Continuum Mech. Thermodyn. (2021) 33:653–660
https://doi.org/10.1007/s00161-020-00937-6

ORIGINAL ARTICLE

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Length-scale effect in stability problems for thin biperiodic cylindrical shells: extended tolerance modelling

Received: 20 September 2020 / Accepted: 7 October 2020 / Published online: 22 October 2020

Abstract Thin linearly elastic Kirchhoff–Love-type circular cylindrical shells of periodically micro-inhomogeneous structure in circumferential and axial directions (*biperiodic* shells) are investigated. The aim of this contribution is to formulate and discuss a new averaged nonasymptotic model for the analysis of selected stability problems for these shells. This, so-called, *general nonasymptotic tolerance model* is derived by applying *a certain extended version of the known tolerance modelling procedure*. Contrary to the starting exact shell equations with highly oscillating, noncontinuous and periodic coefficients, governing equations of the tolerance model have constant coefficients depending also on a cell size. Hence, the model makes it possible to investigate the effect of a microstructure size on the global shell stability (*the length-scale effect*).

Keywords Micro-heterogeneous biperiodic cylindrical shells · Extended tolerance modelling · Stability problems · Length-scale effect

1 Introduction

Thin linearly elastic Kirchhoff–Love-type circular cylindrical shells with a periodically micro-inhomogeneous structure in circumferential and axial directions (*biperiodic shells*) are objects of consideration. By periodic inhomogeneity we shall mean periodically varying thickness and/or periodically varying inertial and elastic properties of the shell material. We restrict our consideration to those biperiodic cylindrical shells, which are

Communicated by Andreas Öchsner.

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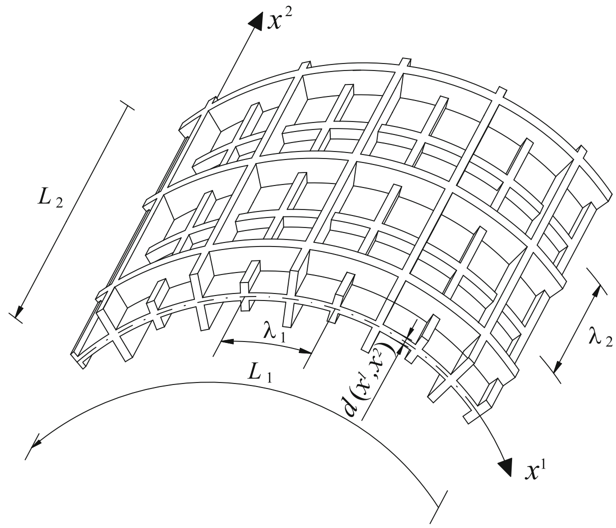

Fig. 1 Fragment of the shell reinforced by two families of biperiodically spaced ribbs

composed of a large number of identical elements. Moreover, every such element, called *a periodicity cell*, can be treated as a thin shell. Typical examples of such shells are presented in Fig. 1 (stiffened shell).

The mechanical problems of periodic structures (shells, plates, beams) are described by partial differential equations with periodic, highly oscillating and discontinuous coefficients. Thus, these equations are too complicated to constitute the basis for investigations of most of the engineering problems. To obtain averaged equations with constant coefficients, many different approximate modelling methods for structures of this kind have been formulated. Periodic cylindrical shells (plates) are usually described using *homogenized models* derived by means of *asymptotic methods*. Unfortunately, the asymptotic procedures are usually restricted to the first approximation, which leads to homogenized models neglecting *the effect of a periodicity cell size* (called *the length-scale effect*) on the overall shell behaviour. The mathematical foundations of this modelling technique can be found in Bensoussan et al. [3] and Jikov et al. [7]. Applications of the asymptotic homogenization procedure to modelling of stationary and nonstationary phenomena for micro-heterogeneous shells (plates) are presented in a large number of contributions. From the extensive list on this subject we can mention monographs by Lewiński and Telega [9] and Andrianov et al. [1].

