Archive for Mathematical Logic
https://doi.org/10.1007/s00153-024-00926-5

Mathematical Logic

Spectral MV-algebras and equispectrality

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Received: 15 December 2022 / Accepted: 15 April 2024

Abstract

In this paper we study the set of MV-algebras with given prime spectrum and we introduce the class of spectral MV-algebras. An MV-algebra is spectral if it is generated by the union of all its prime ideals (or proper ideals, or principal ideals, or maximal ideals). Among spectral MV-algebras, special attention is devoted to bipartite MV-algebras. An MV-algebra is bipartite if it admits an homomorphism onto the MV-algebra of two elements. We prove that both bipartite MV-algebras and spectral MV-algebras can be finitely axiomatized in first order logic. We also prove that there is only, up to isomorphism, a set of MV-algebras with given prime spectrum. A further part of the paper is devoted to some relations between bipartite MV-algebras and their states. Recall that a state on an MV-algebra is a generalization of a probability measure on a Boolean algebra. Particular states are the states with Bayes' property. We show that an MV-algebra admits a state with the Bayes' property if and only if it is bipartite.

Keywords MV-algebra · Prime spectrum · Spectral MV-algebra · Bipartite MV-algebra · State

Mathematics Subject Classification 06D35

1 Introduction

The use of topological invariants in algebra begins with Stone [23], where it is shown that a Boolean algebra can be recovered from its prime spectrum. More precisely,

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Published online: 08 May 2024

Boolean algebras with homeomorphic prime spectrum are isomorphic. As usual in mathematics, we will identify two topological spaces when they are homeomorphic and two algebraic structures when they are isomorphic.

In this paper we are interested in the relation between an MV-algebra and its prime spectrum. Recall that in [9] there is a characterization of prime spectra of MV-algebras as Stone duals of closed homomorphic images of lattices of cylinder polyhedra (a kind of generalization of polyhedra in infinite dimension) in hypercubes; here Stone dual is understood relative to Stone duality for bounded distributive lattices.

In this paper we consider the class of MV-algebras with given spectrum. We call equispectral two MV-algebras with homeomorphic spectrum. For convenience, sometimes we use the abbreviation E-class for equispectrality class.

It is known that an MV-algebra cannot be recovered from its prime spectrum. However, we will prove that there is only a set of MV-algebras with given prime spectrum (up to isomorphism), see Theorem 43. We try to study the set of MV-algebras with given prime spectrum. Note that two MV-algebras have the same prime spectrum if and only if they have the same Belluce lattice, hence, the same lattice of principal ideals. So, these equivalence classes of MV-algebras are quite natural.

We are in search of "nice" representatives of the equispectrality classes. For this reason we introduce the class of spectral MV-algebras. Every MV-algebra is equispectral with a spectral MV-algebra, although not with a unique one.

Among spectral MV-algebras we find some classes of MV-algebras which have been studied before. For instance, we have bipartite MV-algebras, that is the MV-algebras admitting a homomorphism in $\{0, 1\}$. Every bipartite MV-algebra is spectral, but not the other way round. We will see that both bipartite MV-algebras and spectral MV-algebras can be finitely axiomatized in first order logic.

A further part of the paper is devoted to some relations between bipartite MV-algebras and their states. Recall that states on MV-algebras are introduced in [20] in order to generalize the notion of probability on Boolean algebras. This allows one to reason about continuously valued events, like "tomorrow it will rain a lot", rather than on yes-no events. From states and partitions we can define a notion of entropy which can be studied with the methods of analysis.

The paper is organized as follows. In Sect. 3 we introduce spectral and bipartite MV-algebras; in Sect. 3.1 we prove that spectral MV-algebras are finitely axiomatizable in first order logic. In Sect. 3.2 we deal with strongly spectral MV-algebras and we treat the ideals corresponding to spectral MV-algebras, that is, spectral ideals. Section 4 is also devoted to bipartite MV-algebras; in Sect. 4.1 we provide a finite axiomatization of bipartite MV-algebras in first order logic; in Sect. 4.2 we give a relation between bipartite MV-algebras and states on MV-algebras. Section 5 is about what we call the inverse spectrum problem, that is, the problem of investigating MV-algebras with a given prime spectrum; in the same vein, Sect. 5.1 investigates smallest and largest chains with given spectrum. The section contains an example of a property sensitive to the spectrum, the Cantor-Bernstein property. Section 5.2 specializes the two classical equivalences of [18] and [10] to MV-algebras or abelian ℓ-groups with fixed spectrum, Sect. 5.3 studies the categories of MV-algebras of given prime spectrum; the case of maximal spectrum is different and is treated in Sect. 5.4.

