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Fractional heat conduction in solids connected by thin intermediate layer: nonperfect thermal contact

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Abstract We examine the transition region between two solids which state differs from the state of contacting media. Small thickness of the intermediate region allows us to reduce a three-dimensional problem to a two-dimensional one for a median surface endowed with equivalent physical properties. In the present paper, we consider the generalized boundary conditions of nonperfect thermal contact for the time-fractional heat conduction equation with the Caputo derivative and solve the problem for a composite medium consisting of two semi-infinite regions. Numerical results are illustrated graphically.

Keywords Fractional calculus · Caputo derivative · Mittag–Leffler function · Nonperfect thermal contact

1 Introduction

Near the interface between two solids, there arises a transition region which state differs from the state of contacting media owing to different conditions of material–particle interaction. The transition region has its own physical, mechanical and chemical properties, and processes occurring in it differ from those in the bulk. Small thickness of the intermediate region between two solids allows us to reduce a three-dimensional problem to a two-dimensional one for median surface endowed with equivalent physical properties. There are several approaches to reducing three-dimensional equations to the corresponding two-dimensional equations for the median surface. For example, introducing the mixed coordinate system (ξ, η, z) , where ξ and η are the curvilinear coordinates in the median surface and z is the normal coordinate, the linear or polynomial dependence of the considered functions on the normal coordinate can be assumed. This assumption is often used in the theory of elastic shells [1–8]. Similar models were elaborated taking into account elastic and elasto-plastic properties of interfaces considered as two-dimensional zero thickness objects (see, for example, [9–15] and references therein).

For the classical heat conduction equation, which is based on the conventional Fourier law, the reduction of the three-dimensional problem to the simplified two-dimensional one was proposed by Marguerre [16, 17] and later on developed by many authors (see [18–25], among others). In this case, the assumption on linear or polynomial dependence of temperature on the normal coordinate or more general operator method was used. It should be emphasized that numerical modeling of heat conduction in composites with thin interfaces is still a difficult numerical task, and different models of interface as infinitesimal layer described by specific

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transmission conditions including nonlinear effects [26,27] and thin interfaces with large curvature [28–30] still attract the attention of researchers.

Many experimental and theoretical studies testify that in media with complex internal structure the classical Fourier law and the conventional heat conduction equation are no longer sufficiently accurate. This results in formulation of nonclassical theories, in which the standard Fourier law and the parabolic heat conduction equation are replaced by more general equations. For example, the wave equation [31,32] and the telegraph equation [33,34] for temperature were used, the time-nonlocal [35,36] and space-nonlocal [37,38] generalizations of the Fourier law were studied, and the time-fractional and space-fractional generalizations of the heat conduction equation were investigated. The interested reader is referred to the books [39,40] and references therein.

For time-fractional heat conduction, the reduction of the three-dimensional equation to the two-dimensional one was carried out in [40–43]. In the present paper, we consider the generalized boundary conditions of nonperfect thermal contact for the time-fractional heat conduction equation with the Caputo derivative and solve the problem for a composite medium consisting of two semi-infinite regions. Numerical results are illustrated graphically.

2 Time-fractional heat conduction equation

The standard Fourier law

$$\mathbf{q} = -k \operatorname{grad} T \quad (1)$$

states the linear dependence between the heat flux vector \mathbf{q} and the temperature gradient with the proportionality coefficient k being the thermal conductivity of a body.

In combination with the law of conservation of energy

$$C \frac{\partial T}{\partial t} = -\operatorname{div} \mathbf{q}, \quad (2)$$

where C is the heat capacity, the Fourier law (1) results in the parabolic heat conduction equation

$$C \frac{\partial T}{\partial t} = k \Delta T. \quad (3)$$

Fractional calculus (the theory of integrals and derivatives of non-integer order) has many applications in physics, geophysics, rheology, mechanics, engineering, bioengineering, etc. [44–52].

The time-nonlocal dependence between the heat flux vector and the temperature gradient with the “long-tail” power kernel $K(t - \tau)$ [53–56]

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t - \tau)^{\alpha-1} \operatorname{grad} T(\tau) d\tau, \quad 0 < \alpha \leq 1, \quad (4)$$

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha - 1)} \int_0^t (t - \tau)^{\alpha-2} \operatorname{grad} T(\tau) d\tau, \quad 1 < \alpha \leq 2, \quad (5)$$

where $\Gamma(\alpha)$ is the gamma function, can be interpreted in terms of fractional integrals and derivatives

$$\mathbf{q}(t) = -k D_{\text{RL}}^{1-\alpha} \operatorname{grad} T(t), \quad 0 < \alpha \leq 1, \quad (6)$$

$$\mathbf{q}(t) = -k I^{\alpha-1} \operatorname{grad} T(t), \quad 1 < \alpha \leq 2, \quad (7)$$

and results in the time-fractional heat conduction equation with the Caputo fractional derivative

$$C \frac{\partial^\alpha T}{\partial t^\alpha} = k \Delta T, \quad 0 < \alpha \leq 2. \quad (8)$$

The details of deriving the time-fractional heat conduction Eq. (8) from the constitutive Eqs. (6), (7) and the law of conservation of energy (3) can be found in [56].