The length-scale effect can be taken into account using *the nonasymptotic tolerance averaging technique*. This technique is based on the concept of *the tolerance relations* related to the accuracy of the performed measurements or calculations and determined by *tolerance parameters*. The mathematical foundations of this modelling technique can be found in Woźniak and Wierzbicki [23], Woźniak et al. [22] and Ostrowski [10]. For periodic structures, *governing equations of the tolerance models have constant coefficients dependent also on a cell size*. Some applications of this averaging method to the modelling of mechanical and thermomechanical problems for various periodic structures are shown in many works. We can mention here monograph by Tomczyk [12] and papers by Tomczyk and Litawska [16], Tomczyk et al. [13,14], where the length-scale effect in dynamics and stability of micro-periodic cylindrical shells is investigated; paper by Baron [2], where stability problems of medium thickness periodic plates are studied; papers by Jędrysiak [5,6], which deal with stability of thin periodic plates; paper by Tomczyk and Gołąbczak [15], where thermoelasticity problems for periodic shells are analysed. The extended list of references on this subject can be found in [10,12,22,23].

The tolerance averaging technique was also adopted to formulate mathematical models for the analysis of various mechanical and thermomechanical problems *for functionally graded solids*, e.g. for heat conduction in longitudinally graded hollow cylinder by Ostrowski and Michalak [11], for dynamics of transversally graded thin cylindrical shells by Tomczyk and Szczerba [17–19], for stability of longitudinally graded thin cylindrical shells by Tomczyk and Szczerba [20].

In the tolerance modelling technique the crucial role plays the concept of *slowly varying functions*, cf. [10,22,23]. They are functions which can be treated as constant on a cell. Moreover, the products of their derivatives in periodicity directions and characteristic length dimension of the cell are treated as negligibly small. *A certain extended version of the tolerance modelling technique has been proposed by* Tomczyk and

Woźniak [21]. This version is based on a new notion of *weakly slowly varying functions* which is an extension of the classical concept of *slowly varying functions*.

The main aim of this contribution is to formulate and discuss *a new mathematical averaged general tolerance model for the analysis of selected stability problems for the biperiodic cylindrical shells under consideration.* This model makes it possible to investigate stationary stability and dynamic stability as well as parametric vibrations. Contrary to the starting exact equations of the shell stability with periodic, highly oscillating and discontinuous coefficients, *governing equations of the proposed averaged model have constant coefficients depending also on a cell length dimensions*. In order to derive this model we shall apply *the extended tolerance modelling procedure* [21]. Similarities and differences between *the general tolerance model* proposed here and the corresponding *known standard tolerance model* formulated by Tomczyk [12] and derived by applying the more restrictive concept of *slowly varying function* will be discussed. Moreover, a certain asymptotic model will be presented.

It has to be emphasized that the general tolerance model of stability problems for thin linearly elastic Kirchhoff–Love-type circular cylindrical shells having a periodically micro-inhomogeneous structure in the circumferential direction and a constant structure in the axial direction (*uniperiodic shells*), which is proposed by Tomczyk et al. [13], cannot be applied to the analysis of stability problems for *biperiodic shells* considered here.

2 Formulation of the problem: starting equations

We assume that x^1 and x^2 are coordinates parametrizing the shell midsurface M in circumferential and axial directions, respectively. We denote $\mathbf{x} \equiv (x^1, x^2) \in \Omega \equiv (0, L_1) \times (0, L_2)$, where L_1, L_2 are length dimensions of M, cf. Fig. 1. Let $O\,\bar{x}^1\bar{x}^2\bar{x}^3$ stand for a Cartesian orthogonal coordinate system in the physical space E^3 and denote $\bar{\mathbf{x}} \equiv (\bar{x}^1, \bar{x}^2, \bar{x}^3)$. Let us introduce the orthonormal parametric representation of the underformed cylindrical shell midsurface M by means of $M \equiv \left\{ \bar{\mathbf{x}} \in R^3 : \bar{\mathbf{x}} = \bar{\mathbf{r}}\left(x^1, x^2\right),\ \left(x^1, x^2\right) \in \Omega \right\}$, where $\bar{\mathbf{r}}(\cdot)$ is the smooth invertible function such that $\partial\,\bar{\mathbf{r}}/\partial x^1 \cdot \partial\,\bar{\mathbf{r}}/\partial x^2 = 0$, $\partial\,\bar{\mathbf{r}}/\partial x^1 \cdot \partial\,\bar{\mathbf{r}}/\partial x^1 = 1$, $\partial\,\bar{\mathbf{r}}/\partial x^2 \cdot \partial\,\bar{\mathbf{r}}/\partial x^2 = 1$. Note that derivative $\partial\,\bar{\mathbf{r}}/\partial x^\alpha$, $\alpha = 1, 2$, should be understood as differentiation of each component of $\bar{\mathbf{r}} \in E^3$, i.e. $\partial\,\bar{\mathbf{r}}/\partial x^\alpha = [\partial\bar{r}^1/\partial x^\alpha, \partial\bar{r}^2/\partial x^\alpha, \partial\bar{r}^3/\partial x^\alpha]$.

