta w^0 - \bar{D}^{\alpha\beta}_b \partial_\beta \theta^0 - < \mu > a^{\alpha\beta}\ddot{u}^0_\beta = 0,\\
&B^{\alpha\beta\gamma\delta}_g \partial_{\alpha\beta\gamma\delta} w^0 + r^{-1} D^{11\gamma\delta}_h \partial_\delta u^0_\gamma + r^{-2} D^{1111}_h w^0 - r^{-1}\bar{D}^{11}_h \theta^0 + < \mu > \ddot{w}^0 = 0,\\
&\bar{K}^{\alpha\beta}_q \partial_{\alpha\beta}\theta^0 - [< c > + T_0 < \bar{d}^{1\beta}\partial_1 h > G^{-1}_{\beta\eta} < \partial_1 h\, \bar{d}^{1\eta} >]\dot{\theta}^0\\
&\quad = T_0[\bar{D}^{\alpha\beta}_h \partial_\alpha \dot{u}^0_\beta + r^{-1}\bar{D}^{11}_h \dot{w}^0],
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
&D^{\alpha\beta\gamma\delta}_h \equiv < D^{\alpha\beta\gamma\delta} > - < D^{\alpha\beta\eta 1}\ \partial_1 h > G^{-1}_{\eta\xi}\ < \partial_1 h D^{1\xi\gamma\delta} >,\\
&B^{\alpha\beta\gamma\delta}_g \equiv < B^{\alpha\beta\gamma\delta} > - < B^{\alpha\beta 11}\partial_{11} g > E^{-1}\ < \partial_{11} g B^{11\gamma\delta} >,\\
&\bar{D}^{\alpha\beta}_h \equiv < \bar{d}^{\alpha\beta} > - < D^{\alpha\beta\gamma 1}\partial_1 h > G^{-1}_{\gamma\eta} < \partial_1 h\, \bar{d}^{1\eta} >,\\
&\bar{K}^{\alpha\beta}_q \equiv < K^{\alpha\beta} > - < K^{\alpha 1}\partial_1 q > C^{-1} < K^{\beta 1}\partial_1 q > .
\end{aligned} \tag{24}$$

Tensors $D^{\alpha\beta\gamma\delta}_h$, $B^{\alpha\beta\gamma\delta}_g$ are *tensors of effective elastic moduli* for uniperiodic shells considered here.

Tensor $\bar{D}^{\alpha\beta}_h$ is *a tensor of effective elastic-thermal moduli.*

Tensor $\bar{K}^{\alpha\beta}_q$ is *a tensor of effective thermal moduli.*

Since functions $u_\alpha(x,\xi,t)$, $w(x,\xi,t)$, $\theta(x,\xi,t)$ have to be uniquely defined in $\Omega \times \Xi \times$ I, we conclude that u_α, w, θ must take the form

$$\begin{aligned}
&u_\alpha(x,\xi,t) = u^0_\alpha(x,\xi,t) + h(x)U_\alpha(x,\xi,t),\\
&w(x,\xi,t) = w^0(x,\xi,t) + g(x)W(x,\xi,t),\\
&\theta(x,\xi,t) = \theta^0(x,\xi,t) + q(x)\Theta(x,\xi,t),
\end{aligned} \tag{25}$$

with U_α, W, Θ given by (22). We recall that unknowns u^0_α, w^0, U_α, W, θ^0, Θ in (25) are not slowly-varying functions in the sense given by (10), (11). In the asymptotic approach, they are assumed to be bounded and continuous in $\bar{\Omega}$ together with their appropriate derivatives.

Equation (23) for macrodisplacements $u^0_\alpha(x,\xi,t)$, $w^0(x,\xi,t)$ and macrotemperature $\theta^0(x,\xi,t)$ together with expressions (22) for kinematic $U_\alpha(x,\xi,t)$, $W(x,\xi,t)$ and thermal $\Theta(x,\xi,t)$ fluctuation amplitudes, $(x,\xi,t) \in \Omega \times \Xi \times I$, and with expressions (24) for the effective moduli as well as with decomposition (25) represent *the consistent asymptotic model of selected dynamic thermoelasicity problems for the thin uniperiodic cylindrical shells under consideration*.

In the first step of combined modelling it is assumed that within the asymptotic model, solutions $u^0_\alpha(x,\xi,t)$, $w^0(x,\xi,t)$, $\theta^0(x,\xi,t)$, $(x,\xi,t) \in \Omega \times \Xi \times I$, to the thermoelasticity problem under consideration are known. Hence, there are also known functions

$$\begin{aligned}
&u_{0\alpha}(x,\xi,t) = u^0_\alpha(x,\xi,t)) + h(x)U_\alpha(x,\xi,t),\\
&w_0(x,\xi,t) = w^0(x,\xi,t) + g(x)W(x,\xi,t),\\
&\theta_0(x,\xi,t) = \theta^0(x,\xi,t) + q(x)\Theta(x,\xi,t),
\end{aligned} \tag{26}$$

where U_α, W, Θ are given by means of (22).

4.1.1 Discussion of results

The important features of the derived *consistent asymptotic model* are listed below.

- Contrary to starting equations (4), (5) with periodic, highly oscillating and discontinuous coefficients, the asymptotic model equations (23) formulated here *have constant coefficients, but independent of period length*. It means that this model is not able to describe the influence of a cell size on the global shell thermoelasticity.
- Unknown functions u_α^0, U_α, w^0, W and θ^0, Θ of the asymptotic model are demanded to be bounded and continuous in $\bar{\Omega}$ together with their appropriate derivatives. These unknowns are assumed to be independent of parameter $\varepsilon \in (0, 1]$. This is the main difference between the asymptotic approach under consideration and approach, which is used in the known classical homogenization theory, cf. Bensoussan et al. [1], Jikov et al. [2].
- Within the asymptotic model we formulate boundary conditions only for the macrodisplacements u_α^0, w^0 and macrotemperature θ^0. The number and form of these conditions are the same as in the classical shell theory governed by starting equations (4), (5).
- The extra unknown functions U_α, W, Θ called *fluctuation amplitudes* are governed by a system of linear algebraic equations $(21)_{3,4,6}$ and can be always eliminated from the governing equations by means of (22). Hence, the unknowns of final asymptotic model equations (23) are only macrodisplacements u_α^0, w^0 and macrotemperature θ^0.
- The resulting asymptotic model equations (23) are uniquely determined by the postulated *a priori* periodic *displacement fluctuation shape functions* $h \in FS^1(\Omega, \Delta)$, $h \in O(\lambda)$, $g \in FS^2(\Omega, \Delta)$, $g \in O(\lambda^2)$, and *temperature fluctuation shape function* $q \in FS^1(\Omega, \Delta)$, $q \in O(\lambda)$, representing oscillations of displacement and temperature fields inside a cell. These functions can be obtained as exact or approximate solutions to periodic eigenvalue cell problems, cf. [11,23–25]. They can also be regarded as *the shape* functions resulting from the periodic discretization of the cell using, for example, the finite element method. The choice of these functions can also be based on the experience or intuition of the researcher. If the fluctuation shape functions are not derived as solutions to certain periodic eigenvalue problems then *the effective moduli* (24) *of the shell are obtained without specification of the periodic cell problems*. It is a very important advantage of the asymptotic model proposed here, because in most cases obtaining the solutions to the cell problems is not easy and can not be realised in the analytical form. This situation is different from that occurring in the known asymptotic homogenisation approach, cf. e.g. Bensoussan et al. [1], where *only solutions to the periodic cell problems make it possible to define the effective moduli of the structure under consideration.*
- Taking into account that for a homogeneous shell with a constant thickness, $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $\mu(x)$, $\bar{d}^{\alpha\beta}(x)$, $K^{\alpha\beta}(x)$, $c(x)$, $x \in \Omega$, are constant and bearing in mind that $< \partial_1 h >=< \partial_1 g >=< \partial_{11} g >=< \partial_1 b >= 0$ we obtained from (22) that $U_\alpha = W = \Theta = 0$ and from (24) that $D_h^{\alpha\beta\gamma\delta} \equiv D^{\alpha\beta\gamma\delta}$, $B_g^{\alpha\beta\gamma\delta} \equiv B^{\alpha\beta\gamma\delta}$, $\bar{D}_h^{\alpha\beta} = \bar{d}^{\alpha\beta}$, $\bar{K}_q^{\alpha\beta} = K^{\alpha\beta}$. Hence, from decomposition (25) it follows that $u_\alpha = u_\alpha^0$, $w = w^0$, $\theta = \theta^0$. It means that Eq. (23), generated by asymptotically averaged Lagrange function (17) together with asymptotically averaged constitutive equations (18), reduce to the starting equations (4), (5) generated by Lagrange function (7) together with constitutive equations (8) for invariational parameters occurring in (7).

4.2 Step 2. Tolerance modelling

The second step of the combined modelling is based on *the tolerance modelling technique* [8,9].

Let us start with *the tolerance averaging of lagrangian* (7) and independently with *the tolerance averaging of constitutive equations* (8) *for functions* $p^{\alpha\beta}(x, \xi, t)$, $\bar{r}(x, \xi, t)$ being the non-variational parameters of Lagrange function (7).