Here,

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0, \tag{9}$$

is the Riemann–Liouville fractional integral,

$$D_{RL}^\alpha f(t) = \frac{d^n}{dt^n} \left[\frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau \right], \quad n - 1 < \alpha < n, \tag{10}$$

is the Riemann–Liouville fractional derivative, and

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, \quad n - 1 < \alpha < n, \tag{11}$$

denotes the Caputo fractional derivative [57–59].

Recall the Laplace transform rules for fractional integrals and derivatives

$$\mathcal{L} \{ I^\alpha f(t) \} = \frac{1}{s^\alpha} f^*(s), \quad \alpha > 0, \tag{12}$$

$$\mathcal{L} \{ D_{RL}^\alpha f(t) \} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} \frac{d^k}{dt^k} I^{n-\alpha} f(0^+) s^{n-1-k}, \quad n - 1 < \alpha < n, \tag{13}$$

$$\mathcal{L} \left\{ \frac{d^\alpha f(t)}{dt^\alpha} \right\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad n - 1 < \alpha < n. \tag{14}$$

Here the asterisk denotes the transform, s is the Laplace transform variable.

Starting from the pioneering papers [60–62], the time-fractional heat conduction (diffusion) Eq. (8) has attracted considerable attention of researchers. The book [39] presents a picture of the state of the art of investigations in this area.

It should be emphasized that due to the generalized constitutive equations for the heat flux (6) and (7) the corresponding boundary conditions for the time-fractional heat conduction Eq. (8) differ from those for the standard heat conduction equation. Different kinds of boundary conditions for Eq. (8) were analyzed in [39,40,63,64].

The Dirichlet boundary condition specifies the temperature over the surface of the body

$$T \Big|_S = g(\mathbf{x}_S, t). \tag{15}$$

The physical Neumann boundary condition prescribes the boundary value of the heat flux

$$D_{RL}^{1-\alpha} \frac{\partial T}{\partial n} \Big|_S = g(\mathbf{x}_S, t), \quad 0 < \alpha \leq 1, \tag{16}$$

$$I^{\alpha-1} \frac{\partial T}{\partial n} \Big|_S = g(\mathbf{x}_S, t), \quad 1 < \alpha \leq 2, \tag{17}$$

where $\partial/\partial n$ denotes differentiation along the outward-drawn normal at the boundary surface S .

The convective heat exchange between a body and the environment is described by the boundary condition

$$\left(HT + k D_{RL}^{1-\alpha} \frac{\partial T}{\partial n} \right) \Big|_S = g(\mathbf{x}_S, t), \quad 0 < \alpha \leq 1, \tag{18}$$

$$\left(HT + k I^{\alpha-1} \frac{\partial T}{\partial n} \right) \Big|_S = g(\mathbf{x}_S, t), \quad 1 < \alpha \leq 2, \tag{19}$$

where H is the convective heat transfer coefficient.

Let heat conduction in two solids be described by the heat conduction Eq. (8) with the time-fractional derivative of the order α and β , respectively. If the surfaces of these two solids are in perfect thermal contact,

the temperatures on the contact surface and the heat fluxes through the contact surface are the same for both solids, and we obtain the corresponding boundary conditions:

$$T_1 \Big|_S = T_2 \Big|_S, \quad (20)$$

$$k_1 D_{\text{RL}}^{1-\alpha} \frac{\partial T_1}{\partial n} \Big|_S = k_2 D_{\text{RL}}^{1-\beta} \frac{\partial T_2}{\partial n} \Big|_S, \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 2, \quad (21)$$

where subscripts 1 and 2 refer to solids 1 and 2, respectively, and n is the common normal at the contact surface. For the sake of brevity, in (21) $D_{\text{RL}}^{-\alpha} f(t)$ with $\alpha > 0$ is understood as the Riemann–Liouville fractional integral $I^\alpha f(t)$ (see [46,58]).

