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BRIEF NOTE

Block-separable linking constraints in augmented Lagrangian coordination

S. Tosserams · L. F. P. Etman · J. E. Rooda

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Abstract Augmented Lagrangian coordination (ALC) is a provably convergent coordination method for multidisciplinary design optimization (MDO) that is able to treat both linking variables and linking functions (i.e. system-wide objectives and constraints). Contrary to quasi-separable problems with only linking variables, the presence of linking functions may hinder the parallel solution of subproblems and the use of the efficient alternating directions method of multipliers. We show that this unfortunate situation is not the case for MDO problems with block-separable linking constraints. We derive a centralized formulation of ALC for block-separable constraints, which does allow parallel solution of subproblems. Similarly, we derive a distributed coordination variant for which subproblems cannot be solved in parallel, but that still enables the use of the alternating direction method of multipliers. The approach can also be used for other existing MDO coordination strategies such that they can include block-separable linking constraints.

Keywords Multidisciplinary design optimization · Decomposition · Distributed optimization · Linking constraints · Augmented lagrangian

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S. Tosserams (✉) · L. F. P. Etman · J. E. Rooda
Department of Mechanical Engineering,
Eindhoven University of Technology, P.O. Box 513,
5600 MB Eindhoven, The Netherlands
e-mail: s.tosserams@tue.nl

1 Introduction

Many coordination methods have been proposed for the distributed design of large-scale multidisciplinary design optimization (MDO) problems. Examples are collaborative optimization (Braun 1996), bi-level integrated system synthesis (Sobieszczanski-Sobieski et al. 2003), the constraint margin approach of Haftka and Watson (2005), the penalty decomposition methods of DeMiguel and Murray (2006), and augmented Lagrangian coordination (ALC) recently developed by the authors (Tosserams et al. 2008). A major advantage of ALC is that convergence to local Karush-Kuhn-Tucker points can be proven for problems that have both linking variables and linking functions (i.e. objectives and constraints that depend on the variables of more than one subsystem). The other MDO coordination methods with convergence proof typically only apply to so-called quasi-separable problems with linking variables, where linking constraints are not allowed.

Applying the centralized variant of ALC to quasi-separable problems results in subproblems that can be solved in parallel during each iteration of the coordination algorithm (Tosserams et al. 2007). A central master problem coordinates the coupling between the subproblems. This master problem is an unconstrained convex quadratic problem and can be solved analytically. For problems with linking constraints, the convergence proof does not allow subproblems to be solved in parallel anymore. Instead, they have to be solved sequentially. Moreover, the coordinating master problem cannot be solved analytically (Tosserams et al. 2008).

In this note we demonstrate that there exists an important subclass of linking constraints, known as block-separable constraints, for which ALC subproblems can be solved in parallel. The coordinating master problem becomes a convex quadratic programming (QP) problem that can be solved efficiently. Since the relaxed constraints are linear, we can use coordination algorithms based on the alternating direction method of multipliers (Bertsekas and Tsitsiklis 1989). Such algorithms have been shown to be very efficient (Tosserams et al. 2006, 2007; Li et al. 2008).

We also explore whether the distributed coordination variant of ALC (Tosserams et al. 2008) can benefit from the block-separable structure of the constraints. It turns out that nothing can be gained in terms of parallelism, but the formulation does allow the use of the alternating direction method of multipliers.

2 Original problem formulation

The original MDO problem with linking variables and block-separable linking constraints is given by

$$\begin{aligned}
&\min_{\mathbf{y},\mathbf{x}_1,\ldots,\mathbf{x}_M} \sum_{j=1}^{M} f_j(\mathbf{y},\mathbf{x}_j) \\
&\text{subject to } g_{0,i} = \sum_{j\in\mathscr{G}_i} G_{j,i}(\mathbf{y},\mathbf{x}_j) \le 0 \quad i=1,\ldots,m_0^{\mathrm{g}} \\
&\qquad h_{0,i} = \sum_{j\in\mathscr{H}_i} H_{j,i}(\mathbf{y},\mathbf{x}_j) = 0 \quad i=1,\ldots,m_0^{\mathrm{h}} \\
&\qquad \mathbf{g}_j(\mathbf{y},\mathbf{x}_j) \le \mathbf{0} \qquad j=1,\ldots,M \\
&\qquad \mathbf{h}_j(\mathbf{y},\mathbf{x}_j) = \mathbf{0} \qquad j=1,\ldots,M
\end{aligned} \tag{1}$$

Herein M is the number of subsystems, $\mathbf{x}_j \in \mathbb{R}^{n_j^{\mathrm{x}}}$, $j = 1,\ldots,M$ is the vector of local design variables of subsystem j, and $\mathbf{y} \in \mathbb{R}^{n^{\mathrm{y}}}$ is the vector of linking variables. Functions $f_j(\mathbf{y},\mathbf{x}_j) : \mathbb{R}^{n_j} \to \mathbb{R}$, $j = 1,\ldots,M$ are local objectives, and functions $\mathbf{g}_j(\mathbf{y},\mathbf{x}_j) : \mathbb{R}^{n_j} \to \mathbb{R}^{m_j^{\mathrm{g}}}$, and $\mathbf{h}_j(\mathbf{y},\mathbf{x}_j) : \mathbb{R}^{n_j} \to \mathbb{R}^{m_j^{\mathrm{h}}}$, $j = 1,\ldots,M$ are local constraints, where $n_j = n^{\mathrm{y}} + n_j^{\mathrm{x}}$.