In order to do it, we introduce *the extra micro–macro decomposition* of displacement fields $u_\alpha(x, \xi, t)$, $u_\alpha(\cdot, \xi, t) \in TP_\delta^1(\Omega, \Delta)$, $w(x, \xi, t)$, $w(\cdot, \xi, t) \in TP_\delta^2(\Omega, \Delta)$ and temperature field $\theta(x, \xi, t)$, $\theta(\cdot, \xi, t) \in TP_\delta^1(\Omega, \Delta)$, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, superimposed on the known solutions $u_{0\alpha}(x, \xi, t)$, $w_0(x, \xi, t)$, $\theta_0(x, \xi, t)$, cf. (26), obtained within the asymptotic (macroscopic) model. Setting $u_{\hat{h}\alpha} \equiv u_\alpha$, $w_{\hat{g}} \equiv w$, $\theta_{\hat{q}} \equiv \theta$, *the super-*

imposed decomposition has the form

$$\begin{aligned} u_{\hat{h}\alpha}(x,\xi,t) &= u_{0\alpha}(x,\xi,t) + \hat{h}(x)Q_\alpha(x,\xi,t),\\ w_{\hat{g}}(x,\xi,t) &= w_0(x,\xi,t) + \hat{g}(x)V(x,\xi,t),\\ \theta_{\hat{q}}(x,\xi,t) &= \theta_0(x,\xi,t) + \hat{q}(x)\Psi(x,\xi,t), \end{aligned} \tag{27}$$

where

$$\begin{aligned} &Q_\alpha(\cdot,\xi,t),\ \Psi(\cdot,\xi,t) \in SV^1_\delta(\Omega,\Delta), \quad \delta \equiv (\lambda,\delta_0,\delta_1),\\ &V(\cdot,\xi,t) \in SV^2_\delta(\Omega,\Delta), \quad \delta \equiv (\lambda,\delta_0,\delta_1,\delta_2) \end{aligned} \tag{28}$$

for every $\xi \in \Xi$ and $t \in I$.

Displacement fluctuation amplitudes Q_α, V and *temperature fluctuation amplitude* Ψ are *the new unknowns*, which must satisfy conditions (28), i.e. they have to be slowly-varying functions with respect to argument $x \equiv x^1$.

Fluctuation shape functions for displacements $\hat{h} \in FS^1(\Omega,\Delta)$,$\hat{g} \in FS^2(\Omega,\Delta)$ and *fluctuation shape function for temperature* $\hat{q} \in FS^1(\Omega,\Delta)$ are the new, λ-periodic, continuous and strongly oscillating functions, which are assumed to be known in every problem under consideration. They have to satisfy conditions: $\hat{h} \in O(\lambda)$, $\lambda\partial_1\hat{h} \in O(\lambda)$, $\hat{g} \in O(\lambda^2)$, $\lambda\partial_1\hat{g} \in O(\lambda^2)$, $\lambda^2\partial_{11}\hat{g} \in O(\lambda^2)$, $\hat{q} \in O(\lambda)$, $\lambda\partial_1\hat{q} \in O(\lambda)$, $<\mu\hat{h}>=<\mu\hat{g}>=<c\hat{q}>0$. It is assumed that the fluctuation shape functions for displacements and for temperature are either even or odd functions with respect to argument $z \in \Delta$. As in the asymptotic approach, functions $\hat{h}$, $\hat{g}$, $\hat{q}$ from the qualitative point of view describe the expected character of micro-oscillations of displacements or temperature. These micro-oscillations are caused by a periodically heterogeneous structure of the shell. It means that the choice of the fluctuation shape functions depends on the shape of micro-disturbances, which can be expected during every process under consideration. These functions can be obtained as exact or approximate solutions to periodic eigenvalue cell problems, cf. e.g. [11,23–25]. For example, in dynamic processes the fluctuation shape functions are exact or approximate solutions to the periodic eigenvalue problems describing free vibrations of the cell. In this case, they represent either the principal modes of free periodic cell vibrations or physically reasonable approximation of these modes. They can also be derived from the periodic finite element method discretization of the cell. The choice of these functions can also be based on the experience or intuition of the researcher.

Setting $u_{\hat{h}\alpha} \equiv u_\alpha$, $w_{\hat{g}} \equiv w$, $\theta_{\hat{q}} \equiv \theta$, we obtain from (7) lagrangian $L_{\hat{h}\hat{g}\hat{q}}$ having the following form

$$\begin{aligned} L_{\hat{h}\hat{g}\hat{q}} = &-\frac{1}{2}(D^{\alpha\beta\gamma\delta}\partial_\beta u_{\hat{h}\alpha}\partial_\delta u_{\hat{g}\gamma} + 2r^{-1}D^{\alpha\beta 11}w_{\hat{g}}\,\partial_\beta u_{\hat{h}\alpha} + r^{-2}D^{1111}w_{\hat{g}}w_{\hat{g}}\\ &+B^{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w_{\hat{g}}\,\partial_{\gamma\delta}w_{\hat{g}} - K^{\alpha\beta}\partial_\alpha\theta_{\hat{q}}\,\partial_\beta\theta_{\hat{q}} - \mu a^{\alpha\beta}\dot{u}_{\hat{h}\alpha}\dot{u}_{\hat{h}\beta} - \mu(\dot{w}_{\hat{g}})^2)\\ &+p^{\alpha\beta}\partial_\beta u_{\hat{h}\alpha} + \frac{1}{r}p^{11}w_{\hat{g}} + \hat{r}\theta_{\hat{q}}, \end{aligned} \tag{29}$$

where now non-variational parameters $p^{\alpha\beta}(x,\xi,t)$, $\hat{r}(x,\xi,t)$ are determined by the following independent equations

$$\begin{aligned} p^{\alpha\beta} &= \bar{d}^{\alpha\beta}\theta_{\hat{q}},\\ \hat{r} &= c\dot{\theta}_{\hat{q}} + T_0(\bar{d}^{\alpha\beta}\partial_\alpha\dot{u}_{\hat{h}\beta} + r^{-1}\bar{d}^{11}\dot{w}_{\hat{g}}). \end{aligned} \tag{30}$$

Action functional $A(u_{\hat{h}\alpha}, w_{\hat{g}}, \theta_{\hat{q}})$ determined by $L_{\hat{h}\hat{g}\hat{q}}$ is defined by

$$A_{\hat{h}\hat{g}\hat{q}}(u_{\hat{h}\alpha}, w_{\hat{g}}, \theta_{\hat{q}}) = \int_0^{L_1}\int_0^{L_2}\int_{t_0}^{t_1} L_{\hat{h}\hat{g}\hat{q}}\,dt\,d\xi\,dx. \tag{31}$$

We substitute the right-hand sides of (27) into Lagrangian (29) and the constitutive equations (30) for functions $p^{\alpha\beta}(x,\xi,t)$,$\hat{r}\,(x,\xi,t)$. Then, we average the results over the cell applying formula (12) and *tolerance averaging approximation* (13). As a result we obtain function $< L_{\hat{h}\hat{g}\hat{q}} >$ called *the tolerance averaging of lagrangian* (29) *in* $\Delta(x)$ *under superimposed decomposition* (27). Recalling that $u_{0\alpha}$, w_0, θ_0 in (27) are known and under the additional approximation $1+\lambda/r \approx 1$ (i.e. after neglecting terms of an order of λ/r), the final result has the form

$$
\begin{aligned}
< L_{\widehat{hgq}} >= & -\frac{1}{2}[< D^{\alpha\beta\gamma\delta}\partial_\beta u_{0\alpha}\partial_\delta u_{0\gamma} > +2 < D^{\alpha\beta\gamma 1}\partial_1\hbar\partial_\beta u_{0\alpha} > Q_\gamma \\
& + < D^{\alpha 11\gamma}(\partial_1\hbar)^2 > Q_\gamma Q_\alpha + \underline{< D^{\alpha 22\delta}(\hbar)^2 >}\partial_2 Q_\gamma \partial_2 Q_\alpha \\
& +2r^{-1}(< D^{\alpha\beta 11}\partial_\beta u_{0\alpha} w_0 > + < D^{\alpha 111}\partial_1\hbar\, w_0 > Q_\alpha) \\
& +r^{-2} < D^{1111} w_0\, w_0 > + < B^{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w_0\,\partial_{\gamma\delta}w_0 > \\
& +2(< B^{\alpha\beta 11}\partial_{11}\tilde{g}\,\partial_{\alpha\beta}w_0 > V + \underline{< B^{\alpha\beta 22}\tilde{g}\,\partial_{\alpha\beta}w_0 >}\partial_{22}V \\
& +\underline{< B^{1122}\tilde{g}\,\partial_{11}\tilde{g} >}\partial_{22}V\, V) + 4 < \underline{B^{1212}(\partial_1\tilde{g})^2 >}(\partial_2 V)^2 \\
& + < B^{1111}(\partial_{11}\tilde{g})^2 > V^2 + \underline{< B^{2222}(\tilde{g})^2 >}(\partial_{22}V)^2 \\
& - < K^{\alpha\beta}\partial_\alpha\theta_0\,\partial_\beta\theta_0 > -2 < K^{1\beta}\partial_1\tilde{q}\,\partial_\beta\theta_0 > \Psi \\
& -2\underline{< K^{2\beta}\tilde{q}\,\partial_\beta\theta_0 >}\partial_2\Psi - < K^{11}(\partial_1\tilde{q})^2 > \Psi^2 + \underline{< K^{22}(\tilde{q})^2 >}(\partial_2\Psi)^2 \\
& - < \mu a^{\alpha\beta}\dot{u}_{0\alpha}\dot{u}_{0\beta} > - < \mu(\dot{w}_0)^2 > - \underline{< \mu(\hbar)^2 >} a^{\alpha\beta}\dot{Q}_\alpha\dot{Q}_\beta - \underline{< \mu(\tilde{g})^2 >}(\dot{V})^2] \\
& + < p^{\alpha\beta}\partial_\beta u_{0\alpha} > + < p^{\alpha 1}\partial_1\hbar > Q_\alpha + \underline{< p^{\alpha 2}\hbar >}\partial_2 Q_\alpha \\
& +r^{-1} < p^{11}w_0 > + < \hat{r}\theta_0 > + \underline{< \hat{r}\tilde{q} >}\Psi,
\end{aligned}
\tag{32}
$$

