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ORIGINAL ARTICLE

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Extended tolerance modelling of dynamic problems for thin uniperiodic cylindrical shells

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Abstract Dynamic problems of thin linearly elastic Kirchhoff–Love-type circular cylindrical shells having geometrical, elastic and inertial properties densely and periodically varying in circumferential direction (*uniperiodic shells*) are studied. In order to take into account the effect of a cell size on the global dynamic behaviour of such shells (*the length-scale effect*), a new mathematical averaged non-asymptotic model is formulated. This so-called *the general tolerance model* is derived by *applying a certain extended version of the well-known tolerance modelling technique*. Governing equations of this averaged model have constant coefficients depending also on a microstructure size, contrary to the starting exact shell equations with periodic, non-continuous and highly oscillating coefficients (the well-known governing equations of linear Kirchhoff–Love theory of thin elastic cylindrical shells). The effect of a cell size on the transversal free vibrations of an uniperiodic shell strip is studied. It will be shown that within this general tolerance model not only fundamental cell-independent, but also the new additional cell-dependent free vibration frequencies can be derived and analysed. The obtained results will be compared with the corresponding results derived from *the known non-asymptotic standard tolerance model* and *from the asymptotic one*.

Keywords Micro-inhomogeneous · Uniperiodic cylindrical shells · Extended tolerance modelling · Dynamic problems

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1 Introduction

Thin linearly elastic Kirchhoff–Love-type circular cylindrical shells with a periodically micro-inhomogeneous structure in circumferential direction are objects of consideration. Shells of this kind are termed *uniperiodic*. At the same time, the shells have constant structure in axial direction. By periodic inhomogeneity, we shall mean periodically varying thickness and/or periodically varying inertial and elastic properties of the shell material. We restrict our consideration to those uniperiodic cylindrical shells, which are composed of a large number of identical elements. Moreover, every such element, called *a periodicity cell*, can be treated as a thin shell. Typical examples of such shells are presented in Fig. 1 (a shell composed of two kinds of periodically distributed materials) and Fig. 2 (stiffened shell).

Mechanical problems of periodic structures (shells, plates, beams) are described by partial differential equations with periodic, highly oscillating and discontinuous coefficients. Thus, these equations are too complicated to constitute the basis for investigations of the engineering problems. To obtain averaged equations with constant coefficients, many different approximate modelling methods for structures of this kind have been formulated. Periodic cylindrical shells (plates) are usually described using *homogenized models* derived by applying *asymptotic methods*. These asymptotic models represent certain equivalent structures with constant or slowly varying rigidities and averaged mass densities. Unfortunately, the asymptotic procedures are usually restricted to the first approximation, which leads to homogenized models neglecting *the effect of a periodicity cell size* (called *the length-scale effect*) on the overall shell behaviour. The mathematical foundations of this modelling technique can be found in Bensoussan et al. [1], Jikov et al. [2]. Applications of the asymptotic homogenization procedure to modelling of stationary and non-stationary phenomena for microheterogeneous shells (plates) are presented in a large number of contributions. From the extensive list on this subject, we can mention paper by Lutoborski [3] and monographs by Lewiński and Telega [4], Andrianov et al. [5].

The length-scale effect can be taken into account using *the non-asymptotic tolerance averaging technique*. This technique is based on the concept of *the tolerance relations* related to the accuracy of the performed measurements and calculations. The mathematical foundations of this modelling technique can be found in Woźniak and Wierzbicki [6], Woźniak et al. [7,8], Ostrowski [9]. For periodic structures, *governing equations of the tolerance models have constant coefficients dependent also on a cell size*. Some applications of this averaging method to the modelling of mechanical and thermomechanical problems for various periodic structures are shown in many works. We can mention here monograph by Tomczyk [10] and papers by Tomczyk et al. [11,12], where the length-scale effect in mechanics of periodic cylindrical shells is investigated; papers by Baron [13], where dynamic problems of medium thickness periodic plates are studied and by Marczak and Jędrysiak [14], Marczak [15,16], where dynamics of periodic sandwich plates is analysed; papers by Jędrysiak [17–19], which deal with stability of thin periodic plates; papers by Łaciński and Woźniak [20], Rychlewska et al. [21], Ostrowski and Jędrysiak [22], Kubacka and Ostrowski [23], where problems of heat conduction in conductors with periodic structure are analysed. Let us also mention papers by Bagdasaryan et al. [24], Tomczyk and Gołąbczak [25], Tomczyk et al. [26], which deal with coupled thermoelasticity problems, respectively, for multicomponent, multi-layered periodic composites, for thin cylindrical shells with micro-periodic structure in circumferential direction (*uniperiodic shells*) and for thin cylindrical shells with micro-periodic structure in circumferential and axial directions (*biperiodic shells*). The extended list of references on this subject can be found in [6–10].

The tolerance averaging technique was also adopted to formulate mathematical models for analysis of various mechanical and thermomechanical problems for functionally graded solids, e.g. for heat conduction in longitudinally graded hollow cylinder by Ostrowski and Michalak [27,28], for thermoelasticity of transversally graded laminates by Pazera and Jędrysiak [29], Pazera et al. [30], for dynamics for functionally graded annular plates by Wirowski and Rabenda [31], for dynamics or stability of functionally graded thin cylindrical shells by Tomczyk and Szczerba [32–35].

In the tolerance modelling technique, the crucial role plays the concept of *slowly-varying functions*, cf. [6–9]. They are functions, which can be treated as constant on a cell. Moreover, the products of their derivatives in periodicity directions and characteristic length dimension of the cell are treated as negligibly small. A certain extended version of the tolerance modelling technique has been proposed by Tomczyk and Woźniak in [36]. This version is based on a new notion of *weakly slowly-varying functions*, which is a certain extension of the classical concept of *slowly-varying functions*. Note, that following [8,9,36], the notions of *weakly slowly-varying and slowly-varying functions* are recalled in Section 3 of this paper.

New mathematical averaged general tolerance and combined asymptotic-tolerance models of dynamic problems for thin cylindrical shells with two-directional periodic microstructure in directions tangent to the shell midsurface (*biperiodic shells*), derived by means of the concept of *weakly slowly-varying functions*, have been proposed by Tomczyk and Litawska [37,38]. *New mathematical averaged general tolerance models of stability problems* for thin uniperiodic or biperiodic cylindrical shells, derived by applying the notion of *weakly slowly-varying functions*, have been presented by Tomczyk et al. [39,40]. All *the general models* mentioned above are certain generalizations of the corresponding *standard tolerance models* proposed in [10], which have been obtained by using the classical concept of *slowly-varying functions*. Governing equations of *the general and standard models* have constant coefficients depending also on a cell size. However, the general models contain a bigger number of the length-scale terms than the standard models. Moreover, in the framework of general models describing dynamic behaviour of biperiodic shells, we can investigate certain *space-boundary layer phenomena* strictly related to the specific form of boundary conditions imposed on unknown functions describing fluctuations of displacements. These phenomena cannot be analysed within the standard models of biperiodic shells.

The general tolerance model of dynamic problems for *biperiodic shells* does not make it possible to analyse dynamics of *uniperiodic shells*. In tolerance approach, uniperiodic shells are not special cases of biperiodic ones. General tolerance model of uniperiodic shells and that of biperiodic shells have to be led out independently. It follows from the fact that *the modelling physical reliability conditions* for uniperiodic shells are hold only in one periodicity direction, whereas for biperiodic shells these conditions are hold in two periodicity directions tangent to the shell midsurface.

The main aim of this contribution is to formulate and discuss *a new averaged mathematical model constituting a proper tool for the analysis of selected dynamic problems in the uniperiodic cylindrical shells under consideration.* Contrary to the starting exact equations of the shell dynamics with periodic, highly oscillating and discontinuous coefficients, *governing equations of the proposed averaged model have constant coefficients*. Moreover, this model makes it possible to analyse the influence of a cell size on the dynamic shell behaviour (*the length-scale* effect). In order to derive this model, we shall apply *the general* (*extended*) *tolerance modelling procedure* [36]. Similarities and differences between *the general tolerance model* proposed here and the corresponding *known standard tolerance model*, formulated by Tomczyk in [10] and derived by applying the more restrictive concept of *slowly-varying function*, will be discussed.

As example, a certain special length-scale dynamic problem will be analysed in the framework of the proposed model. This problem deals with investigation of transversal free vibrations of a shell strip made of two component materials micro-periodically distributed in circumferential direction. The results obtained from *the general tolerance model* will be compared with those derived from the known *standard tolerance model* including less number of length-scale terms. Moreover, in order to evaluate the length-scale effect, the results obtained from both the tolerance non-asymptotic models will be compared with those derived from *the asymptotic one* being independent of a cell size. It will be shown that in the framework of the general and standard tolerance models, not only *the fundamental lower cell-independent*, but also *the new additional higher-order cell-dependent free vibration frequencies* can be derived and analysed. These cell-dependent frequencies, caused by a periodic structure of the shell, cannot be determined by applying asymptotic models commonly used for investigations of dynamics of heterogeneous shells.

2 Formulation of the problem: starting equations

We assume that x^1 and x^2 are coordinates parametrizing the shell midsurface Min circumferential and axial directions, respectively. We denote $x \equiv x^1 \in \Omega \equiv (0, L_1)$ and $\xi \equiv x^2 \in \Xi \equiv (0, L_2)$, where L_1, L_2 are length dimensions of M, cf. Figs. 1 and 2. Let $O\,\bar{x}^1\bar{x}^2\bar{x}^3$ stand for a Cartesian orthogonal coordinate system in the physical space E^3 and denote $\bar{\mathbf{x}} \equiv (\bar{x}^1, \bar{x}^2, \bar{x}^3)$. Let us introduce the orthonormal parametric representation of the underformed cylindrical shell midsurface $M$by means of $M \equiv \left\{ \bar{\mathbf{x}} \in E^3 : \bar{\mathbf{x}} = \bar{\mathbf{r}}\left(x^1, x^2\right), \left(x^1, x^2\right) \in \Omega \times \Xi \right\}$, where $\bar{\mathbf{r}}(\cdot)$ is the smooth invertible function such that $\partial\,\bar{\mathbf{r}}/\partial x^1 \cdot \partial\,\bar{\mathbf{r}}/\partial x^2 = 0, \partial\,\bar{\mathbf{r}}/\partial x^1 \cdot \partial\,\bar{\mathbf{r}}/\partial x^1 = 1, \partial\,\bar{\mathbf{r}}/\partial x^2 \cdot \partial\,\bar{\mathbf{r}}/\partial x^2 = 1$. Note, that derivative $\partial\,\bar{\mathbf{r}}/\partial x^\alpha, \alpha = 1, 2$, should be understood as differentiation of each component of $\bar{\mathbf{r}} \in E^3$, i.e. $\partial\,\bar{\mathbf{r}}/\partial x^\alpha = [\partial\bar{r}^1/\partial x^\alpha, \partial\bar{r}^2/\partial x^\alpha, \partial\bar{r}^3/\partial x^\alpha]$.

Throughout the paper, indices $\alpha, \beta, \ldots$ run over 1, 2 and are related to midsurface parameters x^1, x^2; summation convention holds. Partial differentiation related to x^α is represented by ∂_α, $\partial_\alpha = \partial/\partial x^\alpha$. Moreover, it is denoted $\partial_{\alpha\ldots\delta} \equiv \partial_\alpha \ldots \partial_\delta$. Differentiation with respect to time coordinate $t \in \mathrm{I} = [t_0, t_1]$ is represented by the overdot. Let $a_{\alpha\beta}$ and $a^{\alpha\beta}$ stand for the covariant and contravariant midsurface first metric tensors, respectively. Under orthonormal parametrization introduced on M, $a_{\alpha\beta} = a^{\alpha\beta}$ are unit tensors. Denote by

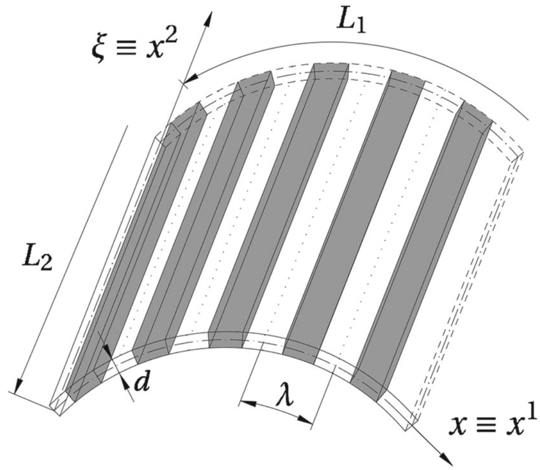

Fig. 1 Fragment of the shell composed of two materials periodically and densely distributed in circumferential direction

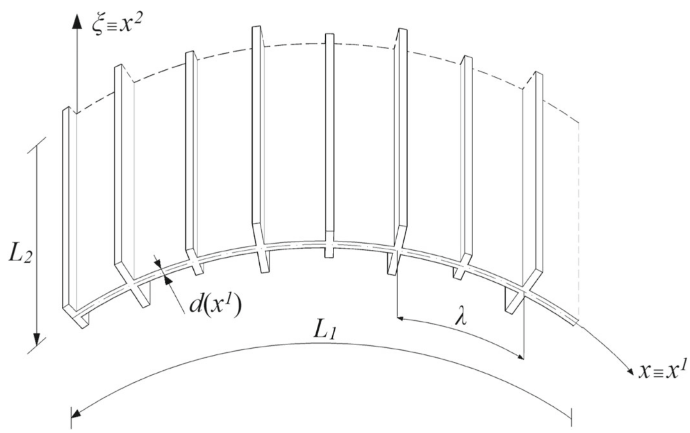

Fig. 2 Fragment of the shell reinforced by two families of uniperiodically spaced ribs

$b_{\alpha\beta}$ the covariant midsurface second metric tensor. Under orthonormal parametrization introduced on M, components of tensor $b_{\alpha\beta}$ are: $b_{22} = b_{12} = b_{21} = 0$, $b_{11} = -r^{-1}$, where r is the midsurface curvature radius.

Let $d(x)$ stand for the shell thickness.

The basic cell Δ and an arbitrary cell $\Delta(x)$ with the centre at point $x \in \Omega_\Delta$ are defined by means of: $\Delta \equiv [-\lambda/2,\ \lambda/2]$, $\Delta(x) \equiv x + \Delta$, $\Omega_\Delta \equiv \{x \in \Omega : \Delta(x) \subset \Omega\}$, where $\lambda \equiv \lambda_1$ is a cell length dimension in $x \equiv x^1$-direction, cf. Figs. 1 and 2. Period λ, called *the microstructure length parameter*, satisfies conditions: $\lambda/\sup d(x) >> 1$ for every $x \in \Omega$, $\lambda/r << 1$ and $\lambda/L_1 << 1$.

It is assumed that the cell Δ has a symmetry axis for $z = 0$, where $z \equiv z^1 \in [-\lambda/2,\ \lambda/2]$. It is also assumed that inside the cell the geometrical, elastic and inertial properties of the shell are described by even functions of argument z.

Denote by $u_\alpha = u_\alpha(x, \xi, t)$, $w = w(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, the shell displacements in directions tangent and normal to M, respectively. The shell stiffness tensors describing elastic properties of the shell are defined by $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $x \in \Omega$. Let $\mu(x)$, $x \in \Omega$, stand for a shell mass density per midsurface unit area. Let $f^\alpha(x, \xi, t)$, $f(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, be external forces per midsurface unit area, respectively, tangent and normal to M.