2 Preliminaries

An *MV-algebra* is a structure $(A, 0, 1, \oplus, \neg)$ of type (0, 0, 2, 1) such that:

1. $(A, 0, \oplus)$ is a monoid (necessarily commutative, see [15]);
2. $\neg\neg x = x$;
3. $1 = \neg 0$;
4. $x \oplus 1 = 1$;
5. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

The relation $x \leq y$ given by $\neg x \oplus y = 1$ is a distributive lattice order in every MV-algebra.

Let $x \odot y = \neg(\neg x \oplus \neg y)$ and $x \ominus y = x \odot \neg y$.

Let $d(x, y)$ be the *Chang distance* function, namely $d(x, y) = (x \ominus y) \oplus (y \ominus x)$.

An *ideal* of an MV-algebra A is a subset I of A which is closed under sum and closed downwards.

Given a subset S of A, we denote by $ideal(S)$ the ideal generated by S.

A *principal ideal* is an ideal generated by one element. It can be shown that every finitely generated ideal in an MV-algebra is principal.

A *prime ideal* P is an ideal different from A such that $x \wedge y \in P$ implies $x \in P$ or $y \in P$.

A *maximal ideal* of A is a proper ideal M such that there is no proper ideal N with $M \subset N$. An MV-algebra is *local* if it has only one maximal ideal.

Recall also that an ideal I of an MV-algebra A is *primary* if $x \odot y \in I$ implies $x^n \in I$ or $y^n \in I$ for some $n \in \mathbb{N}$. It results that an ideal I is primary if and only if the quotient MV-algebra A/I is local.

The *prime spectrum* $Spec(A)$ is the topological space whose universe is the set of all prime ideals of A and whose topology is generated by the open sets

$$U(x) = \{P \in Spec(A) \mid x \notin P\}$$

where $x \in A$. This topology is called the Zariski topology.

Likewise, the maximal spectrum of A is the set of maximal ideals of A with the Zariski topology.

The *Belluce lattice* $Bell(A)$ of an MV-algebra A is the quotient of A modulo the equivalence relation of lying in the same prime ideals, where the lattice structure on the equivalence classes is given by $[x] \wedge [y] = [x \wedge y]$ and $[x] \vee [y] = [x \vee y]$.

The Belluce lattice of an MV-algebra A is isomorphic to the lattice $id_c(A)$ of all principal ideals of A (the c stands for compact, since finitely generated ideals are the compact elements in the lattice of all ideals). As a sketch of proof, note that two elements x, y of an MV-algebra A lie in the same prime ideals if and only if the generate the same principal ideal. In fact, if $ideal(x) = ideal(y)$, then they lie in the same ideals (not only the prime ones). Conversely, if $x \notin ideal(y)$, then the set of all ideals I such that $x \notin I$ and $y \in I$ has a maximal element (by Zorn's lemma) and this element is prime.

If S is a subset of an MV-algebra A, we denote by $Alg(S)$ the subalgebra of A generated by S.

The *radical* $Rad(A)$ of an MV-algebra A is the intersection of all maximal ideals. The radical coincides with the set of *infinitesimal* elements, that is, those elements x such that $nx \leq \neg x$ for every $n \in \mathbb{N}$. Intuitively, an infinitesimal is an element less than $1/n$ for every n, but not quite, since $1/n$ may not exist in A. As an example of MV-algebra with infinitesimals, we can consider the Chang MV-algebra, which is formed by a sequence $0, \epsilon, 2\epsilon, \ldots, 1-2\epsilon, 1-\epsilon, 1$ where ϵ and all its multiples are infinitesimal.

An MV-algebra is *perfect* if it is generated by its radical. An example of perfect MV-algebra is Chang MV-algebra.

The *perfect part* $Perf(A)$ of A is the largest perfect subalgebra of A.

In [10] an equivalence Δ is given between the category of perfect MV-algebras and the category of abelian ℓ-groups.

An MV-algebra is called *local* if it has a unique maximal ideal.

Note that an MV-algebra is Boolean if and only if $x \oplus x = x$ for every x (in which case $\oplus, \odot, \neg$ become the Boolean algebra operations). Every MV-algebra A has a largest Boolean subalgebra $B(A)$, that is, the set of all x such that $x = x \oplus x$.