3 The boundary conditions of nonperfect thermal contact

Consider a composite solid consisting of three domains: the domain 1, the domain 2, and the intermediate domain marked by the index 0. Heat conduction in each domain is described by the time-fractional heat conduction equation:

$$C_1 \frac{\partial^\alpha T_1}{\partial t^\alpha} = k_1 \Delta T_1 \quad \text{in the domain 1,} \quad (22)$$

$$C_2 \frac{\partial^\beta T_2}{\partial t^\beta} = k_2 \Delta T_2 \quad \text{in the domain 2,} \quad (23)$$

$$C_0 \frac{\partial^\gamma T_0}{\partial t^\gamma} = k_0 \Delta T_0 \quad \text{in the intermediate domain.} \quad (24)$$

At the boundary surfaces S_1 and S_2 between the solids and intermediate domain, the conditions of perfect thermal contact are assumed:

$$T_1 \Big|_{S_1} = T_0 \Big|_{S_1}, \quad (25)$$

$$k_1 D_{\text{RL}}^{1-\alpha} \frac{\partial T_1}{\partial n} \Big|_{S_1} = k_0 D_{\text{RL}}^{1-\gamma} \frac{\partial T_0}{\partial n} \Big|_{S_1}, \quad 0 < \alpha \leq 2, \quad 0 < \gamma \leq 2, \quad (26)$$

$$T_2 \Big|_{S_2} = T_0 \Big|_{S_2}, \quad (27)$$

$$k_2 D_{\text{RL}}^{1-\beta} \frac{\partial T_2}{\partial n} \Big|_{S_2} = k_0 D_{\text{RL}}^{1-\gamma} \frac{\partial T_0}{\partial n} \Big|_{S_2}, \quad 0 < \beta \leq 2, \quad 0 < \gamma \leq 2. \quad (28)$$

If the thickness of the intermediate layer is small with respect to two other sizes and is constant, the median surface Σ can be introduced (see Fig. 1). In this case, a three-dimensional equation problem in the intermediate layer can be reduced to a two-dimensional one for the median surface (see Fig. 2). For details, the interested reader is referred to [40]. As a result, we get the following boundary conditions of nonperfect thermal contact at the interface:

$$C_\Sigma \frac{\partial^\gamma (T_1 + T_2)}{\partial t^\gamma} = k_\Sigma \Delta_\Sigma (T_1 + T_2) + 2 \left(k_1 D_{\text{RL}}^{\gamma-\alpha} \frac{\partial T_1}{\partial z} \Big|_{z=0} - k_2 D_{\text{RL}}^{\gamma-\beta} \frac{\partial T_2}{\partial z} \Big|_{z=0} \right), \quad (29)$$

$$C_\Sigma \frac{\partial^\gamma (T_1 - T_2)}{\partial t^\gamma} = k_\Sigma \Delta_\Sigma (T_1 - T_2) + 6 \left(k_1 D_{\text{RL}}^{\gamma-\alpha} \frac{\partial T_1}{\partial z} \Big|_{z=0} + k_2 D_{\text{RL}}^{\gamma-\beta} \frac{\partial T_2}{\partial z} \Big|_{z=0} \right) - \frac{12}{R_\Sigma} (T_1 - T_2), \quad (30)$$

