Optimal design for disc golf by computational fluid dynamics and machine learning

Eero Immonen¹

Received: 2 May 2021 / Revised: 8 October 2021 / Accepted: 21 October 2021 / Published online: 18 December 2021
 © The Author(s) 2021, corrected publication 2022

Abstract

In this article, we introduce a computational methodology for golf disc shape optimization that employs a novel disc shape parameterization by cubic B-splines. Through application of batch Computational Fluid Dynamics simulations and Machine Learning, the disc parameterization yields functional relationships—so-called shape surrogate models—between the flying rotating disc shape and its flight characteristics. The shape surrogate models facilitate free and constrained optimization in both single- and multiobjective settings, such that both aerodynamic (drag and lift) and structural (mass and moment of inertia) features of the disc are addressed simultaneously. Further, the Professional Disc Golf Association rules for permissible golf discs can be cast as nonlinear constraints for the computational optimization problem. The proposed numerical optimization method yields disc drag coefficient values as low as 0.48 (unconstrained) and 0.52 (constrained) and lift coefficient values as high as 0.26 (unconstrained) and 0.19 (constrained). The presented numerical optimization results also describe the many design tradeoffs between the discs that target long flight range (so-called drivers) and the discs that target flight at low speeds (so-called putters). Moreover, novel optimal rule compliant designs are presented for driver-type and putter-type discs, as well as their compromise, the so-called mid-range discs.

Keywords Computational design · Computational fluid dynamics · Disc golf · Drag coefficient · Lift coefficient · Machine learning · Constrained optimization · Multiobjective optimization · Surrogate model

List of symbols

δ	Distance between P- k and X-axis	$\mathcal{M}$	Number of learning cycles in the lsboost algorithm
η	Growth parameter used in parameterization g	$\mathcal{N}$	Number of trees in a Regression Tree Ensemble
γ_{Ω}	Symmetry curve for a disc Ω , i.e., intersection of $\partial\Omega$ and XY -plane	$\mathcal{S}$	A subset of hypersurfaces of $\mathbb{R}^3$
$\hat{N}$	Number of samples in training data for SSM identification	μ	Learning rate parameter in the lsboost algorithm
$\hat{y}$	Prediction for y from parametric model	Ω	Disc domain in $\mathbb{R}^3$
$\mathbf{e}$	Shape Surrogate Model for a given variable e	$\bar{y}$	Sample average of y across training data
$\mathbf{h}$	Vector of PDGA design constraints for shape optimization	$\partial\Omega$	Surface of disc Ω
$\mathbf{L}$	Lower bounds (vector) for disc parameters	ρ	Disc material density kg/m^3
$\mathbf{U}$	Upper bounds (vector) for disc parameters	ρ_a	Density of air kg/m^3
$\mathcal{L}$	Minimum leaf size in a Regression Tree Ensemble	$\mathbf{a}$	Vector of hyperparameters for an SSM
		$\mathbf{x}_{\partial\Omega}$	Parameter vector representing disc Ω
		$\xi_k, \mathbf{a}_k$	Parameters to be optimized during lsboost algorithm
		A_X	Disc projection area in X-direction
		A_Z	Disc projection area in Z-direction
		c_d	Drag coefficient
		c_l	Lift coefficient
		d	Disc diameter, i.e., $2 \times$ maximal distance between γ_{Ω} and X-axis
		D	Latin Hypercube Design matrix ($D_{k,i}$)
		f	Shape Surrogate Model

Responsible Editor: Nathalie Bartoli.

✉ Eero Immonen
eero.immonen@turkuamk.fi

¹ Computational Engineering and Analysis Research Group, Turku University of Applied Sciences, 20520 Turku, Finland

F_d	Drag force [N] in Z-direction
F_l	Lift force [N] in X-direction
g	Parameterization function for disc surfaces
I_X	Moment of inertia about X-axis
$k - \epsilon$	Two equation model involving turbulent kinetic energy and turbulent dissipation
$k - \omega$	Two equation eddy-viscosity model for turbulent flow simulation
L	Precessional rolling moment about Z-axis
l	Disc height [m], i.e., maximal distance between points on $\partial\Omega$ in the X-direction
M	Pitching moment about Y-axis
m	Disc mass [kg]
N	Number of rows in a Latin Hypercube Design
p	Multivariate polynomial
q	Disc rotation speed [rad/s] about Y-axis
r	Disc rotation speed [rad/s] about X-axis
r	Disc rotation speed [rad/s] about Z-axis
R^2	Coefficient of determination
t_m	Minimum disc thickness [m]
v	Disc flight velocity [m/s] along Z-axis
γ_Ω	Any disc characteristic (γ) that only depends on disc shape Ω
AoA	Angle of attack [°]
CAD	Computer-Aided Design
CART	Classification and Regression Trees algorithm
CFD	Computational Fluid Dynamics
cog	Center of gravity
cop	Center of pressure
CV	Cross-Validation
MAE	Mean Absolute Error
ML	Machine Learning
MVP	Multivariate polynomial
P- k	Control points $k = 1, \dots, 19$ for B-spline approximation of γ_Ω
PDGA	Professional Disc Golf Association
RANS	Reynolds-Averaged Navier–Stokes (equations)
RTE	Regression Tree Ensemble
SSM	Shape Surrogate Model
SST	Shear Stress Transport

1 Introduction

1.1 Background

Disc golf is a sport that is similar to traditional golf, but instead of a club and a ball, players use rotating flying discs thrown toward standing baskets in as few intermediate steps as possible. Although perhaps not yet as popular as traditional golf, the disc sport is spreading rapidly across the world. In 2020, the Professional Disc Golf Association (PDGA) reported 71016 active members worldwide, 33%

growth from 2019. Further, the number of recreational disc golf players is considerably larger than this. For example, in Finland alone, single-disc golf courses have reportedly been visited by over 2000 players on a weekly basis in 2020.

While the ball shape and appearance in traditional golf are rather fixed (clubs do vary), considerable shape variation is allowed in golf discs. Indeed, players attempt to choose the most suitable disc for each throw situation and course layout, by exploiting the discs' different flight characteristics. Golf discs are typically categorized as *drivers*, *mid-rangers*, or *putters*, and there are currently over 1300 different-shaped discs approved by PDGA. However, perhaps due to commercial interests or complexity of the physics of rotating flight, surprisingly little academic research has been reported on the disc *design principles and methodologies*, which are the focus of this article.

To describe their products, golf disc manufacturers typically list four integer parameters called *speed*, *glide*, *turn*, and *fade*. The first two relate to the disc's aerodynamic drag and lift characteristics in horizontal flight, whereas the latter two relate to its pitching moment in inclined flight conditions (Lorenz 2007; Menickelli and Pickens 2016). Unfortunately, the reported numerical values of these four parameters are not based on measurements, but opinions of testers nor are such values standardized across different disc manufacturers. Thus, in particular, the listed parameter values do not directly tell a player which disc he or she should use in any given throw situation. The problem is compounded by individual variation: An amateur player (with limited power and technique) might be better off with a different disc than a professional player, for (say) covering the most distance in a single throw (Menickelli and Pickens 2016). These observations make development of a systematic golf disc design methodology both challenging and an important research question, which is addressed herein.

1.2 Contribution and limitations

The purpose of the present article is to introduce a computational methodology for golf disc shape optimization, based on Computational Fluid Dynamics (CFD) and Machine Learning (ML). The proposed methodology employs a novel disc shape parameterization by cubic B-splines that can be harnessed for batch solution of CFD simulation cases. In contrast to the four vague disc shape parameters listed above (speed, glide, turn, fade), the proposed parameterization yields precise functional relationships between the disc shape and its flight characteristics. These relationships, or *shape surrogate models* (SSM), are first identified in the present article through extensive CFD simulations and ML methods and are then used in numerical optimization.

The SSMs facilitate free and constrained optimization in both single- and multiobjective settings, such that

both aerodynamic (drag and lift) and structural (mass and moment of inertia) features of the shape are addressed simultaneously. The upshot is that the PDGA rules for permissible golf discs can be cast as nonlinear constraints for the computational optimization problem.

To demonstrate the efficacy of the proposed methodology, we carry out numerical solution of several optimization problems. They aim to

- Establish *general* principles of rotating flying disc design for improving flight characteristics, without constraints on the shapes beyond a minimum thickness;
- Describe *tradeoffs* between minimum-drag and maximum-lift shape solutions (see Subsect. 2.1);
- Resolve such golf disc shapes that both meet the *PDGA rules*, cast as numerical optimization *constraints*, and yield the longest range (drivers), best flight at low speeds (putters), or their compromise (mid-rangers).