The considerations will be based on the well-known linear Kirchhoff–Love theory of thin elastic shells, cf. Kaliski [41], governed by the following dynamic equilibrium equations

$$\begin{aligned} &\partial_\beta(D^{\alpha\beta\gamma\delta}\partial_\delta u_\gamma) + r^{-1}\partial_\beta(D^{\alpha\beta 11}w) - \mu a^{\alpha\beta}\ddot{u}_\beta + f^\alpha = 0, \\ &r^{-1}D^{\alpha\beta 11}\partial_\beta u_\alpha + \partial_{\alpha\beta}(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w) + r^{-2}D^{1111}w + \mu\ddot{w} - f = 0. \end{aligned} \tag{1}$$

In the above equations, displacements $u_\alpha(x, \xi, t)$, $w(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, are the basic unknowns. For uniperiodic shells, $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $\mu(x)$, $x \in \Omega$, are highly oscillating, discontinuous and periodic functions with respect to argument x. That is why, in the most cases it is impossible to obtain the exact analytical solutions to initial/boundary value problems for Eq. (1) and also numerical problems for these equations are often ill conditioned. Applying *the extended tolerance modelling technique* proposed in [36] to Eq. (1), we will derive *the averaged general tolerance model for the analysis of dynamic problems for the uniperiodic shells considered here*. Governing equations of this model have constant coefficients depending also on a microstructure size.

To make this paper self-consisted, in the subsequent section we shall outline the main concepts and the fundamental assumptions of *the extended tolerance modelling procedure*, which in the general form are given in [8,9,36].

3 Concepts and assumptions of the extended tolerance modelling technique

3.1 Main concepts

The fundamental concepts of the tolerance modelling procedure under consideration are those of *two tolerance relations between points and real numbers determined by tolerance parameters, weakly slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and the averaging operation*. Note, that in the classical approach we deal with not *weakly slowly-varying* but with more restrictive *slowly-varying functions*.

Below, the mentioned above concepts and assumptions will be specified with respect to one-dimensional region $\Omega = (0, L_1)$ defined in this paper.

3.1.1 Tolerance between points

Let λ be a positive real number. Points x, y belonging to $\Omega = (0, L_1) \subset E$ are said to be in tolerance determined by λ, if and only if the distance between points x, y does not exceed λ, i.e. $\|x - y\|_E \leq \lambda$, where $\|\cdot\|$ is the Euclidean norm in E.

3.1.2 Tolerance between real numbers

Let $\tilde{\delta}$ be a positive real number. Real numbers $\mu, \ \nu$ are said to be in tolerance determined by $\tilde{\delta}$, if and only if $|\mu - \nu| \leq \tilde{\delta}$.

The above relations are denoted by: $x \overset{\lambda}{\approx} y$, $\mu \overset{\tilde{\delta}}{\approx} \nu$. Positive parameters $\lambda, \tilde{\delta}$ are called *tolerance parameters*.

3.1.3 Weakly slowly-varying functions

Let F be a function defined in $\bar{\Omega} = [0, L_1] \subset E$, which is continuous, bounded and differentiable in $\bar{\Omega}$ together with its derivatives up to the R-th order. Note, that function Fcan also depend on $\xi \in \bar{\Xi} = [0, L_2]$ and time coordinate t as parameters. It can be observed that function F is said to be differentiable in closed set $\bar{\Omega}$; however, we do not specify how derivatives are defined on its fringe $\partial\Omega$, because differentiation may look differently for any particular problem. Non-negative integer R is assumed to be specified in every problem under consideration. Denote by $\partial_1^k F(\cdot)$, $k = 1, \ldots, R$, the k-th derivative in $\bar{\Omega}$. Let $\delta \equiv (\lambda, \delta_0, \delta_1, \ldots, \delta_R)$ be the set of tolerance parameters. The first of them represents the distances between points in $\bar{\Omega}$. The second one is related to the absolute differences in appropriate space between the values of function $F(\cdot)$ in the points x, y belonging to $\bar{\Omega}$ such that $|x - y| \leq \lambda$. Each tolerance parameter δ_k, $k = 1, \ldots, R$, refers to the absolute differences in appropriate space between the values of derivative $\partial_1^k F(\cdot)$ in the points x, y belonging to $\bar{\Omega}$ such that $|x - y| \leq \lambda$. A function $F(\cdot)$ is said to be *weakly slowly-varying of the R-th kind* with respect to cell Δ and tolerance parameters δ, $F \in WSV_\delta^R(\Omega, \Delta)$, if and only if the following condition is fulfilled

$$\left(\forall (x, y) \in \Omega^2\right) \left[\left(x \overset{\lambda}{\approx} y \right) \Rightarrow |F(x) - F(y)| \leq \delta_0 \ \text{and} \ \left|\partial_1^k F(x) - \partial_1^k F(y)\right| \leq \delta_k \ , \quad k = 1, 2, \ldots, R \right]. \tag{2}$$

Let us recall that the known *slowly-varying function* F, $F \in \mathrm{SV}_{\delta}^{R}(\Omega, \Delta) \subset \mathrm{WSV}_{\delta}^{R}(\Omega, \Delta)$, occurring in the classical tolerance modelling, satisfies not only condition (2) but also the extra restriction

$$(\forall x \in \Omega)[\lambda \left|\partial_1^k F(x)\right| \overset{\delta_k}{\approx} 0, \quad k = 1, 2, \ldots, R\,]. \tag{3}$$

Roughly speaking, from condition (2) it follows that both *the weakly slowly-varying and the slowly-varying functions* can be treated (together with their derivatives up to the *R-th* order) as constant on a cell. From condition (3), it follows that the main difference between *the weakly slowly-varying* and *the slowly-varying functions is that the products of the absolute values of derivatives of slowly-varying functions and microstructure length parameter* λ *are treated as negligibly small*.

It is worth to know that in every problem under consideration, tolerance parameter λ is known *a priori* as a characteristic cell length dimension, whereas values of tolerance parameters $\delta_0, \delta_1, \ldots, \delta_R$ can be determined only *a posteriori*, i.e. after obtaining unique solution to the considered initial-boundary value problem.

3.1.4 Tolerance-periodic functions

An essentially bounded and weakly differentiable function ϑ defined in $\bar{\Omega} = [0, L_1] \subset E$, which can also depend on $\xi \in \bar{\Xi}$ and time coordinate t as parameters, is called *tolerance-periodic of the R-th kind* in reference to cell Δ and tolerance parameters $\delta \equiv (\lambda, \delta_0)$, if for every $x \in \Omega_\Delta$ there exist Δ-periodic function $\tilde{\vartheta}(\cdot)$ defined in E such that $\vartheta\,|_{\Omega_x} \cap \mathrm{Dom}\,\vartheta$ and $\tilde{\vartheta}\,\Big|_{\Omega_x} \cap \mathrm{Dom}\,\tilde{\vartheta}$ are indiscernible in tolerance determined by $\delta \equiv (\lambda, \delta_0)$, where $\Omega_x \equiv \Omega \cap \cup_{z \in \Delta(x)} \Delta(z)$, $x \in \bar{\Omega}$, is a cluster of 2 cells having common sides. Function $\tilde{\vartheta}$ is a Δ-*periodic approximation* of ϑ in $\Delta(x)$. For function $\vartheta(\cdot)$ being tolerance-periodic together with its derivatives up to the R-th order, we shall write $\vartheta \in \mathrm{TP}_{\delta}^{R}(\Omega, \Delta)$, $\delta \equiv (\lambda, \delta_0, \delta_1, \ldots, \delta_R)$.

3.1.5 Fluctuation shape functions

Let h be a continuous, highly oscillating, λ-periodic function defined in $\bar{\Omega} = [0, L_1]$, which has continuous derivatives $\partial_1^k h$, $k = 1, \ldots, R-1$, and either continuous or piecewise continuous bounded derivative $\partial_1^R h$. Function h will be called *the fluctuation shape function of the R-th kind*, $h \in \mathrm{FS}^R(\Omega, \Delta)$, if it satisfies conditions

$$\begin{array}{lll} h \in O(\lambda^R), & \partial_1^k h \in O(\lambda^{R-k}), & k = 1, 2, \ldots, R, \\ \int\limits_{\Delta(x)} \mu(z)\, h(z) \mathrm{d}z = 0, & & \forall x \in \Omega_\Delta, \end{array} \tag{4}$$

where $\mu(\cdot)$ is a certain positive valued λ-periodic function defined in $\bar{\Omega}$. Nonnegative integer R is specified in every discussed problem.

3.1.6 Averaging operation

Let f be a function defined in $\bar{\Omega} \equiv [0, L_1] \subset E$, which is integrable and bounded in every cell $\Delta(x)$, $x \in \Omega_\Delta$. *The averaging operation of* $f(\cdot)$ is defined by

$$< f > (x) \equiv \frac{1}{|\Delta|} \int\limits_{\Delta(x)} f(z) \mathrm{d}z, \qquad x \in \Omega_\Delta, \tag{5}$$

where $|\Delta| = \lambda$. It can be observed that if $f(\cdot)$ is Δ-periodic, then $< f >$ is constant.

3.2 Basic assumptions

The tolerance modelling under consideration is based on three assumptions. The first of them is termed *the tolerance averaging approximation*. The second one is called *the micro–macro decomposition*. The third one is termed *the residual orthogonality assumption*.

3.2.1 Tolerance averaging approximation

For an integrable periodic function f defined in $\bar{\Omega} = [0, L_1] \subset E$ and for $F \in WSV_\delta^R(\Omega, \Delta)$, the following tolerance relations, called *the tolerance averaging approximation*, hold for every $x \in \Omega_\Delta$

$$\begin{aligned} <f\,\partial_1^k F>(x) &= <f>\partial_1^k F(x) + O(\delta_k), \\ & k = 0, 1, \ldots, R, \quad \partial_1^0 F \equiv F, \quad x \in \Omega_\Delta. \end{aligned} \tag{6}$$

The known *slowly-varying functions* $F \in \mathrm{SV}_\delta^R(\Omega, \Delta) \subset \mathrm{WSV}_\delta^R(\Omega, \Delta)$ satisfy not only approximations (6), but also the extra approximate relations

$$\begin{aligned} <f\,\partial_1^k \vartheta>(x) &= <f\,\partial_1^k (hF)>(x) = <f\,\partial_1^k h>F(x) + O(\delta_k), \\ & k = 1, \ldots, R, \quad x \in \Omega_\Delta, \end{aligned} \tag{7}$$

where $\vartheta(\cdot) \equiv h(\cdot)F(\cdot) \in TP_\delta^R(\Omega, \Delta)$, $h \in FS^R(\Omega, \Delta)$, $F \in SV_\delta^R(\Omega, \Delta)$. Note that approximations (7) are also valid for $k = 0$, but then they reduce to proposition expressed with (6), i.e. $<f\,\partial_1^0 \vartheta>(x) \equiv <f\,\vartheta> = <f\,hF>(x) = <f\,h>F(x) + O(\delta_k), \quad x \in \Omega_\Delta$.

In the course of modelling, terms $O(\delta_k)$ in (6) and (7) are neglected. Approximations (6) follow directly from conditions (2) satisfied by *the weakly slowly-varying and slowly-varying* functions. Approximations (7) follow directly from conditions (2) and (3), which hold for *the slowly-varying functions*.

3.2.2 Micro–macro decomposition assumption

The second fundamental assumption, called *the micro–macro decomposition*, states that the displacement fields occurring in the starting equations under consideration can be decomposed into *macroscopic and microscopic parts.* The macroscopic part is represented by *unknown averaged displacements* being *weakly slowly-varying functions* in periodicity direction. The microscopic part is described by *the known highly oscillating periodic fluctuation shape functions multiplied by unknown fluctuation amplitudes weakly slowly-varying with respect to* x. Note, that in the classical tolerance approach, *the weakly slowly-varying functions* are replaced by *the slowly-varying functions*.

Micro–macro decomposition introduced in the dynamic problem discussed in this paper is presented in Sect. 4.1.

3.2.3 Residual orthogonality assumption

It states that for micro–macro decomposition mentioned above, the governing equations of the exact shell theory under consideration do not hold, i.e. there exist residual fields which have to satisfy certain orthogonality conditions.

4 Model equations

4.1 General tolerance model equations

In the problem discussed here, *the micro–macro decomposition* of displacements $u_\alpha(x, \xi, t)$, $u_\alpha(\cdot, \xi, t) \in \mathrm{TP}_\delta^1(\Omega, \Delta)$, $w(x, \xi, t)$, $w(\cdot, \xi, t) \in \mathrm{TP}_\delta^2(\Omega, \Delta)$, $(x, \xi, t) \in \Omega \times \Xi \times I$, being unknowns of Eq. (1), is assumed in the form

$$\begin{aligned} u_\alpha(x, \xi, t) &= u_\alpha^0(x, \xi, t) + h(x)\, U_\alpha(x, \xi, t), \\ w(x, \xi, t) &= w^0(x, \xi, t) + g(x)\, W(x, \xi, t), \end{aligned} \tag{8}$$

where $u_\alpha^0,\ U_\alpha,\ w^0,\ W$ are *weakly slowly-varying functions* with respect to argument $x \in \Omega$, i.e.

$$\begin{aligned} & u_\alpha^0(\cdot,\ \xi, t),\ U_\alpha(\cdot,\ \xi, t) \in WSV_\delta^1(\Omega, \Delta), \quad \delta \equiv (\lambda, \delta_0, \delta_1), \\ & w^0(\cdot,\ \xi, t),\ \ W(\cdot,\ \xi, t) \in WSV_\delta^2(\Omega, \Delta), \quad \delta \equiv (\lambda, \delta_0, \delta_1, \delta_2), \end{aligned} \tag{9}$$

for every $(\xi, t) \in \Xi \times \mathrm{I}$.

Macrodisplacements u_{α}^{0}, w^{0} as well as *displacement fluctuation amplitudes* U_{α}, W are *the new unknowns*.

Fluctuation shape functions $h(\cdot) \in FS^{1}(\Omega, \Delta)$, $g(\cdot) \in FS^{2}(\Omega, \Delta)$ are the known, λ-periodic, continuous and highly oscillating functions representing oscillations inside a cell. Agree with (4), they have to satisfy conditions: $h \in O(\lambda), \lambda\partial_{1}h \in O(\lambda), g \in O(\lambda^{2}), \lambda\partial_{1}g \in O(\lambda^{2}), \lambda^{2}\partial_{11}g \in O(\lambda^{2}), <\mu\, h> = <\mu\, g>0$, where $\mu(\cdot)$ is a shell mass density. In the special case $\mu = const$, the fluctuations shape functions satisfy conditions $<h> = <g>0$. Taking into account that inside the cell the geometrical, elastic and inertial properties of the periodic shell under consideration are described by symmetric (i.e. even) functions of argument $z \equiv z^{1} \in \Delta(x)$, we assume that $h(\cdot)$ is either even or odd function of z. This same restriction is imposed on function $g(\cdot)$. Let $\phi = \phi(z),\ z \in \Delta(x)$, be an even function with respect to z. Under aforementioned restriction, averages $<\phi\, h\, \partial_{1}h>$, $<\phi\, g\, \partial_{1}g>$, $<\phi\, \partial_{1}g\, \partial_{11}g>$, which appear in the course of modelling are equal to zero.