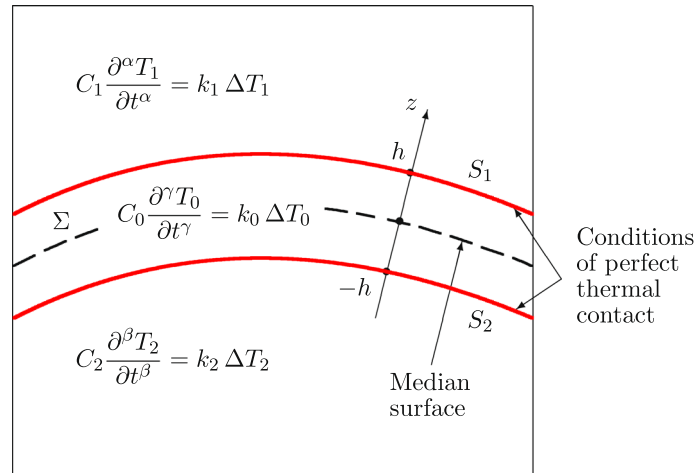


Fig. 1 Thin intermediate layer between two solids

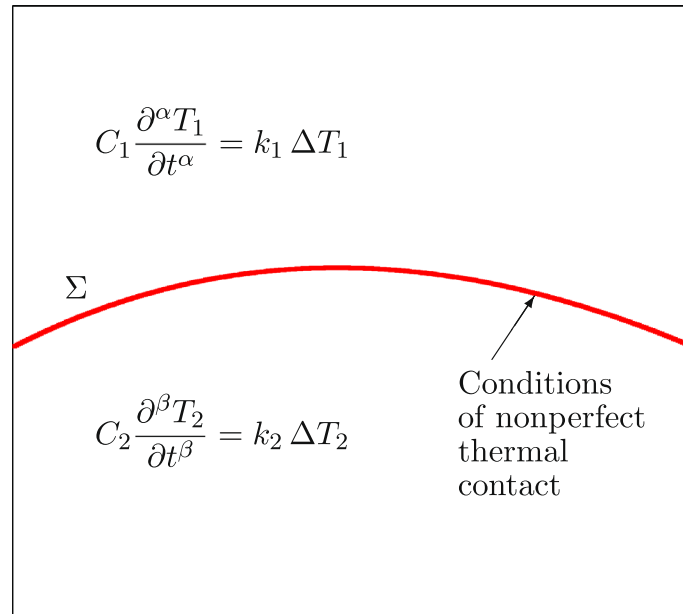


Fig. 2 A contact surface having its own thermophysical characteristics

where Δ_{Σ} is the surface Laplace operator,

$$C_{\Sigma} = 2hC_0, \quad k_{\Sigma} = 2hk_0, \quad R_{\Sigma} = \frac{k_0}{2h} \tag{31}$$

are the reduced heat capacity, reduced thermal conductivity and reduced thermal resistance of the median surface, respectively.

In the case of classical heat conduction ($\alpha = \beta = \gamma = 1$), Eqs. (29) and (30) coincide with the boundary conditions obtained by Podstrigach [22,25].

If the reduced thermal characteristics of the median surface are equal zero

$$C_{\Sigma} = 0, \quad k_{\Sigma} = 0, \quad R_{\Sigma} = 0, \tag{32}$$

then the conditions (29) and (30) reduce to the conditions of perfect thermal contact (20) and (21).

4 Statement and solution of the problem

Consider a composite medium consisting of two semi-infinite regions. Heat conduction in each region in the case of one spatial coordinate is described by the time-fractional heat conduction equation

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1}{\partial x^2}, \quad x > 0, \quad 0 < \alpha \leq 2, \quad (33)$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \frac{\partial^2 T_2}{\partial x^2}, \quad x < 0, \quad 0 < \beta \leq 2, \quad (34)$$

where $a_i = k_i/C_i$, $i = 1, 2$, are the thermal diffusivity coefficients, under the initial conditions

$$t = 0 : T_1 = f_1(x), \quad x > 0, \quad 0 < \alpha \leq 2, \quad (35)$$

$$t = 0 : \frac{\partial T_1}{\partial t} = F_1(x), \quad x > 0, \quad 1 < \alpha \leq 2, \quad (36)$$

$$t = 0 : T_2 = f_2(x), \quad x < 0, \quad 0 < \beta \leq 2, \quad (37)$$

$$t = 0 : \frac{\partial T_2}{\partial t} = F_2(x), \quad x < 0, \quad 1 < \beta \leq 2, \quad (38)$$

and the boundary conditions of nonperfect thermal contact

$$x = 0 : C_\Sigma \frac{\partial^\gamma (T_1 + T_2)}{\partial t^\gamma} = 2 \left(k_1 D_{\text{RL}}^{\gamma-\alpha} \frac{\partial T_1}{\partial x} - k_2 D_{\text{RL}}^{\gamma-\beta} \frac{\partial T_2}{\partial x} \right), \quad (39)$$

$$x = 0 : C_\Sigma \frac{\partial^\gamma (T_1 - T_2)}{\partial t^\gamma} = 6 \left(k_1 D_{\text{RL}}^{\gamma-\alpha} \frac{\partial T_1}{\partial x} + k_2 D_{\text{RL}}^{\gamma-\beta} \frac{\partial T_2}{\partial x} \right) - \frac{12}{R_\Sigma} (T_1 - T_2). \quad (40)$$

In the case of one spatial coordinate, the surface Laplace operator Δ_Σ equals zero. In what follows we restrict ourselves to the particular case when $\alpha = \beta = \gamma$, $R_\Sigma = 0$ and the region $x > 0$ is at initial uniform temperature T_0 and the region $x < 0$ is at initial zero temperature, i.e., we have the time-fractional heat conduction equations

$$\frac{\partial^\alpha T_1(x, t)}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1(x, t)}{\partial x^2}, \quad x > 0, \quad 0 < \alpha \leq 2, \quad (41)$$

$$\frac{\partial^\alpha T_2(x, t)}{\partial t^\alpha} = a_2 \frac{\partial^2 T_2(x, t)}{\partial x^2}, \quad x < 0, \quad 0 < \alpha \leq 2, \quad (42)$$

under the initial conditions

$$t = 0 : T_1(x, t) = T_0, \quad x > 0, \quad 0 < \alpha \leq 2, \quad (43)$$

$$t = 0 : \frac{\partial T_1(x, t)}{\partial t} = 0, \quad x > 0, \quad 1 < \alpha \leq 2, \quad (44)$$