Throughout this analysis, we confine our attention to horizontal disc flight, with zero angle of attack (AoA). Moreover, we assume stationary air (i.e., windless) conditions. It is well known (see Subsect. 2.1) that, even in such ideal conditions, gyroscopic precession tends to change the disc's alignment during rotating flight. Indeed, disc golf players occasionally exploit this tendency by imposing an initial inclination in throw or by choosing a disc with significant curvature in flight trajectory. In the present work, one part of the disc shape optimization problem is to minimize this tendency for alignment change during flight by maximizing the disc's moment of inertia with respect to the axis of forced rotation. Thus, we optimize discs for linear motion, which, by the above, is not always preferred by players in practice. Moreover, we neglect all grip-related preferences in golf disc design and assume that the presented disc flight characteristics can be realized in actual throws.

The approach proposed herein is computational, but we emphasize that the CFD-based golf disc aerodynamics modeling method we utilize is extensively validated by comparing our CFD predictions to measurement data and earlier CFD analyses reported in the literature. Although the optimal disc shape solutions presented in this article comply with first principles of fluid mechanics, they are not guaranteed to be globally optimal ones. This is because the proposed disc shape parameterization does not cover all possible disc shapes and because the optimization algorithms are not guaranteed to yield global solutions for such nonlinear problems. Finally, in this article, we do not address the manufacturability of the presented optimal shape discs beyond the requirement that the minimum surface thickness must always exceed a given threshold of 0.5 mm (see Subsect. 2.5.2).

1.3 Relation to literature

1.3.1 Rotational flight

The dynamics of rotational flight is relevant for many different sports (Goff 2013), among others. Its relative complexity arises from the interplay of fluid mechanics—i.e., lift and drag—and structural mechanics—i.e., moments induced by rotation and translation. At the turn of the millennium, several researchers worldwide started working toward understanding these phenomena for rotating discs, using experimental and (then emerging) computational approaches. The history of flying discs up to that era is well documented in the work of Potts and Crowther (2002) and Potts (2005) who also reported extensive experimental results on the aerodynamics of flying discs. Their results were subsequently complemented by Kamaruddin (2011) who experimentally evaluated the effect of different disc shape parameters on the aerodynamics. One important conclusion of these experimental results, which is also at the core of the optimization approach proposed in this article, is that the *aerodynamic lift and drag coefficients* (cf. Subsect. 2.1) are independent of the *spinning motion* of the disc (Potts and Crowther 2002).

The experimental research cited above was carried out in wind tunnel laboratory conditions. Carrying out experimental work on rotating flying discs outside wind tunnels is not an easy challenge. We mention the recent work of Weizman et al. (2020) and that of Lee et al. (2018). Both proposed sensor architectures and algorithms for estimating the aerodynamic coefficients. Instrumentation was also considered by Mustafin et al. (2012). In the present article, we focus on computational optimization and validate our results by measurements and CFD simulation results reported in the academic literature.

CFD simulations facilitate determination of aerodynamic coefficients for different disc geometries, with good accuracy (Potts and Masters 2015; Lukes et al. 2014; Honeycutt 2020; Jin et al. 2020). Moreover, typically this computational approach requires less time and effort and less expensive research infrastructure than is required for experimental tests. Many other rotating structures immersed in air flow, such as Flettner rotors, have also been successfully studied by CFD methods (De Marco et al. 2016). We utilize these results in selecting and validating the proposed CFD simulation methodology.

If the aerodynamic and structural parameters of a flying rotating disc are known, then computer simulations can be used for predicting its flight trajectory (although validation of such trajectory predictions by experiments is, by the above, difficult). Since the pioneering work of Hummel (2003) and Hubbard and Hummel (2000), several simulation-based approaches have been proposed, including validation (Koyanagi et al. 2012). Crowther and Potts

(2007) described a general six degrees-of-freedom simulation approach; Hummel and Hubbard (2002) proposed an algorithm for estimating the aerodynamic coefficients from measured flight trajectory data. Lissaman and Hubbard (2010) studied the maximum ranges of flying discs, and Kamaruddin (2011) addressed optimal initial launch conditions for maximal flight range. Optimal flight conditions for *discus* were considered in simulations by Hubbard and Cheng (2007). In the present article, we address disc trajectories indirectly by aerodynamic and structural coefficients, which are represented by parametric SSMs used for optimizing the disc shape for linear motion at different disc speeds.

Recently, Kamaruddin et al. (2018) investigated the aerodynamic properties of golf discs using experimental methods and carried out six degrees-of-freedom simulations for resolving disc trajectories. They suggested that, to improve the flight range of a disc, one should maximize its lift-to-drag ratio c_l/c_d (see Subsect. 2.1 for definitions). Moreover, they suggested that, to retain a straight-line flight path for a disc, its coefficient of pitching moment should be minimized. This article complements their findings by providing a computational framework for optimizing disc shapes with respect to several similar objectives, while also addressing shape constraints imposed by PDGA.

1.3.2 CFD-based shape optimization

Several authors have utilized CFD methods for shape optimization. For an overview, we refer the reader to the work of Immonen (2017, 2018), Immonen et al. (2021), and the references therein. While in many practical shape optimization problems, fluid motion is complex [e.g., (Zong et al. 2018)] requiring specialist analysis methods [e.g., (Immonen 2019)], in disc golf the fluid is turbulent single-phase flow, which is elementary for modern CFD solvers. The complexity of the problem here arises from rotational motion and the coupling of structural and aerodynamic properties, which depend on the disc shape. Moreover, these dependencies are nonlinear and can have discontinuities, as demonstrated by Marchand et al. (2017).

The shape optimization method proposed in the present article falls into the category of *parametric surrogate methods*, which have been successfully used for CFD-based optimization for both internal and external flows. Such methods attempt to first establish a computationally simple approximate mathematical relationship, the surrogate model (referred to as SSM in this article), between the design inputs and the optimization objectives, and then carry out optimization on the surrogate. Among the many successful applications in this category, we mention aircraft engine nacelles (Song and Keane 2007), gas turbine pin-fin arrays (Ghosh et al. 2020), wing aerodynamics (Immonen 2017; Zhonghua et al. 2020; Liu et al. 2017), fluidic oscillators

(Jeong and Kim 2018), and elbow draft tubes (Demirel et al. 2017). However, to the author's knowledge, shape optimization for rotating flying discs has not been considered (by any CFD-based method) before. Although potentially requiring extensive simulation data to identify, our motivation for carrying out shape optimization by the surrogate approach, instead of nonparametric alternatives, arises from flexibility: not only can complex nonlinear design constraints be easily included in the analysis (Immonen 2017), but aerodynamic and structural objectives can also be easily treated simultaneously in multiobjective optimization (see Subsect. 3.3.2).

Linear and quadratic polynomial surrogates [see e.g., (Immonen 2017; Mohammad et al. 2021)] are among the most widely used objective function model types in the literature because of their simplicity and because they require relatively little data for model identification. This is beneficial when data generation is carried out by computationally intensive CFD simulations, although methods for addressing efficient data generation have also been presented recently (Yan et al. 2019; Li et al. 2020). Quadratic polynomial surrogates are also able to represent simple nonlinear dependencies between the design inputs and design objectives. However, such simple surrogate models are apparently not sufficient for all CFD-based shape optimization applications. Indeed, Immonen (2018) utilized high-order polynomial surrogates for S-duct shape optimization and many authors are increasingly also utilizing more complex ML-based surrogates. Among others, Jahangirian and Shahrokhi (2011) utilized multilayer perceptron neural networks for describing the ratio c_l/c_d of lift and drag coefficients (cf. Subsect. 2.1) as a function of shape parameters in airfoil shape optimization. Further, Zhang et al. (2021) considered deep neural network surrogate models in the same application domain, and Bouhlel et al. (2020) utilized gradient-enhanced neural networks for airfoil shape optimization at different flight conditions. In the present article, we utilize Regression Tree Ensembles (RTE), with optimized hyperparameters, as ML-based surrogate models (cf. Subsect. 2.4). Ensemble methods have displayed robust performance across a spectrum of CFD-related optimization and prediction applications (Immonen et al. 2020; Molinaro et al. 2021; Umetani and Bickel 2018; Mondal et al. 2019; Wang et al. 2017), although to the author's knowledge they have not been utilized for describing parametric shapes in aerodynamic optimization before.