We substitute the right-hand sides of (8) into (1). For decomposition (8), the governing Eq. (1) do not hold, i.e. there exist residual fields defined by

$$\begin{aligned} p^{\alpha} &\equiv \partial_{\beta}\left(D^{\alpha\beta\gamma\delta}\partial_{\delta}\left(u_{\gamma}^{0} + hU_{\gamma}\right)\right) + r^{-1}\partial_{\beta}\left(D^{\alpha\beta11}\left(w^{0} + gW\right)\right) \\ &\quad -\mu a^{\alpha\beta}\left(\ddot{u}_{\beta}^{0} + h\ddot{U}_{\beta}\right) + f^{\alpha}, \\ \bar{p} &\equiv r^{-1}D^{\alpha\beta11}\partial_{\beta}\left(u_{\alpha}^{0} + hU_{\alpha}\right) + \partial_{\alpha\beta}\left(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}\left(w^{0} + gW\right)\right) \\ &\quad +r^{-2}D^{1111}\left(w^{0} + gW\right) + \mu\left(\ddot{w}^{0} + g\ddot{W}\right) - f. \end{aligned} \tag{10}$$

Following [9,36], we introduce *the residual orthogonality assumption*, which states that residual fields (10) have to satisfy the following orthogonality conditions

$$<p^{\alpha}> = 0, \quad <p^{\alpha}h> = 0, \quad <\bar{p}> = 0, \quad <\bar{p}g> = 0, \tag{11}$$

for almost every $x \in \Omega$ and every $(\xi, t) \in \Xi \times$ I. Averaging operation $< \cdot >$ on cell Δ is defined by (5).

Conditions (11), on the basis of *the tolerance averaging approximation* (6), lead to the system of averaged equations for unknowns u_{α}^{0}, w^{0}, U_{α}, W being *weakly slowly-varying functions in periodicity direction*. Under extra approximation $1 + \lambda/r \approx 1$, this system can be written in the form of:

- the constitutive equations

$$\begin{aligned} N^{\alpha\beta} &= <D^{\alpha\beta\gamma\delta}>\partial_{\delta}u_{\gamma}^{0} + r^{-1}<D^{\alpha\beta11}>w^{0} \\ &\quad +<D^{\alpha\beta\gamma1}\partial_{1}h>U_{\gamma} + \underline{\underline{<D^{\alpha\beta\gamma\delta}h>}}\ \partial_{\delta}U_{\gamma}, \end{aligned} \tag{12}$$

$$\begin{aligned} M^{\alpha\beta} &= <B^{\alpha\beta\gamma\delta}>\partial_{\gamma\delta}w^{0} + <B^{\alpha\beta11}\partial_{11}g>W \\ &\quad +2\underline{\underline{<B^{\alpha\beta\gamma1}\partial_{1}g>}}\ \partial_{\gamma}W + \underline{\underline{<B^{\alpha\beta\gamma\delta}g>}}\ \partial_{\gamma\delta}W, \\ H^{\beta} &= <\partial_{1}h\, D^{\beta1\gamma\delta}>\partial_{\delta}u_{\gamma}^{0} - \underline{<h\, D^{\alpha\beta\gamma\delta}>}\ \partial_{\alpha\delta}u_{\gamma}^{0} \\ &\quad +<D^{\beta11\gamma}(\partial_{1}h)^{2}>U_{\gamma} - \underline{\underline{<D^{\alpha\beta\gamma\delta}(h)^{2}>}}\ \partial_{\alpha\delta}U_{\gamma} \\ &\quad +r^{-1}<\partial_{1}h D^{\beta111}>w^{0}, \\ G &= <\partial_{11}g\, B^{11\alpha\beta}>\ \partial_{\alpha\beta}w^{0} - 2\underline{<\partial_{1}g\, B^{\alpha\beta\gamma1}>}\ \partial_{\alpha\beta\gamma}w^{0} \\ &\quad +\underline{<g\, B^{\alpha\beta\gamma\delta}>}\ \partial_{\alpha\beta\gamma\delta}w^{0} + <(\partial_{11}g)^{2}B^{1111}>W \\ &\quad +(2\underline{\underline{<\partial_{11}g\, B^{11\beta\delta}g>}} - 4\underline{\underline{<(\partial_{1}g)^{2}B^{1\beta1\delta}>}})\ \partial_{\beta\delta}W \\ &\quad +\underline{\underline{<(g)^{2}B^{\alpha\beta\gamma\delta}>}}\ \partial_{\alpha\beta\gamma\delta}W, \end{aligned} \tag{13}$$

- the dynamic equilibrium equations

$$\begin{aligned} &\partial_{\beta}N^{\alpha\beta} - <\mu>a^{\alpha\beta}\ddot{u}_{\beta}^{0} + <f^{\alpha}> = 0, \\ &\partial_{\alpha\beta}M^{\alpha\beta} + r^{-1}N^{11} + <\mu>\ddot{w}^{0} - <f> = 0, \\ &\underline{<\mu\ (h)^{2}>a^{\alpha\beta}}\ \ddot{U}_{\alpha} + H^{\beta}\ - \underline{<f^{\beta}h>} = 0, \\ &\underline{<\mu\ (g)^{2}>}\ \ddot{W} + G - \underline{<fg>} = 0. \end{aligned} \tag{14}$$

The singly and doubly underlined terms in (12)–(14) depend on a cell size λ.

Equations (12)–(14) together with *micro–macro decomposition* (8) and *physical reliability conditions* (9) constitute *the general tolerance model of selected dynamic problems for the thin uniperiodically microheterogeneous shells under consideration*.

The characteristic features of the derived *general tolerance model* are:

- In contrast to starting Eq. (1) with discontinuous, highly oscillating and periodic coefficients, the general tolerance model Eqs. (12)–(14) proposed here *have constant coefficients.* Moreover, *some of these coefficients depend on microstructure length parameter* λ (underlined terms). Hence, the tolerance model makes it possible to *describe the effect of length scale on the global shell behaviour*. Moreover, we can analyse the length-scale effect not only in dynamic but also in stationary problems for the uniperiodic shells under consideration.
- Unknown macrodisplacements u^0_α, w^0 are governed by the system of three partial differential equations $(14)_{1,2}$. The number and form of boundary conditions for averaged variables u^0_α, w^0 are the same as in the classical shell theory governed by Eq. (1). Fluctuation amplitudes U_α, W are governed by the system of three partial differential equations $(14)_{3,4}$. For an open cylindrical shell, the boundary conditions for U_α, W should be defined on all boundaries, i.e. for $x = 0$, $x = L_1$, $\xi = 0$, $\xi = L_2$, cf. Figs. 1 and 2.
- The resulting equations involve terms with time and spatial derivatives of fluctuation amplitudes U_α, W. Hence, these equations describe certain *time-boundary layer and space-boundary layer phenomena* strictly related to the specific form of initial and boundary conditions imposed on the fluctuation amplitudes.
- Decomposition (8) and hence, also resulting tolerance model Eqs. (12)–(14) are uniquely determined by the postulated *a priori* λ-periodic *fluctuation shape functions*, $h(\cdot) \in FS^1(\Omega, \Delta)$, $h \in O(\lambda)$ and $g(\cdot) \in FS^2(\Omega, \Delta)$, $g \in O(\lambda^2)$, which represent oscillations inside a cell. These functions can be obtained as exact or approximate solutions to certain periodic eigenvalue problems describing free periodic vibrations of the cell, cf., e.g. Tomczyk [10], Jędrysiak [19]. It means that they represent either the principal modes of free periodic vibrations of the cell or physically reasonable approximations of these modes. These functions can also be regarded as *the shape functions* resulting from the periodic discretization of the cell using for example the finite element method. The choice of these functions may also be based on the experience or intuition of the researcher.
- It has to be emphasized that solutions to selected initial-boundary value problems formulated in the framework of the tolerance model have a physical sense only if conditions (9) hold for the pertinent tolerance parameters δ, i.e. if unknown macrodisplacements u^0_α, w^0 and fluctuation amplitudes U_α, W of the general tolerance model equations are *weakly slowly-varying functions* in the periodicity direction. These conditions can also be used for the *a posteriori* evaluation of tolerance parameters δ and hence, for the verification of the physical reliability of the obtained solutions.
- For a homogeneous shell with a constant thickness, $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $\mu(x)$, $x \in \Omega$, are constant and because $<\mu h> = <\mu g> = 0$, we obtain $<h> = <g> = 0$, and hence, $<\partial_1 h> = <\partial_1 g> = <\partial_{11} g> = 0$. In this case, Eq. $(14)_{1,2}$ reduce to the well-known shell equations of motion for averaged displacements $u^0_\alpha(x, \xi, t)$, $w^0(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times I$, and independently for fluctuation amplitudes $U_\alpha(x, \xi, t)$, $W(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times I$, we arrive at the system of equations, which under condition $<f^\beta h> = <fg> = 0$ and under homogeneous initial conditions for U_α and W, has only trivial solution $U_\alpha = W = 0$. Hence, from decomposition (8) it follows that $u_\alpha = u^0_\alpha$, $w = w^0$. It means that Eqs. (12)–(14) reduce to the starting Eq. (1).

4.2 Standard tolerance model equations

Let us compare *the general tolerance model* proposed here with the corresponding known *standard tolerance model* presented and discussed in [10], which was derived under assumption that the unknown functions $u^0_\alpha(x, \xi, t)$, $w^0(x, \xi, t)$, $U_\alpha(x, \xi, t)$, $W(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times I$, in micro–macro decomposition (8) are *slowly-varying*. We recall that *family of slowly-varying functions* being a subset of *the weakly slowly-varying functions' set* is defined by means of (2) and (3). For *the slowly-varying functions* approximate relations (6), (7) hold.

Following [10], *the standard tolerance model* consists of:

- Micro–macro decomposition (8) with physical reliability conditions (9), in which *weakly slowly-varying functions* $u^0_\alpha(\cdot, \xi, t)$, $U_\alpha(\cdot, \xi, t) \in WSV^1_\delta(\Omega, \Delta)$, $w^0(\cdot, \xi, t)$, $W(\cdot, \xi, t) \in WSV^2_\delta(\Omega, \Delta)$, $(\xi, t) \in \Xi \times I$, are replaced by *slowly-varying functions* $u^0_\alpha(\cdot,\xi,t),\ U_\alpha(\cdot,\xi,t)\in SV^1_\delta(\Omega,\Delta),\ w^0(\cdot,\xi,t),\ W(\cdot,\xi,t)\in SV^2_\delta(\Omega,\Delta),\ (\xi,t)\in\Xi\times I$,

- Constitutive equations

$$\begin{aligned}
N^{\alpha\beta} &= <D^{\alpha\beta\gamma\delta}>\partial_\delta u^0_\gamma + r^{-1}<D^{\alpha\beta 11}>w^0 \\
&\quad +<D^{\alpha\beta\gamma 1}\partial_1 h>U_\gamma \ + \underline{\underline{<D^{\alpha\beta\gamma 2}h>}}\,\partial_2 U_\gamma, \\
M^{\alpha\beta} &= <B^{\alpha\beta\gamma\delta}>\partial_{\gamma\delta}w^0 + <B^{\alpha\beta 11}\partial_{11}g>W \\
&\quad +2\underline{<B^{\alpha\beta\gamma 1}\partial_1 g>}\,\partial_\gamma W + \underline{\underline{<B^{\alpha\beta 22}g>}}\,\partial_{22}W,
\end{aligned}\tag{15}$$

$$\begin{aligned}
H^{\beta} &= <\partial_1 h\, D^{\beta 1\gamma\delta}>\partial_\delta u^0_\gamma - \underline{<h\, D^{\alpha\beta\gamma\delta}>}\,\partial_{\alpha\delta}u^0_\gamma \\
&\quad +<D^{\beta 11\gamma}(\partial_1 h)^2>U_\gamma - \underline{\underline{<D^{2\beta\gamma 2}(h)^2>}}\,\partial_{22}U_\gamma \\
&\quad +r^{-1}<\partial_1 h D^{\beta 111}>w^0, \\
G &= +<\partial_{11}g\, B^{11\alpha\beta}>\,\partial_{\alpha\beta}w^0 - 2\underline{<\partial_1 g\, B^{\alpha\beta\gamma 1}>}\,\partial_{\alpha\beta\gamma}w^0 \\
&\quad +\underline{<g\, B^{\alpha\beta\gamma\delta}>}\,\partial_{\alpha\beta\gamma\delta}w^0 + <(\partial_{11}g)^2 B^{1111}>W \\
&\quad +(2\underline{\underline{<\partial_{11}g\, B^{1122}g>}} - 4\underline{\underline{<(\partial_1 g)^2 B^{1212}>}})\,\partial_{22}W \\
&\quad +\underline{<(g)^2 B^{2222}>}\,\partial_{2222}W,
\end{aligned}\tag{16}$$

where the singly and doubly underlined terms depend on the period length λ, and where the doubly underlined terms are different from the corresponding terms in constitutive Eqs. (12), (13) of the general tolerance model,

- The dynamic equilibrium equations

$$\begin{aligned}
&\partial_\beta N^{\alpha\beta} - <\mu>a^{\alpha\beta}\ddot{u}^0_\beta + <f^\alpha> = 0, \\
&\partial_{\alpha\beta}M^{\alpha\beta} + r^{-1}N^{11} + <\mu>\ddot{w}^0 - <f> = 0, \\
&\underline{<\mu\,(h)^2>}a^{\alpha\beta}\,\ddot{U}_\alpha \quad + H^\beta - \underline{<f^\beta h>} = 0, \\
&\underline{<\mu\,(g)^2>}\,\ddot{W} + G - \underline{<fg>} = 0.
\end{aligned}\tag{17}$$

The main similarities and differences between the general and standard tolerance models are:

- Both the general and standard tolerance models *have constant coefficients*. Moreover, *some of them depend on a period length* λ (underlined terms). Hence, the tolerance models allow us to investigate *the length-scale effect in dynamical and stationary problems*. From comparison of the doubly underlined terms in constitutive relations (12), (13) of *the general tolerance model* with the corresponding doubly underlined terms in constitutive relations (15), (16) of *the standard model*, it follows that *general constitutive relations* (12), (13) contain a bigger number of terms depending on the microstructure size than *the standard constitutive relations* (15), (16). Thus, from the analytical results it follows that *the general model proposed here makes it possible to investigate the length-scale effect in more detail*.
- Unknown macrodisplacements u^0_α, w^0 and fluctuation amplitudes U_α, W of the general tolerance model equations must be *weakly slowly-varying functions* in periodicity direction, i.e. they have to satisfy condition (2). Unknowns u^0_α, w^0, U_α, W of the standard tolerance model equations must be *slowly-varying functions* in x, i.e. they have to satisfy conditions (2) and (3).
- Contrary to constitutive relations (12), (13) of *the general tolerance model*, constitutive Eqs. (15), (16) of *the standard tolerance model* do not involve derivatives of fluctuation amplitudes U_α, W with respect to argument x. It arises from tolerance relations (7), which hold for the slowly-varying functions. Hence, in the framework of *the standard tolerance model*, the boundary conditions for unknown fluctuation amplitudes U_α, W should be defined only on boundaries $\xi = 0$, $\xi = L_2$, whereas in the framework of *the general tolerance model* the boundary conditions for U_α, W should be defined on all boundaries of the shell. It means that for an open cylindrical shell, applying *the general model* Eqs. (12)–(14) we can investigate *the space-boundary layer phenomena* near all boundaries of the shell, whereas within *the standard model* Eqs. (15)–(17) we can analyse these phenomena only near boundaries $\xi = 0$, $\xi = L_2$. We recall that the space-boundary layer phenomena are strictly related to the specific form of boundary conditions imposed on unknown fluctuation amplitudes.