The linking constraints $\mathbf{g}_0 = [g_{0,1},\ldots,g_{0,m_0^{\mathrm{g}}}]^T : \mathbb{R}^n \to \mathbb{R}^{m_0^{\mathrm{g}}}$ and $\mathbf{h}_0 = [h_{0,1},\ldots,h_{0,m_0^{\mathrm{h}}}]^T : \mathbb{R}^n \to \mathbb{R}^{m_0^{\mathrm{h}}}$, $n = n^{\mathrm{y}} + \sum_{j=1}^{M} n_j^{\mathrm{x}}$ are block-separable (i.e. $\mathbf{g}_0$ and $\mathbf{h}_0$ are separable in terms of $G_{j,i}(\mathbf{y},\mathbf{x}_j) : \mathbb{R}^{n_j} \to \mathbb{R}$ and $H_{j,i}(\mathbf{y},\mathbf{x}_j) : \mathbb{R}^{n_j} \to \mathbb{R}$, but the functions $G_{j,i}$ and $H_{j,i}$ themselves do not need to be separable in $\mathbf{y}$ and $\mathbf{x}_j$). Sets $\mathscr{G}_i \subseteq \{1,2,\ldots,M\}$ and $\mathscr{H}_i \subseteq \{1,2,\ldots,M\}$ contain the indices of subsystems on whose variables system-wide constraints $g_{0,i}$ and $h_{0,i}$ depend. Since these constraints couple multiple subsystems, sets $\mathscr{G}_i$ and $\mathscr{H}_i$ should contain at least two elements: $|\mathscr{G}_i| \ge 2$ and $|\mathscr{H}_i| \ge 2$, where $|\mathscr{X}|$ is the cardinality of set $\mathscr{X}$.

Block-separable linking constraints can for example be encountered in MDO problems where each subsystem represents a component of a larger system such as structural optimization problems. The total mass, volume, or budget for the whole system then is a sum of component contributions, where each subsystem term may depend nonlinearly on a subsystem's design variables. A constraint on such a system quantity, e.g. mass, would give rise to a so-called block-separable linking constraint where the $G_{j,i}$ and $H_{j,i}$ functions represent component contributions.

To arrive at subproblems that can be solved in parallel, we need to work around the coupling of local subsystem variables $\mathbf{x}_j$ present in the block-separable linking constraints. To this end, we introduce a support variable for each block separable term $G_{j,i}$ and $H_{j,i}$. Then, the linking constraints only couple these support variables, and no longer the local variables $\mathbf{x}_j$. By treating the support variables as linking variables, we are able to use the ALC method for quasi-separable problems of Tosserams et al. (2007) with the difference that we have to include the linking constraints in terms of the support variables in the coordinating master problem.

The first step to the above approach is the introduction of a support variable $s_{j,i} \in \mathbb{R}$ for each component $G_{j,i}$. Similarly, we introduce a support variable $t_{j,i} \in \mathbb{R}$ for each component $H_{j,i}$. These support variables then assume the role of the corresponding $G_{j,i}$ and $H_{j,i}$ in the linking constraints $\mathbf{g}_0$ and $\mathbf{h}_0$. Additional constraints are introduced to force $s_{j,i} = G_{j,i}$ and $t_{j,i} = H_{j,i}$. Let $\mathbf{s}_i = [s_{j,i} | j \in \mathscr{G}_i]^T \in \mathbb{R}^{|\mathscr{G}_i|}$, and $\mathbf{t}_i = [t_{j,i} | j \in \mathscr{H}_i]^T \in \mathbb{R}^{|\mathscr{H}_i|}$ be the vectors of all elements $s_{j,i}$ and $t_{j,i}$ associated with constraint $g_{0,i}$ and $h_{0,i}$, respectively. Then (1) becomes

$$\begin{aligned}
&\min_{\mathbf{y},\mathbf{x},\mathbf{s},\mathbf{t}} \sum_{j=1}^{M} f_j(\mathbf{y},\mathbf{x}_j) \\
&\text{subject to } g_{0,i}(\mathbf{s}_i) = \sum_{j\in\mathscr{G}_i} s_{j,i} \le 0, \qquad i=1,\ldots,m_0^{\mathrm{g}} \\
&\qquad h_{0,i}(\mathbf{t}_i) = \sum_{j\in\mathscr{H}_i} t_{j,i} = 0, \qquad i=1,\ldots,m_0^{\mathrm{h}} \\
&\qquad \mathbf{g}_j(\mathbf{y},\mathbf{x}_j) \le \mathbf{0} \qquad j=1,\ldots,M \\
&\qquad \mathbf{h}_j(\mathbf{y},\mathbf{x}_j) = \mathbf{0} \qquad j=1,\ldots,M \\
&\qquad s_{j,i} = G_{j,i}(\mathbf{y},\mathbf{x}_j) \quad j\in\mathscr{G}_i,\ i=1,\ldots,m_0^{\mathrm{g}} \\
&\qquad t_{j,i} = H_{j,i}(\mathbf{y},\mathbf{x}_j) \quad j\in\mathscr{H}_i,\ i=1,\ldots,m_0^{\mathrm{h}} \\
&\text{where } \mathbf{x} = \left[\mathbf{x}_1^T,\ldots,\mathbf{x}_M^T\right]^T, \mathbf{s} = \left[\mathbf{s}_1^T,\ldots,\mathbf{s}_{m_0^{\mathrm{g}}}^T\right]^T, \\
&\qquad \mathbf{t} = \left[\mathbf{t}_1^T,\ldots,\mathbf{t}_{m_0^{\mathrm{h}}}^T\right]^T
\end{aligned} \tag{2}$$