1.4 Organization of the article

This article is organized as follows. In Sect. 2, we introduce elements of the computational methodology: The kinematic disc characteristics (Subsect. 2.1), CFD simulation models for resolving them (Subsect. 2.2), disc shape parameterization (Subsect. 2.3), SSM representations (Subsect. 2.4),

and optimization (Subsect. 2.5). We then apply this computational methodology to flying disc shape optimization in Sect. 3, with optimization stages described in Subsect. 3.1 and results from SSM identification and numerical optimization covered in Subsect. 3.2 and Subsect. 3.3, respectively. Golf disc design principles are discussed in Subsect. 3.4, along with a detailed analysis of the optimal disc shape solutions. Finally, the conclusions of this study are presented in Sect. 4.

2 Methodology

2.1 Kinematic characteristics of flying discs

In this subsection, we mathematically describe those characteristics affecting the flight of rotating discs that will be later used in shape optimization. Consider, initially, discs denoted by Ω and defined as homogeneous (density ρ [kg/m³]) solid bodies that are rotationally symmetric about the X -axis, as in Fig. 1. In general, the disc motion is both translational and rotational: it is linear along the Z -axis at velocity v [m/s] and rotational about the X -axis at angular velocity r [rad/s]. These considerations imply that the disc's AoA is 0°, a standing assumption for the remainder of this article.

Air resistance induces a drag force F_d [N] along the negative Z -axis that reduces the disc's linear velocity. It is defined by

$$F_d = -\frac{1}{2}\rho_a v^2 c_d A_Z, \quad (1)$$

where ρ_a kg/m³ is the density of air, A_Z [m²] is the disc's projection area in the Z -direction, and $c_d > 0$ is the dimensionless *drag coefficient* which depends on the shape of the

disc. Similarly, denoting the disc's projection area in the X -direction by A_X [m²], the disc motion is affected by a lift force F_l [N] in the X -direction. It is defined by

$$F_l = \frac{1}{2}\rho_a v^2 c_l A_X, \quad (2)$$

where $c_l \in \mathbb{R}$ is the dimensionless *lift coefficient*. It, also, depends on the shape of the disc and can be positive or negative. In the X -direction, the disc motion is also affected by the disc mass m [kg], which is a design parameter.

Several studies have shown that c_d and c_l are independent of r (Potts and Crowther 2002; Potts 2005; Kamaruddin 2011). Consequently, disc shape can be optimized with respect to c_d and c_l by assuming $r = 0$. We make this standing assumption in the remainder of the article. The shape optimization problem thus becomes simpler both computationally and conceptually, as the effects of structural mechanics and fluid mechanics are less coupled in the analysis. In the remainder of this subsection, we discuss the implications of assuming $r = 0$ and how the present article addresses them.

By setting $r = 0$, we neglect the deceleration of the rotation speed r due to air viscosity and the sideways force in the Y -direction. Potts and Crowther (2002) demonstrated that the sideways forces are negligible, compared to forces in the X - and Z -directions, at least for $\frac{\pi dr}{v} < 0.7$, where d [m] is the disc diameter. At higher rotation speeds, the Magnus effect becomes more visible, but it is not considered herein.

For nonzero v , the pressure distribution on the surfaces of the disc is not symmetric about the axis of rotation (Potts 2005; Kamaruddin 2011); see also Fig. 4b. Thus, the corresponding center of pressure (cop) does not coincide with the center of gravity (cog). Consequently, the lift force, acting through the cop in the X -direction, induces a nonzero pitching moment M [Nm] that attempts to change the disc's alignment at rate q [rad/s] and, as a result, the flight trajectory. Whenever $r > 0$, gyroscopic effects would dictate that this pitching moment M also results in a precessional rolling moment L [Nm] at rate p [rad/s]. Potts (2005) and Kamaruddin (2011) derived the following simplified equations for p and q :

$$p = -\frac{2M}{rI_X}, \quad q = \frac{2L}{rI_X}, \quad \text{for } r > 0, \quad (3)$$

where I_X [kg m²] is the disc's moment of inertia about the X -axis:

$$I_X = \rho \iiint_{\Omega} (y^2 + z^2) dV \quad (4)$$

Here, M and L depend crucially on the flow conditions around the disc: They vary throughout the flight and are difficult to control. The moment of inertia I_X , on the other hand, is a static design parameter that only depends on the disc's

Fig. 1 Flying rotating disc: Linear motion is along the positive Z -axis and rotation is around X -axis, which passes through the disc's center of gravity

shape. It can, thus, be maximized by shape design in order to minimize the rates p and q in Eq. (3), which improves the disc's linear flight characteristics; this approach is taken in the present article. In summary, the disc characteristics relevant for shape optimization in the present article are as follows: c_d and c_l (aerodynamical characteristics) as well as I_X and m (structural characteristics).

2.2 CFD analysis of golf discs

In this section, we describe the CFD simulation methodology utilized throughout this article for predicting the drag and lift coefficients, c_d and c_l , corresponding to a disc shape profile.

2.2.1 Virtual wind tunnel CFD model

For aerodynamics modeling, the flying disc setup described in Subsect. 2 is equivalent to a stationary disc with an incoming flow at speed v along the Z-axis. Thus, we consider the rectangular virtual wind tunnel arrangement as in Fig. 2.

Similar to the approach of De Marco et al. (2016) and Immonen (2017, 2018); Immonen et al. (2021), here the disc is modeled as a (stationary) no-slip wall. The virtual wind tunnel contains a velocity inlet surface (velocity v m/s, normal to boundary) and a corresponding pressure outlet surface (0 Pa) at the opposing end, representing an opening. The four other planar surfaces of the virtual wind tunnel are symmetry zones, which imply, in particular, that there is no boundary layer development (that would possibly introduce a pressure disturbance on the disc) on these surfaces. Determination of c_d and c_l , as in Eqs. (1) and (2), respectively, for any given disc shape involves regressing the drag and lift forces F_d and F_l , obtained from

CFD simulation results, against the corresponding flow velocities v .

2.2.2 Mesh generation

For shape optimization, it is important to establish the aerodynamic performance of the different shape candidates *relative to each other*. As described by Immonen (2017, 2018) and Immonen et al. (2021), this can be achieved by keeping the simulation conditions, including the computational mesh and iteration count, as constant (i.e., *equivalent*) as possible across the shape candidates. The upshot is that establishing relative performance is possible even if the CFD model accuracy, in *absolute terms*, is coarse. As shape optimization in the present context would involve thousands of simulation cases, in the mesh selection process, we favored a reasonable total computation time (i.e., smaller number of cells) over a high accuracy in absolute terms (i.e., larger number of cells). Another practical requirement was that a large number of equivalent meshes, for varying disc shapes, were to be generated without errors in an automated procedure. In the present context, this means that the local mesh dimensions, e.g., maximum edge length, must be compatible with the disc geometries obtained from the parameterization (described in Subsect. 2.3).

The resulting meshes used throughout this study consist of approximately 360000 cells each. There are polyhedral cells in the bulk flow domain and a prismatic boundary layer of 5 layers, at 1.3 growth rate, spread around the disc cavity. An example of such mesh structure is displayed in Fig. 3. Equivalent meshes with this structure were generated in ANSYS Fluent Meshing by an automated batch process utilizing CAD geometry files created in SpaceClaim. During the course of this process, the mass m and moment of inertia I_X were recorded from SpaceClaim for each disc shape.

Fig. 2 Virtual wind tunnel geometry for CFD simulations

Fig. 3 Mesh structure in the vicinity of the disc for a disc geometry candidate. The disc interior is not included in the CFD model

2.2.3 CFD simulations

In numerical CFD simulations, we used ANSYS Fluent 2019R3 with double precision arithmetic, SIMPLE pressure–velocity coupling and second-order discretization for resolving the Reynolds-Averaged Navier–Stokes (RANS) equations. Since the flow around a disc is turbulent (Lukes et al. 2014; Kamaruddin et al. 2018; Potts and Masters 2015), a suitable turbulence closure equation for the RANS system was required. Based on the earlier observations of Immonen (2017, 2018), Immonen et al. (2021) and Potts and Masters (2015) from similar simulation cases, the $k-\omega$ SST model was chosen for resolving turbulence in the present work.

2.2.4 Validation studies

We conclude this section by presenting validation results for the proposed aerodynamic CFD modeling methodology. Table 1 presents a comparison of c_d and c_l predictions from CFD simulations to values reported in previous studies, for three different disc shapes. Here, c_d and c_l were calculated based on the single given reference velocity only, i.e., without regression.