- The governing equations of both *the general and standard models* include terms with time derivatives of the fluctuation amplitudes. Hence, the governing equations of both models describe certain time-boundary layer phenomena strictly related to the specific form of initial conditions imposed on unknown fluctuation amplitudes.

4.3 Asymptotic model equations

The asymptotic model equations can be obtained directly from either the general tolerance model Eqs. (12)–(14) or the standard model Eqs. (15)–(17) by the formal limit passage $\lambda \to 0$, i.e. after neglecting terms depending on a cell size λ. Asymptotic model consists of partial differential equations for macrodisplacements $u^0_\alpha,\ w^0$ coupled with linear algebraic equations for kinematic fluctuation amplitudes $U_\alpha,\ W$. After eliminating fluctuation amplitudes from the governing equations by means of

$$\begin{aligned} U_\gamma &= -G^{-1}_{\gamma\eta}\ [<\partial_1 h D^{1\eta\mu\vartheta}>\ \partial_\vartheta u^0_\mu + r^{-1}<\partial_1 h D^{1\eta 11}>w^0],\\ W &= -E^{-1}<\partial_{11} g B^{11\gamma\delta}>\partial_{\gamma\delta} w^0, \end{aligned} \tag{18}$$

where $G_{\alpha\gamma} = <D^{\alpha 1\gamma 1}(\partial_1 h)^2>$, $E = <B^{1111}(\partial_{11} g)^2>$, $G_{\alpha\gamma} G^{-1}_{\gamma\eta} = \delta_{\alpha\eta}$ ($\delta_{\alpha\eta}$ is an unit tensor) we arrive finally at *the asymptotic model equations expressed only in macrodisplacements* $u^0_\alpha,\ w^0$

$$\begin{aligned} &D_h^{\alpha\beta\gamma\delta}\partial_{\beta\delta}u^0_\gamma + r^{-1}D_h^{\alpha\beta 11}\partial_\beta w^0 - <\mu>a^{\alpha\beta}\ddot{u}^0_\beta + <f^\alpha> = 0,\\ &B_g^{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma\delta}w^0 + r^{-1}D_h^{11\gamma\delta}\partial_\delta u^0_\gamma + r^{-2}D_h^{1111}w^0\\ &+<\mu>\ddot{w}^0 - <f> = 0, \end{aligned} \tag{19}$$

where

$$\begin{aligned} D_h^{\alpha\beta\gamma\delta} &\equiv <D^{\alpha\beta\gamma\delta}> - <D^{\alpha\beta\eta 1}\ \partial_1 h>G^{-1}_{\eta\xi}<\partial_1 h D^{1\xi\gamma\delta}>,\\ B_g^{\alpha\beta\gamma\delta} &\equiv <B^{\alpha\beta\gamma\delta}> - <B^{\alpha\beta 11}\partial_{11} g>E^{-1}\ <\partial_{11} g B^{11\gamma\delta}>. \end{aligned} \tag{20}$$

Tensors $D_h^{\alpha\beta\gamma\delta}$, $B_g^{\alpha\beta\gamma\delta}$ are *tensors of effective elastic moduli* for uniperiodic shells considered here.

Asymptotic model Eq. (19) *have constant coefficients but independent of a period length*. It means that this model is not able to describe the influence of a cell size on the global shell dynamics. Within the asymptotic model, we formulate boundary conditions only for the macrodisplacements u^0_α, w^0. The number and form of these conditions are the same as in the classical shell theory governed by starting Eq. (1).

Application of *the new non-asymptotic general tolerance model* Eqs. (12)–(14) proposed here to a certain special dynamic problem for the micro-periodic shells under consideration will be shown in the next section. The results obtained from this model will be compared with those derived from the known non-asymptotic more simple *standard tolerance model* governed by Eqs. (15)–(17) and from *the asymptotic one* given by Eq. (19).

5 Example of application

5.1 Description of the problem

In this subsection, transversal free vibrations of a thin simply supported shell strip with span $L \equiv L_1$ along the circumferential $x \equiv x^1$-coordinate and with r, d as its midsurface curvature radius and constant thickness, respectively, are discussed. The shell strip is made of two homogeneous elastic isotropic materials, which are perfectly bonded on interfaces and periodically densely distributed along x-coordinate. The elastic and inertial properties of the shell strip are constant in axial direction. A fragment of such a shell strip is shown in Fig. 1, where in the problem under consideration length dimension L_2 of the shell midsurface along $\xi \equiv x^2$-coordinate is assumed to be infinite.

The basic cell Δ is defined by: $\Delta \equiv [-\lambda/2,\ \lambda/2]$, where λ is a cell length dimension in $x \equiv x^1$-direction, cf. Figs. 1 and 3. We recall that *the microstructure length parameter* λ has to satisfy conditions: $\lambda/d >> 1$, $\lambda/r << 1$ and $\lambda/L_1 << 1$. Setting $z \equiv z^1 \in [-\lambda/2,\ \lambda/2]$, we assume that the cell has a

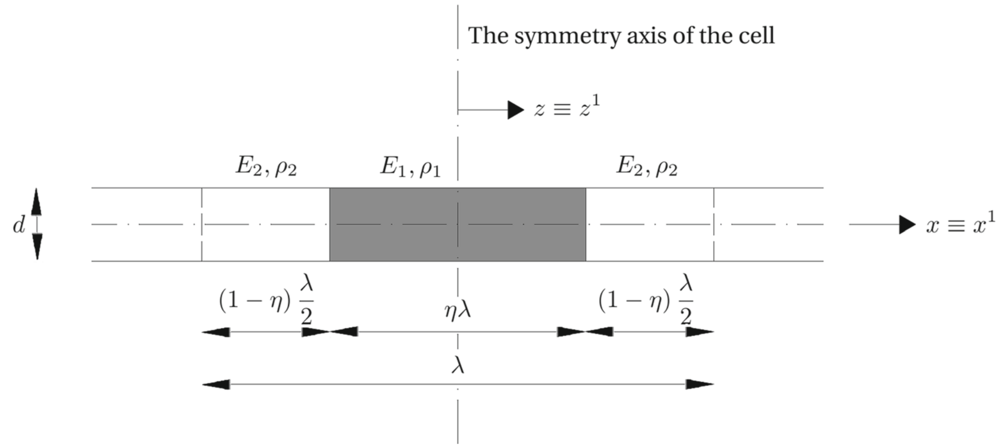

Fig. 3 Basic cell $\Delta \equiv [-\lambda/2,\ \lambda/2]$ of the uniperiodic shell

symmetry axis for $z = 0$. Inside the cell, the geometrical and elastic properties of the shell are described by symmetric (i.e. even) functions of argument z.

Properties of the component materials are described by: Young's moduli E_1, E_2, Poisson's ratios ν_1, ν_2 and mass densities ρ_1, ρ_2, cf. Fig. 3. It is assumed that elastic $E(x)$ and inertial $\rho(x)$ properties of the composite shell are periodic functions in x, $x \equiv x^1 \in \Omega$, but Poisson's ratio $\nu \equiv \nu_1 = \nu_2$ is constant. Inside the cell, functions $E(z)$, $\rho(z)$, $z \in \Delta$, take the form

$$E(z),\ \rho(z) = \begin{cases} E_1,\ \rho_1 & for\ \ z \in (-\eta\,\lambda/2,\ \eta\ \lambda/2), \\ E_2,\ \rho_2 & for\ \ z \in [-\lambda/2,\ -\eta\,\lambda/2] \cup [\eta\,\lambda/2,\ \lambda/2], \end{cases} \tag{21}$$

where $\eta \in [0, 1]$ is a parameter describing distribution of material properties in the cell, cf. Fig. 3.

The shell's stiffnesses $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $x \in \Omega$, are λ-periodic functions in argument x. Inside the cell, the rigidities $D^{\alpha\beta\gamma\delta}(z)$, $B^{\alpha\beta\gamma\delta}(z)$, $z \in \Delta$, of the shell are described by: $D^{\alpha\beta\gamma\delta}(z) = H^{\alpha\beta\gamma\delta}\ E(z)d/(1-\nu^2)$, $B^{\alpha\beta\gamma\delta}(z) = H^{\alpha\beta\gamma\delta}\ E(z)d^3/(12(1-\nu^2))$, where $E(z)$, $z \in \Delta$, is given by (21) and the nonzero components of tensor $H^{\alpha\beta\gamma\delta}$ are: $H^{1111} = H^{2222} = 1$, $H^{1122} = H^{2211} = \nu$, $H^{1212} = H^{1221} = H^{2121} = H^{2112} = (1-\nu)/2$.

The shell's mass density $\mu(x)$, $x \in \Omega$, per midsurface unit area is λ-periodic function in argument x. Inside the cell, the shell mass density $\mu(z)$ per midsurface unit area is given by $\mu(z) = \rho(z)\,d$, where $\rho(z)$, $z \in \Delta$, is given by (21).

The considerations will be based on general (extended) tolerance model Eqs. (12)–(14), standard tolerance model Eqs. (15)–(17) and asymptotic model Eq. (19).

In order to investigate free vibrations, we assume that external forces f^α, f in the averaged model equations mentioned above are equal to zero.

Moreover, the forces of inertia in directions tangential to the shell midsurface are neglected.

We also neglect fluctuating parts hU_α of displacements u_α.

Bearing in mind that the fluctuation shape function $g \in FS^2(\Omega, \Delta)$, $g \in O(\lambda^2)$ should approximate the expected principal modes of shell free vibrations on the cell and based on the knowledge of these principal modes in thin heterogeneous shells and plates, cf., e.g. Ostrowski [9], Tomczyk [10], Jędrysiak [19], we shall postulate the fluctuation shape function as: $g(z) = \lambda^2[\cos(2\pi z/\lambda) + c]$, $z \in \Delta$, where constant c, calculated from condition $<\mu g> = 0$, is equal to: $c = -(\rho_1 - \rho_2)\sin(\eta\pi)[\pi(\eta\rho_1 + (1-\eta)\rho_2)]^{-1}$. We recall that $\eta \in [0, 1]$ is a parameter describing distribution of material properties in the cell, cf. Fig. 3.

This dynamic problem is treated to be independent of the ξ-coordinate. Hence, $u_2^0 = 0$ and the remaining unknowns u_1^0, w^0, W of the tolerance and asymptotic models are only functions of x-midsurface parameter and t-coordinate.

Bearing in mind assumptions given above, the effect of a cell size on the transversal free vibrations' frequencies of the considered shell strip will be evaluate by comparison of results obtained from the three averaged models under consideration, i.e. from:

(a) the new general tolerance model represented by equations of motion (14) with constitutive relations (12), (13),

(b) the known standard tolerance model governed by Eqs. (15)–(17) including less number of length-scale terms than the extended one,
(c) the asymptotic model given by Eq. (19) being independent of a cell size.

Moreover, the effects of the differences between the elastic properties and also between the inertial properties of the component materials in the cell on these frequencies will be studied.

5.2 Analysis in the framework of the general tolerance model

Under assumptions given in Sect. 5.1, the system of *extended tolerance model* Eq. (14) reduces to the following system of three equations for $u_1^0(x,t)$, $w^0(x,t)$, $W(x,t)$, $(x,t) \in \Omega \times I$,

$$\begin{aligned}
&<D^{1111}>(\partial_{11}u_1^0 + r^{-1}\partial_1 w^0) = 0,\\
&r^{-1}<D^{1111}>(\partial_1 u_1^0 + r^{-1}w^0) + <B^{1111}>\partial_{1111}w^0\\
&\quad +<B^{1111}\partial_{11}g>\partial_{11}W + \lambda^2<B^{1111}\bar{g}>\partial_{1111}W + <\mu>\ddot{w}^0 = 0,\\
&\quad +<\partial_{11}g\,B^{1111}>\partial_{11}w^0 + \lambda^2<B^{1111}\bar{g}>\partial_{1111}w^0\\
&\quad +<(\partial_{11}g)^2B^{1111}>W + \left(2\lambda^2 < \partial_{11}g\,B^{1111}\bar{g}> - 4\lambda^2<(\partial_1\tilde{g})^2B^{1111}>\right)\partial_{11}W\\
&\quad +\lambda^4<(\bar{g})^2B^{1111}>\partial_{1111}W + \lambda^4<\mu(\bar{g})^2>\ddot{W} = 0,
\end{aligned} \tag{22}$$

where $\bar{g}(\cdot) = \lambda^{-2}g(\cdot)$, $\tilde{g}(\cdot) = \lambda^{-1}g(\cdot)$. We recall that derivative $\partial_{11}g(x)$, $x \in \Omega$, of fluctuation shape function $g(x)$ is independent of λ as parameter. Some terms in (22) depend explicitly on microstructure length parameter λ. All coefficients in (22) are constant.

Solutions to Eq. (22) satisfying the boundary conditions for the shell strip simply supported on edges $x = 0$, $x = L$ can be assumed in the form, cf. Kaliski [41],

$$\begin{aligned}
u_1^0(x,t) &= A\cos(\alpha x)\cos(\omega t),\\
w^0(x,t) &= B\sin(\alpha x)\cos(\omega t),\\
W(x,t) &= C\sin(\alpha x)\cos(\omega t),
\end{aligned} \tag{23}$$

where A, B, C are arbitrary constants different from zero, $\alpha = \pi/L$ is a wave number, ω is a smallest frequency of transverse free vibrations. Functions $\cos(\alpha x)$, $\sin(\alpha x)$ relate to the lowest free vibration modes.

Substituting the right-hand sides of (23) into Eq. (22), we obtain the system of three linear homogeneous algebraic equations for A, B, C. For a non-trivial solution, the determinant of this system must be equal to zero. In this manner, we arrive at the characteristic equation for frequency $(\omega^{\text{gtm}})_{\text{uni}} \equiv \omega$ of the transverse free vibrations of the shell strip under consideration.