3 Centralized coordination

When the support variables $\mathbf{s}$ and $\mathbf{t}$ are seen as linking variables, problem (2) resembles a quasi-separable problem with only linking variables. To illustrate this, let $\mathbf{y}^a = [\mathbf{y}^T, \mathbf{s}^T, \mathbf{t}^T]^T$ be the vector of linking variables augmented with the support variables. Then (2) can be written as

$$\begin{aligned}
&\min_{\mathbf{y}^a, \mathbf{x}} \sum_{j=1}^{M} f_j(\mathbf{y}^a, \mathbf{x}_j) \\
&\text{subject to } \mathbf{g}_0(\mathbf{y}^a) \leq \mathbf{0} \\
&\qquad \mathbf{h}_0(\mathbf{y}^a) = \mathbf{0} \\
&\qquad \mathbf{g}_j(\mathbf{y}^a, \mathbf{x}_j) \leq \mathbf{0} \qquad j = 1, \dots, M \\
&\qquad \mathbf{h}_j(\mathbf{y}^a, \mathbf{x}_j) = \mathbf{0} \qquad j = 1, \dots, M \\
&\qquad h^{g}_{j,i}(\mathbf{y}^a, \mathbf{x}_j) = 0 \qquad j \in \mathcal{G}_i, \quad i = 1, \dots, m_0^{g} \\
&\qquad h^{h}_{j,i}(\mathbf{y}^a, \mathbf{x}_j) = 0 \qquad j \in \mathcal{H}_i, \quad i = 1, \dots, m_0^{h} \\
&\quad \text{where } \mathbf{x} = \left[\mathbf{x}_1^T, \dots, \mathbf{x}_M^T\right]^T, \mathbf{y}^a = \left[\mathbf{y}^T, \mathbf{s}^T, \mathbf{t}^T\right]^T \\
&\quad h^{g}_{j,i}(\mathbf{y}^a, \mathbf{x}_j) = s_{j,i} - G_{j,i}(\mathbf{y}, \mathbf{x}_j) \\
&\quad h^{h}_{j,i}(\mathbf{y}^a, \mathbf{x}_j) = t_{j,i} - H_{j,i}(\mathbf{y}, \mathbf{x}_j)
\end{aligned} \tag{3}$$

No linking constraints that depend on the local variables of more than one subproblem are present. The constraints $h^{g}_{j,i}$ and $h^{h}_{j,i}$ depend only on shared variables $\mathbf{y}^a$ and local variables $\mathbf{x}_j$ and can thus be seen as local constraints to subsystem j.

Following the ALC variant for quasi-separable problems of Tosserams et al. (2007), we introduce linking variable copies $\mathbf{y}_j$ for $\mathbf{y}$ at each subsystem $j = 1, \dots, M$, as well as consistency constraints $\mathbf{c}^{y}_j(\mathbf{y}, \mathbf{y}_j) = \mathbf{y} - \mathbf{y}_j = \mathbf{0}$, $j = 1, \dots, M$, to force these copies equal to their originals. Similarly, we also introduce support variable copies $\hat{\mathbf{s}}_i \in \mathbb{R}^{|\mathcal{G}_i|}$ and $\hat{\mathbf{t}}_i \in \mathbb{R}^{|\mathcal{H}_i|}$ for $\mathbf{s}_i$ and $\mathbf{t}_i$, respectively, at the subsystems together with consistency constraints $c^{s}_{j,i} = s_{j,i} - \hat{s}_{j,i} = 0$, $j \in \mathcal{G}_i$, $i = 1, \dots, m_0^{g}$, and $c^{t}_{j,i} = t_{j,i} - \hat{t}_{j,i} = 0$, $j \in \mathcal{H}_i$, $i = 1, \dots, m_0^{h}$. To arrive at separable constraint sets, the linking variable copies $\mathbf{y}_j$ assume the role of the original linking variables in the local constraints $\mathbf{g}_j$, $\mathbf{h}_j$, $h^{g}_{j,i}$ and $h^{h}_{j,i}$. The linking constraints $\mathbf{g}_0$ and $\mathbf{h}_0$ depend on the original support variables $\mathbf{s}$ and $\mathbf{t}$ such that they can be included in the coordinating master problem.