In all simulations, $AoA = 0^\circ$ and $\rho_a = 1.225 \text{ kg/m}^3$. Further, 1000 steady-state iterations were sufficient in all cases to ensure solution convergence by the residual curves and the forces were calculated as averages from the last 100 iterations. All simulations were carried out using 40 cores in parallel. The total simulation time for each CFD case was approximately 10 minutes. Figure 4 illustrates the flow field prediction obtained from CFD simulations for a commercial disc geometry used as part of the validation study (Table 1, row 3).

Clearly, the CFD simulation results are in reasonably good agreement with the values reported in the literature. Besides insufficient mesh resolution, the differences can be attributed to lack of geometry details and potential accuracy variations in measurements. Indeed, Potts (2005) described the difficulty of measuring forces on flexible, deforming, and rotating discs in high-speed flows.

Remark In the validation studies, the area A_x was used in both drag force equation (1) and lift force equation (2). Consequently, the values reported for c_d in Table 1 are lower than those reported later in this article for optimal disc shapes.

2.3 Computational shape parameterization

Let $\mathcal{S}$ denote the set of all closed hypersurfaces of $\mathbb{R}^3$ with the following two properties: (i) each $\partial\Omega \in \mathcal{S}$ is

Fig. 4 CFD prediction of the flow field on the XZ-plane for the Floater disc validation case (Table 1, row 3). Flow ($v = 27.5$, m/s) is from left to right. The rendered disc surface is shown to highlight that the simulation is carried out in 3D

symmetric with respect to all rotations about the X -axis and (ii) the curve γ_Ω on the intersection of $\partial\Omega$ and the XY -plane is twice continuously differentiable and non-self-intersecting. In this article, every $\partial\Omega \in \mathcal{S}$ denotes a disc surface and we seek to find those discs with the best flight characteristics. To do that, we assume a parameterization $g : \mathbb{R}^P \rightarrow \mathcal{S} : \mathbf{x}_{\partial\Omega} \rightarrow \partial\Omega = g(\mathbf{x}_{\partial\Omega})$, by which $\partial\Omega$ is represented by P real numbers.

For optimization, a computational representation of the function g is required. Here it is obtained from the coordinates of 19 points, labeled P- k ($k = 1, \dots, 19$), on a plane. These so-called *control points* yield the disc's intersecting curve γ_Ω via a cubic B-spline approximation, as illustrated in Fig. 5a–c. The curve γ_Ω is revolved around the vertical axis to produce the surface $\partial\Omega$ (and hence the disc Ω) in 3D for CFD analyses.

Figure 5d illustrates the admissible regions for the control points and the shown asymptotic average disc shape is obtained from uniformly sampling these regions. Here, the 5 control points P-8...P-12 vary in the (overlapping) rectangular regions shown, whereas the other points are constrained: the point P-1 is always at (100, 0), the points (P- k , P-20- k), $k = 1, \dots, 7$ always share the same (and fixed) horizontal coordinate point, and are at an equal vertical distance (thickness parameter t) from each other. Moreover, the vertical locations of the constrained points are assumed to follow a power law δ^η , where δ is the distance from vertical axis and η is a growth parameter. No self-intersections are allowed for the spline curves. This

Table 1 Model validation: Our CFD predictions c_d and c_l are compared to measured values (superscript *meas*) or CFD predictions reported in the literature (superscript *ref*)

#	References	Geometry	r	v	c_d	c_l	c_d^{ref}	c_l^{ref}	c_d^{meas}	c_l^{meas}	Remarks
1	(Potts 2005) p. 146-147	$d = 0.275$, $l/d = 0.14$	152	20	0.09	0.08	N/A	N/A	0.11	0.14	Disc rotation not included in CFD
2	(Potts and Crowther 2002) p. 10	As in #1	0	20	0.09	0.08	N/A	N/A	0.1	0.13	The same CFD simulation as in #1
3	(Lukes et al. 2014) p. 688	Floater by Dynamic Discs	0	27.5	0.05	0.11	0.06-0.08	0.15-0.17	N/A	N/A	Geometry drawn by hand approximation (Fig. 4)
4	(Lukes et al. 2014) p. 688	$d = 0.2$, $l/d = 0.1$	0	27.5	0.10	0.02	0.08-0.09	0-0.01	0.125	0.06	Reference CFD: $k-\epsilon$, unstructured grid
5	(Potts and Masters 2015) p. 404	As in #4	0	20	0.12	0.02	0.11	neg.	0.13	0.05	Reference CFD: $k-\omega$ SST, unstructured grid

If known, the table lists the disc diameter d [m] and height l [m]

Fig. 5 **a–c** Example disc shapes obtained from parameterization g by varying the location of points P-1 to P-19 (coordinates in millimeters). **d** The admissible regions for points P-1 to P-19 displayed in colored areas. The average disc shape from uniform sampling of these regions is displayed in black for reference

computational parameterization yields, in total, $\mathcal{P} = 12$ design parameters for the admissible disc shapes.

2.4 Shape surrogate models

Let y_Ω be a unique characteristic that only depends on the shape of the disc Ω , such as any one of the 4 characteristics c_d, c_l, m, I_X listed in Subsect. 2.1. An SSM is a function $f : \mathbb{R}^{\mathcal{P}} \rightarrow \mathbb{R} : \mathbf{x}_{\partial\Omega} \rightarrow y_\Omega = f(\mathbf{x}_{\partial\Omega})$, for $\partial\Omega = g(\mathbf{x}_{\partial\Omega})$. Typically the functions f employ so-called *hyperparameters*, which are specified during SSM identification, as explained below.

2.4.1 Multivariate polynomial SSMs

Multivariate polynomials (MVP) have been utilized as SSMs by several authors, e.g., Immonen (2017, 2018). Their identification by least-squares optimization is straightforward and there are many systematic methods for adding and removing terms in MVP models based on their statistical significance in explaining the response variable (Desboulets 2018). Moreover, unlike many ML-based models, MVP model predictions can, especially for low-order polynomials, be interpreted by polynomial coefficients and term interactions.

MVPs thus establish a baseline performance which more complex SSMs should improve to merit attention. The general representation of a MVP $p : \mathbb{R}^p \rightarrow \mathbb{R}$ yields a prediction $\hat{y}_\Omega$ for y_Ω as:

$$\hat{y}_\Omega = f(\mathbf{x}_{\partial\Omega}) = p(x_1, \dots, x_p) \quad (5)$$

$$= \sum_{i_1=0}^{n_1} \dots \sum_{i_p=0}^{n_p} a_{i_1, \dots, i_p} x_1^{i_1} \dots x_p^{i_p} \quad (6)$$

where the coefficients $a_{i_1, \dots, i_p}$ are to be identified. The degree of the polynomial p is a design hyperparameter.

2.4.2 Regression tree ensemble SSMs

A regression tree model (RTM) is a decision tree, obtained through supervised recursive binary partitioning (branching), that attempts to represent the response y_Ω as a function of $\mathbf{x}_{\partial\Omega}$ by a sequence of if-then conditions. The well-known CART algorithm of Breiman et al. (Breiman et al. 1984) can be used for RTM identification. As in many ML models, it has parameters learned from data (such as node split points) and hyperparameters (such as node split criterion), called $\mathbf{a}$, that are chosen by the user based on the literature or direct optimization. For CART, they were summarized by Mantovani et al. (2018).

A Regression Tree Ensemble (RTE) is a predictive model composed of a weighted combination of multiple ($\mathcal{N} > 1$) regression trees, which attempt to combine results from many weak learners into one higher-quality ensemble predictor. Among the algorithms for generating ensembles, each containing hyperparameters that influence the prediction outcome in the present article, we consider bootstrap aggregation [*bagging*, cf. (Breiman 1996)] and least-squares boosting with gradient descent [*lsboost*, cf. (Hastie et al. 2009)].

Bagging involves generating bootstrap replicas, i.e., randomly resampling the data set, and training regression trees on the replicas, as in the so-called *Random Forest* concept (Breiman 2001). The hyperparameters controlling the procedure are the minimum leaf size $\mathcal{L}$, i.e., the minimum number of observations per leaf node in a regression tree and the number of trees $\mathcal{N}$ in the ensemble. The prediction of the bagged RTE is:

$$\hat{y}_\Omega = \hat{f}(\mathbf{x}_{\partial\Omega}, \mathbf{a}) = \frac{1}{\mathcal{N}} \sum_{k=0}^{\mathcal{N}} f_k(\mathbf{x}_{\partial\Omega}, \mathbf{a}) \quad (7)$$

where $f_k(\mathbf{x}_{\partial\Omega}, \mathbf{a})$ are the individual regression tree predictions, utilizing the hyperparameters $\mathbf{a}$.