with averaged non-variational parameters given by

$$
\begin{aligned}
& < p^{\alpha\beta}\partial_\beta u_{0\alpha} >=< \bar{d}^{\alpha\beta}\theta_0\partial_\beta u_{0\alpha} > + \underline{< \bar{d}^{\alpha\beta}\tilde{q}\,\partial_\beta u_{0\alpha} >}\Psi, \\
& < p^{\alpha 1}\partial_1\hbar >=< \bar{d}^{\alpha 1}\partial_1\hbar\,\theta_0 > + \underline{< \bar{d}^{\alpha 1}\tilde{q}\,\partial_1\hbar >}\Psi, \\
& < p^{\alpha 2}\hbar >=< \bar{d}^{\alpha 2}\hbar\,\theta_0 > + \underline{< \bar{d}^{\alpha 2}\tilde{q}\,\hbar >}\Psi, \\
& < p^{11}w_0 >=< \bar{d}^{11}w_0\theta_0 >, \\
& < \hat{r}\theta_0 >=< c\theta_0\dot{\theta}_0 > + T_0[< \bar{d}^{\alpha\beta}\theta_0\,\partial_\alpha\dot{u}_{0\beta} > + < \bar{d}^{1\beta}\partial_1\hbar\,\theta_0 > \dot{Q}_\beta \\
& \quad + \underline{< \bar{d}^{2\beta}\hbar\theta_0 >}\partial_2\dot{Q}_\beta + r^{-1} < \bar{d}^{11}\theta_0\,\dot{w}_0 >], \\
& < \hat{r}\tilde{q} >= \underline{< c\tilde{q}^2 >}\dot{\Psi} + T_0[\underline{< \tilde{q}\,\bar{d}^{\alpha\beta}\partial_\alpha\dot{u}_{0\beta} >} + \underline{< \tilde{q}\,\bar{d}^{1\beta}\partial_1\hbar >}\dot{Q}_\beta \\
& \quad + < \tilde{q}\,\bar{d}^{2\beta}\hbar > \partial_2\dot{Q}_\beta].
\end{aligned}
\tag{33}
$$

The underlined terms in (32), (33) depend on a period length λ.

Action functional

$$
A_{\widehat{hgq}}(Q_\alpha, V, \Psi) = \int_0^{L_1}\int_0^{L_2}\int_{t_0}^{t_1} < L_{\widehat{hgq}} > dt\,d\xi\,dx,
\tag{34}
$$

with $< L_{\widehat{hgq}} >$ given by (32) and with expressions (33) for averaged invariable parameters occurring in (32), is called *the tolerance averaging of action functional* (31) *under superimposed decomposition* (27).

The extended principle of stationary action applied to (34) leads to the following system of Euler–Lagrange equations for $Q_\alpha(x,\xi,t),\ V(x,\xi,t),\ \Psi(x,\xi,t),\ (x,\xi,t)\in\Omega\times\Xi\times I$,

$$
\begin{aligned}
& \frac{\partial}{\partial t}\frac{\partial < L_{\widehat{hgq}} >}{\partial\dot{Q}_\alpha} - \frac{\partial < L_{\widehat{hgq}} >}{\partial Q_\alpha} + \partial_2\frac{\partial < L_{\widehat{hgq}} >}{\partial(\partial_2 Q_\alpha)} = 0, \\
& \frac{\partial}{\partial t}\frac{\partial < L_{\widehat{hgq}} >}{\partial\dot{V}} - \frac{\partial < L_{\widehat{hgq}} >}{\partial V} + \partial_2\frac{\partial < L_{\widehat{hgq}} >}{\partial(\partial_2 V)} - \partial_{22}\frac{\partial < L_{\widehat{hgq}} >}{\partial(\partial_{22} V)} = 0, \\
& -\frac{\partial < L_{\widehat{hgq}} >}{\partial\Psi} + \partial_2\frac{\partial < L_{\widehat{hgq}} >}{\partial(\partial_2\Psi)} = 0.
\end{aligned}
\tag{35}
$$

Combining (35) with (32) and (33) we obtain finally the explicit form of *the superimposed tolerance microscopic model equations*

$$
\begin{aligned}
& - < D^{\beta 11\gamma}(\partial_1\hbar)^2 > Q_\gamma + \underline{\underline{< D^{\beta 22\gamma}(\hbar)^2 >}}\partial_{22}Q_\gamma + \underline{< \bar{d}^{\beta 1}\partial_1\hbar\tilde{q} >}\Psi \\
& -\underline{< \mu(\hbar)^2 >} a^{\alpha\beta}\ddot{Q}_\alpha = r^{-1} < \partial_1\hbar\, D^{\beta 111}w_0 > - < \partial_1\hbar\,\bar{d}^{\beta 1}\theta_0 > + < \partial_1\hbar D^{\beta 1\gamma\delta}\partial_\delta u_{0\gamma} >,
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
&< (\partial_{11}\breve{g})^2 B^{1111} > V + (\underline{\underline{2 < \partial_{11}\breve{g} B^{1122}\breve{g} >}} - 4\underline{\underline{< (\partial_1\breve{g})^2 B^{1212} >}})\, \partial_{22} V \\
&+ \underline{\underline{< (\breve{g})^2 B^{2222} >}}\, \partial_{2222} V + \underline{< \mu(\breve{g})^2 >}\, \ddot{V} = - < \partial_{11}\breve{g} B^{11\alpha\beta} \partial_{\alpha\beta} w_0 >,
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&< K^{11}(\partial_1\breve{q})^2 > \Psi - \underline{\underline{< K^{22}(\breve{q})^2 >}}\, \partial_{22}\Psi + \underline{< c(\breve{q})^2 >}\, \dot{\Psi} \\
&\quad + T_0[\underline{< \bar{d}^{1\beta}\breve{q}\, \partial_1\breve{h} >}\, \dot{Q}_\beta + \underline{\underline{< \bar{d}^{2\beta}\breve{q}\, \breve{h} >}}\, \partial_2 \dot{Q}] \\
&= -T_0[\underline{< \breve{q}\, \bar{d}^{\alpha\beta} \partial_\alpha \dot{u}_{0\beta} >}] - < K^{1\beta}\partial_1\breve{q}\, \partial_\beta\theta_0 > .
\end{aligned} \tag{38}
$$

Let us observe that after application of the *extended principle of stationary action*, terms occurring in lagrangian (32), which do not contain fluctuation amplitudes and terms $< K^{2\beta}\breve{q}\, \partial_\beta\theta_0 > \partial_2\Psi$, $< p^{\alpha 2}\breve{h} > \partial_2 Q_\alpha$ dropped out from the modelling.

Equations (36)–(38) together with the *superimposed micro–macro decomposition* (27) and *physical reliability conditions* (28) constitute *the superimposed tolerance model* (i.e. microscopic model imposed on the macroscopic one obtained in the first step of combined modelling) *for the analysis of selected dynamic thermoelasticity problems for the thin uniperiodic cylindrical shells under consideration.* Averages $< \cdot >$ occurring in (36)–(38) are *constant* and some of them *involve microstructure length parameter* λ (singly and doubly underlined terms).

Let us observe that we have obtained system of two equations (36) for displacement fluctuation amplitudes $Q_\alpha(x, \xi, t)$ coupled with Eq. (38) for temperature fluctuation amplitude $\Psi(x, \xi, t)$, and independent equation (37) for displacement fluctuation amplitude $V(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$.

4.2.1 Discussion of results

The important features of the derived *tolerance microscopic model* are listed below.