Let $\mathbf{y}^a_j = [\mathbf{y}_j, \hat{s}_{j,i} | j \in \mathcal{G}_i, \hat{t}_{j,i} | j \in \mathcal{H}_i]$ be the auxiliary copies associated with subsystem j, and let $\mathbf{c}_j = [\mathbf{c}^{y}_j, c^{s}_{j,i} | j \in \mathcal{G}_i, c^{t}_{j,i} | j \in \mathcal{H}_i]$ be the consistency constraints for subsystem j, then the modified problem is given by

$$\begin{aligned}
&\min_{\mathbf{y}^a, \mathbf{x}, \mathbf{y}^a_1, \dots, \mathbf{y}^a_M} \sum_{j=1}^{M} f_j(\mathbf{y}_j, \mathbf{x}_j) \\
&\text{subject to } g_{0,i}(\mathbf{s}_i) = \sum_{j \in \mathcal{G}_i} s_{j,i} \leq 0 \qquad i = 1, \dots, m_0^{g} \\
&\qquad h_{0,i}(\mathbf{t}_i) = \sum_{j \in \mathcal{H}_i} t_{j,i} = 0 \qquad i = 1, \dots, m_0^{h} \\
&\qquad \mathbf{g}_j(\mathbf{y}_j, \mathbf{x}_j) \leq \mathbf{0} \qquad j = 1, \dots, M \\
&\qquad \mathbf{h}_j(\mathbf{y}_j, \mathbf{x}_j) = \mathbf{0} \qquad j = 1, \dots, M \\
&\qquad \hat{s}_{j,i} = G_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \qquad j \in \mathcal{G}_i, \quad i = 1, \dots, m_0^{g} \\
&\qquad \hat{t}_{j,i} = H_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \qquad j \in \mathcal{H}_i, \quad i = 1, \dots, m_0^{h} \\
&\qquad \mathbf{c}_j(\mathbf{y}^a, \mathbf{y}^a_j) = \mathbf{0} \qquad j = 1, \dots, M \\
&\quad \text{where } \mathbf{x} = \left[\mathbf{x}_1^T, \dots, \mathbf{x}_M^T\right]^T, \mathbf{y}^a = \left[\mathbf{y}^T, \mathbf{s}^T, \mathbf{t}^T\right]^T \\
&\quad \mathbf{y}^a_j = [\mathbf{y}_j, \hat{s}_{j,i} | j \in \mathcal{G}_i, \hat{t}_{j,i} | j \in \mathcal{H}_i] \\
&\quad \mathbf{c}_j = \left[\mathbf{c}^{y}_j, c^{s}_{j,i} | j \in \mathcal{G}_i, c^{t}_{j,i} | j \in \mathcal{H}_i\right]
\end{aligned} \tag{4}$$

All consistency constraints $\mathbf{c}_j$ are relaxed with an augmented Lagrangian penalty function $\phi_j(\mathbf{c}_j) = \mathbf{v}_j^T \mathbf{c}_j + \|\mathbf{w}_j \circ \mathbf{c}_j\|_2^2$. All relaxed consistency constraints are linear, hence algorithms that use the alternating direction method of multipliers can be used to coordinate the decomposed problem (Bertsekas and Tsitsiklis 1989). The relaxed problem becomes

$$\begin{aligned}
&\min_{\mathbf{y}^a, \mathbf{x}, \mathbf{y}^a_1, \dots, \mathbf{y}^a_M} \sum_{j=1}^{M} f_j(\mathbf{y}_j, \mathbf{x}_j) + \sum_{j=1}^{M} \phi_j\big(\mathbf{c}_j(\mathbf{y}^a, \mathbf{y}^a_j)\big) \\
&\text{subject to } g_{0,i}(\mathbf{s}_i) = \sum_{j \in \mathcal{G}_i} s_{j,i} \leq 0 \qquad i = 1, \dots, m_0^{g} \\
&\qquad h_{0,i}(\mathbf{t}_i) = \sum_{j \in \mathcal{H}_i} t_{j,i} = 0 \qquad i = 1, \dots, m_0^{h} \\
&\qquad \mathbf{g}_j(\mathbf{y}_j, \mathbf{x}_j) \leq \mathbf{0} \qquad j = 1, \dots, M \\
&\qquad \mathbf{h}_j(\mathbf{y}_j, \mathbf{x}_j) = \mathbf{0} \qquad j = 1, \dots, M \\
&\qquad \hat{s}_{j,i} = G_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \quad j \in \mathcal{G}_i, \quad i = 1, \dots, m_0^{g} \\
&\qquad \hat{t}_{j,i} = H_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \quad j \in \mathcal{H}_i, \quad i = 1, \dots, m_0^{h} \\
&\quad \text{where } \mathbf{x} = \left[\mathbf{x}_1^T, \dots, \mathbf{x}_M^T\right]^T, \mathbf{y}^a = \left[\mathbf{y}^T, \mathbf{s}^T, \mathbf{t}^T\right]^T \\
&\quad \mathbf{y}^a_j = [\mathbf{y}_j, \hat{s}_{j,i} | j \in \mathcal{G}_i, \hat{t}_{j,i} | j \in \mathcal{H}_i] \\
&\quad \mathbf{c}_j = [\mathbf{c}^{y}_j, c^{s}_{j,i} | j \in \mathcal{G}_i, c^{t}_{j,i} | j \in \mathcal{H}_i]
\end{aligned} \tag{5}$$

The decomposed problem consists of a central master problem P_0 and M subproblems P_j, $j = 1, \dots, M$.