The lsboost algorithm, on the other hand, progresses sequentially such that the ensemble iteratively (during up to $\mathcal{M}$ learning cycles) fits a new learner (regression tree)

to the difference between the observed response and the aggregated prediction of all learners grown previously. The ensemble fits to minimize mean-squared error through gradient descent controlled by a hyperparameter called learning rate μ . The prediction of the lsboost RTE is obtained recursively as:

$$F_0(\mathbf{x}_{\partial\Omega}) = \overline{y_\Omega}, \quad (8a)$$

$$F_k(\mathbf{x}_{\partial\Omega}) = F_{k-1}(\mathbf{x}_{\partial\Omega}) + \xi_k f_k(\mathbf{x}_{\partial\Omega}, \mathbf{a}_k), \quad k \geq 1 \quad (8b)$$

$$\hat{y}_\Omega = F_{\mathcal{M}}(\mathbf{x}_{\partial\Omega}) \quad (8c)$$

where $\overline{y_\Omega}$ is the sample average of y_Ω across shapes in the training data and $f_k(\mathbf{x}_{\partial\Omega}, \mathbf{a}_k)$ are the individual regression tree predictions. The values $(\xi_k, \mathbf{a}_k)$ are obtained sequentially by the lsboost algorithm.

2.5 Optimization

2.5.1 SSM hyperparameter optimization

Hyperparameter values control the learning process in RTE identification and they can be optimized analogously to ordinary parameters, in order to obtain a better fit for the SSM representations. In the present work, we used Bayesian optimization (Snoek et al. 2012) in MATLAB 2020b for obtaining the hyperparameters that minimize $\log[1 + \text{CV}(5)]$, where $\text{CV}(5)$ is the fivefold cross-validation loss corresponding to the model's output with the hyperparameters. The optimal hyperparameters are, by definition, those of the model which achieves the best cross-validation score. To keep the optimization time manageable, in practical optimization runs, the CART algorithm was used with default values and the optimization was carried out for the ensemble method (bagging vs. lsboost), $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$, and μ .

2.5.2 Shape optimization

In this article, we consider both free and PDGA-constrained shape optimization problems. In the former, the only requirement is disc manufacturability, which is here imposed as a requirement for the minimum-disc surface thickness t_m to always exceed 0.5 mm. For computational analysis, t_m is defined as the minimum pairwise distance between non-neighboring points on the B-spline approximation for γ_Ω . The PDGA-constrained optimization problem refers to meeting the PDGA global approval conditions (PDGA 2021). It is a set of 15 design constraints for golf discs approved for all competitions, consisting of maximum disc weight (200 g) and several dimensional constraints, summarized concisely in Fig. 6 for reference.

Fig. 6 Graphical summary of PDGA rules for golf discs and the associated terminology, reproduced from PDGA (2021) for completeness

Remark There is some room for interpretation in the PDGA rules, which makes their strict numerical implementation difficult. For example, the linear measurement tolerances are up to 1 mm, which can be over 6% of the disc height in practice. Further, referring to Figure 6, PDGA does not define the “uniform” flight plate. The PDGA rules also do *not* require a straight vertical disc section to be present for inner rim depth measurement, even though such is displayed in Fig. 6.

The general optimization problem considered here is:

$$\min_{\mathbf{L} \leq \mathbf{x}_{\partial\Omega} \leq \mathbf{U}} [SSM_1(\mathbf{x}_{\partial\Omega}), \dots, SSM_K(\mathbf{x}_{\partial\Omega})] \quad (9a)$$

$$\text{subject to } t_m(\mathbf{x}_{\partial\Omega}) \geq 0.5 \text{ mm} \quad (9b)$$

$$\text{and } \mathbf{m}(\mathbf{x}_{\partial\Omega}) \leq 200 \text{ g} \quad (9c)$$

$$\text{and } \mathbf{h}(\mathbf{x}_{\partial\Omega}) \leq \mathbf{0} \quad (9d)$$

where K is the number of shape surrogate models considered in the optimization. In free-shape optimization, the only constraint is (9b), which is the minimum thickness requirement.

For practical optimization, the PDGA dimensional constraints were programmed to the function $\mathbf{h}$ in Equation (9d). The PDGA disc weight constraint in Equation (9c) was included via a SSM $\mathbf{m}$ for disc mass obtained from the disc volumes by assuming a constant polyethylene disc material ($\rho = 920 \frac{\text{kg}}{\text{m}^3}$). In numerical optimization, we used $\mathbf{L} = 0.7 \min_i(\mathbf{x}_i)$ and $\mathbf{U} = 1.4 \max_i(\mathbf{x}_i)$, where the minimum and maximum are componentwise and taken across the designs i used for SSM identification (see Fig. 5d and Sect. 3). Numerical optimization was carried out in MATLAB 2020b using the built-in solver functions, as listed in Table 2. The choice of solvers was based on experimentation and recommendations given in the software documentation. We emphasize that although the proposed solvers are

Table 2 List of MATLAB 2020b optimization functions used for numerical optimization

	Unconstrained	Constrained
Single objective	patternsearch	surrogateopt
Multiobjective	paretosearch	paretosearch

We refer the reader to the software documentation for details on the underlying algorithms

shown to yield reasonable optimal solutions, they are not claimed to be superior to other algorithms; establishing the *best* optimization method for the present application is left for future work.

3 Flying disc shape optimization

3.1 Stages of optimization

The proposed disc shape optimization method progresses in two stages: (1) training data of $\hat{N}$ samples are generated for SSM identification and (2) the SSMs are used for parametric optimization. The steps within these two stages are described in detail below and are illustrated schematically in Fig. 7.

Stage 1: Training data generation

1a) Generate a Latin Hypercube Design $D \in [0, 1]^{N \times 12}$ for a large N . Set $k = 1$.

Repeat until $k = \hat{N}$:

1b) Define the parameter vector $\mathbf{x}_{\partial\Omega}^k = (x_1^k, \dots, x_{12}^k)^T \in \mathbb{R}^{12}$ from components $x_i^k = \min_i(\mathbf{x}_i) + D_{k,i} \cdot [\max_i(\mathbf{x}_i) - \min_i(\mathbf{x}_i)]$. Here $i = 1, \dots, 12$ and the minima and maxima are componentwise, as in Fig. 5d and Subsect. 2.5.2.

Fig. 7 Flow chart of the two-stage shape optimization process, with key concepts displayed in dashed sections

- 1c) Construct the plane curve γ_Ω^k by cubic B-splines from the parameter vector $\mathbf{x}_{\partial\Omega}^k$ as in Subsect. 2.3. If the curve is non-self-intersecting, add $\mathbf{x}_{\partial\Omega}^k$ to training data and proceed to 1d). Otherwise increment k by 1 and go back to 1b).
- 1d) Revolve γ_Ω^k to a 3D surface $\partial\Omega_k$ and create a virtual wind tunnel geometry as in Subsect. 2.2.1.
- 1e) Create an equivalent computational mesh as in Subsect. 2.2.2.
- 1f) Resolve the forces F_d^k and F_l^k for $\partial\Omega_k$ using CFD for 4 different velocities $v = 1, 5, 15$ and $35 \frac{m}{s}$, as in Subsect. 2.2.3 and Subsect. 2.2.4.
- 1g) Calculate k^{th} samples $c_{d,k}, c_{l,k}, m_k$ and $I_{X,k}$ corresponding to the shape $\partial\Omega_k$ as in Subsect. 2.2.1 and Subsect. 2.2.2.
- 1h) Increment k by 1.

Stage 2: Parametric optimization

- 2a) Identify the SSMs by regression and ML methods as in Subsect. 2.4 and Subsect. 2.5.1, using the samples collected in 1g).
- 2b) Find optimal shapes by numerical optimization as in Subsect. 2.5.2.
- 2c) Validate optimal solutions by CFD as in Subsect. 2.2.3 and Subsect. 2.2.4.

After Stage 1, there are $\hat{N}$ samples of c_d, c_l, m and I_X , corresponding to different disc shape profiles $\partial\Omega$, for SSM identification such that the underlying parameters $\mathbf{x}_{\partial\Omega}^k$ and $k = 1, \dots, \hat{N}$ are orthogonal (because D is orthogonal). In this study, we carried out batch training data generation such that $\hat{N} = 700$, which resulted in 2800 CFD cases to be solved. The number $\hat{N}$ was chosen as a compromise: high

enough for identifying MVP SSMs up to degree 3 (cubic) as baseline models and low enough to result in manageable total computation time (about 1 week). The remainder of this section describes the results obtained from the parametric optimization stage, using that training data.