Since MV-algebras are defined equationally, they form a variety and free objects on any cardinal k exist. The free MV-algebra over k is the set of McNaughton functions from $[0,1]^k$ to $[0,1]$, that is, the continuous functions which are piecewise affine with integer coefficients.

The order of an element $x \in A$ is the smallest integer m such that $mx = 1$, or ∞ if this m does not exist. In this paper we will use the following terminology, for $x \in A$:

1. x is small if x has infinite order;
2. x is large if it is not small;
3. x is cosmall if $\neg x$ is small;
4. x is colarge if $\neg x$ is large.

Recall the following well known theorem:

Theorem 1 *The following items are equivalent for every MV-algebra A:*

1. $x \oplus x = x$ *for all* $x \in A$*;*
2. $x \odot x = x$ *for all* $x \in A$*;*
3. $x \wedge \neg x = 0$ *for all* $x \in A$*;*
4. *A is a Boolean algebra.*

Proof See [5, Theorem 1.5.3]. □

A kind of structure closely related to MV-algebras is given by (abelian) ℓ-groups.

An ℓ-group (lattice ordered group) is a group G with a lattice order such that $x \leq y$ implies $x + z \leq y + z$.

In this paper ℓ-groups will always be abelian.

The absolute value of an element $x \in G$ is $\mid x \mid = x \vee -x$.

A *strong unit* of an ℓ-group G is an element $u \in G$ which is positive and such that for every $x \in G$ there is n such that $x \leq nu$.

In [18] an equivalence Γ is given between the category of MV-algebras and the category of lattice ordered abelian groups with strong unit. Namely $\Gamma(G, u)$ is an MV-algebra with domain $[0, u]$ and operations $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$.

Note that the MV-algebra $\Gamma(G, u)$ is definable, in first order logic, in the unital ℓ-group (G, u).

The motivation behind the functor Γ comes from the study of AF C^*-algebras in quantum mechanics.

For example we have:

Definition 1 The Chang MV-algebra is defined as

$$C = \Gamma(\mathbb{Z}\, lex\, \mathbb{Z}, (1, 0))$$

where Γ is the Mundici functor of [18] and lex denotes lexicographic product of groups.

Let us call Δ the equivalence of [10] between abelian ℓ-groups and perfect MV-algebras. Namely

$$\Delta(G) = \Gamma(\mathbb{Z}\, lex\, G, (1, 0)).$$

3 Spectral MV-algebras

In this section we study spectral MV-algebras. We begin with the following definition.

Definition 2 Given an MV-algebra A, the *spectral skeleton* $Ss(A)$ of A is the subalgebra of A generated by the union of $Spec(A)$.

A characterization of the spectral skeleton is the following:

Proposition 2 *For every MV-algebra A, $Ss(A)$ coincides with*

1. *the MV-algebra $Umax(A)$ generated by the union of all maximal ideals of A;*
2. *the MV-algebra $Uprop(A)$ generated by the union of all proper ideals of A;*
3. *the MV-algebra $Uprinc(A)$ generated by the union of all proper principal ideals of A;*
4. *the MV-algebra generated by the small elements of A.*

Proof For the first point, every maximal ideal is prime and every prime ideal is contained in a maximal ideal, so $Umax(A) = Ss(A)$.

For the second point, every maximal ideal is proper and every proper ideal is contained in a maximal ideal, so $Umax(A) = Uprop(A) = Ss(A)$.

For the third point, every proper ideal is a union of proper principal ideals, so $Uprop(A) = Uprinc(A) = Ss(A)$.

The fourth point follows from the third, because an element is small if and only if it belongs to a proper principal ideal. □

Definition 3 An MV-algebra A is called *spectral* if $A = Ss(A)$.

Proposition 3 *For every MV-algebra A:*

1. $Ss(A)$ *has the same proper ideals as* A;
2. $Ss(A)$ *is spectral;*
3. $Ss(A)$ *has the same principal proper ideals as* A;
4. $id_c(A)$ *and* $id_c(Ss(A))$ *are isomorphic;*
5. $Spec(A)$ *and* $Spec(Ss(A))$ *are homeomorphic.*

Proof (1) First let I be a proper ideal of A. Then I is included in $Ss(A)$, it is stable under sum and is stable downwards in $Ss(A)$, and $1 \notin I$. So I is a proper ideal in $Ss(A)$.