The coordinating master problem P_0 solves for $\mathbf{y}^{\mathrm{a}} = [\mathbf{y}^T, \mathbf{s}^T, \mathbf{t}^T]^T$. Only the functions that depend on these variables have to be included, and the master problem P_0 is given by

$$\begin{aligned}
&\min_{\mathbf{y}^{\mathrm{a}}} \sum_{j=1}^{M} \phi_j(\mathbf{c}_j(\mathbf{y}^{\mathrm{a}}, \mathbf{y}_j^{\mathrm{a}})) \\
&\text{subject to } g_{0,i}(\mathbf{s}_i) = \sum_{j \in \mathcal{G}_i} s_{j,i} \leq 0 \qquad i = 1, \ldots, m_0^{\mathrm{g}} \\
&\qquad h_{0,i}(\mathbf{t}_i) = \sum_{j \in \mathcal{H}_i} t_{j,i} = 0 \qquad i = 1, \ldots, m_0^{\mathrm{h}} \\
&\quad \text{where } \mathbf{y}^{\mathrm{a}} = \left[\mathbf{y}^T, \mathbf{s}^T, \mathbf{t}^T\right]^T \\
&\qquad \mathbf{c}_j = \left[\mathbf{c}_j^{\mathrm{y}}, c_{j,i}^{\mathrm{s}} | j \in \mathcal{G}_i, c_{j,i}^{\mathrm{t}} | j \in \mathcal{H}_i\right]
\end{aligned} \tag{6}$$

Since the augmented Lagrangian functions ϕ_j are quadratic and strictly convex for $\mathbf{w} > \mathbf{0}$, problem P_0 is a convex QP problem, which is separable into three uncoupled problems in terms of variables $\mathbf{y}$, $\mathbf{s}$, and $\mathbf{t}$, respectively. In $\mathbf{y}$ we only have to minimize the penalties on $\mathbf{c}^{\mathrm{y}}$, for which the analytical solution of Tosserams et al. (2007) can be used. In $\mathbf{s}$ we have a convex QP with inequality constraints $\mathbf{g}_0 \leq \mathbf{0}$, and in $\mathbf{t}$ an equality constrained convex QP with $\mathbf{h}_0 = \mathbf{0}$ has to be solved.

Each of the M subproblems P_j solves for $\mathbf{y}_j$, $\mathbf{x}_j$, $\hat{s}_{j,i} | j \in \mathcal{G}_i$, and $\hat{t}_{j,i} | j \in \mathcal{H}_i$. The support variable copies $\hat{s}_{j,i} | j \in \mathcal{G}_i$, and $\hat{t}_{j,i} | j \in \mathcal{H}_i$ are eliminated from the subproblem formulation using the equality constraints $\hat{s}_{j,i} = G_{j,i}(\mathbf{y}_j, \mathbf{x}_j)$ and $\hat{t}_{j,i} = H_{j,i}(\mathbf{y}_j, \mathbf{x}_j)$. For subproblem j all constraints that include a block-term that depends on $\mathbf{y}_j$ and $\mathbf{x}_j$ are included. Let $\mathcal{I}_j^{\mathrm{g}} = \{i | j \in \mathcal{G}_i\}$ and $\mathcal{I}_j^{\mathrm{h}} = \{i | j \in \mathcal{H}_i\}$ be the set of indices i of functions $\mathbf{g}_0$ and $\mathbf{h}_0$ that contain a block-term associated with subsystem j. Subproblem P_j is given by

$$\begin{aligned}
&\min_{\mathbf{y}_j, \mathbf{x}_j} f_j(\mathbf{y}_j, \mathbf{x}_j) + \phi_j(\mathbf{c}_j(\mathbf{y}^{\mathrm{a}}, \mathbf{y}_j^{\mathrm{a}})) \\
&\text{subject to } \mathbf{g}_j(\mathbf{y}_j, \mathbf{x}_j) \leq \mathbf{0} \\
&\qquad \mathbf{h}_j(\mathbf{y}_j, \mathbf{x}_j) = \mathbf{0} \\
&\quad \text{where } \hat{s}_{j,i} = G_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \qquad i \in \mathcal{I}_j^{\mathrm{g}} \\
&\qquad \hat{t}_{j,i} = H_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \qquad i \in \mathcal{I}_j^{\mathrm{h}} \\
&\qquad \mathbf{y}^{\mathrm{a}} = \left[\mathbf{y}^T, \mathbf{s}^T, \mathbf{t}^T\right]^T \\
&\qquad \mathbf{c}_j = \left[\mathbf{c}_j^{\mathrm{y}}, c_{j,i}^{\mathrm{s}} | j \in \mathcal{G}_i, c_{j,i}^{\mathrm{t}} | j \in \mathcal{H}_i\right]
\end{aligned} \tag{7}$$

Since subproblems P_j, $1, \ldots, M$ do not depend on each other's variables, they can be solved in parallel. Overall, the solution costs for a subproblem with block-separable terms are expected to be similar to those for quasi-separable problems since the number of variables in P_j is equal to the number of variables of subproblems for its quasi-separable counterpart. Only $\mathbf{y}_j$ and $\mathbf{x}_j$ remain after elimination of the support variables $\hat{s}_{j,i}$ and $\hat{t}_{j,i}$. However, the shape of functions $G_{j,i}$ and $H_{j,i}$ may incur additional nonlinearities, and hence computational costs when compared to the quasi-separable formulation.