- The microscopic model equations (36)–(38) have *constant coefficients*. Moreover, *some of these coefficients depend on a cell size* λ (underlined coefficients). Hence, *the above model is able to describe the effect of a microstructure size on the thermoelastic shell behaviour*. Moreover, we can analyse the length-scale effect not only in non-stationary but also in stationary problems for the uniperiodic shells considered here.
- The right-hand sides of (36)–(38) are known under assumption that $u_{0\alpha}, w_0, \theta_0$ were determined in the first step of the combined modelling.
- Governing equations (36)–(38) contain spatial derivatives of Q_α, V, Ψ with respect to argument $\xi \in \Xi$ only. Hence, the boundary conditions for these unknown fluctuation amplitudes should be defined only on boudaries $\xi = 0, \xi = L_2$.
- Decomposition (27) and hence also the resulting Eqs. (36)–(38) are uniquely determined by the postulated *a priori*, λ-periodic, continuous and highly oscillating *fluctuation shape functions for displacements* $\breve{h} \in FS^1(\Omega, \Delta)\breve{h} \in O(\lambda), \breve{g} \in FS^2(\Omega, \Delta), \breve{g} \in O(\lambda^2)$, and *for temperature* $\breve{q} \in FS^1(\Omega, \Delta), \breve{q} \in O(\lambda)$, which represent oscillations of displacement and temperature fields inside a cell. These functions can be derived as solutions to periodic eigenvalue cell problems. In the most cases, approximate forms of these solutions are taken into account, cf. e.g. [11,23–25]. The choice of these functions can be also based on the experience or intuition of the researcher.
- The basic unknowns $Q_\alpha(x, \xi, t)$, $V(x, \xi, t)$, $\Psi(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, of the microscopic model equations must be *the slowly-varying functions in periodicity direction,* i.e. $Q_\alpha(\cdot, \xi, t)$, $\Psi(\cdot, \xi, t) \in SV^1_\delta(\Omega, \Delta)$, $V(\cdot, \xi, t) \in SV^2_\delta(\Omega, \Delta)$ for every $(\xi, t) \in \Xi \times I$. This requirement can be verified only *a posteriori* and it determines the range of the physical applicability of the model.

Now, let us discuss an important modification of Eqs. (36)–(38). Let us replace fluctuation shape functions $\breve{h}, \breve{g}, \breve{q}$ in (36)–(38) by fluctuation shape functions h, g, q, respectively. We recall that h, g, q are fluctuation shape functions occurring in the asymptotic model equations derived in the first step of the combined modelling. On the basis of results shown in the most of the known publications dealing with the thermoelasticity problems for microheterogeneous structures, cf. e.g. [36], terms with zero stress temperature T_0 in heat conduction equation (38) can be treated as negligibly small. These terms are responsible for the connection between the displacements in directions tangent to the shell midsurface and the temperature. Omission of these terms means that the displacements are dependent on the temperature, but the temperature is not dependent on the displacements. That also means that the temperature in the shell can be calculated independently of displacements and can therefore be treated as a thermal load of the elastic shell. Thus, assuming that the impact of the displacement fields on the temperature in the dynamic thermoelasticity problem under consideration doesn't exist or is negligibly small, we will neglect the terms with zero stress temperature T_0 in (38). Note, that for each problem investigated, the introduction of this assumption should be preceded by a numerical analysis dealing with comparison of the solutions to Eqs. (36), (38), obtained using the terms with T_0, with solutions which do not take into account these terms.

Under assumptions given above, Eqs. (36)–(38) reduce to the following form

$$\begin{aligned} &-< D^{\beta 11\gamma}(\partial_1 h)^2 > Q_\gamma + \underline{< D^{\beta 22\gamma}(h)^2 >}\partial_{22}Q_\gamma + \underline{< \bar{d}^{\beta 1}\partial_1 h\, q >}\Psi \\ &\quad -\underline{< \mu(h)^2 >}a^{\alpha\beta}\ddot{Q}_\alpha = r^{-1} < \partial_1 h\, D^{\beta 111}w_0 > - < \partial_1 h\, \bar{d}^{\beta 1}\theta_0 > + < \partial_1 h D^{\beta 1\gamma\delta}\partial_\delta u_{0\gamma} >, \end{aligned} \tag{39}$$

$$\begin{aligned} &< (\partial_{11}g)^2 B^{1111} > V + (\underline{2 < \partial_{11}g B^{1122}\tilde{g} >} - 4\underline{< (\partial_1 g)^2 B^{1212} >})\,\partial_{22}V \\ &\quad +\underline{< (g)^2 B^{2222} >}\partial_{2222}V + \underline{< \mu(g)^2 >}\ddot{V} = - < \partial_{11}g B^{11\gamma\delta}\partial_{\gamma\delta}w_0 >, \end{aligned} \tag{40}$$

$$< K^{11}(\partial_1 q)^2 > \Psi - \underline{< K^{22}(q)^2 >}\partial_{22}\Psi + \underline{< c(q)^2 >}\dot{\Psi} = - < K^{1\beta}\partial_1 q\,\partial_\beta\theta_0 > . \tag{41}$$

By means of the consistent asymptotic modelling used to the right-hand sides of (39)–(41), we obtain

$$\begin{aligned} &r^{-1} < D^{\alpha 111}\partial_1 h\, w_0 > + < D^{\alpha 1\gamma\delta}\partial_1 h\, \partial_\delta u_{0\gamma} > - < \partial_1 h\, \bar{d}^{\alpha 1}\theta_0 > \\ &\quad = r^{-1} < D^{\alpha 111}\partial_1 h > w^0 + < D^{\alpha 1\gamma\delta}\partial_1 h > \partial_\delta u^0_\gamma + < \partial_1 h\, D^{\alpha 1\gamma 1}\,\partial_1 h > U_\gamma \\ &\quad - < \partial_1 h\, \bar{d}^{1\beta} > \theta^0, \end{aligned} \tag{42}$$

$$< \partial_{11}g\, B^{11\gamma\delta}\,\partial_{\gamma\delta}w_0 > = < \partial_{11}g\, B^{11\gamma\delta} > \partial_{\gamma\delta}w^0 + < \partial_{11}g\, B^{1111}\partial_{11}g > W, \tag{43}$$

$$< K^{1\beta}\partial_1 q\,\partial_\beta\theta_0 > = < K^{1\beta}\partial_1 q > \partial_\beta\theta^0 + < \partial_1 q\, K^{11}\,\partial_1 q > \Theta. \tag{44}$$

We recall that in the consistent asymptotic modelling procedure, unknowns u^0_α, U_α, θ^0, Θ are assumed to be continuous and bounded in $\bar{\Omega}$ together with their first derivatives and unknowns w^0, W are assumed to be continuous and bounded in $\bar{\Omega}$ together with their derivatives up to the second order, cf. Sect. 4.1. We also recall that under limit passage $\varepsilon \to 0$, every continuous and bounded function of argument $z \in \Delta_\varepsilon(x)$, tends to function of argument $x \in \bar{\Omega}$, cf. Sect. 4.1 herein or monograph [8]. Hence, during the modelling procedure, the above-mentioned unknown functions are moved outside the averaging operator.

From comparison of the right-hand sides of (42), (43), (44) with asymptotic model equations $(21)_3$, $(21)_4$, $(21)_6$, respectively, it follows that the right-hand sides of (42)–(44) are equal to zero. Accordingly, the left-hand sides of (42)–(44), which coincide with the right-hand sides of (39)–(41), are also equal to zero, and we arrive finally to the following equations for unknown *slowly-varying fluctuation amplitudes for displacements* $Q_\alpha(x,\xi,t)$, $V(x,\xi,t)$ *and temperature* $\Psi(x,\xi,t)$, $(x,\xi,t) \in \Omega \times \Xi \times I$,

$$\begin{aligned} &- < D^{\beta 11\gamma}(\partial_1 h)^2 > Q_\gamma + \underline{< D^{\beta 22\gamma}(h)^2 >}\partial_{22}Q_\gamma + \underline{< \bar{d}^{\beta 1}\partial_1 h\, q >}\Psi \\ &\quad -\underline{< \mu(h)^2 >}a^{\alpha\beta}\ddot{Q}_\alpha = 0, \end{aligned} \tag{45}$$

$$\begin{aligned} &< (\partial_{11}g)^2 B^{1111} > V + (\underline{2 < \partial_{11}g B^{1122}\tilde{g} >} - 4\underline{< (\partial_1 g)^2 B^{1212} >})\,\partial_{22}V \\ &\quad +\underline{< (g)^2 B^{2222} >}\partial_{2222}V + \underline{< \mu(g)^2 >}\ddot{V} = 0, \end{aligned} \tag{46}$$

$$< K^{11}(\partial_1 q)^2 > \Psi - \underline{< K^{22}(q)^2 >}\partial_{22}\Psi + \underline{< c(q)^2 >}\dot{\Psi} = 0. \tag{47}$$

Equations (45)–(47) are independent of solutions $u_{0\alpha}$, w_0, θ_0 obtained in the first step of combined modelling, i.e. in the framework of *the macroscopic (asymptotic) model* and *hence make it possible to separate the microscopic description of some special dynamic or thermal or coupled dynamic thermoelasticity problems from macroscopic description of these problems.* Let us observe that Eq. (45), which are conjugated with Eq. (47), allow us to investigate some thermoelasticity problems dealing with the coupling of the cell-dependent circumferential and axial displacement micro-fluctuations with the cell-dependent temperature micro-fluctuations. Equation (46) makes it possible to analyse micro-dynamic problems, e.g. the cell-dependent transversal free vibrations. Using Eq. (47) we can study some thermal problems related to cell-dependent fluctuations of the temperature field. We recall that the underlined terms in (45)–(47) depend on the microstructure size.