Throughout the paper, indices $\alpha, \beta, \ldots$ run over 1, 2 and are related to midsurface parameters x^1, x^2, summation convention holds. Partial differentiation related to x^α is represented by ∂_α, $\partial_\alpha = \partial/\partial x^\alpha$. Moreover, it is denoted $\partial_{\alpha\ldots\delta} \equiv \partial_\alpha \ldots \partial_\delta$. Let $a_{\alpha\beta}$ and $a^{\alpha\beta}$ stand for the covariant and contravariant midsurface first metric tensors, respectively. Denote by $b_{\alpha\beta}$ the covariant midsurface second metric tensor. Under orthonormal parametrization introduced on M, $a_{\alpha\beta} = a^{\alpha\beta}$ are unit tensors, and components of tensor $b_{\alpha\beta}$ are: $b_{22} = b_{12} = b_{21} = 0$, $b_{11} = -r^{-1}$. We denote $t \in I = [t_0, t_1]$ as the time coordinate. Let $d(\mathbf{x})$, r stand for the shell thickness and the midsurface curvature radius, respectively.

The basic cell Δ and an arbitrary cell $\Delta(\mathbf{x})$ with the centre at point $\mathbf{x} \in \Omega_\Delta$ are defined by means of: $\Delta \equiv [-\lambda_1/2,\ \lambda_1/2] \times [-\lambda_2/2,\ \lambda_2/2]$, $\Delta(\mathbf{x}) \equiv \mathbf{x} + \Delta$, $\Omega_\Delta \equiv \{\mathbf{x} \in \Omega : \Delta(\mathbf{x}) \subset \Omega\}$, where λ_1 and λ_2 are the period lengths of the shell structure, respectively, in x^1- and x^2-directions. The diameter $\lambda \equiv \sqrt{(\lambda_1)^2 + (\lambda_2)^2}$ of Δ, called *the microstructure length parameter*, is assumed to satisfy conditions: $\lambda/\sup d(\mathbf{x}) >> 1$ for every $\mathbf{x} \in \Omega$, $\lambda/r << 1$ and $\lambda/\min(L_1, L_2) << 1$.

Setting $\mathbf{z} \equiv (z^1, z^2) \in [-\lambda_1/2,\ \lambda_1/2] \times [-\lambda_2/2,\ \lambda_2/2]$, we assume that the cell Δ has two symmetry axes: for $z^1 = 0$ and $z^2 = 0$. It is also assumed that inside the cell not only the geometrical but also elastic and inertial properties of the shell are described by symmetric (i.e. even) functions of $\mathbf{z}$.

Denote by $u_\alpha = u_\alpha(\mathbf{x}, t)$, $w = w(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, $t \in I$, the shell displacements in directions tangent and normal to M, respectively. Elastic properties of the shell are described by shell stiffness tensors $D^{\alpha\beta\gamma\delta}(\mathbf{x})$, $B^{\alpha\beta\gamma\delta}(\mathbf{x})$. Let $\mu(\mathbf{x})$ stand for a shell mass density per midsurface unit area. We denote by $\overline{N}^{\alpha\beta}(t)$ the time-dependent compressive membrane forces.