$$t = 0 : T_2(x, t) = 0, \quad x < 0, \quad 0 < \alpha \leq 2, \quad (45)$$

$$t = 0 : \frac{\partial T_2(x, t)}{\partial t} = 0, \quad x < 0, \quad 1 < \alpha \leq 2, \quad (46)$$

and under the boundary conditions

$$x = 0 : T_1(x, t) = T_2(x, t), \quad (47)$$

$$x = 0 : C_\Sigma \frac{\partial^\alpha [T_1(x, t) + T_2(x, t)]}{\partial t^\alpha} = 2 \left[k_1 \frac{\partial T_1(x, t)}{\partial x} - k_2 \frac{\partial T_2(x, t)}{\partial x} \right]. \quad (48)$$

The conditions at infinity are also assumed

$$\left. \frac{\partial T_1(x, t)}{\partial x} \right|_{x \rightarrow \infty} = 0, \quad \left. \frac{\partial T_2(x, t)}{\partial x} \right|_{x \rightarrow -\infty} = 0. \quad (49)$$

The Laplace transform with respect to time t leads to two ordinary differential equations

$$s^\alpha T_1^* - s^{\alpha-1} T_0 = a_1 \frac{d^2 T_1^*}{dx^2}, \quad x > 0, \quad (50)$$

$$s^\alpha T_2^* = a_2 \frac{d^2 T_2^*}{dx^2}, \quad x < 0, \quad (51)$$

having the solutions satisfying the conditions at infinity (49)

$$T_1^* = \frac{1}{s} T_0 + B_1 \exp\left(-\sqrt{\frac{s^\alpha}{a_1}} x\right) \quad x > 0, \quad (52)$$

$$T_2^* = B_2 \exp\left(-\sqrt{\frac{s^\alpha}{a_2}} |x|\right), \quad x < 0. \quad (53)$$

The integration constants B_1 and B_2 are obtained from the boundary conditions of nonperfect thermal contact (47) and (48)

$$B_1 = -\frac{1}{1+\mu} \frac{T_0}{s} + \frac{1-\mu}{2(1+\mu)} T_0 \frac{s^{\alpha/2-1}}{s^{\alpha/2} + b}, \quad (54)$$

$$B_2 = \frac{\mu}{1+\mu} \frac{T_0}{s} + \frac{1-\mu}{2(1+\mu)} T_0 \frac{s^{\alpha/2-1}}{s^{\alpha/2} + b}, \quad (55)$$

where

$$\mu = \frac{k_1 \sqrt{a_2}}{k_2 \sqrt{a_1}}, \quad b = \frac{k_1 \sqrt{a_2} + k_2 \sqrt{a_1}}{C_\Sigma \sqrt{a_1 a_2}}. \quad (56)$$

Taking into account Eq. (A.3) from appendix A and Eq. (B.3) from Appendix B and the convolution theorem, we obtain the solution

$$T_1(x, t) = T_0 - \frac{T_0}{1+\mu} \int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a_1} \tau^{\alpha/2}}\right) \times \left\{ \frac{1}{\Gamma(\alpha/2)} - \frac{1-\mu}{2} E_{\alpha/2, \alpha/2}[-b(t-\tau)^{\alpha/2}] \right\} d\tau, \quad x > 0, \quad (57)$$

$$T_2(x, t) = \frac{T_0}{1+\mu} \int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{|x|}{\sqrt{a_2} \tau^{\alpha/2}}\right) \times \left\{ \frac{\mu}{\Gamma(\alpha/2)} + \frac{1-\mu}{2} E_{\alpha/2, \alpha/2}[-b(t-\tau)^{\alpha/2}] \right\} d\tau, \quad x < 0. \quad (58)$$

For the standard heat conduction equation ($\alpha = 1$) we get

$$T_1(x, t) = T_0 - \frac{T_0}{1+\mu} \operatorname{erfc}\left(\frac{x}{2\sqrt{a_1 t}}\right) + \frac{(1-\mu)T_0}{2(1+\mu)} \exp\left(\frac{bx}{\sqrt{a_1}} + b^2 t\right) \operatorname{erfc}\left(\frac{x}{2\sqrt{a_1 t}} + b\sqrt{t}\right), \quad x > 0, \quad (59)$$

$$T_2(x, t) = \frac{\mu T_0}{1+\mu} \operatorname{erfc}\left(\frac{|x|}{2\sqrt{a_2 t}}\right) + \frac{(1-\mu)T_0}{2(1+\mu)} \exp\left(\frac{b|x|}{\sqrt{a_2}} + b^2 t\right) \operatorname{erfc}\left(\frac{|x|}{2\sqrt{a_2 t}} + b\sqrt{t}\right), \quad x < 0, \quad (60)$$

where $\operatorname{erfc}(x)$ is the complementary error function.