4.3 Combined asymptotic-tolerance model

Summarizing results obtained in *Step 1* and *Step 2* we conclude that *the combined asymptotic-tolerance model of selected dynamic thermoelasticity problems for the thin uniperiodically microheterogeneous cylindrical shells under consideration* derived here is represented by:

- *Macroscopic model* defined by Eq. (23) for macrodisplacements $u_\alpha^0(x, \xi, t)$, $w^0(x, \xi, t)$ and macrotemperature $\theta^0(x, \xi, t)$ together with expressions (22) for kinematic $U_\alpha(x, \xi, t)$, $W(x, \xi, t)$ and thermal $\Theta(x, \xi, t)$ fluctuation amplitudes and with expressions (24) for the effective moduli, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$. This model is obtained by means of *the consistent asymptotic modelling* and is independent of the microstructure size.
- *Superimposed microscopic model equations* (36)–(38) for new kinematic $Q_\alpha(x, \xi, t)$, $V(x, \xi, t)$ and thermal $\Psi(x, \xi, t)$ fluctuation amplitudes, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, together with *micro–macro decomposition* (27) and physical reliability conditions (28). This model is derived by means of *the tolerance (non-asymptotic) modelling*. Some coefficients of tolerance model equations (underlined terms) depend on the microstructure length parameter λ. Microscopic and macroscopic models are conjugated with themselves under assumption that in the framework of the macroscopic model the solutions (26) to the problem under consideration are known.
- Total decomposition having the following form

$$
\begin{aligned}
&u_\alpha(x, \xi, t) = u_\alpha^0(x, \xi, t) + h(x)U_\alpha(x, \xi, t) + \bar{h}(x)Q_\alpha(x, \xi, t),\\
&w(x, \xi, t) = w^0(x, \xi, t) + g(x)W(x, \xi, t) + \bar{g}(x)V(x, \xi, t),\\
&\theta(x, \xi, t) = \theta^0(x, \xi, t) + q(x)\Theta(x, \xi, t) + \bar{q}(x)\Psi(x, \xi, t),\\
&(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}.
\end{aligned}
\tag{48}
$$

The characteristic features of the derived *combined asymptotic-tolerance model* are:

- In contrast to starting equations (4), (5) with discontinuous, highly oscillating and periodic coefficients, the combined model equations proposed here *have constant coefficients*. Moreover, s*ome coefficients of the superimposed microscopic model equations depend on a cell size* λ. Thus, the combined model can be applied to the analysis of many phenomena caused by the length-scale effect.
- The solutions to initial-boundary value problems formulated in the framework of the combined asymptotic-tolerance model have a physical sense only if unknown macrodisplacements u_α^0, w^0 and macrotemperature θ^0 as well as kinematic U_α, W and thermal Θ fluctuation amplitudes of *the asymptotic model* are *continuous and bounded* in $\bar{\Omega}$ together with their pertinent derivatives, and if unknown kinematic Q_α, W and thermal Ψ fluctuation amplitudes of *the superimposed microscopic tolerance model*are *slowly-varying* with respect to periodicity cell and pertinent tolerance parameters.
- The resulting combined model equations are uniquely determined by the strongly oscillating periodic *fluctuation shape functions* for displacements and temperature, which have to be known in every problem under consideration. In general case, the fluctuation shape functions of both the macroscopic and the microscopic models are different. Under assumption that the fluctuation shape functions of both the models coincide as well as under assumption that the terms with zero stress temperature T_0 in conductivity equation (38) can be treated as negligibly small, we have derived superimposed microscopic model equations (45)–(47), which are independent of the solutions obtained in the framework of the macroscopic model. Taking into account this result we can conclude that *an important advantage of the combined model is that it makes it possible to separate the macroscopic description of some special problems from their microscopic description.*
- Microscopic model equations (45)–(47) can be applied to the analysis of certain *initial-boundary layer and space-boundary layer phenomena* strictly related to the specific form of initial and boundary conditions imposed on the kinematic and thermal micro-fluctuation amplitudes. That is why, these equations are referred to as *the boundary layer equations*, where the term "boundary" is related both to time and space. A certain space-boundary layer problem is shown in Sect. 5.
- Applying *the tolerance modelling* directly to the total decomposition (48) we also obtain the system of equations for u_α^0, w^0, θ^0, U_α, Q_α, W, V, Θ, Ψ. However, this system is much more complicated than the system obtained in the framework of the combined modelling.

4.4 Comparison of asymptotic-tolerance models for uniperiodic and biperiodic shells

We recall that the asymptotic-tolerance model for the thin uniperiodic cylindrical shells formulated here is represented by asymptotic (macroscopic) model equations (23) for macrodisplacements u_α^0, w^0 and macrotemperature θ^0 with expressions (22) for kinematic U_α, W and thermal Θ fluctuation amplitudes and by tolerance (microscopic) model equations (36)–(38) for kinematic Q_α, V and thermal Ψ fluctuation amplitudes as well as by total decomposition (48). Asymptotic and tolerance models are combined together under assumption that in the framework of the asymptotic model the solutions (26) to the problem under consideration are known.

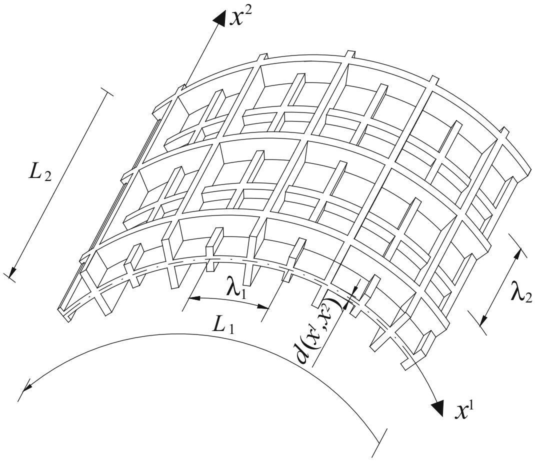

Fig. 3 Fragment of the shell reinforced by two families of biperiodically spaced ribs

Let us compare the asymptotic-tolerance model for the thin uniperiodic cylindrical shells derived here with the corresponding asymptotic-tolerance model for the thin cylindrical shells with a periodic structure in circumferential and axial directions (*biperiodic shells*) proposed and discussed by Tomczyk et al. [32]. An example of such a shell is presented in Fig. 3.

For the biperiodic shells, the region Ω is defined by: $\Omega \equiv (0, L_1) \times (0, L_2)$. *The basic cell* Δ and an arbitrary cell $\Delta(\mathbf{x})$ with the centre at point $\mathbf{x} \equiv (x^1, x^2) \in \Omega_\Delta$ are defined by means of: $\Delta \equiv [-\lambda_1/2,\ \lambda_1/2] \times [-\lambda_2/2,\ \lambda_2/2]$, $\Delta(\mathbf{x}) \equiv \mathbf{x} + \Delta$, $\Omega_\Delta \equiv \{\mathbf{x} \in \Omega : \Delta(\mathbf{x}) \subset \Omega\}$, where $\lambda_1$and λ_2 are the period lengths of the shell structure respectively in x^1- and x^2-directions, cf. Fig. 3. The diameter $\lambda \equiv \sqrt{(\lambda_1)^2 + (\lambda_2)^2}$ of Δ, called *the microstructure length parameter*, is assumed to satisfy conditions: $\lambda / \sup_{\mathbf{x}\in\Omega} d(\mathbf{x}) \gg 1$, $\lambda/r << 1$ and $\lambda/\min(L_1, L_2) << 1$.

For the biperiodic shells under consideration, elastic stiffness tensors $D^{\alpha\beta\gamma\delta}$, $B^{\alpha\beta\gamma\delta}$, the shell mass density μ, membrane thermal stiffness tensor $\bar{d}^{\alpha\beta}$, tensor of heat conductivity $K^{\alpha\beta}$ and specific heat c are periodic, highly oscillating and non-continuous functions not only with respect to argument $x \equiv x^1 \in (0, L_1)$, but also with respect to argument $\xi \equiv x^2 \in (0, L_2)$.

Following [32], the asymptotic-tolerance model for the analysis of dynamic thermoelasticity problems for the biperiodic shells under consideration consists of:

- *Macroscopic model* equations (23), expressions (22), (24) and solutions (26), in which unknowns u^0_α, U_α, w^0, W, θ^0, Θ are assumed to be bounded and continuous in $\bar{\Omega} \equiv [0, L_1] \times [0, L_2]$ together with their appropriate derivatives and where fluctuation shape functions h, g, q are periodic not only with respect to midsurface parameter $x \equiv x^1 \in (0, L_1)$ but also with respect to $\xi \equiv x^2 \in (0, L_2)$.
- *Superimposed microscopic model equations* for new kinematic $Q_\alpha(x^1, x^2, t)$, $V(x^1, x^2, t)$ and thermal $\Psi(x^1, x^2, t)$ fluctuation amplitudes, $(x^1, x^2, t) \in (0, L_1) \times (0, L_2) \times I$, derived by means of *the tolerance (non-asymptotic) modelling*

$$\begin{aligned} &-<\partial_\beta \tilde{h}\, D^{\alpha\beta\gamma\delta}\partial_\gamma \tilde{h}> Q_\delta + \underline{<\tilde{g}\, \bar{d}^{\alpha\beta}\partial_\beta \tilde{h}>}\Psi - \underline{<\mu(\tilde{h})^2>}\, a^{\alpha\beta}\ddot{Q}_\beta \\ &= r^{-1}<D^{\alpha\beta 11}\partial_\beta \tilde{h}\, w_0> + \underline{<D^{\alpha\beta\gamma\delta}\partial_\delta \tilde{h}\, \partial_\beta u_{0\gamma}>} - <\partial_\beta \tilde{h}\, \bar{d}^{\alpha\beta}\theta_0>, \end{aligned} \tag{49}$$