Conversely, let J be a proper ideal of $Ss(A)$. Let K be the ideal of A generated by J. Then K is the set of elements of A which are below some element of J. Since $1 \notin J$, we have also $1 \notin K$ and K is a proper ideal of A. Then K is included in $Ss(A)$ and $J = K$. So proper ideals of A and $Ss(A)$ coincide.

(2) $Ss(A)$ is generated by the proper ideals of A, so by 1), $Ss(A)$ is generated by the proper ideals of $Ss(A)$ itself, and is spectral.

(3) For the third point, let I be a proper principal ideal of A, generated by x. Then $I \subseteq Ss(A)$ and is generated by x, so I is a principal ideal of $Ss(A)$. Since $1 \notin I$, I is a proper principal ideal of $Ss(A)$.

Conversely, let J be a proper principal ideal of $Ss(A)$. Let K be the ideal generated by J in A. Then K is the set of all elements of A below some element of J, and since $1 \notin J$, we have $1 \notin K$ and K is proper, so $K \subseteq Ss(A)$ and $K = J$ and J is proper principal in A.

(4) For the fourth point, we have an isomorphism between the lattices of principal ideals of A and $Ss(A)$, which is the identity outside the top elements of the lattices and maps the top of the one lattice to the top of the other.

(5) For the fifth point, Recall the isomorphism $Bell(A) \cong id_c(A)$, valid for every MV-algebra A. So, by item 3, the lattices $Bell(A)$ and $Bell(Ss(A))$ are isomorphic. So $Spec(Bell(A))$ and $Spec(Bell(Ss(A)))$ are homeomorphic. Since the Belluce lattice operator preserves the spectrum, we conclude that $Spec(A)$ and $Spec(Ss(A))$ are homeomorphic. □

The above proposition shows the relevance of spectral MV-algebras for the spectrum problem. Recall from [9] that the spectrum problem is the (informal) problem of characterizing the prime spectrum of an MV-algebra or abelian ℓ-group. This problem has a long story, essentially beginning with [14], where spectra of commutative rings with unity are characterized as spectral spaces.

Note also:

Proposition 4 *For every MV-algebra A we have:*

1. $Rad(A) \subseteq Ss(A)$.
2. $B(A) \subseteq Ss(A)$.

Proof The first point holds because $Rad(A)$ is always a proper ideal of A.

For the second point, let $x \in B(A)$. If $x \neq 1$, then x generates a proper ideal of A. If $x = 1$, then $x = \neg 0$ and 0 belongs to a proper ideal of A. □

From the previous proposition it follows:

Corollary 1 *1. Every perfect MV-algebra is spectral;*
2. every Boolean algebra is spectral.

Proposition 5 *Let A be an MV-algebra in the variety generated by the Chang MV-algebra C. Then A is spectral.*

Proof By [10, Corollary 5.2], for every MV-algebra A in the variety generated by C, and for every maximal ideal M of A, we have $A = M \cup \neg(M)$. So A is generated by any maximal ideal M and is spectral. □

Let us consider MV-chains. Note that ideal theory of MV-chains is much simpler than the general case of MV-algebras. For instance, every ideal is prime, and ideals are totally ordered by inclusion. The spectrum problem for the particular case of MV-chains has been solved by [6].

Proposition 6 *Let A be an MV-chain. Then $Ss(A) = Perf(A)$.*

Proof The union of the spectrum of an MV-chain is the set of all elements belonging to a proper ideal of A, that is, the radical of A. Note that $Perf(A)$ is the subalgebra generated by the radical of A. □

Proposition 7 *Let A, B be two MV-chains and $f : A \to B$ be a homomorphism. Then f restricts to a homomorphism $g : Perf(A) \to Perf(B)$.*

Proof It is known that f preserves the radical, and $Perf(A)$ is generated by the radical. □

Instead, for non-linearly ordered MV-algebras, homomorphisms do not respect the spectral skeleta (not even for finite MV-algebras).

Example 1 Let $A = \{0, 1/2, 1\} \times \{0, 1\}$, $B = \{0, 1/2, 1\}$. Let $f(x, y) = x$. Then $(1/2, 1) \in Ss(A)$ but $f(1/2, 1) = 1/2 \notin Ss(B)$.

In the following theorem we denote by $Spec$ the functor from the category of MV-algebras to the category of topological spaces which associates to every MV-algebra A its spectrum.