4 Distributed coordination

Next, we explore opportunities for parallelism in the distributed coordination variant of ALC (Tosserams et al. 2008), and start from the all-in-one problem with additional support variables (2). Auxiliary variables $\mathbf{y}_j \in \mathbb{R}^{n^y}$ are introduced at each subsystem $j = 1, \ldots, M$. To be able to eliminate the support variables from the subproblem formulations, we do not introduce auxiliary copies for $\mathbf{s}_i$ and $\mathbf{t}_i$ for the distributed approach. Instead, the linking constraints are relaxed directly, allowing the elimination of all support variables $\mathbf{s}_i$ and $\mathbf{t}_i$.

Following ALC, linearly independent consistency constraints

$$\mathbf{c}_{jn}(\mathbf{y}_j, \mathbf{y}_n) = \mathbf{y}_j - \mathbf{y}_n = \mathbf{0} \quad n \in \mathcal{N}_j | n > j, \quad j = 1, \ldots, M \tag{8}$$

are introduced that force $\mathbf{y}_1 = \mathbf{y}_2 = \ldots = \mathbf{y}_M$. Here, $\mathcal{N}_j$ is the set of neighbors to which subsystem j is connected through the consistency constraints. The modified problem with auxiliary variables and consistency constraints is given by

$$\begin{aligned}
&\min_{\mathbf{x}, \mathbf{s}, \mathbf{t}, \mathbf{y}_1, \ldots, \mathbf{y}_M} \sum_{j=1}^{M} f_j(\mathbf{y}_j, \mathbf{x}_j) \\
&\text{subject to } g_{0,i}(\mathbf{s}_i) = \sum_{j \in \mathcal{G}_i} s_{j,i} \leq 0 \qquad i = 1, \ldots, m_0^{\mathrm{g}} \\
&\qquad h_{0,i}(\mathbf{t}_i) = \sum_{j \in \mathcal{H}_i} t_{j,i} = 0 \qquad i = 1, \ldots, m_0^{\mathrm{h}} \\
&\qquad \mathbf{g}_j(\mathbf{y}_j, \mathbf{x}_j) \leq \mathbf{0} \qquad j = 1, \ldots, M \\
&\qquad \mathbf{h}_j(\mathbf{y}_j, \mathbf{x}_j) = \mathbf{0} \qquad j = 1, \ldots, M \\
&\qquad s_{j,i} = G_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \quad j \in \mathcal{G}_i, \quad i = 1, \ldots, m_0^{\mathrm{g}} \\
&\qquad t_{j,i} = H_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \quad j \in \mathcal{H}_i, \quad i = 1, \ldots, m_0^{\mathrm{h}} \\
&\qquad \mathbf{c}_{jn} = \mathbf{y}_j - \mathbf{y}_n = \mathbf{0} \quad n \in \mathcal{N}_j | n > j, \ j = 1, \ldots, M \\
&\quad \text{where } \mathbf{x} = \left[\mathbf{x}_1^T, \ldots, \mathbf{x}_M^T\right]^T, \mathbf{s} = \left[\mathbf{s}_1^T, \ldots, \mathbf{s}_{m_0^{\mathrm{g}}}^T\right]^T, \\
&\qquad \mathbf{t} = \left[\mathbf{t}_1^T, \ldots, \mathbf{t}_{m_0^{\mathrm{h}}}^T\right]^T
\end{aligned} \tag{9}$$

The consistency constraints and linking constraints are relaxed with an augmented Lagrangian penalty function ϕ. A slack variable $z_i \in \mathbb{R}$, $i = 1, \ldots, m_0^{\mathrm{g}}$ is introduced for each system-wide inequality constraint. Since all relaxed constraints are linear, the alternating directions method of multipliers can be used to solve the decomposed problem. The relaxed problem becomes

$$\begin{aligned}
\min_{\mathbf{x},\mathbf{s},\mathbf{t},\mathbf{y}_1,\ldots,\mathbf{y}_M,\mathbf{z}} \quad & \sum_{j=1}^{M} f_j(\mathbf{y}_j, \mathbf{x}_j) + \sum_{j=1}^{M-1} \sum_{n\in\mathcal{N}_j|n>j} \phi(\mathbf{c}_{jn}(\mathbf{y}_j, \mathbf{y}_n)) \\
& + \sum_{i=1}^{m_0^{\mathrm{g}}} \phi\left(\sum_{j\in\mathcal{G}_i} s_{j,i} + z_i^2\right) + \sum_{i=1}^{m_0^{\mathrm{h}}} \phi\left(\sum_{j\in\mathcal{H}_i} t_{j,i}\right) \\
\text{subject to } \quad & \mathbf{g}_j(\mathbf{y}_j, \mathbf{x}_j) \leq \mathbf{0} \qquad j = 1, \ldots, M \\
& \mathbf{h}_j(\mathbf{y}_j, \mathbf{x}_j) = \mathbf{0} \qquad j = 1, \ldots, M \\
& s_{j,i} = G_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \quad j \in \mathcal{G}_i, \quad i = 1, \ldots, m_0^{\mathrm{g}} \\
& t_{j,i} = H_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \quad j \in \mathcal{H}_i, \quad i = 1, \ldots, m_0^{\mathrm{h}} \\
\text{where } \quad & \mathbf{x} = \left[\mathbf{x}_1^T, \ldots, \mathbf{x}_M^T\right]^T, \mathbf{s} = \left[\mathbf{s}_1^T, \ldots, \mathbf{s}_{m_0^{\mathrm{g}}}^T\right]^T, \\
& \mathbf{t} = \left[\mathbf{t}_1^T, \ldots, \mathbf{t}_{m_0^{\mathrm{h}}}^T\right]^T \mathbf{z} = \left[z_1, \ldots, z_{m_0^{\mathrm{g}}}\right]^T
\end{aligned} \tag{10}$$