$$<\partial_{\alpha\beta}\tilde{g}\, B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}\tilde{g}> V \underline{+ <\mu(\tilde{g})^2>}\, \ddot{V} = -<B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}\tilde{g}\, \partial_{\alpha\beta}w_0>, \tag{50}$$

$$\begin{aligned} &<K^{\alpha\beta}\partial_\alpha \tilde{q}\, \partial_\beta \tilde{q}>\Psi + \underline{<c(\tilde{q})^2>}\dot{\Psi} + T_0\underline{<\bar{d}^{\alpha\beta}\tilde{q}\, \partial_\alpha \tilde{h}>}\dot{Q}_\beta \\ &= -<K^{\alpha\beta}\partial_\alpha \tilde{q}\, \partial_\beta \theta_0> - T_0\underline{<\tilde{q}\, \bar{d}^{\alpha\beta}\partial_\alpha \dot{u}_{0\beta}>}, \end{aligned} \tag{51}$$

where unknown fluctuation amplitudes Q_α, V, Ψ are slowly-varying functions in $x^1 \in (0, L_1)$ and $x^2 \in (0, L_2)$, i.e. Q_α, $\Psi \in SV^1_\delta(\Omega, \Delta)$, $V \in SV^2_\delta(\Omega, \Delta)$, and where fluctuation shape functions $\tilde{h}$, $\tilde{g}$, $\tilde{q}$ are periodic with respect to x^1 and x^2, i.e. $\tilde{h} \in FS^1(\Omega, \Delta)$, $\tilde{h} \in O(\lambda)$, $\tilde{g} \in FS^2(\Omega, \Delta)$, $\tilde{g} \in O(\lambda^2)$, $\tilde{q} \in FS^1(\Omega, \Delta)$, $\Omega \equiv (0, L_1) \times (0, L_2)$, $\Delta \equiv [-\lambda_1/2,\ \lambda_1/2] \times [-\lambda_2/2,\ \lambda_2/2]$, $\lambda \equiv \sqrt{(\lambda_1)^2 + (\lambda_2)^2}$. Underlined terms depend on a cell size.

- Under assumption that *the fluctuation shape functions* $h,\ g,\ q$ introduced in the first step of combined modelling coincide with those introduced in the second step as well as under assumption that the terms with zero stress temperature T_0 in conductivity equation (51) can be treated as negligibly small, the superimposed microscopic model equations, which *are independent of the solutions obtained in the framework of the macroscopic model*, were derived in [32]:

$$-<\partial_\beta h\, D^{\alpha\beta\gamma\delta}\partial_\gamma h>Q_\delta+\underline{<\bar{q}\,\bar{d}^{\alpha\beta}\partial_\beta h>}\Psi-\underline{<\mu(h)^2>}a^{\alpha\beta}\ddot{Q}_\beta=0, \tag{52}$$

$$<\partial_{\alpha\beta}g\, B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}g>V\underline{+<\mu(g)^2>}\ddot{V}=0, \tag{53}$$

$$<K^{\alpha\beta}\partial_\alpha q\,\partial_\beta q>\Psi+\underline{<c(q)^2>}\dot{\Psi}=0. \tag{54}$$

  Equations (52)–(54) make it possible to separate the microscopic description of some special dynamic or thermal or coupled dynamic thermoelasticity problems from macroscopic description of these problems.
- Total decomposition having form of (48). Obviously, the functions in (48) which are bounded and continuous in $\bar{\Omega}\equiv[0,L_1]$, slowly-varying or periodic in x^1 must be replaced by corresponding functions bounded and continuous in $\bar{\Omega}\equiv[0,L_1]\times[0,L_2]$, slowly-varying or periodic in $x^1\in(0,L_1)$ and $x^2\in(0,L_2)$.

  The main differences and similarities between both models are:

- In contrast to starting equations (4), (5) with discontinuous, highly oscillating and periodic coefficients, equations of both the combined models *have constant coefficients*. Moreover, s*ome coefficients of the tolerance models' equations derived in the second step of combined modelling depend on a cell size*λ. Thus, both the combined models can be applied to the analysis of many phenomena caused by the length-scale effect.
- Microscopic model equations (36)–(38) for uniperiodic shells *are more complicated* than microscopic model equations (49)–(51) for biperiodic shells and contain a lot of length-scale terms (doubly underlined terms), which do not have counterparts in the equations for biperiodic shells. The occurrence of these terms is strictly related to the fact that physical reliability conditions (28), imposed on unknown micro-fluctuation amplitudes Q_α, V, Ψ in the framework of the combined model for uniperiodic shells, *are less restrictive* than pertinent conditions imposed on Q_α, V, Ψ within the combined model for biperiodic shells. In combined model for uniperiodic shells the micro-fluctuation amplitudes are slowly-varying in only *one direction* (i.e. with respect to argument $x\equiv x^1\quad x^1\in(0,L_1)$), whereas in model for biperiodic shells these functions are slowly-varying in *two directions* (i.e. with respect to arguments x^1, x^2, $x^1\in(0,L_1)$, $x^2\in(0,L_2)$).
- In the framework of *the uniperiodic shell combined model*, unknown kinematic micro-fluctuation amplitudes $Q_\alpha,\ V$ and thermal micro-fluctuation amplitude Ψ are governed by *partial differential equations* (36)–(38), whereas within *the biperiodic shell combined model* these unknowns are governed by *ordinary differential equations* (49)–(51) *involving only time derivatives of* $Q_\alpha,\ V,\ \Psi$. Hence, there are no extra boundary conditions for unknowns $Q_\alpha,\ V,\ \Psi$ of the biperiodic shell model and that is why they play the role of kinematic or thermal internal variables. On the other hand, the boundary conditions for unknown $Q_\alpha,\ V,\ \Psi$ of the uniperiodic shell model should be defined only on boundaries $x^2=0,\ x^2=L_2$.
- Both the models are uniquely determined by the postulated *a priori* $\lambda-$periodic *fluctuations shape functions for displacements* $h\in O(\lambda), g\in O(\lambda^2), \bar{h}\in O(\lambda), \bar{g}\in O(\lambda^2)$ *and for temperature* $q\in O(\lambda), \bar{q}\in O(\lambda)$. In the combined model for uniperiodic shells we deal with fluctuation shape functions, which are periodic in one direction only (i.e. with respect to argument $x^1\in(0,L_1)$), whereas in the other one these functions are periodic in two directions (i.e. with respect to arguments $x^1\in(0,L_1)$, $x^2\in(0,L_2)$.
- Under assumption that *the fluctuation shape functions* introduced in the first step of combined modelling, which is based on the consistent asymptotic procedure, coincide with those introduced in the second step based on the tolerance approach as well as under assumption that the terms with zero stress temperature T_0 in conductivity equations (38) (*uniperiodic shells*) and (51) (*biperiodic shells*) can be treated as negligibly small, we derive superimposed microscopic model equations (45)–(47) for uniperiodic shells and (52)–(54) for biperiodic shells, which *are independent of the solutions obtained in the framework of the macroscopic models*. Taking into account this result we can conclude that *an important advantage of the combined models is that they make it possible to separate the macroscopic description of some special problems from their microscopic description.* Moreover, under special initial and boundary conditions, microscopic model equations (45)–(47) for uniperiodic shells can be applied to the analysis of *space-boundary layer and time-boundary layer phenomena*. Microscopic model equations (52)–(54) for biperiodic shells, under

special initial conditions are also referred to as the boundary layer equations, but the term "boundary" is related only to time.

- The solutions to selected initial-boundary value problems formulated within both the combined models have a physical sense only if the basic unknowns are *slowly-varying functions* either in argument x^1 (*uniperiodic shells*) or in arguments x^1, x^2 (*biperiodic shells*).

5 Examples of applications

5.1 Description of the problems

The *biggest advantage of the asymptotic-tolerance model derived in this paper is that it makes it possible to separate the macroscopic description of some special dynamic or thermal or coupled dynamic thermoelasticity problems from microscopic description of these problems*, cf. Eqs. (45)–(47). For that reason, in this section, we shall study two special length-scale problems applying micro-dynamic equation (46) and micro-thermal equation (47). The first of them refers to the derivation of formula for the frequency of the cell-dependent transversal free micro-vibrations. The second one deals with study of the effect of a microstructure size on the shape of initial distributions of temperature micro-fluctuations.

The object of consideration is a thin circular closed cylindrical shell with $r, L_1 = 2\pi r, L_2, d$ as its midsurface curvature radius, circumferential length, axial length and constant thickness, respectively. The shell is composed of two kinds of homogeneous, elastic, isotropic materials periodically and densely distributed in the circumferential direction and perfectly bonded on interfaces. Such a shell is shown in Fig. 2, where in the problem analysed here length dimension $L_1 = 2\pi r$.

In agreement with considerations in Sect. 2, we define λ as the period length of the shell structure in $x \equiv x^1$-direction, cf. Fig. 2. *The microstructure length parameter* λhas to satisfy conditions: $\lambda/d \gg 1$, $\lambda/r \ll 1$ and $\lambda/L_1 \ll 1$. *The periodicity cell* is defined by: $\Delta \equiv [-\lambda/2,\ \lambda/2]$. Setting $z \equiv z^1 \in [-\lambda/2,\ \lambda/2]$, we assume that the cell has a symmetry axis for $z \equiv z^1 = 0$. It is also assumed that inside the cell, the geometrical, elastic, inertial and thermal properties of the shell are described by symmetric (i.e. even) functions in argument z.