Theorem 8 *Let A, B be two MV-chains and $f : A \to B$ be a homomorphism. Let g be as in Proposition 7. Then $Spec(f)$ coincides with $Spec(g)$. So every homomorphism of MV-chains has the same Spec as a homomorphism of perfect MV-chains.*

Proof Claim: consider any $Q \in Spec(Perf(B))$, or equivalently $Q \in Spec(B)$. We have $f^{-1}(Q) \in Spec(A) = Spec(Perf(A))$, so $f^{-1}(Q) \subseteq Perf(A) = dom\ g$. So if $x \in f^{-1}(Q)$ then $f(x) \in Q$ and $x \in dom\ g$, hence $g(x) = f(x) \in Q$ and $f^{-1}(Q) \subseteq g^{-1}(Q)$. Clearly also $g^{-1}(Q) \subseteq f^{-1}(Q)$, so $f^{-1}(Q) = g^{-1}(Q)$ and $Spec(f) = Spec(g)$. □

3.1 Axiomatizing spectral MV-algebras

In this subsection we continue the study of spectral MV-algebras by building an axiomatization of spectral MV-algebras in first order logic. We give first a number of preliminary results.

Lemma 1 *An MV-algebra A is spectral if and only if $A = Alg(S)$, where S is the set of small elements of A.*

Proof Suppose that A is spectral. Then A is generated by elements lying in proper ideals of A, which are small. Conversely, if A is generated by small elements, then it is generated by proper ideals, since every small element generates a proper ideal. □

Lemma 2 *An MV-algebra A is spectral if and only if for every $x \in A$, either x is a finite sup of small elements, or $\neg x$ is a finite sup of small elements.*

Proof If every element of A is finite sup of small elements or negation of finite sup of small elements, then A is generated by small elements, so A is spectral.

Conversely, suppose A is spectral. By the previous lemma, for every element $x \in A$ there is an MV-term t such that $x = t(s_1, \ldots, s_n)$, where s_i are small elements. We can prove that x is finite sup of small or negation of finite sup of small by induction on t.

In fact, if t is a variable then x itself is small.

If $t = \neg u$, then the thesis follows easily.

If $t = u \oplus v$, we apply the inductive hypothesis to u and v. If u and v are sup of small then $t = u \oplus v \leq 2u \vee 2v$, so t is below a finite sup of small (note that if u is small then $2u$ is small) and t is itself a finite sup of small. If $\neg u$ is a sup of small then $u = u_1 \wedge \ldots \wedge u_k$ is an inf of cosmall. Then $u \oplus v = (u_1 \oplus v) \wedge \ldots \wedge (u_k \oplus v)$ is an inf of cosmall, so $\neg(u \oplus v)$ is a sup of small. Similarly, if $\neg v$ is a sup of small then $\neg(u \oplus v)$ is a sup of small. So, in any case, $t = u \oplus v$ verifies the inductive step. This completes the inductive argument. □

Lemma 3 *Let A be a spectral MV-algebra. If $x \in A$ is a finite supremum of small elements, then x is the supremum of* at most two *small elements.*

Proof By [6, 7] we can suppose A is included in a power U^H, where U is an ultrapower of $[0, 1]$ and H is a set.

Suppose $x = s_1 \vee \ldots \vee s_k$ where $s_1, \ldots, s_k$ are small and k is the least possible. Let $i < j \leq k$. If for every n there is a component $h \in H$ such that $s_i^h, s_j^h \leq 1/n$ then $s_i \vee s_j$ is small, which is not possible. So, there is n such that for every h, $s_i^h \geq 1/n$ or $s_j^h \geq 1/n$, that is, $s_i^h \vee s_j^h \geq 1/n$, that is $s_i \vee s_j \geq 1/n$ and $n(s_i \vee s_j) = ns_i \vee ns_j = 1$ so $x \leq ns_i \vee ns_j$ and x is the supremum of *two* small elements, that is, $x \wedge ns_i$ and $x \wedge ns_j$. □

By the two previous lemmas we obtain:

Proposition 9 *A is spectral if and only if for every element x, either x or its negation are supremum of at most two small elements.*

Alternatively we have:

Proposition 10 *A is spectral if and only if either all elements are small or cosmall, or* 1 *is the sup of two small elements.*

Proof Clearly, if all elements are small or cosmall, or if 1 is the sup of two small, then A is spectral.