Denoting

$$\begin{aligned}
&\varepsilon \equiv \lambda/L, \quad \bar{B} \equiv B^{1111}, \quad \widehat{\mu} \equiv <\mu>, \quad \bar{\mu} \equiv <\mu(\bar{g})^2>,\\
&\bar{b} \equiv <(\partial_{11}g)^2\bar{B}> + 2(\pi\varepsilon)^2[2<(\partial_1\tilde{g})^2\bar{B}> - <\bar{g}\,\partial_{11}g\,\bar{B}>] + (\pi\varepsilon)^4<(\bar{g})^2\bar{B}>,\\
&\bar{c} \equiv <\partial_{11}g\,\bar{B}>^2 + 2(\pi\varepsilon)^2<\bar{g}\,\bar{B}><\partial_{11}g\,\bar{B}> + (\pi\varepsilon)^4<\bar{g}\,\bar{B}>^2,
\end{aligned} \tag{24}$$

and recalling that $\bar{g} = \lambda^{-2}g$, $\tilde{g} = \lambda^{-1}g$, from the characteristic equation mentioned above we derive the following formulae for *the fundamental lower free vibration frequency* $(\omega_-^{\text{gtm}})_{\text{uni}}$ *and for the new additional higher free vibration frequency* $(\omega_+^{\text{gtm}})_{\text{uni}}$, caused by a periodic structure of the shell strip

$$\begin{aligned}
(\omega_-^{\text{gtm}})^2_{\text{uni}} &= \frac{1}{2}\left(\frac{\bar{b}}{(L\varepsilon)^4\bar{\mu}} + \frac{\alpha^4<\bar{B}>}{\widehat{\mu}}\right)\\
&\quad -\frac{\bar{b}}{2(L\varepsilon)^4\bar{\mu}}\sqrt{1 + \frac{2\bar{\mu}}{\widehat{\mu}(\bar{b})^2}\left(2\bar{c} - <\bar{B}>\bar{b}\right)(\pi\varepsilon)^4 + \left(\frac{\bar{\mu}<\bar{B}>}{\widehat{\mu}\,\vec{b}}\right)^2(\pi\varepsilon)^8},
\end{aligned} \tag{25}$$

$$\begin{aligned}
(\omega_+^{\text{gtm}})^2_{\text{uni}} &= \frac{1}{2}\left(\frac{\bar{b}}{(L\varepsilon)^4\bar{\mu}} + \frac{\alpha^4<\bar{B}>}{\widehat{\mu}}\right)\\
&\quad +\frac{\bar{b}}{2(L\varepsilon)^4\bar{\mu}}\sqrt{1 + \frac{2\bar{\mu}}{\widehat{\mu}(\bar{b})^2}\left(2\bar{c} - <\bar{B}>\bar{b}\right)(\pi\varepsilon)^4 + \left(\frac{\bar{\mu}<\bar{B}>}{\widehat{\mu}\,\vec{b}}\right)^2(\pi\varepsilon)^8}.
\end{aligned} \tag{26}$$

Results (25), (26) depend on dimensionless microstructure length parameter $\varepsilon \equiv \lambda/L$.

It can be observed that there are not terms involving rigidity D^{1111} in formulae (25), (26); these terms dropped out when deriving the characteristic frequency equation. It means that in the framework of *the general tolerance model*, the effect of stiffness D^{1111} on the transversal free vibrations of the shell strip under consideration is omitted.

5.3 Analysis in the framework of standard tolerance model

Let us investigate the above dynamic problem in the framework of the standard tolerance model containing fewer terms depending on a period length λ than the general tolerance model.

Under assumptions given in Sect. 5.1, the system of *standard tolerance model* Eq. (17) reduces to the following system of three equations for $u_1^0(x,t)$, $w^0(x,t)$, $W(x,t)$, $(x,t) \in \Omega \times I$,

$$
\begin{aligned}
&<D^{1111}>(\partial_{11}u_1^0 + r^{-1}\partial_1 w^0) = 0,\\
&r^{-1}<D^{1111}>(\partial_1 u_1^0 + r^{-1}w^0) + <B^{1111}>\partial_{1111}w^0\\
&+<B^{1111}\partial_{11}g>\partial_{11}W + <\mu>\ddot{w}^0 = 0,\\
&+<\partial_{11}g\, B^{1111}>\partial_{11}w^0 + <(\partial_{11}g)^2 B^{1111}>W + \lambda^4<\mu(\bar{g})^2>\ddot{W} = 0,
\end{aligned}
\tag{27}
$$

where $\bar{g}(\cdot) = \lambda^{-2}g(\cdot)$, Some terms in (27) depend explicitly on microstructure length parameter λ. All coefficients in (27) are constant.

Solutions to Eq. (27) satisfying the boundary conditions for the shell strip simply supported on edges $x = 0$, $x = L$ are assumed in the form of (23). Substituting the right-hand sides of (23) into Eq. (27), we obtain the system of three linear homogeneous algebraic equations for A, B, C, which has a non-trivial solution provided that its determinant is equal to zero. In this manner, we arrive at the characteristic equation for frequency $(\omega^{\text{stm}})_{\text{uni}} \equiv \omega$ of the transverse free vibrations of the shell strip under consideration.

Using denotations $(24)_{1-4}$ as well as denoting

$$
\tilde{b} \equiv <(\partial_{11}g)^2\bar{B}>, \quad \tilde{c} \equiv <\partial_{11}g\,\bar{B}>^2,
\tag{28}
$$

from the characteristic equation mentioned above we derive the following formulae for *the fundamental lower free vibration frequency* $(\omega_-^{\text{stm}})_{\text{uni}}$ *and for the new additional higher free vibration frequency* $(\omega_+^{\text{stm}})_{\text{uni}}$, caused by a periodic structure of the shell strip

$$
\begin{aligned}
(\omega_-^{\text{stm}})^2_{\text{uni}} = \frac{1}{2}\left(\frac{\tilde{b}}{(L\varepsilon)^4\bar{\mu}} + \frac{\alpha^4<\bar{B}>}{\widehat{\mu}}\right)&\\
-\frac{\tilde{b}}{2(L\varepsilon)^4\bar{\mu}}\sqrt{1 + \frac{2\bar{\mu}}{\widehat{\mu}(\tilde{b})^2}\left(2\tilde{c} - <\bar{B}>\tilde{b}\right)(\pi\varepsilon)^4 + \left(\frac{\bar{\mu}<\bar{B}>}{\widehat{\mu}\,\tilde{b}}\right)^2(\pi\varepsilon)^8}&,
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
(\omega_+^{\text{stm}})^2_{\text{uni}} = \frac{1}{2}\left(\frac{\tilde{b}}{(L\varepsilon)^4\bar{\mu}} + \frac{\alpha^4<\bar{B}>}{\widehat{\mu}}\right)&\\
+\frac{\tilde{b}}{2(L\varepsilon)^4\bar{\mu}}\sqrt{1 + \frac{2\bar{\mu}}{\widehat{\mu}(\tilde{b})^2}\left(2\tilde{c} - <\bar{B}>\tilde{b}\right)(\pi\varepsilon)^4 + \left(\frac{\bar{\mu}<\bar{B}>}{\widehat{\mu}\,\tilde{b}}\right)^2(\pi\varepsilon)^8}&,
\end{aligned}
\tag{30}
$$

Results (29), (30) depend on dimensionless microstructure length parameter $\varepsilon \equiv \lambda/L$.

There are not terms involving rigidity D^{1111} in formulae (29), (30); these terms dropped out when deriving the characteristic frequency equation. It means that in the framework of *the standard tolerance model*, the effect of stiffness D^{1111} on the transversal free vibrations of the shell strip under consideration is omitted.

5.4 Analysis in the framework of asymptotic model

In order to evaluate results obtained in the framework of general and standard models, which take into account the length-scale effect, let us consider the modelling problem within the asymptotic model (19) being independent of a cell size.

Now, asymptotic model Eq. (19) reduce to the following form

$$\begin{aligned} &<D^{1111}>(\partial_{11}u_1^0 + r^{-1}\partial_1 w^0) = 0, \\ &r^{-1}<D^{1111}>\left(\partial_1 u_1^0 + r^{-1}w^0\right) \\ &\quad +<B^{1111}>\partial_{1111}w^0 + <B^{1111}\partial_{11}g>\partial_{11}W + <\mu>\ddot{w}^0 = 0, \\ &<\partial_{11}g\, B^{1111}>\partial_{11}w^0 + <(\partial_{11}g)^2 B^{1111}>W = 0, \end{aligned} \tag{31}$$

All coefficients in (31) are constant.

Note, that (31) can also be directly derived from governing Eqs. (22) or (27) by neglecting terms depending explicitly on microstructure length parameter λ.

Solutions to Eq. (31) satisfying the boundary conditions for the shell strip simply supported on edges $x = 0$, $x = L$ are assumed in the form of (23). Substituting the right-hand sides of (23) into Eq. (31), we obtain the system of three linear homogeneous algebraic equations for A, B, C and then from the comparison of determinant of this system to zero, we derive the characteristic equation for frequency $(\omega^{\mathrm{am}})_{\mathrm{uni}} \equiv \omega$, from which we obtain the following formula for transversal free vibration frequency $(\omega^{\mathrm{am}})_{\mathrm{uni}}$

$$(\omega^{\mathrm{am}})^2_{\mathrm{uni}} = \frac{\alpha^4}{<\mu>}\left[<B^{1111}> - \frac{<\partial_{11}g\, B^{1111}>^2}{<B^{1111}\,(\partial_{11}g)^2>}\right]. \tag{32}$$

This frequency is independent of a cell size. There are not terms involving rigidity D^{1111} in formula (32); these terms dropped out when deriving the frequency equation. It means that in the framework of *the asymptotic model*, the effect of stiffness D^{1111} on the transversal free vibrations of the shell strip under consideration is omitted.

5.5 Numerical calculations

The numerical analysis is based on results (25), (26) and (29), (30) obtained from the general and standard tolerance models, respectively, and on result (32) derived from the asymptotic model.

Let us define the following dimensionless free vibration frequencies

$$\left(\Omega_-^{\mathrm{gtm}}\right)^2_{\mathrm{uni}} \equiv \frac{(1-\nu^2)\,\rho_1 L^2}{E_1}(\omega_-^{\mathrm{gtm}})^2_{\mathrm{uni}}, \tag{33}$$

$$\left(\Omega_+^{\mathrm{gtm}}\right)^2_{\mathrm{uni}} \equiv \frac{(1-\nu^2)\,\rho_1 L^2}{E_1}(\omega_+^{\mathrm{gtm}})^2_{\mathrm{uni}}, \tag{34}$$

$$\left(\Omega_-^{\mathrm{stm}}\right)^2_{\mathrm{uni}} \equiv \frac{(1-\nu^2)\,\rho_1 L^2}{E_1}(\omega_-^{\mathrm{stm}})^2_{\mathrm{uni}}, \tag{35}$$

$$\left(\Omega_+^{\mathrm{stm}}\right)^2_{\mathrm{uni}} \equiv \frac{(1-\nu^2)\,\rho_1 L^2}{E_1}(\omega_+^{\mathrm{stm}})^2_{\mathrm{uni}}, \tag{36}$$

$$\left(\Omega^{\mathrm{am}}\right)^2_{\mathrm{uni}} \equiv \frac{(1-\nu^2)\,\rho_1\, L^2}{E_1}(\omega^{\mathrm{am}})^2_{\mathrm{uni}}, \tag{37}$$

where frequencies $(\omega_-^{\mathrm{gtm}})_{\mathrm{uni}}$, $(\omega_+^{\mathrm{gtm}})_{\mathrm{uni}}$, $(\omega_-^{\mathrm{stm}})_{\mathrm{uni}}$, $(\omega_+^{\mathrm{stm}})_{\mathrm{uni}}$, $(\omega^{\mathrm{am}})_{\mathrm{uni}}$ are determined by formulae (25), (26), (29), (30) and (32), respectively.

The calculations are made for $L/r = \pi/2$, Poisson's ratio $\nu = 0.3$, for different values of geometrical parameter $\eta \in [0.1,\ 0.9]$ describing distribution of elastic and inertial properties in the cell, for different values of dimensionless microstructure length parameter $\varepsilon \equiv \lambda/L \in [0.01,\ 0.1]$, ($L = \mathrm{const}$), for various ratios $d/L \in [0.001,\ 0.01]$ ($L = const$), $E_2/E_1 \in [0.01,\ 1.0]$ (ratio describing differences between values of Young's moduli of component materials in the cell, $E_1 = \mathrm{const}$) and $\rho_2/\rho_1 \in [0.01,\ 1.0]$ (ratio describing differences between values of mass densities of component materials in the cell, $\rho_1 = \mathrm{const}$). It can be observed that values of ratio d/L imply values of the ratio of shell thickness d to microstructure length parameter λ, i.e. $d/\lambda = (d/L)\,\varepsilon^{-1}$ (the shell thickness $d$varies together with parameter ε). We recall that ratio d/λ has to satisfy condition: $d/\lambda << 1$.

In Figs. 4a and b, there are diagrams of *dimensionless lower free vibration frequencies* given by (33), (35), (37) versus dimensionless microstructure length parameter $\lambda/L \in [0.01,\ 0.1]$. These diagrams are made for $\eta = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$, $d/L = 0.001$ (hence $d/\lambda \in [0.01,\ 0.1]$) and for a) $E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$, b) $E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$. From the results shown in Fig. 4, it follows that from the computational point of view there are no differences between values of the dimensionless *lower free vibration frequencies* $(\Omega_-^{\text{gtm}})_{\text{uni}}$, $(\Omega_-^{\text{stm}})_{\text{uni}}$, $(\Omega^{\text{am}})_{\text{uni}}$ derived from the general and standard tolerance models and from the asymptotic one.

In Fig. 5a and b, there are diagrams of *dimensionless higher free vibration frequencies* determined by (34), (36) versus dimensionless microstructure length parameter $\lambda/L \in [0.01,\ 0.1]$. These diagrams are made for $\eta = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$, $d/L = 0.001$ (hence $d/\lambda \in [0.01,\ 0.1]$) and for a) $E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$, b) $E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$. From the results shown in Fig. 5, it follows that from the computational point of view there are no differences between values of the dimensionless *higher free vibration frequencies* $(\Omega_+^{\text{gtm}})_{\text{uni}}$, $(\Omega_+^{\text{stm}})_{\text{uni}}$ derived from the general and standard tolerance models.

Plots of *dimensionless lower free vibration frequencies* $(\Omega_-^{\text{gtm}})_{\text{uni}}$ (33), $(\Omega_-^{\text{stm}})_{\text{uni}}$ (35), $(\Omega^{\text{am}})_{\text{uni}}$ (37) versus ratio $E_2/E_1 \in [0.01,\ 1]$ performed for $\eta = 0.25$, $\rho_2/\rho_1 = \{0.1,\ 0.5,\ 0.9,\ 1\}$, $\lambda/L = 0.1$, $d/L = 0.01$ (hence $d/\lambda = 0.1$) are presented in Fig. 6.

Plots of *dimensionless higher free vibration frequencies* $(\Omega_+^{\text{gtm}})_{\text{uni}}$ (34), $(\Omega_+^{\text{stm}})_{\text{uni}}$ (36) versus ratio $E_2/E_1 \in [0.01,\ 1]$ performed for $\eta = 0.25$, $\rho_2/\rho_1 = \{0.1,\ 0.5,\ 0.9,\ 0.98\}$ and for a) $\lambda/L = 0.1$, $d/L = 0.01$ (hence $d/\lambda = 0.1$), b) $\lambda/L = 0.01$, $d/L = 0.001$ (hence $d/\lambda = 0.1$) are presented in Fig. 7a and b.

In Fig. 8, there are shown diagrams of *dimensionless lower free vibration frequencies* $(\Omega_-^{\text{gtm}})_{\text{uni}}$ (33), $(\Omega_-^{\text{stm}})_{\text{uni}}$ (35), $(\Omega^{\text{am}})_{\text{uni}}$ (37) versus ratio $\rho_2/\rho_1 \in [0.01,\ 1]$, made for $\eta = 0.25$, $E_2/E_1 = \{0.1,\ 0.5,\ 0.9,\ 1\}$, $\lambda/L = 0.1$, $d/L = 0.01$.