The considerations will be based on the well-known linear Kirchhoff–Love second-order theory of thin elastic shells, cf. Brush and Almroth [4], Kaliski [8], governed by the following dynamic equilibrium equations

$$\begin{aligned}&\partial_\beta\left(D^{\alpha\beta\gamma\delta}\partial_\delta u_\gamma\right) + r^{-1}\partial_\beta\left(D^{\alpha\beta 11}w\right) - \mu a^{\alpha\beta}\ddot{u}_\beta = 0,\\&r^{-1}D^{\alpha\beta 11}\partial_\beta u_\alpha + \partial_{\alpha\beta}\left(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w\right) + r^{-2}D^{1111}w + \bar{N}^{\alpha\beta}\partial_{\alpha\beta}w + \mu\ddot{w} = 0.\end{aligned} \tag{1}$$

For periodic shells, coefficients $D^{\alpha\beta\gamma\delta}(\mathbf{x})$, $B^{\alpha\beta\gamma\delta}(\mathbf{x})$, $\mu(\mathbf{x})$ in Eq. (1) are periodic, highly oscillating and noncontinuous functions in $\mathbf{x}$. That is why, in the most cases it is impossible to obtain the exact analytical solutions to initial/boundary value problems for equations (1) and also numerical problems for these equations are ill conditioned. Applying *the extended tolerance modelling technique* proposed in [21] to Eq. (1), we will derive the averaged *general tolerance model* for the analysis of stationary and dynamic stability problems for the biperiodic shells considered here. Governing equations of this model have constant coefficients depending also on a microstructure size.

To make this paper self-consisted, in the subsequent section we shall outline the main concepts and the fundamental assumptions of *the extended tolerance modelling procedure*, which in the general form are given in [10,21–23].

3 Concepts and assumptions of the extended tolerance modelling technique

The fundamental concepts of the tolerance modelling approach under consideration are those of *two tolerance relations between points and real numbers determined by tolerance parameters, weakly slowly varying functions, tolerance-periodic functions, fluctuating shape functions and the averaging operation*. Some of them are recalled below.

Let F be a function defined in $\bar{\Omega} \equiv [0, L_1] \times [0, L_2]$, which is continuous, bounded and differentiable in $\bar{\Omega}$ together with its gradients up to the Rth order. Nonnegative integer R is assumed to be specified in every problem under consideration. Let $\delta \equiv (\lambda, \delta_0, \delta_1, \ldots, \delta_R)$ be the set of tolerance parameters. The first of them represents the distances between points in $\bar{\Omega}$. The second one and the kth one, $k = 1, 2, \ldots, R$, are related, respectively, to the absolute differences between the values of function $F(\cdot)$ and its gradient $\partial_\alpha^k F(\cdot)$, $\alpha = 1, 2$, in points $\mathbf{x}, \mathbf{y}$ belonging to $\bar{\Omega}$ such that $|\mathbf{x} - \mathbf{y}| \leq \lambda$. A function $F(\cdot)$ is said to be *weakly slowly varying of the Rth kind* with respect to cell Δ and tolerance parameters δ, $F \in WSV_\delta^R(\Omega, \Delta)$, if and only if the following condition holds

$$(\forall(\mathbf{x}, \mathbf{y}) \in \Omega^2)\left[(\mathbf{x} \overset{\lambda}{\approx} \mathbf{y}) \Rightarrow F(\mathbf{x}) \overset{\delta_0}{\approx} F(\mathbf{y}) \;\; \text{and} \;\; \partial_\alpha^k F(\mathbf{x}) \overset{\delta_k}{\approx} \partial_\alpha^k F(\mathbf{y}), \quad k = 1, 2, \ldots, R, \quad \alpha = 1,\, 2\right], \tag{2}$$

where symbols $\overset{\lambda}{\approx}$ and $\overset{\delta_0}{\approx}$ (or $\overset{\delta_k}{\approx}$) denote tolerance relations between points and real numbers, respectively.

Roughly speaking, *the weakly slowly varying function* can be treated (together with its gradients up to the Rth order) as constant on an arbitrary cell. Note that the main difference between *the weakly slowly varying* and the well-known *slowly varying functions* occurring in the classical tolerance modelling is that *the products of derivatives of weakly slowly varying functions and microstructure length parameter* λ *are not negligibly small*.

An essentially bounded and weakly differentiable function φ defined in $\bar{\Omega} \equiv [0, L_1] \times [0, L_2]$ is called *tolerance-periodic of the Rth kind* with respect to cell Δ and tolerance parameters δ, $\varphi \in TP_\delta^R(\Omega, \Delta)$, if it can be treated (together with its gradients up to the *Rth* order) as periodic on an arbitrary cell.