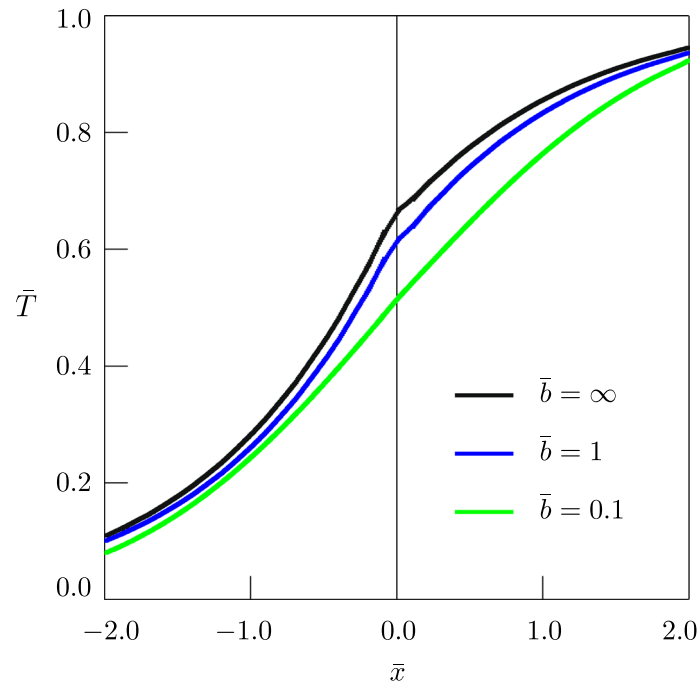


Fig. 3 Dependence of temperature on distance for $\alpha = 0.5$ and $\mu = 2$

Another particular case of the solution corresponds to the so-called ballistic heat conduction ($\alpha = 2$):

$$T_1(x, t) = \begin{cases} \frac{T_0}{1 + \mu} \left\{ \mu + \frac{1 - \mu}{2} \exp \left[-b \left(t - \frac{x}{\sqrt{a_1}} \right) \right] \right\}, & 0 \leq x < \sqrt{a_1}t, \\ T_0, & \sqrt{a_1}t < x < \infty, \end{cases} \quad (61)$$

$$T_2(x, t) = \begin{cases} \frac{T_0}{1 + \mu} \left\{ \mu + \frac{1 - \mu}{2} \exp \left[-b \left(t - \frac{|x|}{\sqrt{a_2}} \right) \right] \right\}, & -\sqrt{a_2}t < x \leq 0, \\ 0, & -\infty < x < -\sqrt{a_2}t. \end{cases} \quad (62)$$

Figures 3, 4, 5, 6, 7 and 8 show the dependence of temperature on distance for typical values of the orders of fractional derivatives. In calculations we have used the following nondimensional quantities:

$$\bar{T} = \frac{T}{T_0}, \quad \bar{b} = bt^{\alpha/2}, \quad \bar{x} = \frac{x}{\sqrt{a_1}t^{\alpha/2}} \quad \text{for } x > 0, \quad \bar{x} = \frac{x}{\sqrt{a_2}t^{\alpha/2}} \quad \text{for } x < 0. \quad (63)$$

5 Concluding remarks

We have investigated heat conduction in a composite medium consisting of two regions being in nonperfect thermal contact, which in the general case is characterized by the reduced heat capacity, reduced thermal conductivity and reduced thermal resistance.