In Fig. 9a and b, there are shown diagrams of *dimensionless higher free vibration frequencies* $(\Omega_+^{\text{gtm}})_{\text{uni}}$ (34), $(\Omega_+^{\text{stm}})_{\text{uni}}$ (36) versus ratio $\rho_2/\rho_1 \in [0.01,\ 1]$, made for $\eta = 0.25$, $E_2/E_1 = \{0.1,\ 0.5,\ 0.9,\ 0.98\}$ and for a) $\lambda/L = 0.1$, $d/L = 0.01$, b) $\lambda/L = 0.01$, $d/L = 0.001$.

Plots of *dimensionless lower free vibration frequencies* $(\Omega_-^{\text{gtm}})_{\text{uni}}$ (33), $(\Omega_-^{\text{stm}})_{\text{uni}}$ (35), $(\Omega^{\text{am}})_{\text{uni}}$ (37) versus both ratios $E_2/E_1 \in [0.01,\ 1]$ and $\rho_2/\rho_1 \in [0.01,\ 1]$ are presented in Fig. 10. These plots are performed for $\eta = 0.25$, $\lambda/L = 0.1$, $d/L = 0.01$.

Plots of *dimensionless higher free vibration frequencies* $(\Omega_+^{\text{gtm}})_{\text{uni}}$ (34), $(\Omega_+^{\text{stm}})_{\text{uni}}$ (36) versus both ratios $E_2/E_1 \in [0.01,\ 1]$ and $\rho_2/\rho_1 \in [0.01,\ 1]$ are presented in Fig. 11a and b. These plots are performed for $\eta = 0.25$ and for a) $\lambda/L = 0.1$, $d/L = 0.01$, b) $\lambda/L = 0.01$, $d/L = 0.001$.

In Fig. 12, there are diagrams of *dimensionless lower free vibration frequencies* $(\Omega_-^{\text{gtm}})_{\text{uni}}$, $(\Omega_-^{\text{stm}})_{\text{uni}}$, $(\Omega^{\text{am}})_{\text{uni}}$ given by (33), (35), (37), respectively, versus ratio $d/L \in [0.001,\ 0.01]$. These diagrams are made for $\eta = 0.25$, $\lambda/L = 0.1$ (hence, $d/\lambda \in [0.01,\ 0.1]$) and for three pairs of ratios: ($E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$), ($E_2/E_1 = 0.5$, $\rho_2/\rho_1 = 0.5$), ($E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$)

In Fig. 13, there are diagrams of *dimensionless higher free vibration frequencies* $(\Omega_+^{\text{gtm}})_{\text{uni}}$, $(\Omega_+^{\text{stm}})_{\text{uni}}$ given by (34), (36), respectively, versus ratio $d/L \in [0.001,\ 0.01]$. These diagrams are made for $\eta = 0.25$, $\lambda/L = 0.1$ and for three pairs of ratios: ($E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$), ($E_2/E_1 = 0.5$, $\rho_2/\rho_1 = 0.5$), ($E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$).

5.6 Discussion of analytical and numerical results

On the basis of analytical results (25), (26), (29), (30), (32) and computational results shown in Figs. 4, 5, 6, 7, 8, 9, 10, 11, 12 and 13, the following important conclusions can be formulated:

- *In the framework of the general and standard tolerance models, not only the fundamental lower* $(\omega_-^{\text{gtm}})_{\text{uni}}$ (25), $(\omega_-^{\text{stm}})_{\text{uni}}$ (29), *but also the new additional higher* $(\omega_+^{\text{gtm}})_{\text{uni}}$ (26), $(\omega_+^{\text{stm}})_{\text{uni}}$ (30) *free vibration frequencies can be derived and analysed*. The higher free vibration frequencies are caused by a periodic microstructure of the shell strip under consideration, and hence, they depend on a microstructure length parameter λ. These frequencies cannot be determined using the asymptotic model.
- Free vibration frequencies derived in the framework of general tolerance model contain a bigger number of terms depending on a cell size than those obtained from the standard model.

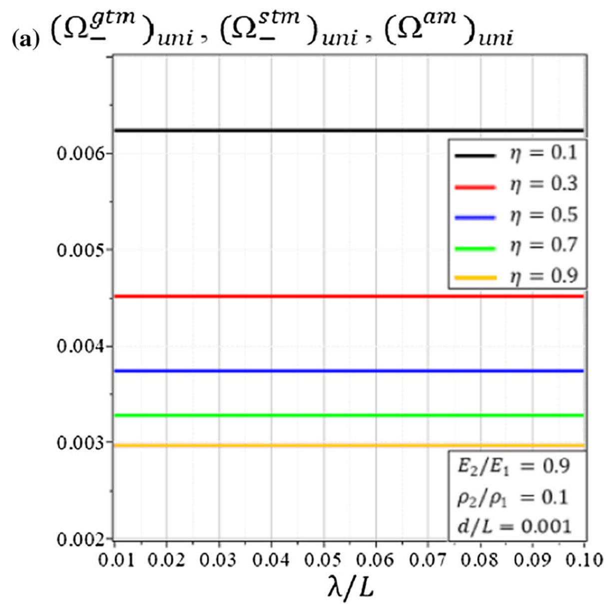

Fig. 4 Diagrams of dimensionless lower free vibration frequencies $(\Omega_-^{gtm})_{uni}$ (33), $(\Omega_-^{stm})_{uni}$ (35), $(\Omega^{am})_{uni}$ (37) versus dimensionless microstructure length parameter λ/L, made for $\eta = \{0.1, 0.3, 0.5, 0.7, 0.9\}$, $d/L = 0.001$ and for **a** $E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$, **b** $E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$

- From the results shown in Figs. 4, 6, 8, 10 and 12, it follows that *differences between values of the dimensionless lower free vibration frequencies* $(\Omega_-^{gtm})_{uni}$ (33), $(\Omega_-^{stm})_{uni}$ (35) *determined in the framework of both the general and standard tolerance models are negligibly small.* Moreover, *differences between values of these lower free vibration frequencies and free vibration frequency* $(\Omega^{am})_{uni}$ (37) *obtained from the asymptotic model are also negligibly small.* Thus, the effect of microstructure length parameter λ on the shell's fundamental lower free vibration frequencies can be neglected in the dynamic problem under consideration. It means that asymptotic model governed by Eq. (31), being more simple than the non-asymptotic tolerance models, is sufficient to determine and investigate free vibration frequencies of the uniperiodic shell strip considered here.

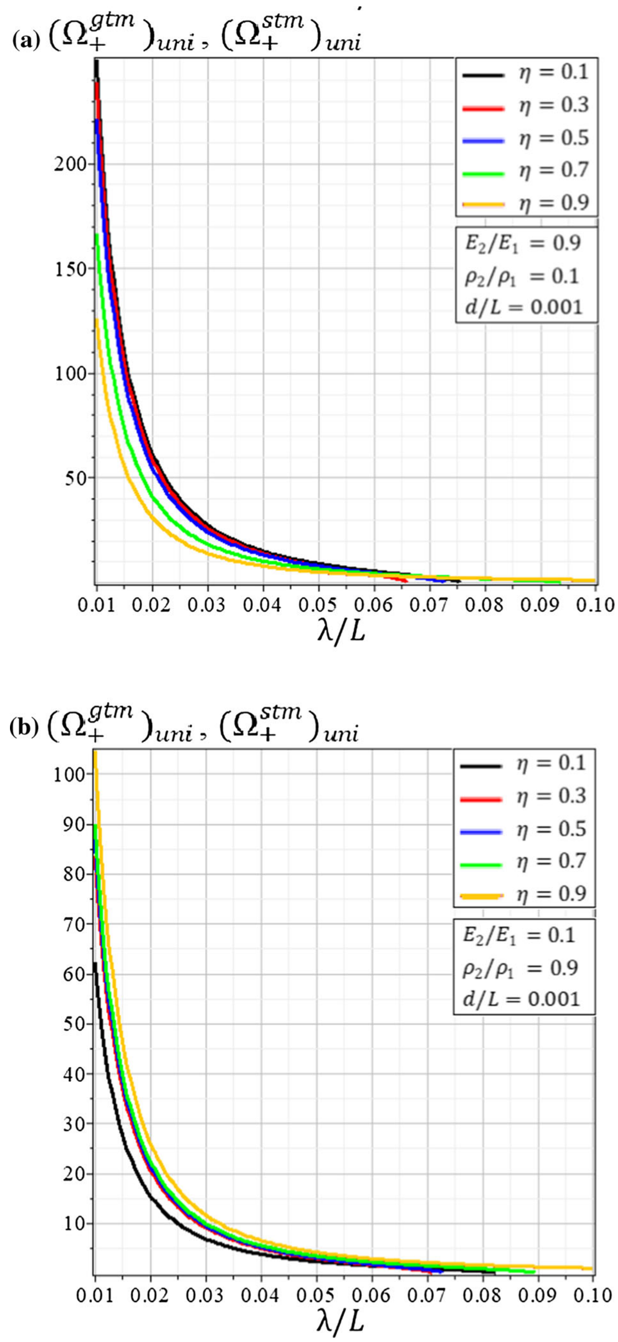

Fig. 5 Diagrams of dimensionless higher free vibration frequencies $(\Omega_{+}^{\mathrm{gtm}})_{\mathrm{uni}}$ (34), $(\Omega_{+}^{\mathrm{stm}})_{\mathrm{uni}}$ (36) versus dimensionless microstructure length parameter λ/L, made for $\eta = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$, $d/L = 0.001$ and for **a** $E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$, **b** $E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$

- From the results shown in Figs. 5, 7, 9, 11 and 13, it follows that *differences between values of the dimensionless higher free vibration frequencies* $(\Omega_{+}^{\mathrm{gtm}})_{\mathrm{uni}}$ (34), $(\Omega_{+}^{\mathrm{stm}})_{\mathrm{uni}}$ (36) *determined in the framework of both the general and standard tolerance models are negligibly small.* Hence, the simpler standard tolerance model is sufficient to determine and study the cell-dependent higher free vibration frequencies of the shell strip under consideration.
- Analysing the results presented in Fig. 4, it can be seen that for every value of geometrical parameter $\eta = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$ describing distribution of material properties in the cell and for a fixed ratio

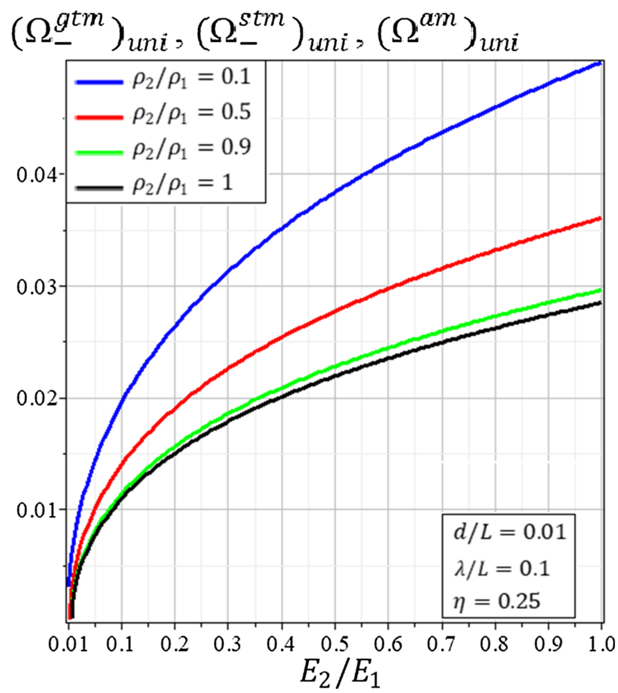

Fig. 6 Diagrams of dimensionless lower free vibration frequencies $(\Omega_-^{gtm})_{uni}$ (33), $(\Omega_-^{stm})_{uni}$ (35), $(\Omega^{am})_{uni}$ (37) versus ratio E_2/E_1, made for $\eta = 0.25$, $\rho_2/\rho_1 = \{0.1,\ 0.5,\ 0.9,\ 1\}$, $\lambda/L = 0.1$, $d/L = 0.01$

$d/L = 0.001$ (but under condition $d/\lambda << 1$), *the dimensionless lower free vibration frequencies* $(\Omega_-^{gtm})_{uni}$ (33), $(\Omega_-^{stm})_{uni}$ (35), $(\Omega^{am})_{uni}$ (37) are independent of dimensionless microstructure length parameter $\varepsilon \equiv \lambda/L \in [0.01,\ 0.1]$. We also observe that for every value of parameter $\eta = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$, values of these cell-independent free vibration frequencies are greater for the pair of ratios ($E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$), i.e. for a shell strip with very strong inertial heterogeneity and very weak elastic inhomogeneity, than for the pair of ratios ($E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$), i.e. for a shell strip with very strong elastic heterogeneity and very weak inertial inhomogeneity. For the fixed $d/L = 0.001$ and pair of ratios ($E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$), the greatest and smallest values of frequencies $(\Omega_-^{gtm})_{uni}$ (33), $(\Omega_-^{stm})_{uni}$ (35), $(\Omega^{am})_{uni}$ (37) are obtained for $\eta = 0.1$ and $\eta = 0.9$, respectively, cf. Fig. 4a. For the fixed $d/L = 0.001$ and pair of ratios ($E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$), the greatest and smallest values of these frequencies are obtained for $\eta = 0.9$ and $\eta = 0.1$, respectively, cf. Fig. 4b. We recall that $\eta\lambda$ describes a length of the cell with component material of bigger values of Young's modulus and mass density.