For the considered shell made of elastic, isotropic constituents, the bending rigidities $B^{1111} = B^{2222}$, B^{1122}, B^{1212} and the shell mass density $\mu(\cdot)$ per midsurface unit area occurring in (46) as well as the tensor of heat conductivity $K^{11} = K^{22}$ and the specific heat c existing in (47) take constant values in each shell component material.

We assume that fluctuation shape function for displacements $g \in FS^2(\Omega, \Delta)$ in Eq. (46) and fluctuation shape function for temperature $q \in FS^1(\Omega, \Delta)$ in Eq. (47) are either even or odd functions with respect to argument $z \in [-\lambda/2,\ \lambda/2]$.

Let the investigated problems be rotationally symmetric with a period λ/r. Hence, unknown slowly-varying micro-fluctuation amplitudes $V(x, \xi, t), \Psi(x, \xi, t), (x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, of Eqs. (46) and (47), respectively, are independent of argument x. Obviously, fluctuation shape functions g and q are λ-periodic functions in argument x. Hence, the micro-fluctuations of displacements and of temperature given by $g(x)V(\xi, t)$ and $q(x)\Psi(\xi, t)$, respectively, are functions not only of arguments $\xi,\ t$, but also of argument x, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$.

The subsequent analysis will be based on equations (46) and (47), in which micro-fluctuation amplitudes $V(x, \xi, t), \Psi(x, \xi, t), (x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, are replaced by $V(\xi, t)$ and $\Psi(\xi, t)$, $(\xi, t) \in \Xi \times \mathrm{I}$, respectively, i.e. on the following equations

$$
\begin{aligned}
&< (\partial_{11}g)^2 B^{1111} > V(\xi, t) + (\underline{2 < \partial_{11}g B^{1122} g >} - 4\underline{< (\partial_1 g)^2 B^{1212} >})\, \partial_{22} V(\xi, t) \\
&+ \underline{< (g)^2 B^{2222} >}\, \partial_{2222} V(\xi, t) + \underline{< \mu(g)^2 >}\, \ddot{V}(\xi, t) = 0, \qquad (55)
\end{aligned}
$$

$$
< K^{11}(\partial_1 q)^2 > \Psi(\xi, t) - \underline{< K^{22}(q)^2 >} \partial_{22}\Psi(\xi, t) + \underline{< c(q)^2 >}\, \dot{\Psi}(\xi, t) = 0. \qquad (56)
$$

5.2 Example 1: The cell-dependent transversal free micro-vibrations

In this subsection, the cell-dependent frequency of the transversal free micro-vibrations of the uniperiodically shell under consideration will be derived. This micro-dynamic problem will be based on Eq. (55).

It is assumed that the edges $\xi = 0$, $\xi = L_2$ of the shell are simply supported, i.e. they are hinged with the support free, cf. Kaliski [43]. Solution to Eq. (55) satisfying these boundary conditions can be assumed in the form

$$V(\xi, t) = A \sin(k\,\xi) \cos(\omega_* t), \tag{57}$$

where A is an arbitrary constant different from zero, $k = \pi/L_2$ is a wave number and ω_* is a frequency of the transversal free micro-vibrations.

Substituting the right-hand side of (57) into Eq. (55), under extra denotations

$$\begin{aligned} &\bar{a} \equiv \pi^4 (L_2)^{-4} < B^{2222} (\bar{g})^2 >, \\ &\bar{b} \equiv 2\pi^2 (L_2)^{-2} (< B^{1122} \bar{g}\, \partial_{11} g > -2 < B^{1212} (\partial_1 \tilde{g})^2 >), \\ &\bar{c} \equiv < B^{1111} (\partial_{11} g)^2 >, \quad \bar{\mu} \equiv < \mu (\bar{g})^2 >, \end{aligned} \tag{58}$$

where $\bar{g} = \lambda^{-2} g$, $\tilde{g} = \lambda^{-1} g$, we arrive at the following formula for frequency ω_* of the transversal free micro-vibrations

$$\omega_*^2 = \frac{\bar{a}}{\bar{\mu}} - \frac{\bar{b}}{\lambda^2\,\bar{\mu}} + \frac{\bar{c}}{\lambda^4\,\bar{\mu}}. \tag{59}$$

The free micro-vibration frequency derived above depends on a cell size λ. It has to be emphasized that under assumptions given in Sect. 5.1, values of all averages $< \cdot >$ occurring in (58) are greater than zero. We recall that these values are calculated by means of (12). It has to be emphasized that this special micro-dynamic problem can be studied neither in the framework of the asymptotic models for the uniperiodic shells under consideration nor within the known commercial numerical models based on the finite element method or the finite difference method.

It should be noted that the cell-dependent free micro-vibrations occurring in periodic structures are very significant in some special problems of dynamics or dynamical stability, cf. e.g. [11,19,25]. Let us also mention paper [39], where the effect of micro-vibrations on the space-boundary layer phenomena, observed in elastodynamics of microheterogeneous cylindrical shells, is analysed. Note, that by the space-boundary layer phenomena we mean phenomena dealing with strongly exponentially disappearing displacement fluctuations near one of the boundaries of the shell.

5.3 Example 2: Special length-scale thermal boundary value problem

In this subsection, the effect of a cell size λ on the initial distributions of temperature micro-fluctuations in the uniperiodic shells under consideration will be analysed. This micro-thermal problem will be based on Eq. (56).

We shall investigate the problem of time decaying of the temperature fluctuation amplitude $\Psi(\xi, t)$, $(\xi, t) \in \Xi \times I$, setting

$$\Psi(\xi, t) = \Psi^*(\xi) \exp(-\gamma t), \quad t \geq 0,$$

with $\gamma > 0$ as *a time decaying coefficient*. Unit of coefficient γ is $[s^{-1}]$. Function $\Psi^*(\xi)$ represents an initial distribution of temperature micro-fluctuations, i.e. $\Psi(\xi, t = 0) = \Psi^*(\xi)$.

Hence, under denotations

$$\tilde{k}^2 \equiv \frac{< K^{11} (\partial_1 q)^2 >}{\lambda^2 < K^{22} (\bar{q})^2 >}, \qquad \gamma_* \equiv \frac{< K^{11} (\partial_1 q)^2 >}{\lambda^2 < c\,(\bar{q})^2 >},$$

where $\bar{q}(\cdot) = \lambda^{-1} q(\cdot)$, Eq. (56) yields

$$\partial_{22} \Psi^*(\xi) - \tilde{k}^2 [1 - (\gamma/\gamma_*)] \Psi^*(\xi) = 0, \tag{60}$$

where γ_* is *a certain new time decaying coefficient depending on microstructure length parameter* λ. Unit of coefficient γ_* is $[s^{-1}]$. Because averages $< K^{11} (\partial_1 q)^2 >$, $< K^{22} (\bar{q})^2 >$, $< c(\bar{q})^2 >$ are greater than zero then $\tilde{k}^2 > 0$ and $\gamma_* > 0$. The boundary conditions for $\Psi^*(\xi)$ are assumed in the form

$$\Psi^*(\xi = 0) = \Psi_0^*, \quad \Psi^*(\xi = L_2) = 0,$$

where Ψ_0^* is the known constant.

The solution to Eq. (60) depends on relations between time decaying coefficients γ and γ_*. The following special cases can be taken into account:

1^0) If $0 < \gamma < \gamma_*$ and setting $\tilde{k}_\gamma^2 \equiv \tilde{k}^2[1 - (\gamma/\gamma_*)]$ then

$$\Psi^*(\xi) = \Psi_0^*[\exp(-\tilde{k}_\gamma \xi)(1 - \exp(-2\tilde{k}_\gamma L_2))^{-1} + \exp(\tilde{k}_\gamma \xi)(1 - \exp(2\tilde{k}_\gamma L_2))^{-1}];$$

the initial temperature micro-fluctuations decay exponentially.

If $0 < \gamma << \gamma_*$ then the following approximate solution to Eq. (60) can be taken into account

$$\Psi^*(\xi) = \Psi_0^* \exp(-\tilde{k}_\gamma \xi);$$

in this case the initial temperature micro-fluctuations are strongly exponentially decaying near the boundary $\xi = 0$. It means that the micro-fluctuations can be treated as equal to zero outside a certain narrow layer near boundary $\xi = 0$. Thus, Eq. (56) being a starting point in the thermal problem under consideration makes it possible to investigate *the space-boundary layer phenomena*.

2^0) If $\gamma = \gamma_*$ then

$$\Psi^*(\xi) = \Psi_0^*(1 - \xi/L_2);$$

we deal with a linear decaying of initial temperature micro-fluctuation amplitude.

3^0) If $\gamma > \gamma_*$ and setting $\kappa^2 \equiv \tilde{k}^2[(\gamma/\gamma_*) - 1] \neq (n\pi)^2(L_2)^{-2}$ then

$$\Psi^*(\xi) = \Psi_0^* \sin(\kappa(L_2 - \xi))(\sin(\kappa L_2))^{-1};$$

the temperature micro-fluctuations oscillate.