For subsystem j, an optimization subproblem P_j in $\mathbf{y}_j$, $\mathbf{x}_j$, $s_{j,i}|j \in \mathcal{G}_i$, and $t_{j,i}|j \in \mathcal{H}_i$ can be defined by including only those terms of (10) that depend on these variables. Again, the support variables $s_{j,i}$ and $t_{j,i}$ are eliminated with constraints $s_{j,i} = G_{j,i}(\mathbf{y}_j, \mathbf{x}_j)$, and $t_{j,i} = H_{j,i}(\mathbf{y}_j, \mathbf{x}_j)$. Each slack variable in $\mathbf{z} = [z_1, \ldots, z_{m_0^{\mathrm{g}}}]$ is assigned to one of the subsystems. Note that one does not need to assign all $\mathbf{z}$ to the same subsystem as done in Tosserams et al. (2008). Let $\mathbf{z}_j$ be the (possibly empty) subset of slack variables $\mathbf{z}$ assigned to subsystem j, then subproblem P_j is given by

$$\boxed{\begin{aligned}
\min_{\mathbf{y}_j,\mathbf{x}_j,\mathbf{z}_j} \quad & f_j(\mathbf{y}_j, \mathbf{x}_j) + \sum_{n\in\mathcal{N}_j|n>j} \phi(\mathbf{c}_{jn}(\mathbf{y}_j, \mathbf{y}_n)) \\
& + \sum_{n<j|n\in\mathcal{N}_j} \phi(\mathbf{c}_{nj}(\mathbf{y}_n, \mathbf{y}_j)) \\
& + \sum_{i\in\mathcal{I}_j^{\mathrm{g}}} \phi\left(\sum_{k\in\mathcal{G}_i} s_{k,i} + z_i^2\right) + \sum_{i\in\mathcal{I}_j^{\mathrm{h}}} \phi\left(\sum_{k\in\mathcal{H}_i} t_{k,i}\right) \\
\text{subject to } \quad & \mathbf{g}_j(\mathbf{y}_j, \mathbf{x}_j) \leq \mathbf{0} \\
& \mathbf{h}_j(\mathbf{y}_j, \mathbf{x}_j) = \mathbf{0} \\
\text{where } \quad & s_{j,i} = G_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \qquad i \in \mathcal{I}_j^{\mathrm{g}} \\
& t_{j,i} = H_{j,i}(\mathbf{y}_j, \mathbf{x}_j) \qquad i \in \mathcal{I}_j^{\mathrm{h}}
\end{aligned}} \tag{11}$$

For the distributed case, only subproblems that are not coupled through any of the penalty terms can be solved in parallel. Thus, subsystem j can be solved in parallel with subsystem p if $p \notin \mathcal{N}_j$, and $p \notin \mathcal{G}_i | j \in \mathcal{G}_i$, and $p \notin \mathcal{H}_i | j \in \mathcal{H}_i$. This amount of parallelism also applies to general linking constraints, and therefore nothing is gained in terms of parallelism for the distributed coordination variant. However, begin able to use an alternating direction approach is an advantage when compared to the general case.

5 Numerical results

To illustrate the numerical benefits of the proposed approach, we modify Example 4 of Tosserams et al. (2006) such that it has a block-separable constraint. This nonconvex problem deals with finding the dimensions of a structure consisting of three beams that are clamped at one end, while the free ends are connected by two tensile rods. A vertical load is applied to the free end of the lowest beam. The goal of the original formulation is to minimize the total weight of the structure while satisfying stress, force, and deflection constraints. If we instead minimize the deflection of the loaded node, and constrain the total mass, we arrive at a mass allocation problem where the mass constraint is block-separable.

The total mass is limited to 7 kg, and the remaining problem parameters are as in Tosserams et al. (2006). As a reference, the all-in-one problem was solved from 1000 random starting points with Matlab's SQP solver `fmincon` (Mathworks 2008) with default settings using finite difference gradients. Three local solutions were observed with optimal deflections of 2.70, 2.72, and 2.74 cm, respectively.