3.2 Results from SSM identification

For parametric flying disc shape optimization, separate SSMs are required for describing c_d, c_l, m , and I_X . Table 3 summarizes results from their numerical identification. Clearly quadratic MVP surrogates provide good fits for m and I_X , in terms of both coefficient of determination R^2 and mean absolute error (MAE). However, ML-based models with optimized hyperparameters perform significantly better at representing c_l and c_d . It is also clear that, for c_d and c_l , the MVP representation does not improve by increasing the model complexity beyond polynomial degree 2. This indicates that the nonlinear dependencies of c_d and c_l on the flying disc shape parameters cannot be described by simple polynomial expressions and the proposed ML-based SSMs indeed merit attention. The SSM model specifications displayed in boldface in Table 3 are chosen for parametric optimization and are denoted by $\mathbf{c}_d, \mathbf{c}_l, \mathbf{m}$, and $\mathbf{I}_X$ in what follows.

3.3 Results from shape optimization

In this subsection, we describe the numerical results from shape optimization. The corresponding optimal shape profiles are analyzed and compared in detail in Subsect. 3.4.

Table 3 List of SSMs considered and their goodness of fit

Variable	SSM type	Hyperparameters	R^2	MAE	Remark
c_d	MVP	Polynomial degree 1	0.33	0.0894	R^2 calculated by leave-one-out cross-validation
c_d	MVP	Polynomial degree 2	0.61	0.0588	R^2 calculated by leave-one-out cross-validation
c_d	MVP	Polynomial degree 3	0.37	0.0254	R^2 calculated by leave-one-out cross-validation
c_l	MVP	Polynomial degree 1	0.53	0.931	R^2 calculated by leave-one-out cross-validation
c_l	MVP	Polynomial degree 2	0.792	0.454	R^2 calculated by leave-one-out cross-validation
c_l	MVP	Polynomial degree 3	0.64	0.20	R^2 calculated by leave-one-out cross-validation
c_d	RTE/lsboost	$\mathcal{L} = 1, \mathcal{M} = 24, \mu = 0.21$	0.85	0.0024	Optimized hyperparameters with fivefold cross-validation
c_l	RTE/lsboost	$\mathcal{L} = 52, \mathcal{M} = 495, \mu = 0.30$	0.99	0.001	Optimized hyperparameters with fivefold cross-validation
m	MVP	Polynomial degree 1	0.825	0.091	R^2 calculated by leave-one-out cross-validation
m	MVP	Polynomial degree 2	0.999	0.0015	R^2 calculated by leave-one-out cross-validation
I_X	MVP	Polynomial degree 1	0.807	0.124	R^2 calculated by leave-one-out cross-validation
I_X	MVP	Polynomial degree 2	0.999	0.0057	R^2 calculated by leave-one-out cross-validation

Here, R^2 represents the coefficient of determination and MAE is the mean absolute error. The SSMs shown in boldface font were chosen for parametric shape optimization

3.3.1 Free optimization

Numerical results from free-shape optimization are presented in Table 4 and Fig. 8. These results describe the general minimum-drag and maximum-lift disc shapes, without reference to the PDGA rules. In Table 4, the design objectives are displayed on the first column; for multiobjective problems (last 4 rows), the objectives are listed in a row vector. The corresponding optimal shape solution figures are referenced in the last column. In the numerical results, the superscripts *SSM*, *CFD*, and *CAD* denote the origin of the result, i.e., SSM, CFD prediction, or SpaceClaim 3D CAD geometry, respectively. The second column describes the focus of the optimization problem: Drag and lift can be optimized separately or simultaneously in single-objective problems (first 3 rows), whereas multiobjective optimization facilitates deciding upon the tradeoff from the Pareto efficient frontier. For multiobjective problems, Table 4 describes minimum-drag and maximum-lift solutions only, for brevity, as they are the most relevant for golf disc design.

3.3.2 PDGA-constrained optimization

Numerical results from PDGA-constrained shape optimization are presented in Table 5 and Fig. 9, which also display CFD prediction of the static pressure near the discs. Optimization results for driver, putter, and mid-ranger type discs are included. In each of the presented 3 cases, the optimization objective is $\min[\mathbf{c}_d, -\mathbf{c}_l, -\mathbf{I}_X]$, as in the last 2 rows of Table 4. Each disc profile presented in Table 5 satisfies the PDGA rules (Subsect. 2.5.2). The terminology used in Table 5 is analogous to that for Table 4.

3.4 Analysis of the optimal disc shapes

We now compare and analyze in detail the optimal shape solutions presented in Tables 4 and 5. In general, the numerical accuracies of the SSM predictions reported for m and I_X are better than those for c_d and c_l . Moreover, the mean absolute error, between SSM and CFD predictions, is lower for c_l than for c_d . These observations are in agreement with the SSM model fitness values reported in Table 3. Although there are some numerical inaccuracies in the prediction data, the corresponding optimal disc shapes display distinct

Table 4 Numerical results from free-shape optimization

Objective(s)	Focus	$\mathbf{c}_l^{SSM}$	c_l^{CFD}	$\mathbf{c}_d^{SSM}$	c_d^{CFD}	$\mathbf{m}^{SSM}$	m^{CAD}	$\mathbf{I}_X^{SSM}$	I_X^{CAD}	Figures
min $\mathbf{c}_d$	Drag	0.09	0.04	0.53	0.48	0.21	0.21	$1.8e-3$	$1.7e-3$	Figure 8a
max $\mathbf{c}_l$	Lift	0.29	0.26	0.84	0.80	0.05	0.05	$2.2e-4$	$3.3e-4$	Figure 8b
min $0.5\mathbf{c}_d - 0.5\mathbf{c}_l$	Drag and lift	0.22	0.17	0.64	0.69	0.10	0.10	$4.1e-4$	$7.9e-4$	Figure 8c
min $[\mathbf{c}_d, -\mathbf{c}_l]$	Drag	0.13	0.02	0.57	0.51	0.24	0.24	$1.7e-3$	$1.8e-3$	Figure 8d
min $[\mathbf{c}_d, -\mathbf{c}_l]$	Lift	0.28	0.21	0.84	0.78	0.08	0.08	$5.9e-4$	$5.5e-4$	Figure 8e
min $[\mathbf{c}_d, -\mathbf{c}_l, -\mathbf{I}_X]$	Drag	0.14	0.04	0.61	0.48	0.20	0.20	$1.7e-3$	$1.7e-3$	Figure 8f
min $[\mathbf{c}_d, -\mathbf{c}_l, -\mathbf{I}_X]$	Lift	0.22	0.20	0.68	0.81	0.14	0.14	$1.2e-3$	$1.2e-3$	Figure 8g

Fig. 8 Optimal shape profiles from free optimization. The control points P-1 to P-19 for cubic B-splines are shown for reference, except for (8c) where they are omitted for clarity

characteristics that are logical and compliant with the principles of fluid mechanics, as explained in the next subsections.

3.4.1 Free optimization (Table 4)

Focusing only on minimizing drag (row 1) or maximizing lift (row 2) yields shapes that display the best performance (across the presented optimal solutions) with respect to these objectives, respectively, on the SSMs.

The minimum-drag disc shape (Fig. 8a) has a clear and horizontally almost symmetrical wedge toward both the upstream and downstream flows, which is in line with the well-known fact that minimum-drag shapes resemble the American football (Immonen et al. 2021). The minimum-drag disc shape also hosts a relatively wide rim section (Fig. 6). Such are often seen in driver-type golf discs targeting long-distance throws. A wide rim affects the disc flight in two distinct ways: it increases gyroscopic stability by

increasing the disc mass away from the axis of rotation and it also delays the onset of flow separation from the bottom surface. On golf balls, the latter effect is produced by surface dimples, which are designed to enhance the flight distance.