4^0) If $\gamma > \gamma_*$ and $\kappa^2 \equiv \tilde{k}^2[(\gamma/\gamma_*) - 1] = (n\pi)^2(L_2)^{-2}$ then *the solution doesn't exist.*

The above effects cannot be analysed in the framework of the asymptotic models commonly used for investigations of thermoelastic problems for micro-periodically shells under consideration. It can be observed that within the asymptotic models neglecting the length-scale terms, Eq. (56) reduces to equation $< K^{11}(\partial_1 q)^2 > \Psi = 0$, which has only trivial solution $\Psi = 0$.

Notice, that in the problem under consideration, for an arbitrary but fixed time argument t the form of temperature micro-fluctuation amplitude $\Psi(\xi, t)$ is the same as the form of initial temperature micro-fluctuation amplitude $\Psi^*(\xi)$.

6 Final remarks and conclusions

The following remarks and conclusions can be formulated:

- The objects of analysis are thin linearly thermoelastic Kirchhoff–Love-type circular cylindrical shells having a periodically micro-heterogeneous structure in the circumferential direction (uniperiodic shells), cf. Figs. 1 and 2. By periodic inhomogeneity we shall mean periodically variable shell thickness and/or periodically variable inertial, elastic and thermal properties of the shell material. At the same time, the shells have constant structures in the axial direction.
- The starting equations are the well-known governing equations of linear Kirchhoff–Love theory of thin elastic cylindrical shells combined with Duhamel–Neumann thermoelastic constitutive relations and coupled with the known linearized Fourier heat conduction equation, in which the heat sources are neglected, cf. [43–47]. These starting equations are given by (4) and (5). For uniperiodic shells, they have *highly oscillating, non-continuous and periodic coefficients*. That is why, the direct application of these equations to investigations of specific thermoelasticity problems is non-effective even using computational methods.
- The aim of this contribution was to formulate and discuss *a new mathematical averaged model for the analysis of selected dynamic thermoelasticity problems for the uniperiodic cylindrical shells under consideration.* This so-called *combined asymptotic-tolerance model* was formulated by applying *a combined modelling* including *the consistent asymptotic and the tolerance non-asymptotic modelling techniques*, which are combined together into *a single new procedure*, cf. [8,9]. However, the combined modelling was not used directly to Eqs. (4), (5), but to the integral functional determined by Lagrange function describing the thermoelastic behaviour of the shells under consideration. *The appropriate form of this function was implied by the thermoelasticity equations* (4), (5). The variational formulation of the thermoelasticity problem under consideration was based on *the extended principle of stationary action*, cf. [8,9]. Note, that the classical stationary action principle could not be applied, because the heat conduction is described with *the odd order partial differential equation with respect to the time coordinate*, cf. Eq. (5).

- The combined modelling technique is realized in two steps. In the first step, we apply *the consistent asymptotic averaging technique to starting lagrangian* (7) describing thermoelastic behaviour of the shells under consideration and *independently to constitutive equations* (8) *for non-variational parameters*. The asymptotic modelling is determined by asymptotic decomposition (14). The resulting asymptotically averaged lagrangian is given by (17) with asymptotically averaged non-variational parameters (18). Then using *the extended stationary action principle*, we obtain *the asymptotically averaged Euler–Lagrange equations* (20), which explicit form is given by (21). Equation (21) consist of partial differential equations for *macrodisplacements* u^0_α, w^0 and *macrotemperature* θ^0 coupled with linear algebraic equations for *displacement fluctuation amplitudes* U_α, W and *temperature fluctuation amplitude* Θ. After eliminating fluctuation amplitudes from the governing equations by means of (22), we arrive finally at *the asymptotic model equations* (23) *expressed only in macrodisplacements* u^0_α, w^0 and macrotemperature θ^0. Unknowns of this model must be continuous and bounded functions in periodicity directions. Coefficients of asymptotic model equations (23) are constant, but independent of a cell size. Hence, the model obtained in the first step of the combined modelling is referred to as *the macroscopic model*. Assuming that in the framework of macroscopic model the solutions (26) to the problem under consideration are known, we can pass to the second step. This step is based on *the tolerance averaging of lagrangian* (7) and *independently* on *the tolerance averaging of non-variational parameters* (8). The tolerance modelling is determined by *the extra micro–macro decomposition* (27) imposed on the known solutions (26) obtained within the macroscopic model. The resulting tolerantly averaged lagrangian is given by (32) with tolerantly averaged non-variational parameters (33). Then, applying *the extended stationary action principle*, we obtain *the averaged Euler–Lagrange equations* (35) for *kinematic* Q_α, V and *thermal* Ψ *fluctuation amplitudes* being *the new unknowns*. These unknowns must satisfy conditions (28), i.e. they have to be slowly-varying functions with respect the cell and tolerance parameters. Explicit form of Euler–Lagrange equations (35) is given by (36)–(38). Coefficients of the resulting equations are constant. Moreover, some of them depend on the microstructure size (underlined terms). Hence, the model obtained in the second step of the combined modelling is referred to as *the microscopic model*. Summing up, the *new combined asymptotic-tolerance model proposed here is represented by macroscopic* (asymptotic) *cell-independent model equations* (23) for *macrodisplacements* $u^0_\alpha(x,\xi,t)$, $w^0(x,\xi,t)$ and *macrotemperature* $\theta^0(x,\xi,t)$ together with expressions (22) for *kinematic* $U_\alpha(x,\xi,t)$, $W(x,\xi,t)$ and *thermal* $\Theta(x,\xi,t)$ *fluctuation amplitudes*, $(x,\xi,t)\in\Omega\times\Xi\times\mathrm{I}$, as well as by *superimposed microscopic* (*tolerance*) *cell-dependent model equations* (36)–(38) for unknown *microscopic kinematic* $Q_\alpha(x,\xi,t)$, $V(x,\xi,t)$ and *thermal* $\Psi(x,\xi,t)$ *fluctuation amplitudes*, $(x,\xi,t)\in\Omega\times\Xi\times I$, and by *total decomposition* (48). Macro- and microscopic model equations are coupled to each other under assumption that in the framework of the macroscopic model, the solutions (26) to the problem under consideration are known.
- The resulting combined model equations are uniquely determined by the highly oscillating periodic *fluctuation shape functions* describing oscillations of displacement and temperature fields inside the cell. These functions have to be known in every problem under consideration. They can be obtained as exact or approximate solutions to periodic eigenvalue cell problems, cf. [11,23–25]. They can also be regarded as *the shape* functions resulting from the periodic discretization of the cell using, for example, the finite element method. The choice of these functions can also be based on the experience or intuition of the researcher. In general case, the fluctuation shape functions of both the macroscopic and the microscopic models are different. Assuming that the fluctuation shape functions of both the models coincide and that the impact of the displacement fields on the temperature in the dynamic thermoelasticity problem under consideration doesn't exist or is negligibly small, we have derived superimposed microscopic model equations (45)–(47), which are independent of the solutions obtained in the framework of the macroscopic model. Hence, *an important advantage of the averaged combined model proposed here is that it makes it possible to separate the macroscopic description of some special thermoelasticity problems from their microscopic description.*
- As illustrative examples, certain two special length-scale problems were discussed on the basis of the micro-dynamic equation (46) and micro-thermal equation (47) derived in the second step of the combined modelling. These equations are independent of solutions obtained in the first step of combined modelling, i.e. in the framework of asymptotic model. Hence, it makes it possible to investigate the shell's microscopic behaviour independently of the shell's macroscopic behaviour. The object of consideration of both problems was a thin circular closed cylindrical shell made of two kinds of homogeneous elastic isotropic materials periodically and densely distributed in the circumferential direction. Fragment of such a shell is shown in Fig. 2. The first of these problems dealt with the cell-dependent transversal free micro-vibrations caused

by a microheterogeneous structure of the uniperiodic cylindrical shell under consideration. The resulting free micro-vibration frequency (59) depends on a cell size λ. The second length-scale problem dealt with the effect of a microstructure size λ on the character of the initial distributions of temperature micro-fluctuations. It was shown that in the uniperiodic shells under consideration, the form of initial temperature micro-fluctuations depends on relations between the given time decaying coefficient $\gamma > 0$ and *a certain time decaying coefficient* γ_* *depending on microstructure length parameter* λ. The initial temperature micro-fluctuations *decay exponentially* for $0 < \gamma < \gamma_*$. They *decay linearly* for $\gamma = \gamma_*$. If $\gamma > \gamma_*$ then the temperature micro-fluctuations *oscillate*. Moreover, if $0 < \gamma << \gamma_*$ then the micro-fluctuation amplitude is strongly decaying near the boundary $\xi = 0$. It means that the temperature micro-fluctuations can be treated as equal to zero outside a certain narrow layer near boundary $\xi = 0$. Thus, we have shown that the tolerance model formulated in the second step of the combined modelling, which is independent of solutions obtained in the first step of the combined modelling (i.e. within asymptotic model), makes it possible to analyse *the space-boundary layer phenomena. The length-scale problems discussed in this distribution can be studied neither in the framework of the asymptotic models for the uniperiodic shells under consideration nor within the known commercial numerical models based on the finite element method or the finite difference method*.

Some applications of the new asymptotic-tolerance tolerance model to the analysis of various dynamical or thermal or coupled dynamical thermoelasticity problems for the thin uniperiodic cylindrical shells under consideration are reserved for the forthcoming papers.

Declarations

Conflict of interest The authors declare that they have no known conflict of interest.

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