- Analysing results presented in Fig. 5, it can be seen that for every value of geometrical parameter $\eta = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$ and for the fixed ratio $d/L = 0.001$ (but under condition $d/\lambda << 1$), *the dimensionless higher free vibration frequencies* $(\Omega_+^{gtm})_{uni}$ (34), $(\Omega_+^{stm})_{uni}$ (36) obtained from general and standard tolerance models, respectively, decrease with the increasing of values of dimensionless microstructure length parameter $\varepsilon \equiv \lambda/L \in [0.01,\ 0.1]$, i.e. with the decreasing of differences between period length λ and the length dimension L of the shell midsurface in circumferential direction, $L = \text{const}$. The strongest decrease in the dimensionless higher free vibration frequencies takes place for $\varepsilon \equiv \lambda/L \in [0.01,\ 0.02]$. We also observe that for every value of parameter $\eta = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$, values of these cell-dependent free vibration frequencies are greater for the pair of ratios ($E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$) than for the pair of ratio ($E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$). For the fixed $d/L = 0.001$ and pair of ratios ($E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$), the greatest and smallest values of frequencies $(\Omega_+^{gtm})_{uni}$ (34), $(\Omega_+^{stm})_{uni}$ (36) are obtained for ($\lambda/L = 0.01$, $\eta = 0.1$) and ($\lambda/L = 0.1$, $\eta = 0.9$), respectively, cf. Fig. 5a. For the fixed $d/L = 0.001$ and pair of ratios ($E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$), the greatest and smallest values of these frequencies are obtained for ($\lambda/L = 0.01$, $\eta = 0.9$) and ($\lambda/L = 0.1$, $\eta = 0.1$), respectively, cf. Fig. 5b.
- By studying the results given in Figs. 6, 7, 10 and 11, it can be observed that for fixed $\eta = 0.25$ and for either $\lambda/L = 0.1$, $d/L = 0.01$ (hence $d/\lambda = 0.1$), cf. Figs. 6, 7a, 10, 11a, or $\lambda/L = 0.01$, $d/L = 0.001$ (in this case d/λ is also equal to 0.1), cf. Fig. 7b and 11b, values of *dimensionless lower* $(\Omega_-^{gtm})_{uni}$ (33),

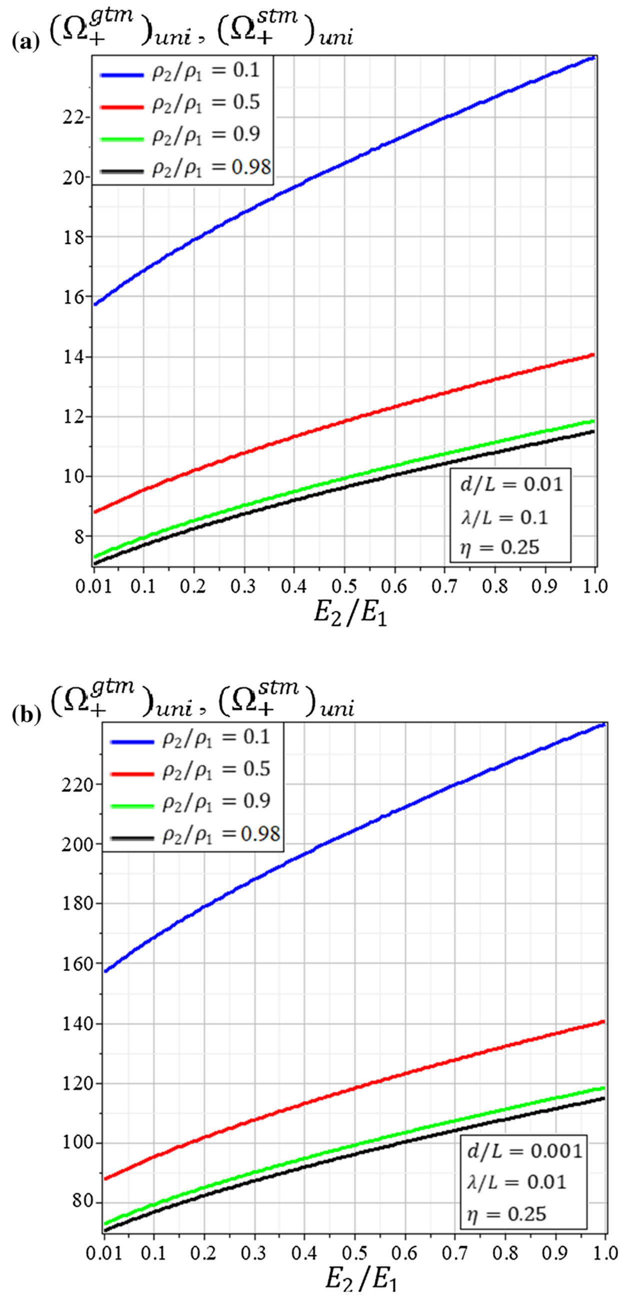

Fig. 7 Diagrams of dimensionless higher free vibration frequencies $(\Omega_+^{gtm})_{uni}$ (34), $(\Omega_+^{stm})_{uni}$ (36) versus ratio E_2/E_1, made for $\eta = 0.25$, $\rho_2/\rho_1 = \{0.1,\ 0.5,\ 0.9,\ 0.98\}$ and for **a** $\lambda/L = 0.1$, $d/L = 0.01$, **b** $\lambda/L = 0.01$, $d/L = 0.001$

$(\Omega_-^{stm})_{uni}$ (35), $(\Omega^{am})_{uni}$ (37) *and higher* $(\Omega_+^{gtm})_{uni}$ (34), $(\Omega_+^{stm})_{uni}$ (36) *free vibration frequencies* increase with the increasing of ratio $E_2/E_1 \in [0.01,\ 1]$, i.e. with the decreasing of differences between *elastic properties* of the shell component materials. Because the value of Young's module E_1 for the elastically stronger material is fixed, then these differences decrease if values of E_2 for the elastically weaker material tend to value of E_1. For an arbitrary but fixed ratio E_2/E_1, *values of the lower and higher free vibration frequencies under consideration increase* with the decreasing of ratio $\rho_2/\rho_1 = \{0.1,\ 0.5,\ 0.9,\ 1\}$ (Fig. 6), $\rho_2/\rho_1 = \{0.1,\ 0.5,\ 0.9,\ 0.98\}$ (Fig. 7) or $\rho_2/\rho_1 \in [0.01,\ 1]$ (Figs. 10 and 11), i.e. *with the increasing of inertial heterogeneity*.

- Analysing the results presented in Figs. 8, 9, 10 and 11, it can be observed that for a fixed $\eta = 0.25$ and for either $\lambda/L = 0.1$, $d/L = 0.01$ (hence $d/\lambda = 0.1$), cf. Figs. 8, 9a, 10 and 11a, or $\lambda/L = 0.01$, $d/L = 0.001$

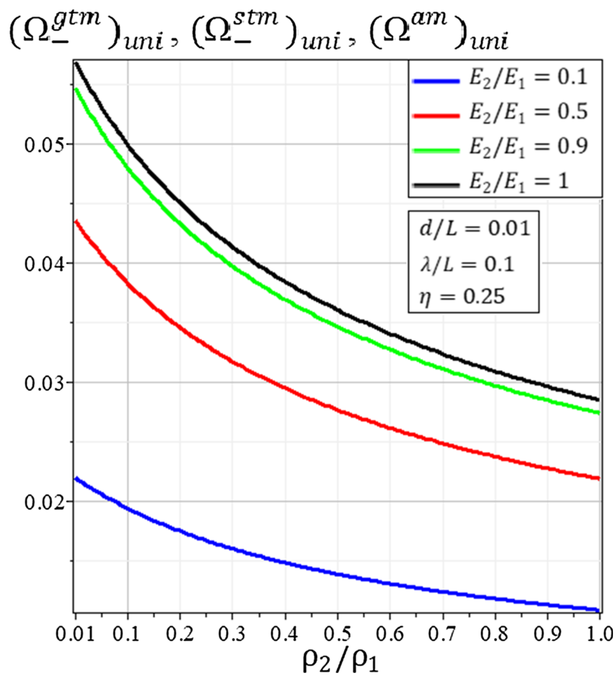

Fig. 8 Diagrams of dimensionless lower free vibration frequencies $(\Omega_-^{\text{gtm}})_{\text{uni}}$ (33), $(\Omega_-^{\text{stm}})_{\text{uni}}$ (35), $(\Omega^{\text{am}})_{\text{uni}}$ (37) versus ratio ρ_2/ρ_1, made for $\eta = 0.25$, $E_2/E_1 = \{0.1,\ 0.5,\ 0.9,\ 1\}$, $\lambda/L = 0.1$, $d/L = 0.01$

(in this case d/λ is also equal to 0.1), cf. Figs. 9b and 11b, values of *dimensionless lower* $(\Omega_-^{\text{gtm}})_{\text{uni}}$ (33), $(\Omega_-^{\text{stm}})_{\text{uni}}$ (35), $(\Omega^{\text{am}})_{\text{uni}}$ (37) *and higher* $(\Omega_+^{\text{gtm}})_{\text{uni}}$ (34), $(\Omega_+^{\text{stm}})_{\text{uni}}$ (36) *free vibration frequencies* decrease with the increasing of ratio $\rho_2/\rho_1 \in [0.01,\ 1]$, i.e. with the decreasing of differences between *inertial properties* of the shell component materials. Because the value of mass density ρ_1 for the stronger material is fixed, then these differences decrease if values of ρ_2 for the weaker material tend to value of ρ_1. For an arbitrary but fixed ratio ρ_2/ρ_1, *values of the lower and higher free vibration frequencies under consideration increase* with the increasing of ratio $E_2/E_1 = \{0.1,\ 0.5,\ 0.9,\ 1\}$ (Fig. 8), $E_2/E_1 = \{0.1,\ 0.5,\ 0.9,\ 0.98\}$ (Fig. 9) or $E_2/E_1 \in [0.01,\ 1]$ (Figs. 10 and 11), i.e. *with the decreasing of elastic heterogeneity*.

- On the basis of computational results shown in Figs. 6, 7, 8, 9, 10 and 11, we conclude that for a fixed $\eta = 0.25$ and for either $\lambda/L = 0.1$, $d/L = 0.01$ (hence $d/\lambda = 0.1$), cf. Figs. 6, 7a, 8, 9a, 10 and 11a, or $\lambda/L = 0.01$, $d/L = 0.001$ (in this case d/λ is also equal to 0.1), cf. Figs. 7b, 9b and 11b, *the highest values of dimensionless lower* $(\Omega_-^{\text{gtm}})_{\text{uni}}$ (33), $(\Omega_-^{\text{stm}})_{\text{uni}}$ (35), $(\Omega^{\text{am}})_{\text{uni}}$ (37) *and higher* $(\Omega_+^{\text{gtm}})_{\text{uni}}$ (34), $(\Omega_+^{\text{stm}})_{\text{uni}}$ (36) *free vibration frequencies* are obtained for the pair of ratios $(E_2/E_1 = 1,\quad \rho_2/\rho_1 = 0.01)$, i.e.. *for a shell strip with a very strong inertial heterogeneity and with elastic homogeneous structure*. *The smallest values of these free vibration frequencies* are obtained for the pair of ratios $(E_2/E_1 = 0.01,\quad \rho_2/\rho_1 = 1.0)$, i.e.. *for a shell strip with a very strong elastic heterogeneity and with inertial homogeneous structure*.
- For a fixed dimensionless microstructure length parameter $\varepsilon \equiv \lambda/L = 0.1$, fixed $\eta = 0.25$ and for every pair of ratios $(E_2/E_1 = 0.1,\ \rho_2/\rho_1 = 0.9)$, $(E_2/E_1 = 0.5,\ \rho_2/\rho_1 = 0.5)$, $(E_2/E_1 = 0.9,\ \rho_2/\rho_1 = 0.1)$, values of *dimensionless lower* $(\Omega_-^{\text{gtm}})_{\text{uni}}$ (33), $(\Omega_-^{\text{stm}})_{\text{uni}}$ (35), $(\Omega^{\text{am}})_{\text{uni}}$ (37) *and higher free vibration frequencies* $(\Omega_+^{\text{gtm}})_{\text{uni}}$ (34), $(\Omega_+^{\text{stm}})_{\text{uni}}$ (36) *increase with the increasing of ratio* $d/L \in [0.001,\ 0.01]$, $L = \text{const}$, cf. Figs. 12 and 13, i.e. with the increasing of ratio $d/\lambda \in [0.01,\ 0.1]$, $\lambda = \text{const}$, i.e. with the decreasing of differences between the shell thickness $d$and microstructure length parameter λ. For every $d/L \in [0.001,\ 0.01]$, the biggest values of these frequencies are obtained for pair of ratios $(E_2/E_1 = 0.9,\ \rho_2/\rho_1 = 0.1)$.

6 Final remarks and conclusions

The objects of analysis are thin linearly elastic Kirchhoff–Love-type circular cylindrical shells having a periodically microheterogeneous structure in circumferential direction (uniperiodic shells), cf. Figs. 1 and 2.

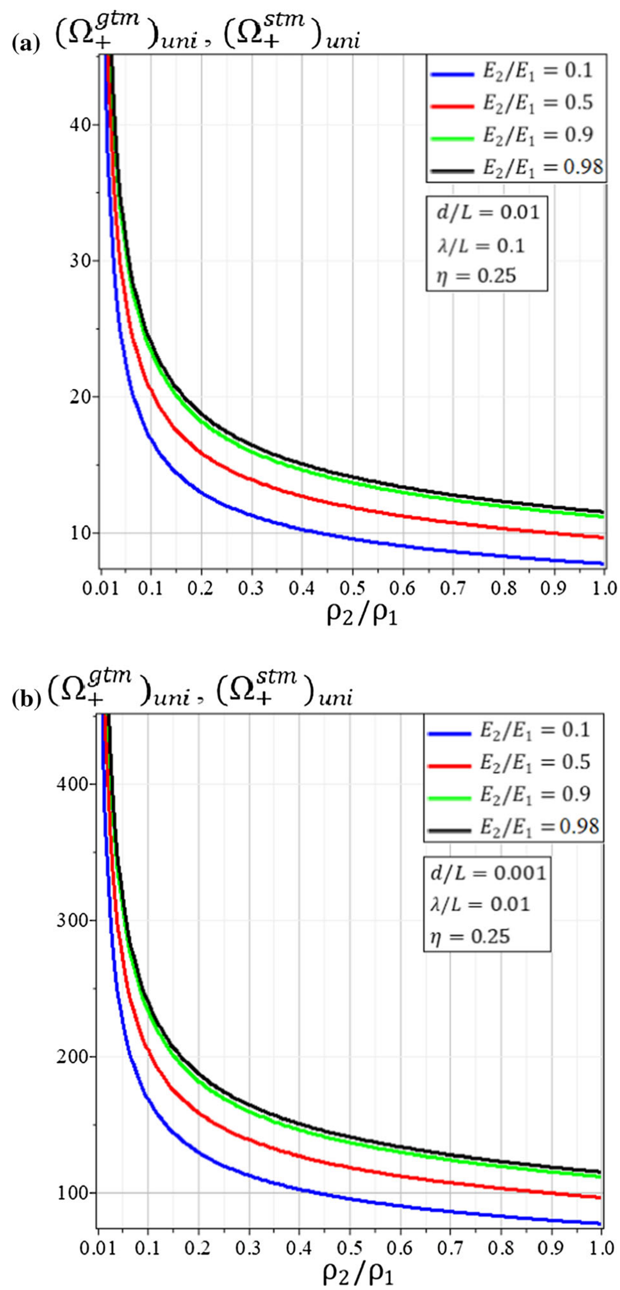

Fig. 9 Diagrams of dimensionless higher free vibration frequencies $(\Omega_+^{\mathrm{gtm}})_{\mathrm{uni}}$ (34), $(\Omega_+^{\mathrm{stm}})_{\mathrm{uni}}$ (36) versus ratio ρ_2/ρ_1, made for $\eta = 0.25$, $E_2/E_1 = \{0.1,\ 0.5,\ 0.9,\ 0.98\}$ and for **a** $\lambda/L = 0.1$, $d/L = 0.01$, **b** $\lambda/L = 0.01$, $d/L = 0.001$

Considerations are based on the known Kirchhoff–Love theory of thin elastic shells governed by Eq. (1). For periodic shells, these equations involve periodic, highly oscillating and non-continuous coefficients. That is why, the direct application of these equations to investigations of special dynamic problems is non-effective even using computational methods.