Let f be a function defined in $\bar{\Omega} \equiv [0, L_1] \times [0, L_2]$, which is integrable and bounded in every cell $\Delta(\mathbf{x})$, $\mathbf{x} \in \Omega_\Delta$. *The averaging operation of* $f(\cdot)$ is defined by

$$\langle f \rangle(\mathbf{x}) \equiv \frac{1}{|\Delta|} \int\limits_{\Delta(\mathbf{x})} f(\mathbf{z}) \mathrm{d}\mathbf{z}, \quad \mathbf{x} \in \Omega_\Delta. \tag{3}$$

It should be noted that if f is a Δ-periodic function, then $\langle f \rangle$ is constant, but if f is a tolerance-periodic function, then $\langle f \rangle(\mathbf{x})$ is a slowly varying function with respect to $\mathbf{x}$.

Let h be λ-periodic, highly oscillating and continuous function in $\bar{\Omega}$ having continuous gradients $\partial^k h$, $k = 1, \ldots, R - 1$, and continuous or a piecewise continuous bounded gradient $\partial^R h$. Function $h(\cdot)$ will be called *the fluctuation shape function of the Rth kind,* $h \in FS^R(\Omega, \Delta)$, if it satisfies conditions: $h \in O(\lambda^R)$, $\partial_\alpha^k h \in O(\lambda^{R-k})$, $k = 1, 2, \ldots, R$, $\alpha = 1,\, 2$, $\langle \mu h \rangle = 0$, where μ is a shell mass density.

The tolerance modelling under consideration is based on three assumptions. The first of them is termed *the tolerance averaging approximation*. The second one is called *the micro–macro-decomposition*. The third one is termed *the residual orthogonality assumption*.

Let f be an arbitrary integrable and periodic or tolerance-periodic function defined in $\bar{\Omega}$, and let $F \in WSV_\delta^1(\Omega, \Delta)$. *The tolerance averaging approximation has the form*

$$\langle f\, F\rangle(\mathbf{x}) = \langle f\rangle(\mathbf{x})F(\mathbf{x}) + O(\delta),$$
$$\langle f\, \partial_\alpha^k F\rangle(\mathbf{x}) = \langle f\rangle(\mathbf{x})\,\partial_\alpha^k F(\mathbf{x}) + O(\delta), \quad k = 1, \ldots, R, \quad \alpha = 1,\ 2. \tag{4}$$

In the course of modelling, terms $O(\delta)$ are neglected.

The second fundamental assumption, called *the micro–macro-decomposition*, states that the displacement fields occurring in the starting equations under consideration can be decomposed into *macroscopic and microscopic parts*. The macroscopic part is represented by *unknown averaged displacements* being weakly slowly varying functions in periodicity directions. The microscopic part is described by *the known highly oscillating periodic fluctuation shape functions multiplied by unknown displacement fluctuation amplitudes* weakly slowly varying with respect to $\mathbf{x}$. Note that in the classical tolerance approach, *the weakly slowly varying functions* are replaced by *the slowly varying functions*.

The third fundamental assumption, called *the residual orthogonality assumption*, states that for micro–macro decomposition mentioned above, the governing equations of the exact shell theory under consideration do not hold, i.e. there exist residual fields which have to satisfy certain orthogonality conditions.

4 General tolerance model

4.1 Modelling procedure

In the problem discussed here, *the micro–macro-decomposition* of displacements $u_\alpha = u_\alpha(\mathbf{x}, t) \in TP_\delta^1(\Omega, \Delta)$, $w = w(\mathbf{x}, t) \in TP_\delta^2(\Omega, \Delta)$, $\mathbf{x} \in \Omega$, $t \in I$, being unknowns of Eq. (1), is assumed in the form

$$u_\alpha(\mathbf{x}, t) = u_\alpha^0(\mathbf{x}, t) + h(\mathbf{x})U_\alpha(\mathbf{x}, t),$$
$$w(\mathbf{x}, t) = w^0(\mathbf{x}, t) + g(\mathbf{x})W(\mathbf{x}, t), \tag{5}$$

where $u_\alpha^0,\ U_\alpha,\ w^0,\ W$ are *weakly slowly varying functions* with respect to argument $\mathbf{x} \in \Omega$, i.e.

$$u_\alpha^0,\ U_\alpha \in WSV_\delta^1(\Omega, \Delta), \quad w^0,\ W \in WSV_\delta^2(\Omega, \Delta). \tag{6}$$

Macrodisplacements u_α^0, w^0 as well as *displacement fluctuation amplitudes* U_α, W are *the new unknowns*.