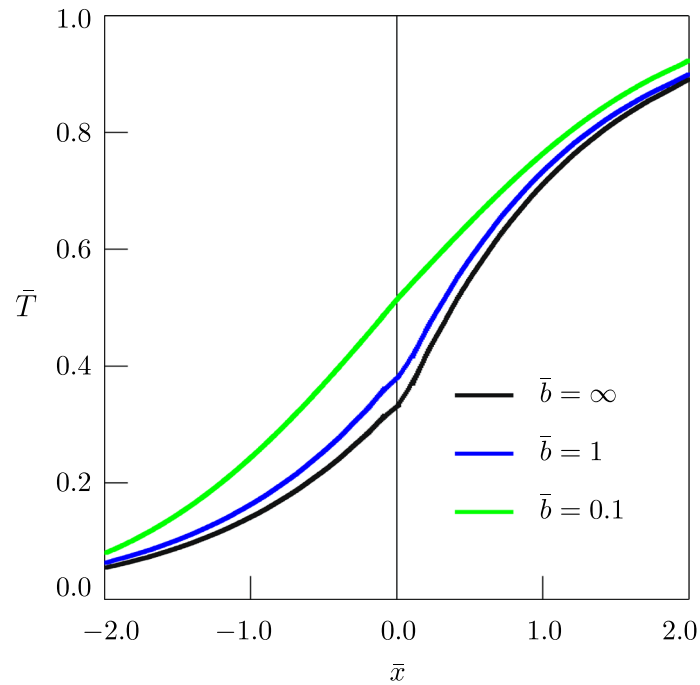


Fig. 4 Dependence of temperature on distance for $\alpha = 0.5$ and $\mu = 0.5$

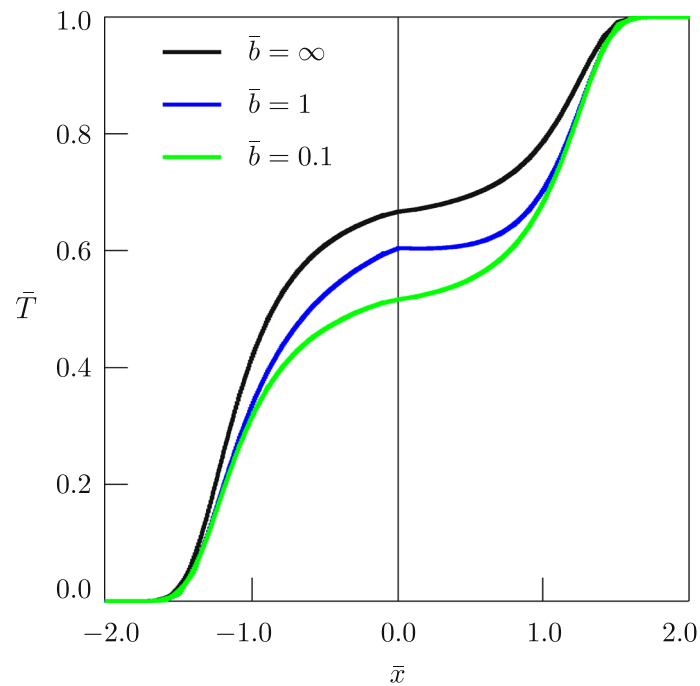


Fig. 5 Dependence of temperature on distance for $\alpha = 1.8$ and $\mu = 2$

In the case $0 < \alpha < 1$, the time-fractional heat conduction equation interpolates the elliptic Helmholtz equation ($\alpha \rightarrow 0$) and the parabolic heat conduction equation ($\alpha = 1$). When $1 < \alpha < 2$, the time-fractional heat conduction equation interpolates the standard heat conduction equation ($\alpha = 1$) and the hyperbolic wave