The maximum-lift disc shape (Fig. 8b), on the other hand, has virtually no rim and has a uniformly curved upper section. Such curvature, also present on the top side of airfoils, increases the length of the streamlines traversing on the disc's top surface, which results in higher velocities—and hence a lower static pressure—there. The absence of a rim results in immediate flow separation on the bottom of the disc, slowing down the flow there through turbulent circulation and thus inducing higher static pressure underneath the disc. The net result of these effects is a higher pressure difference across the disc and thus a higher upward force, which is expected by the optimization objective. It should also be noted that the upper surface profile of a maximum-lift disc is strictly decreasing along the disc radius, whereas

Fig. 9 Optimal golf disc profiles satisfying the PDGA rules (left) and CFD predictions of the associated static pressure fields [Pa] at the XZ-planes (right), for $v = 15$ m/s (disc flying toward right). The disc

surfaces are not shown, in order to highlight the uniformity (or lack thereof) of static pressure

Table 5 Numerical results from PDGA-constrained multiobjective shape optimization. In each case, the objective is $\min [c_d, -c_l, -I_x]$ and the optimal disc profiles meet the PDGA regulations

Disc type	Focus	c_l^{SSM}	c_l^{CFD}	c_d^{SSM}	c_d^{CFD}	m^{SSM}	m^{CAD}	I_x^{SSM}	I_x^{CAD}	Figures
Driver	Drag	0.16	0.09	0.6	0.52	0.2	0.2	$1.7e-3$	$1.8e-3$	Figure 9a–b
Putter	Lift	0.23	0.19	0.69	0.61	0.16	0.16	$1.4e-3$	$1.4e-3$	Figure 9c–d
Mid-ranger	Drag and lift	0.18	0.16	0.65	0.59	0.15	0.15	$1.3e-3$	$1.3e-3$	Figure 9e–f

for the minimum-drag disc profile (Fig. 8a), the top side has a local peak (reaching above the flight plate) at approximately the inner diameter location.

By the above, it is clear that minimization of the drag coefficient and maximization of the lift coefficient are contradictory objectives for flying rotating discs—a well-known fact for airfoils (Immonen 2017). In analogy to the recommendation of Kamaruddin et al. (2018) to maximize the lift-to-drag ratio c_l/c_d , we consider minimizing $ac_d - (1-a)c_l$ for $a \in [0, 1]$ to prioritize low drag ($a \approx 1$) or high lift ($a \approx 0$). The disc shape shown in Fig. 8c (solid line) and row 3 represents the equal-weight solution $a = 0.5$. The cases $a = 0.15$ and $a = 0.85$ are included in Fig. 8c for illustration. Obviously the optimal solution at $a = 0.5$ is not a simple average of the optimal solutions at $a = 0.15$ and $a = 0.85$, indicating nonlinearity in the optimization problem. The

extent to which one objective can be improved without making the others worse is addressed by Pareto optimality in multiobjective optimization.

The Pareto set resulting from multiobjective optimization is a collection of optimal solutions in which no one criterion can be improved without making the others worse. Choice of a point from the Pareto set then reflects preference for some design objectives relative to the others. The optimal solutions shown in Fig. 8d (row 4) and e (row 5) represent preference for drag and lift, respectively, on the same Pareto set. The difference between Fig. 8a and d is that the latter solution cannot be improved with respect to drag without reducing lift. Similarly, the difference between Fig. 8b and e is that the latter solution cannot be improved with respect to lift without increasing drag. The last two rows (6 and 7) continue by including the

effect of moment of inertia in the optimization problem. Figure 8f shows a minimum-drag profile which cannot be improved without reducing either lift or moment of inertia (or both). Similarly, Fig. 8g shows a maximum-lift profile which cannot be improved without either increasing drag or reducing moment of inertia (or both). It is essentially such an extension of the high-lift optimal disc shape shown in Fig. 8e that also maximizes gyroscopic stability by the added mass on the rim section.

3.4.2 PDGA-constrained optimization (Table 5)

All 3 optimal solutions have remarkably smooth corners throughout their cross-section profiles, which is in contrast to many existing golf discs in practice. Smooth corners are aerodynamically better, but sharp corners may be preferred for a better grip, which is not addressed in the present study.

The optimal *driver* disc (targeting long flight range) on row 1 is one which minimizes c_d and this feature cannot be improved at the associated I_x and c_l levels. Similarly, the optimal *putter* disc on row 2 (targeting flight at slow velocities) is one which maximizes lift and cannot be improved without compromising drag or moment of inertia. The optimal *mid-ranger* on row 3, finally, is a compromise of low drag and high lift at the given moment of inertia.

In each of the 3 cases, the moment of inertia with respect to the axis of rotation (I_x) is maximized as part of the optimization problem, to minimize the perturbation effects induced by moments on the flight trajectory as in Eq. (3). This is seen in the wide rim sections for all 3 disc types, including the putter. Many existing putter-type discs only have a thin rim section, especially in comparison to driver-type discs; here, the difference in rim widths is small. However, similar to existing putter-type discs, the curvature on the top side of the putter disc is higher than for the driver and the mid-ranger.

For the driver disc profile, the static pressure profile (Fig. 9b) is remarkably uniform both above and below the disc, indicating closeness of cop and cog. Consequently, the moments that would change the disc's alignment via Eq. (3) during flight in practice, are small. This is a desirable characteristic when attempting a long throw with a linear trajectory, but may be difficult to attain in practice (see e.g., Fig. 4b). Here, flight stability is further enhanced by high moment of inertia relative to the axis of rotation. For the putter (Fig. 9d) and mid-ranger (Fig. 9f) discs, the net vertical pressure difference across the disc is higher than for the driver, indicating a higher upward force at the same disc velocity. This is reflected in the highest c_l value for the putter disc: it is the most capable of maintaining flight in low speed throws.

4 Conclusion

In this article, we have introduced a novel computational methodology for golf disc shape optimization, addressing structural and fluid mechanical characteristics simultaneously. The flying disc shapes arise here from parametric cubic B-spline representations. Together with an extensive set of CFD simulations they yield ML-based parametric descriptions—shape surrogate models or SSMs—for the main shape characteristics (c_d , c_l , m , I_x) that affect the disc flight. These SSMs are can then directly be used in parametric disc shape optimization.

In the present article, we have carried out both single objective and multiobjective disc shape optimization, with and without design constraints. The design constraints considered herein are the PDGA rules for admissible competitive golf discs. The numerical optimization results display both general principles for golf disc design, targeting either long throws (driver-type discs) via a small drag coefficient or flight at low velocities (putter-type discs) via a large lift coefficient. The presented numerical results on constrained optimization display three novel golf disc profiles that meet the PDGA rules and target these objectives or their compromise (mid-ranger-type discs).

There are several interesting directions for future work. Applying new and powerful optimization algorithms instead of the off-the-shelf ones employed herein could potentially improve the optimal disc designs further. Such an analysis should also carefully address convergence and parameter sensitivity of the optimal solutions, which is not considered herein beyond the observation that the optimal disc shapes display design nonlinearity. Since the pitching and rolling moments are available from numerical fluid flow simulations, besides merely maximizing the moment of inertia, one could attempt to directly minimize pitching and rolling rates by identifying surrogate models for the moments. The physics of flight could also be made more realistic. In this article, we have confined our attention to zero angle of attack conditions. It would, thus, be interesting to carry out disc shape optimization such that it is robust with respect to nonzero angle of attack. Due to the high number of CFD simulation cases required, this would also highlight the need to incorporate intelligent (and adaptive) methods for data generation for SSM identification. Moreover, future work should be directed toward addressing sensitivity of the results to nonstationary air conditions. Finally, future work should focus on ensuring manufacturability, e.g., by 3D printing, of the optimal disc shape profiles, as well as discussing potential design modifications for practical throw situations (e.g., grip). The latter is only possible via physical prototypes, not considered herein, that would also make experimental validation of the methods of this work possible.

Acknowledgements The author thanks the anonymous reviewers for pointing out several ways for improving the article.

Funding Open access funding provided by Turku University of Applied Sciences. The author thankfully acknowledges Academy of Finland who provided partial funding for this research via project Adaptive Fidelity Digital Twins for Robust and Intelligent Control Systems (335512).

Declarations

Conflict of interest The author states that there is no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals.