Fluctuation shape functions $h(\cdot) \in FS^1(\Omega, \Delta)$, $g(\cdot) \in FS^2(\Omega, \Delta)$ are the known, λ-periodic, continuous and highly oscillating functions. They have to satisfy conditions: $h \in O(\lambda)$, $\lambda\partial_\alpha h \in O(\lambda)$, $g \in O(\lambda^2)$, $\lambda\partial_\alpha g \in O(\lambda^2)$, $\lambda^2\partial_{\alpha\beta} g \in O(\lambda^2)$, $\langle\mu\, h\rangle = \langle\mu\, g\rangle = 0$, where $\mu(\cdot)$ is a shell mass density. In the special case $\mu = \text{const}$, the fluctuations shape functions satisfy conditions $\langle h\rangle = \langle g\rangle = 0$. Taking into account that inside the cell the geometrical, elastic and inertial properties of the periodic shell under consideration are described by symmetric (i.e. even) functions of argument $\mathbf{z} \in \Delta(\mathbf{x})$, we assume that $h(\cdot)$ is either even or odd function of $\mathbf{z}$. The same restriction is imposed on function $g(\cdot)$.

We substitute the right-hand sides of (5) into (1). For decomposition (5), governing equation (1) does not hold, i.e. there exist residual fields defined by

$$p^\alpha \equiv \partial_\beta\left(D^{\alpha\beta\gamma\delta}\partial_\delta\left(u_\gamma^0 + hU_\gamma\right)\right) + r^{-1}\partial_\beta\left(D^{\alpha\beta 11}\left(w^0 + gW\right)\right)$$
$$-\mu a^{\alpha\beta}\left(\ddot{u}_\beta^0 + h\ddot{U}_\beta\right),$$
$$p \equiv r^{-1}D^{\alpha\beta 11}\partial_\beta\left(u_\alpha^0 + hU_\alpha\right) + \partial_{\alpha\beta}\left(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}(w^0 + gW)\right)$$
$$+r^{-2}D^{1111}(w^0 + gW) + \bar{N}^{\alpha\beta}\partial_{\alpha\beta}(w^0 + gW) + \mu(\ddot{w}^0 + g\ddot{W}). \tag{7}$$

Following [21], we introduce *the residual orthogonality assumption* which states that residual fields (7) have to satisfy the following orthogonality conditions

$$\langle p^\alpha\rangle = 0, \quad \langle p^\alpha h\rangle = 0, \quad \langle p\rangle = 0, \quad \langle pg\rangle = 0, \tag{8}$$

for almost every $\mathbf{x} \in \Omega$ and every $t \in I$. Averaging operation $\langle\cdot\rangle$ on cell Δ is defined by (3).

Conditions (8), on the basis of *tolerance averaging approximation* (4) lead to the system of averaged equations for unknowns u_α^0, w^0, U_α, W being *weakly slowly varying functions in periodicity directions*. Under extra approximation $1 + \lambda/r \approx 1$, this system can be written in the form of:

- the constitutive equations

$$
\begin{aligned}
N^{\alpha\beta} &= \langle D^{\alpha\beta\gamma\delta}\rangle \partial_\delta u^0_\gamma + r^{-1}\langle D^{\alpha\beta 11}\rangle w^0 + \langle D^{\alpha\beta\gamma\delta}\partial_\delta h\rangle U_\gamma \ + \underline{\underline{\langle D^{\alpha\beta\gamma\delta} h\rangle}}\, \partial_\delta U_\gamma,\\
M^{\alpha\beta} &= \langle B^{\alpha\beta\gamma\delta}\rangle \partial_{\gamma\delta} w^0 + \langle B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta} g\rangle W + 2\underline{\underline{\langle B^{\alpha\beta\gamma\delta}\partial_\delta g\rangle}}\, \partial_\gamma W + \underline{\underline{\langle B^{\alpha\beta\gamma\delta} g\rangle}}\, \partial_{\gamma\delta} W,\\
H^{\beta} &= \langle \partial_\alpha h\, D^{\alpha\beta\gamma\delta}\rangle \partial_\delta u^0_\gamma - \underline{\langle h\, D^{\alpha\beta\gamma\delta}\rangle}\, \partial_{\alpha\delta} u^0_\gamma + \langle D^{\alpha\beta\gamma\delta}\partial_\alpha h\, \partial_\delta h\rangle U_\gamma\\
&\quad - \underline{\underline{\langle D^{\alpha\beta\gamma\delta}(h)^2\rangle}}\, \partial_{\alpha\delta} U_\gamma + r^{-1}\langle \partial_\alpha h D^{\alpha\beta 11}\rangle w^0,\\
G &= \langle \partial_{\gamma\delta} g\, B^{\alpha\beta\gamma\delta}\rangle\, \partial_{\alpha\beta} w^0 - 2\underline{\langle \partial_\delta g\, B^{\alpha\beta\gamma\delta}\rangle}\, \partial_{\alpha\beta\gamma} w^0\\
&\quad + \underline{\langle g\, B^{\alpha\beta\gamma\delta}\rangle}\, \partial_{\alpha\beta\gamma\delta} w^0 \ + \langle \partial_{\alpha\beta} g B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta} g\rangle\, W + (2\underline{\underline{\langle \partial_{\gamma\delta} g\, B^{\alpha\beta\gamma\delta} g\rangle}}\\
&\quad - 4\underline{\underline{\langle \partial_\gamma g B^{\alpha\beta\gamma\delta}\partial_\delta g\rangle}})\, \partial_{\alpha\beta} W + \underline{\underline{\langle (g)^2 B^{\alpha\beta\gamma\delta}\ \rangle}}\, \partial_{\alpha\beta\gamma\delta} W,
\end{aligned}
\tag{9}
$$

- the dynamic equilibrium equations

$$
\begin{aligned}
&\partial_\beta N^{\alpha\beta} - \langle\mu\rangle a^{\alpha\beta}\ddot{u}^0_\beta = 0,\\
&\partial_{\alpha\beta} M^{\alpha\beta} + r^{-1}N^{11} + \bar{N}^{\alpha\beta}(t)\left[\partial_{\alpha\beta} w^0 + \underline{\underline{\langle g\rangle}}\, \partial_{\alpha\beta} W\right] + \langle\mu\rangle \ddot{w}^0 = 0,\\
&\underline{\langle \mu\ (h)^2\rangle}\, a^{\alpha\beta}\, \ddot{U}_\alpha + H^\beta \ = 0,\\
&\underline{\langle \mu\ (g)^2\ \rangle}\, \ddot{W} + G - \bar{N}^{\alpha\beta}(t)\underline{\langle \partial_\alpha g\, \partial_\beta g\rangle}\, W\\
&\quad + \bar{N}^{\alpha\beta}(t)\left[\underline{\underline{\langle (g)^2\rangle}}\, \partial_{\alpha\beta} W + \underline{\underline{\langle g\rangle}}\, \partial_{\alpha\beta} w^0\right] = 0.
\end{aligned}
\tag{10}
$$

In Eqs. (9), (10) the singly and doubly underlined terms depend on a cell size λ.

Equations (9), (10) together with *micro–macro-decomposition* (5) and *physical reliability conditions* (6) constitute *the general tolerance model of selected stability problems for the micro-heterogeneous biperiodic shells under consideration*. This model makes it possible to analyse stationary stability and dynamic stability as well as parametric vibrations.

5 Discussion of results

The characteristic features of the derived *general tolerance model* are:

- In contrast to exact stability shell equations (1) with periodic, discontinuous and highly oscillating coefficients, general tolerance model equations (9), (10) proposed here *have constant coefficients*. Moreover, *some of them depend on a period length* λ (underlined terms). Hence, the tolerance model makes it possible to *describe the effect of a microstructure size on the global stability shell behaviour*.
- Unknown macrodisplacements u^0_α, w^0 and fluctuation amplitudes U_α, W of the general tolerance model equations must be *weakly slowly varying functions* in periodicity directions. *This requirement can be verified only a posteriori*, and *it determines the range of the physical applicability of the model.*
- The number and form of boundary/initial conditions for the basic unknowns of the tolerance model are the same as in the classical shell theory governed by equations (1).
- Decomposition (5) and hence also governing equations (9), (10) of the general tolerance model are uniquely determined by the given a priori highly oscillating periodic *fluctuations shape functions* $h(\cdot) \in FS^1(\Omega, \Delta)$, $h \in O(\lambda)$, $g(\cdot) \in FS^2(\Omega, \Delta)$, $g \in O(\lambda^2)$, which represent oscillations of displacement fields inside a cell. These functions can be obtained as exact or approximate solutions to certain periodic eigenvalue cell problems, cf. [12]. These functions can also be regarded as *the shape functions* resulting from the periodic discretization of the cell using, for example, the finite element method. The choice of these functions can be also based on the experience or intuition of the researcher.
- Neglecting in (9), (10) the underlined terms, we obtained the *asymptotic model* of the shells under consideration. This model is not able to describe the length-scale effect on the overall shell stability being independent of the cell size. It is necessary to observe that equations $(10)_{3,4}$ for the fluctuation amplitudes are linear algebraic equations now.