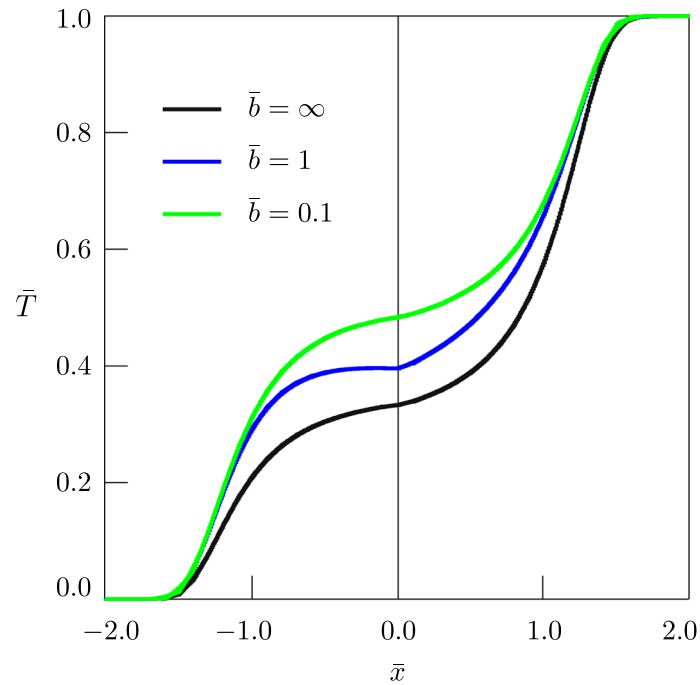


Fig. 6 Dependence of temperature on distance for $\alpha = 1.8$ and $\mu = 0.5$

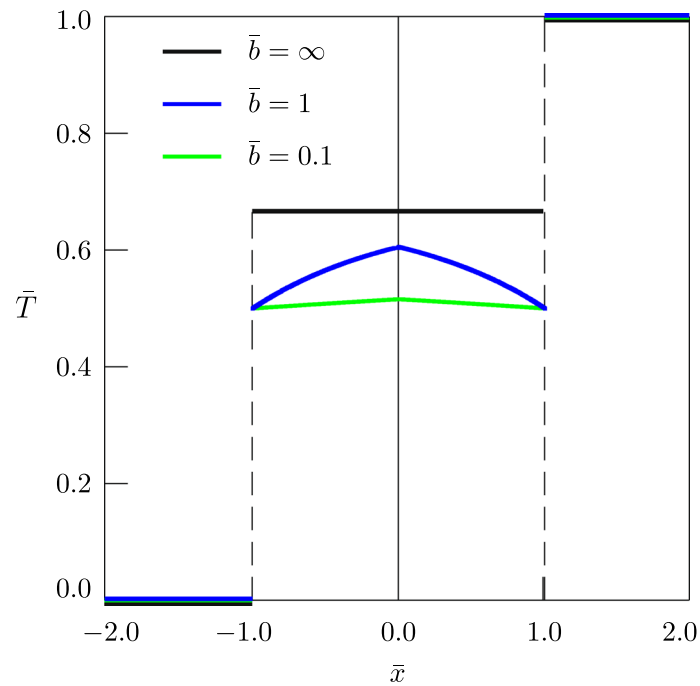


Fig. 7 Dependence of temperature on distance for $\alpha = 2$ and $\mu = 2$

equation ($\alpha = 2$). In the case of ballistic heat conduction ($\alpha = 2$), there appear the wave fronts (Figs. 7 and 8). The solutions were obtained in terms of integrals with integrands expressed in terms of Mittag–Leffler and Wright functions; for calculation of these functions we have used algorithms proposed in [65] and [66].

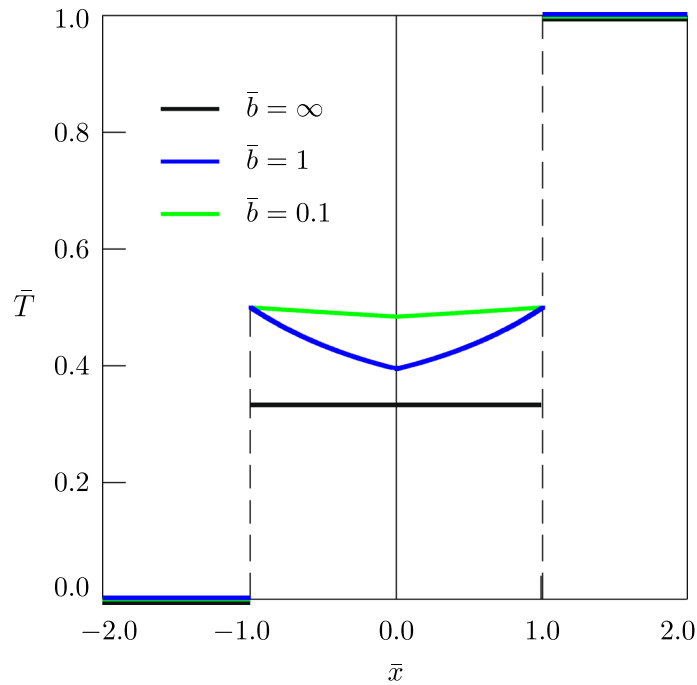


Fig. 8 Dependence of temperature on distance for $\alpha = 2$ and $\mu = 0.5$

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Appendix A: Mittag–Leffler functions

The Mittag–Leffler functions [57–59,67] play the essential role in fractional calculus. The Mittag–Leffler function in one parameter

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \quad z \in C, \tag{A.1}$$

is a generalization of the exponential function.

The Mittag–Leffler-type function in two parameters α and β is defined by the series representation

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta > 0, \quad z \in C. \tag{A.2}$$

The following formula for the inverse Laplace transform is fulfilled:

$$\mathcal{L}^{-1} \left\{ \frac{s^{\alpha-\beta}}{s^{\alpha} + b} \right\} = t^{\beta-1} E_{\alpha,\beta}(-bt^{\alpha}). \tag{A.3}$$

Appendix B: Wright function and Mainardi function

The Wright function $W(\alpha, \beta; z)$ is defined as [58,59]

$$W(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\alpha k + \beta)}, \quad \alpha > -1, \quad z \in C. \tag{B.1}$$

The Mainardi function $M(\alpha; z)$ is the particular case of the Wright function

$$\begin{aligned} M(\alpha; z) &= W(-\alpha, 1 - \alpha; -z) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k! \Gamma[-\alpha k + (1 - \alpha)]}, \quad 0 < \alpha < 1, \quad z \in \mathbb{C}. \end{aligned} \quad (\text{B.2})$$

We use the formula for the inverse Laplace transform

$$\mathcal{L}^{-1} \{s^{\alpha-1} \exp(-\lambda s^\alpha)\} = \frac{1}{t^\alpha} M(\alpha; \lambda t^{-\alpha}), \quad 0 < \alpha < 1, \quad \lambda > 0. \quad (\text{B.3})$$

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