For the distributed optimization experiments, we follow the partition of Tosserams et al. (2006) to arrive at three subsystems, each associated with one part of the design problem. Three coordination variants are selected to solve the partitioned problem. The first two follow a traditional centralized ALC structure (following Tosserams et al. 2008) with an inner loop that is solved either exact or inexact. Due to the coupling introduced by the mass constraints, subproblems cannot be solved in parallel for these two variants. The third variant, labeled ALC-BS AD, follows the centralized formulation for block-separable constraints of (6)–(7) with the alternating direction method of multipliers, and has subproblem that can be solved in parallel.

Table 1 displays the optimal deflections and the required number of subproblem optimizations for the three variants (outer loop termination tolerance is set to 10^{-2}). The results for each variant are based on 10 experiments, each with a different randomly selected initial design. The obtained solutions for the three

Table 1 Optimal deflections and number of subproblem optimizations

Coordination variant	Optimal deflection (in cm)	Subproblem optimizations
All-in-one	2.70–2.74	–
ALC exact	2.68–2.78	223.5
ALC inexact	2.62–2.72	60.6
ALC-BS AD	2.67–2.73	27.1

variants are all feasible and close to the reference all-in-one solutions (within tolerance). The results indicate that the proposed block-separable ALC variant with alternating direction method of multipliers yields substantially lower costs for this example. A factor 10 is gained when compared to the exact variant, and a factor 2 with respect to the inexact variant.

We observe that the cost increase for solving subproblems due to the additional penalty terms associated with the block-separable constraints is small for this example. The average number of function evaluations per subproblem optimization for the AD variant is 45, which is of the same order as was observed for quasi-separable subproblems.

6 Conclusions and implications for other coordination methods

We have proposed an ALC approach for MDO problems with block-separable linking constraints that allows subproblems to be solved in parallel. In centralized form, a convex QP master problem is obtained to coordinate subproblems that can be solved in parallel. For the distributed approach, nothing is gained in terms of parallelism due to the coupling between subproblems through the linking constraints. Therefore, the centralized approach with a convex QP master problem appears to be most suitable to coordinate MDO problems with block-separable constraints. For both the centralized and the distributed structures, the relaxed constraints are linear, and solution algorithms based on the alternating direction method of multipliers can be used to solve the decomposed problems.

Other existing coordination approaches such as collaborative optimization (CO) (Braun 1996), the Penalty Decomposition (PD) methods of DeMiguel and Murray (2006), and the Constraint Margin approach (CM) of Haftka and Watson (2005) can be extended in a similar fashion to coordinate block-separable linking constraints, while maintaining parallel solution of the subproblems. For CO and PD, support variables $\mathbf{s}_i$ and $\mathbf{t}_i$ and their associated copies $\hat{\mathbf{s}}_i$ and $\hat{\mathbf{t}}_i$ have to be introduced, as well as the consistency constraints between them. The linear linking constraints $g_{0,i} = \sum_{j\in\mathcal{G}_i} s_{j,i} \leq 0$ and $h_{0,i} = \sum_{j\in\mathcal{H}_i} t_{j,i} = 0$ are then added to the CO and PD master problems, while the subproblems are given by (7). For CM, only the support variables $\mathbf{s}_i$ and $\mathbf{t}_i$ are introduced, and the linear linking constraints $g_{0,i} = \sum_{j\in\mathcal{G}_i} s_{j,i} \leq 0$ and $h_{0,i} = \sum_{j\in\mathcal{H}_i} t_{j,i} = 0$ as well as the support variables are included in the CM master problem. Values for the support variables from the master problem are sent to the CM subproblems as fixed parameters, while the subproblems also try to maximize the margins with respect to equality constraints $h^{\mathrm{g}}_{j,i}$ and $h^{\mathrm{h}}_{j,i}$.

The approach presented in this paper can even be extended to linking objectives or constraints of the more general form:

$$f_0(F_1(\mathbf{y}, \mathbf{x}_1), F_2(\mathbf{y}, \mathbf{x}_2), \ldots, F_M(\mathbf{y}, \mathbf{x}_M)) \tag{12}$$

$$g_{0,i}(G_{1,i}(\mathbf{y}, \mathbf{x}_1), G_{2,i}(\mathbf{y}, \mathbf{x}_2), \ldots, G_{M,i}(\mathbf{y}, \mathbf{x}_M)) \leq 0 \tag{13}$$

$$h_{0,i}(H_{1,i}(\mathbf{y}, \mathbf{x}_1), H_{2,i}(\mathbf{y}, \mathbf{x}_2), \ldots, H_{M,i}(\mathbf{y}, \mathbf{x}_M)) = 0 \tag{14}$$

For the linking objective, additional support variables $\mathbf{r} = [r_1, \ldots, r_M]$ and consistency constraints $r_i = F_1(\mathbf{y}, \mathbf{x}_i)$ need to be introduced and relaxed, similar to the linking constraints case. Instead of a QP master problem P_0, one would instead have a nonlinear master problem. Its objective would have a convex quadratic part (the penalty terms on $\mathbf{y}$, $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$), and a non-linear part associated with the linking objective f_0 that depends on the support variables $\mathbf{r}$. Its constraints are non-linear, and depend on $\mathbf{s}$ and $\mathbf{t}$ the same way as they depend on $G_{j,i}$ and $H_{j,i}$. Again, this coordinating problem would be separable into smaller problems in $\mathbf{y}$, $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$, respectively.

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