- Neglecting the doubly underlined terms in (9) and (10) we obtain constitutive relations and dynamic equilibrium equations of the known *standard tolerance model of biperiodic shells under consideration* proposed by Tomczyk [12]. This *standard model* was derived under assumption that the unknown functions $u^0_\alpha,\ w^0, U_\alpha,\ W$ in decomposition (5) are *slowly varying*. We recall that *the slowly varying functions* are a subclass of *the weakly slowly varying* functions. The main difference between *the weakly slowly varying* and the well-known *slowly varying functions* is that the products of derivatives of *weakly slowly varying functions* and *microstructure length parameter* λ *are not negligibly small*. From comparison of both *the general* and *the standard tolerance models it follows that the general model equations contain a bigger number of terms depending on the microstructure size than the standard model equations.* Thus, we can conclude that the general model proposed in the contribution allows us to investigate the length-scale effect in more detail. However, this conclusion must be confirmed by numerical results. It can be observed that within the framework of the general model, unknown fluctuation amplitudes U_α, W are governed by a system of *partial differential equations* $(10)_{3,4}$, whereas within the framework of the standard model these unknowns are governed by a system of *ordinary differential equations* involving only time derivatives. Hence, there are no extra boundary conditions for unknowns $U_\alpha,\ W$ of the standard model.
- For a homogeneous shell with a constant thickness, $D^{\alpha\beta\gamma\delta}(\mathbf{x})$, $B^{\alpha\beta\gamma\delta}(\mathbf{x})$, $\mu(\mathbf{x})$ are constant, and because $\langle\mu h\rangle = \langle\mu g\rangle = 0$, we obtain $\langle h\rangle = \langle g\rangle = 0$, and hence $\langle\partial_\alpha h\rangle = \langle\partial_\alpha g\rangle = \langle\partial_{\alpha\beta} g\rangle = 0$. In this case equations $(10)_{1,2}$ reduce to the well-known shell equations of motion for averaged displacements $u^0_\alpha(\mathbf{x}, t),\ w^0(\mathbf{x}, t)$, $(\mathbf{x}, t) \in \Omega \times I$. Independently, for fluctuation amplitudes $U_\alpha(\mathbf{x}, t),\ W(\mathbf{x}, t)$, $(\mathbf{x}, t) \in \Omega \times I$, we arrive at the system of equations, which under homogeneous initial conditions for U_α, W has only trivial solution $U_\alpha = W = 0$. Hence, from decomposition (5) it follows that $u_\alpha = u^0_\alpha,\ w = w^0$. It means that equations (9), (10) reduce to starting equations (1).
- After neglecting in (10) the stability terms, we obtain the general tolerance model, which makes it possible to investigate *only dynamic problems* for the micro-heterogeneous biperiodic shells under consideration. This model was formulated by Tomczyk and Litawska [16].

6 Final remarks

The tolerance modelling technique based on the notion of *the weakly slowly varying function*, cf. Tomczyk and Woźniak [21], is proposed as a tool to derive a new mathematical nonasymptotic averaged model for the analysis of selected stability problems for thin linearly elastic cylindrical shells with micro-periodic structure in circumferential and axial directions. Contrary to "exact" shell equations (1) with highly oscillating, non-continuous and periodic coefficients, tolerance model equations (9), (10) have constant coefficients depending also on a cell size. Hence, this model makes it possible *to describe the effect of a length scale on the global shell stability*.

Some applications of the new general tolerance model proposed here to the analysis of stability problems for cylindrical shells with two-directional micro-periodic structure will be shown in the forthcoming papers.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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