Replication of results This article presents results obtained from CFD simulations and computational optimization. Each software utility used in the process is described in this article. Additionally, the shape optimization procedure and all the associated parameters have been listed herein. The reader can thus reproduce the optimization results with the stated information. Since the data files, including mesh files and CFD simulation result files take over 200 Gb, they are not included in the supplementary material.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- Bouhlef MA, He S, Martins JR (2020) Scalable gradient-enhanced artificial neural networks for airfoil shape design in the subsonic and transonic regimes. *Struct Multidisc Optim* 61:1363–1376
- Breiman L (1996) Bagging predictors. *Mach Learn* 24(2):123–140
- Breiman L (2001) Random forests. *Mach Learn* 45(1):5–32
- Breiman L, Friedman J, Stone CJ, Olshen RA (1984) Classification and regression trees. CRC Press, Boca Raton
- Crowther W, Potts J (2007) Simulation of a spinstabilised sports disc. *Sports Eng* 10(1):3–21
- De Marco A, Mancini S, Pensa C, Calise G, De Luca F (2016) Flettner rotor concept for marine applications: a systematic study. *Int J Rotating Mach*. <https://doi.org/10.1155/2016/3458750>
- Demirel G, Acar E, Celebioglu K, Aradag S (2017) CFD-driven surrogate-based multi-objective shape optimization of an elbow type draft tube. *Int J Hydrogen Energy* 42(28):17601–17610
- Desboulets LDD (2018) A review on variable selection in regression analysis. *Econometrics* 6(4):45
- Ghosh S, Mondal S, Fernandez E, Kapat JS, Roy A (2020) Parametric shape optimization of pin-fin arrays using a surrogate model-based Bayesian method. *J Thermophys Heat Transf* 35:245–255
- Goff JE (2013) A review of recent research into aerodynamics of sport projectiles. *Sports Eng* 16(3):137–154
- Hastie T, Tibshirani R, Friedman J (2009) The elements of statistical learning: data mining, inference, and prediction. Springer, Berlin
- Honeycutt DB (2020) Prediction of disc wing aerodynamic characteristics using computational fluid dynamics. Master's thesis. The University of North Carolina at Charlotte
- Hubbard M, Cheng K (2007) Optimal disc trajectories. *J Biomech* 40(16):3650–3659
- Hubbard M, Hummel S (2000) Simulation of frisbee flight. In: 5th Conference on Mathematics and Computers in Sport. pp 124–134
- Hummel S (2003) Frisbee flight simulation and throw biomechanics. Master's thesis. University of California, Davis
- Hummel S, Hubbard M (2002) Identification of frisbee aerodynamic coefficients using flight data. In: 4th International Conference on the Engineering of Sport, vol 20
- Immonen E (2017) 2d shape optimization under proximity constraints by CFD and response surface methodology. *Appl Math Model* 41:508–529
- Immonen E (2018) Shape optimization of annular s-ducts by CFD and high-order polynomial response surfaces. *Eng Comput*
- Immonen E (2019) A parametric morphing method for generating structured meshes for marine free surface flow applications with plane symmetry. *J Comput Design Eng* 6(3):348–353
- Immonen E, Sovela J, Ranta S, Murashko K, Immonen P (2020) Retrofit optimization of battery air cooling by CFD and machine learning. In: ECMS Proceedings. pp 139–145
- Immonen E, Eriksson K, Haavisto J (2021) Computational hydrodynamic optimization of galvanic anode shapes for tunnel thrusters. *Mar Syst Ocean Technol* 16(1):23–28
- Jahangirian A, Shahrokhi A (2011) Aerodynamic shape optimization using efficient evolutionary algorithms and unstructured CFD solver. *Comput Fluids* 46(1):270–276
- Jeong H-S, Kim K-Y (2018) Shape optimization of a feedback-channel fluidic oscillator. *Eng Appl Comput Fluid Mech* 12(1):169–181
- Jin C, Wang G, He R, Chen X. Numerical simulation of discus to evaluate the trajectory and flight dynamic characteristics. In *AIAA Aviat. F.*, page 2715, (2020)
- Kamaruddin N M (2011) Dynamics and performance of flying discs. PhD thesis, University of Manchester (UK)
- Kamaruddin N, Potts J, Crowther W (2018) Aerodynamic performance of flying discs. *Aircr Eng Aerosp Technol*. <https://doi.org/10.1108/AEAT-09-2016-0143>
- Koyanagi R, Seo K, Ohta K, Ohgi Y (2012) A computer simulation of the flying disc based on the wind tunnel test data. *Proc Eng* 34:80–85
- Lee J, Lee B, Song JW, Lee YJ, Sung S (2018) Study on improved flight coefficient estimation and trajectory analysis of a flying disc through onboard magnetometer measurements. *Sensors* 18(10):3564
- Li J, Zhang M, Martins JR, Shu C (2020) Efficient aerodynamic shape optimization with deep-learning-based geometric filtering. *AIAA Journal* 58(10):4243–4259
- Lissaman P, Hubbard M (2010) Maximum range of flying discs. *Proc Eng* 2(2):2529–2535
- Liu J, Song W-P, Han Z-H, Zhang Y (2017) Efficient aerodynamic shape optimization of transonic wings using a parallel infilling strategy and surrogate models. *Struct Multidisc Optim* 55(3):925–943
- Lorenz RD (2007) Spinning flight: dynamics of frisbees, boomerangs, samaras, and skipping stones. Springer, Berlin
- Lukes R, Hart J, Potts J, Haake S (2014) A CFD analysis of flow around a disc. *Proc Eng* 72:685–690
- Mantovani RG, Horváth T, Cerri R, Junior SB, Vanschoren J, de Carvalho ACPdLF (2018) An empirical study on hyperparameter tuning of decision trees. *arXiv preprint arXiv:1812.02207*

- Marchand J-B, Astolfi JA, Bot P (2017) Discontinuity of lift on a hydrofoil in reversed flow for tidal turbine application. *Eur J Mech B/Fluids* 63:90–99
- Menickelli J, Pickens R (2016) *Definitive guide to disc golf*. Triumph Books, Chicago
- Mohammad A, Mourad A, Al-Marzouqi A, El-Naas M, Van der Bruggen B, Al-Marzouqi M, Alnaimat F, Suleiman M, Al Musharfy M (2021) CFD and statistical approach to optimize the average air velocity and air volume fraction in an inert-particles spouted-bed reactor (ipsbr) system. *Heliyon* 7(3):e06369
- Molinaro R, Singh J-S, Catsoulis S, Narayanan C, Lakehal D (2021) Embedding data analytics and CFD into the digital twin concept. *Comput Fluids* 214:104759
- Mondal S, Joly MM, Sarkar S (2019) Multi-fidelity global-local optimization of a transonic compressor rotor. *Turbo Expo*. American Society of Mechanical Engineers, New York
- Mustafin R, Wehner J, Sattler W, Gohlke K (2012) Tfo: tangible flying objects. In: *Proceedings of the Sixth International Conference on Tangible, Embedded and Embodied Interaction*. pp 193–196
- PDGA (2021) Disc golf equipment manufacturer guidelines. <https://www.pdga.com/technical-standards/manufacturer-guidelines>. Accessed 01 Mar, 2021
- Potts J, Crowther W (2002) Frisbee (tm) aerodynamics. In: 20th AIAA applied aerodynamics conference
- Potts J R (2005) *Disc-wing aerodynamics*. PhD thesis, University of Manchester (UK)
- Potts JR, Masters D (2015) Validation of the aerodynamic loading on basic flying disc geometries derived from CFD simulations. *Proc Eng* 112:400–405
- Snoek J, Larochelle H, Adams RP (2012) Practical bayesian optimization of machine learning algorithms. *arXiv preprint* [arXiv:1206.2944](https://arxiv.org/abs/1206.2944)
- Song W, Keane AJ (2007) Surrogate-based aerodynamic shape optimization of a civil aircraft engine nacelle. *AIAA Journal* 45(10):2565–2574
- Umetani N, Bickel B (2018) Learning three-dimensional flow for interactive aerodynamic design. *ACM Trans Graph (TOG)* 37(4):1–10
- Wang J-X, Wu J, Ling J, Iaccarino G, Xiao H (2017) A comprehensive physics-informed machine learning framework for predictive turbulence modeling. *arXiv preprint* [arXiv:1701.07102](https://arxiv.org/abs/1701.07102)
- Weizman Y, Tan AM, Fuss FK (2020) Measurement of flight dynamics of a frisbee using a triaxial mems gyroscope. *Multid Dig Publ Inst Proc* 49:66–89
- Yan X, Zhu J, Kuang M, Wang X (2019) Aerodynamic shape optimization using a novel optimizer based on machine learning techniques. *Aerosp Sci Technol* 86:826–835
- Zhang X, Xie F, Ji T, Zhu Z, Zheng Y (2021) Multi-fidelity deep neural network surrogate model for aerodynamic shape optimization. *Comput Methods Appl Mech Eng* 373:113485
- Zhonghua H, Chenzhou X, Zhang L, Zhang Y, Zhang K, Wenping S (2020) Efficient aerodynamic shape optimization using variable-fidelity surrogate models and multilevel computational grids. *Chin J Aeronaut* 33(1):31–47
- Zong Z, Hong Z, Wang Y, Hefazi H (2018) Hull form optimization of trimaran using self-blending method. *Appl Ocean Res* 80:240–247

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.