The main aim of this paper was to formulate and discuss *a new mathematical non-asymptotic averaged model for the analysis of dynamic problems for the uniperiodic shells under consideration*. This, so-called, *general tolerance model* was derived by applying *the extended tolerance modelling technique* proposed by Tomczyk and Woźniak [36]. The mentioned above extended version of the well-known tolerance modelling [6,9] is based on a new notion of *weakly slowly-varying function*, which is a certain extension of the well-known concept of *slowly-varying function* occurring in the classical tolerance modelling procedure. Both

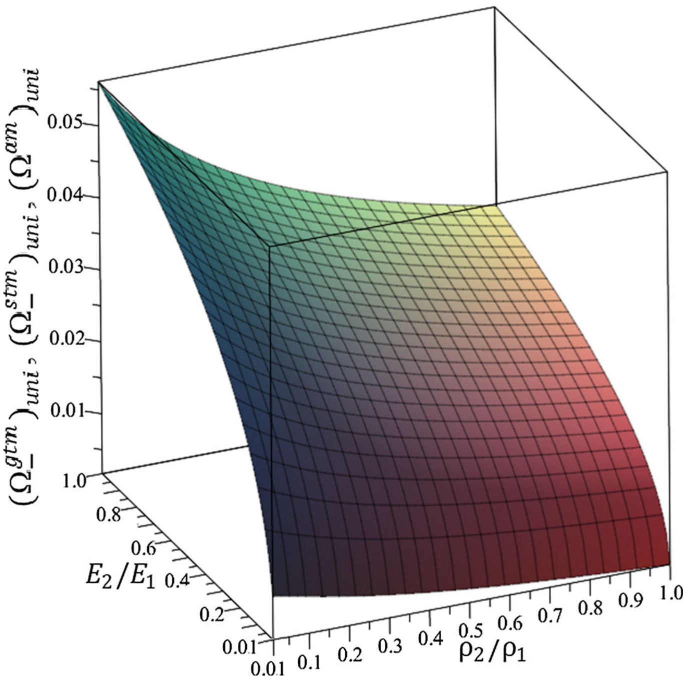

Fig. 10 Diagrams of dimensionless lower free vibration frequencies $(\Omega_-^{\mathrm{gtm}})_{\mathrm{uni}}$ (33), $(\Omega_-^{\mathrm{stm}})_{\mathrm{uni}}$ (35), $(\Omega^{\mathrm{am}})_{\mathrm{uni}}$ (37) versus ratios E_2/E_1 and ρ_2/ρ_1, made for $\eta = 0.25$, $\lambda/L = 0.1$, $d/L = 0.01$

the *weakly slowly-varying and the slowly-varying functions* can be treated as constant on a periodicity cell. The main difference between *the weakly slowly-varying and the classical slowly-varying functions* is that the products of derivatives of *weakly slowly-varying functions* in periodicity directions and microstructure length parameter (i.e. characteristic length dimension of the cell) are not assumed to be negligibly small. *It means that the concept of a weakly slowly-varying function is less restrictive than the concept of a slowly-varying function. It also means that the averaged general model equations obtained by using the extended* (*general*) *tolerance modelling procedure contain a bigger number of terms dependent on the microstructure size than the averaged standard model equations derived by applying the classical tolerance modelling technique based on the notion of a slowly-varying function*. Definition of *weakly slowly-varying functions* is given by (2). The classical *slowly-varying functions* are defined by (2) and (3). The other basic concepts of the extended tolerance modelling technique as *tolerance parameters, averaging operation, tolerance-periodic functions, fluctuation shape functions are the same as in the classical tolerance modelling procedure*, cf. Sect. 3.1. As the standard tolerance modelling, *the general tolerance modelling* is based on three assumptions: *the tolerance averaging approximation*, *the micro–macro decomposition and the residual orthogonality assumption*, cf. Sect. 3.2. Obviously, in the assumptions mentioned above, the unknown *slowly-varying functions* occurring in the classical tolerance modelling are replaced by *the weakly slowly-varying functions*.

The general tolerance model for the analysis of dynamic problems for the uniperiodic shells under consideration derived here is represented by constitutive relations (12), (13) and dynamic equilibrium Eq. (14) with *constant coefficients depending also on a cell size*. Hence, this model takes into account the effect of a microstructure size on the global shell dynamics (*the length-scale effect*).

The governing model equations are uniquely determined by continuous, periodic and highly oscillating *fluctuation shape functions* representing disturbances of displacement fields inside a cell. These functions are assumed to be known in every problem under consideration. They can be obtained as exact or approximate solutions to certain periodic eigenvalue problems describing free periodic vibrations of the cell. These functions can also be regarded as *the shape functions* resulting from the periodic discretization of the cell using for example the finite element method. The choice of these functions may also be based on the experience or intuition of the researcher.

Solutions to the initial-boundary value problems formulated in the framework of the general tolerance model have a physical sense only if the basic unknowns of this model are *weakly slowly-varying functions with respect to the periodicity cell and pertinent tolerance* parameters. This requirement can be verified only

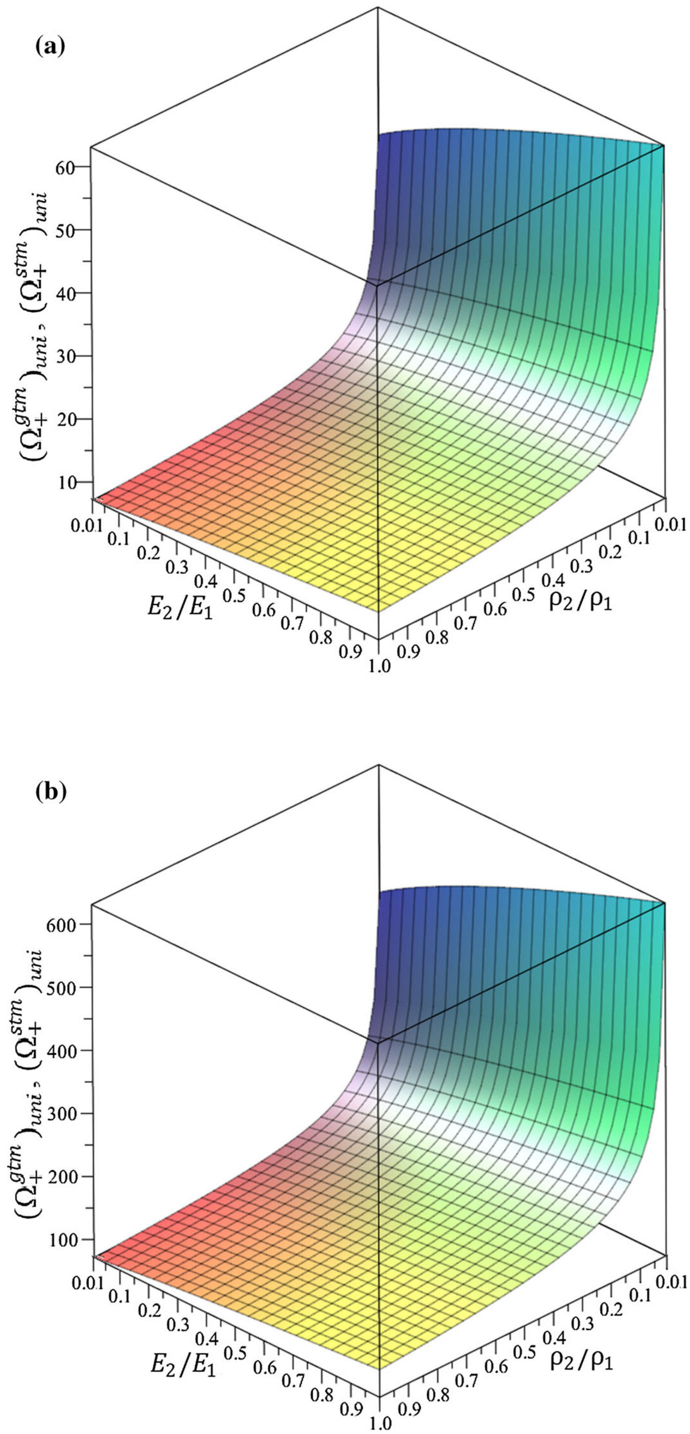

Fig. 11 Diagrams of dimensionless higher free vibration frequencies $(\Omega_+^{gtm})_{uni}$ (34), $(\Omega_+^{stm})_{uni}$ (36) versus ratios E_2/E_1 and ρ_2/ρ_1, made for $\eta = 0.25$ and for **a** $\lambda/L = 0.1$, $d/L = 0.01$, **b** $\lambda/L = 0.01$, $d/L = 0.001$

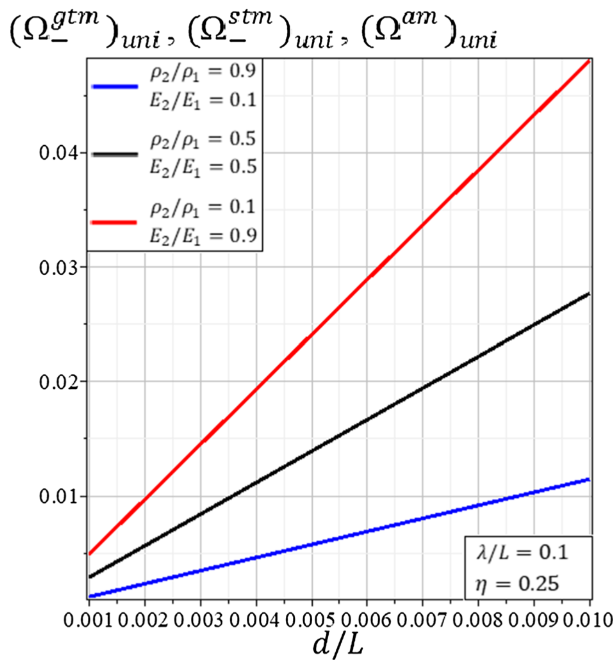

Fig. 12 Diagrams of dimensionless lower free vibration frequencies $(\Omega_-^{gtm})_{uni}$ (33), $(\Omega_-^{stm})_{uni}$ (35), $(\Omega^{am})_{uni}$ (37) versus ratio d/L, made for $\eta = 0.25$, $\lambda/L = 0.1$ and for three pairs of ratios: ($E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$), ($E_2/E_1 = 0.5$, $\rho_2/\rho_1 = 0.5$), ($E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$)

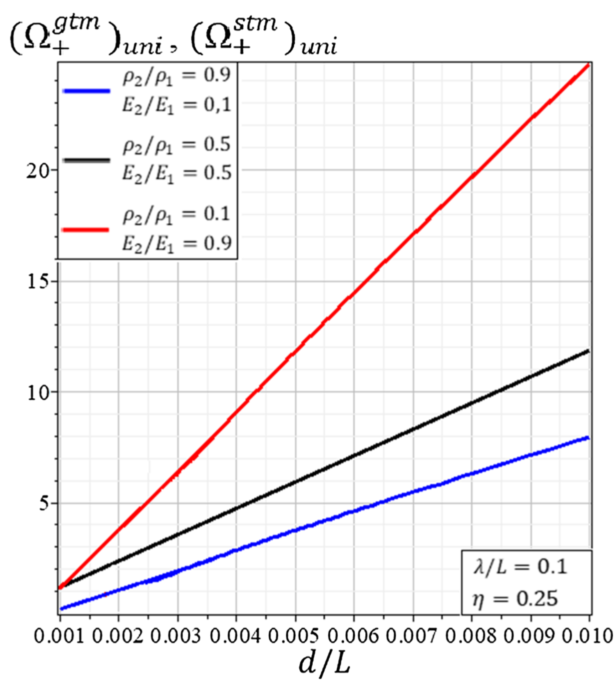

Fig. 13 Diagrams of dimensionless higher free vibration frequencies $(\Omega_+^{gtm})_{uni}$ (34), $(\Omega_+^{stm})_{uni}$ (36) versus ratio d/L, made for $\eta = 0.25$, $\lambda/L = 0.1$ and for three pairs of ratios: ($E_2/E_1 = 0.1$, $\rho_2/\rho_1 = 0.9$), ($E_2/E_1 = 0.5$, $\rho_2/\rho_1 = 0.5$), ($E_2/E_1 = 0.9$, $\rho_2/\rho_1 = 0.1$)

a posteriori, and it imposes certain restrictions on the class of problems described by the general tolerance model proposed here.

The general model formulated here includes a bigger number of terms depending on a microstructure size than *the known standard one* derived by Tomczyk in [10] and recalling here by means of (15)–(17). Thus, from the theoretical results it follows that the general model makes it possible to analyse dispersion phenomena in more detail.

As illustrative example, a special length-scale dynamic problem was discussed in the framework of general tolerance model (12)–(14). This problem dealt with investigations of transversal free vibrations of a simply supported shell strip made of two component materials periodically and densely distributed in circumferential direction. The results were compared with the corresponding results obtained from standard tolerance model (15)–(17) and from asymptotic one (19), which neglects the length-scale effect. It was shown that in the framework of the general and standard tolerance models, not only *the fundamental cell-independent lower* (25), (29), but also *the new additional cell-dependent higher* (26), (30) *free vibration frequencies* can be determined and analysed. The higher free vibration frequencies, caused by a periodic structure of the shell strip, cannot be determined using the asymptotic model. Formulae for free vibration frequencies derived in the framework of the general tolerance model contain a bigger number of terms depending on a cell size than those obtained from the standard tolerance model. However, from the computational results it follows that the differences between values of *fundamental lower free vibration frequency* (29) derived from the standard tolerance model and values of *fundamental lower free vibration frequency* (25) obtained from the general tolerance one are negligibly small. Moreover, from the computational results it follows that the differences between values of *fundamental lower free vibration frequencies* (25), (29) derived from the general and standard tolerance models, respectively, and values of *free vibration frequency* (32) obtained from the asymptotic one are also negligibly small. It means that from the computational point of view, *the lower free vibration frequencies* (25), (29) are independent of a microstructure size. It also means, the effect of the periodicity cell size on the *fundamental lower free vibration frequencies* of the shell strip under consideration can be neglected. Hence, *the asymptotic model* (19) *being more simple than the tolerance non-asymptotic ones is sufficient from the point of view of calculations made for the dynamic problem under consideration*. From the computational results, it also follows that the differences between values of *higher cell-dependent free vibration frequency* (26) derived from the general tolerance model and values of *higher cell-dependent free vibration frequency* (30) obtained from the standard tolerance one are negligibly small. Thus, in order to determine and investigate these cell-dependent frequencies, the standard tolerance model (15)–(17), being more simple than the general tolerance one (12)–(14), can be applied. Values of the lower and higher free vibration frequencies derived from the general or standard tolerance models or from the asymptotic ones increase with the decreasing of differences between elastic properties of the shell component materials and decrease with decreasing differences between inertial properties of the shell component materials. *The highest values of lower and higher free vibration frequencies are obtained for a shell strip with a very strong inertial heterogeneity and with elastic homogeneous structure*. *The smallest values of these free vibration frequencies are obtained for a shell strip with a very strong elastic heterogeneity and with inertial homogeneous structure*. Values of *the lower and higher free vibration frequencies increase with the decreasing of differences between the shell thickness and microstructure length parameter* λ; *the biggest values of these frequencies are obtained for a shell strip with a very strong inertial heterogeneity and with very weak elastic inhomogeneity. The higher free vibration frequencies* obtained from the general or standard tolerance models *decrease with the increasing of values of dimensionless microstructure length parameter* $\varepsilon \equiv \lambda/L_1 \in [0.01,\ 0.1]$*, i.e. with the decreasing of differences between period length* λ *and the length dimension* L_1 *of the shell midsurface in circumferential direction.* The strongest decrease in the values of the higher free vibration frequencies takes place for $\varepsilon \equiv \lambda/L_1 \in [0.01,\ 0.02]$.

Some other applications of the general tolerance model proposed here to the analysis of dynamic problems for cylindrical shells with one-directional micro-periodic structure will be shown in the forthcoming papers.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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