Conversely, suppose A spectral. Suppose x is neither small nor cosmall. By the previous proposition, one of x or $\neg x$ is sup of two small. Suppose $x = y \vee z$ is the sup of two small elements (if $\neg x$ is the sup of two small elements the argument is the same). Then $1 = nx = ny \vee nz = 1$, so 1 is the sup of two small elements ny and nz. □

In order to find nonspectral MV-algebras we observe that [0, 1] is not spectral, and more generally:

Proposition 11 *A local MV-algebra is spectral if and only if it is perfect.*

Proof Let A be local and perfect. Then A is generated by its maximal ideal, so it is spectral.

Conversely, let A be local and non-perfect. The small elements are in the maximal ideal, so they generate the perfect part of A. Hence, A is not spectral. □

The following theorem clarifies the relation between spectral and local MV-algebras:

Theorem 12 *Every non-spectral MV-algebra is local.*

Before proving the theorem let us put a lemma:

Lemma 4 *In every MV-algebra the following are equivalent:*

1. 1 *is the sup of two small elements;*
2. 1 *is the sum of two small elements;*
3. *there is an element which is both small and cosmall.*

Proof (of the lemma) If $1 = s_1 \vee s_2$, with s_1, s_2 small, then $1 = s_1 \oplus s_2$. Conversely if $1 = s_1 \oplus s_2$, with s_1, s_2 small, then $1 = (2s_1) \vee (2s_2)$, and $2s_1$ and $2s_2$ are still small.

Suppose now that $1 = s_1 \oplus s_2$ with s_1, s_2 small; then $\neg s_1 \leq s_2$ so $\neg s_1$ is small and s_1 is cosmall. So s_1 is small and cosmall. Conversely, if s is small and cosmall then $1 = s \oplus \neg s$ is the sum of two small. □

Now let us prove the theorem.

Proof By Proposition 10, if A is not spectral, then 1 is not the sup of two small elements. Hence by the previous lemma, there is no element both small and cosmall. So every element is large or colarge. Then A is local, see for instance [4]. □

From Proposition 11 and Theorem 12 it follows:

Corollary 2 *An MV-algebra is spectral if and only if it is non-local or perfect.*

By the previous corollary we can say that an MV-algebra is not spectral if and only if it is local non perfect. Thus we have seen that the elements of an MV-algebra which are not small and not cosmall play a special role in the structure of a non spectral MV-algebra. Let us stress that in [8] not small and not cosmall elements of a local MV-algebra are called *finite elements*. Thus, for a local MV-algebra A, the set $Fin(A)$ is defined as the set of all finite elements of A. Then by [8, Proposition 3.9] we get:

Corollary 3 *An MV-algebra A is not spectral if and only if it is generated by the union of the set of all large and colarge elements and the set of all infinitesimal elements.*

Corollary 2 can be further elaborated so to obtain a finitary axiomatization of spectral MV-algebras in first order logic. In fact, let us prove the following theorem:

Theorem 13 *Spectral MV-algebras are finitely axiomatizable in first order logic.*

Proof Let us consider two lemmas. □

Lemma 5 (see [3]) *Perfect MV-algebras are finitely axiomatizable in first order logic.*

Then for local MV-algebras we have:

Lemma 6 (see [8]) *Local MV-algebras are finitely axiomatizable in first order logic.*

Putting together Corollary 2, Lemmas 5 and 6 we obtain the theorem.

3.2 Strongly spectral MV-algebras and spectral ideals

In this section we study strongly spectral MV-algebras, another particular case of spectral MV-algebras.

Definition 4 An MV-algebra is called *strongly spectral* if every element is small or cosmall.

Proposition 14 *Every strongly spectral MV-algebra is spectral.*

Proof Every small element lies in a proper ideal, and every cosmall element is the negation of an element lying in a proper ideal. □

Conversely we have the following collapse:

Proposition 15 *A local MV-algebra is spectral if and only if it is strongly spectral.*

Proof Both conditions are equivalent to the non-existence of elements x such that $1/n \leq x \leq 1 - 1/n$ for some n. □

In more general MV-algebras the two notions do not collapse, for instance:

Proposition 16 *Every MV-algebra* $[0, 1]^I$*, with* $| I | > 1$*, is spectral but not strongly spectral.*

Proof The MV-algebra is non local, so it is spectral. However, the constant function 1/2 is large and colarge. □

Note also:

Proposition 17 *Every free MV-algebra is strongly spectral.*

Proof Let F_k be the free MV-algebra on k elements. Let $f \in F_k$ be a McNaughton function. Then $f : [0, 1]^k \to [0, 1]$ and $f = p(\pi_1, \ldots, \pi_k)$ where p is an MV-polynomial and $\pi_1, \ldots, \pi_k$ are the projections. Let $\overline{0}$ be the vertex zero of $[0, 1]^k$. It can be shown by induction on p that $f(\overline{0}) \in \{0, 1\}$. So, f is not contained between $1/n$ and $1 - 1/n$ since it assumes at least one of the values 0 and 1. □

Often in MV-algebra theory, properties of quotient MV-algebras A/I correspond to properties of the ideal I.

In this vein, we could say that an ideal I of an MV-algebra A is *spectral* if A/I is a spectral MV-algebra, and I is *perfect* if A/I is perfect. We describe explicitly these ideals as follows:

Proposition 18 *1. I is spectral if and only if I is not primary or I is perfect.*
2. I is perfect if and only if for every $a \in A, n \in \mathbb{N}$ $n(a \wedge \neg a) \ominus (a \vee \neg a) \in I$.

Proof The first point follows because an MV-algebra is spectral if and only if it is not local or perfect. The second point follows because an MV-algebra A is perfect if and only if for every $a \in A$, $a \wedge \neg a$ is infinitesimal. □

4 Bipartite MV-algebras

We recall the class of bipartite MV-algebras, studied in [12]. Bipartite MV-algebras are a particular case of spectral MV-algebras, despite the definition is based on maximal ideals, whereas the definition of spectral MV-algebras is based on prime ideals.

Definition 5 An MV-algebra A is called *bipartite* if A contains an ideal M such that A/M has two elements. We call *BP* the class of bipartite MV-algebras.

We denote by BP the class of bipartite MV-algebras.

Among the results proved in [12] we recall the following:

1. $A \in BP$ iff A is embeddable into a direct product $\prod_{i \in I} A_i$ such that for at least one $j \in I$, A_j is perfect.
2. $A \notin BP$ iff $Alg(Inf(A)) = A$, where $Inf(A) = \{x \wedge \neg x \mid x \in A\}$.

Finally, we recall that in [1] we have a classification of algebras in *BP*.

Proposition 19 *Every bipartite MV-algebra is spectral.*

Proof Let A be bipartite, suppose M is an ideal and A/M has two elements. Clearly $1 \notin M$. Moreover, let x be any element of A. If $x \notin M$, then x and 1 are in the same class modulo M, so $d(x, 1) \in M$, where d is Chang distance. But $d(x, 1) = \neg x$, so for every $x \notin M$ we have $\neg x \in M$, and since $x = \neg\neg x$, M generates A. □

Corollary 4 *Every free MV-algebra is bipartite.*

Proof Let A be the free MV-algebra on a set X. Then A is isomorphic to the MV-algebra M_X of McNaughton functions from $[0, 1]^X$ to $[0, 1]$. Let $\underline{0}$ be the constant function zero from X to $[0, 1]$. Every element of M_X has value 0 or 1 on $\underline{0}$. The set I of all elements $f \in M_X$ such that $f(\underline{0}) = 0$ is an ideal of M_X and, since $f(\underline{0}) \in \{0, 1\}$ for every $f \in M_X$, it follows that M_X/I has two elements. So M_X is bipartite and, by the previous proposition, M_X is spectral. □

Now we need the following definition.

Definition 6 Let A be an MV-algebra, $M \subseteq A$ an ideal, then M is called *super maximal* if for all $x \in A$, $x \in M$ or $\neg x \in M$.

We get $A \in BP$ if and only if A admits a super maximal ideal.

It is clear that a super maximal ideal M is a maximal ideal. Equivalently, M is super maximal iff $A/M \cong \{0, 1\}$. Also, if M is a super maximal ideal of A, then M generates A as an MV-algebra.

By Propositions 19 and 11, not every MV-algebra has a super maximal ideal.

Proposition 20 *Suppose $A \subseteq \prod_i A_i$ subdirectly. If some A_i is in BP, then A is in BP.*

Proof Let $A' = \prod_i A_i$. Denote by $(x_i)_i$ a tuple belonging to A'. Suppose each A_j has a supermaximal ideal M_j. Let

$$M' = \{(x_i)_i \in A' \mid x_j \in M_j \ for\ all\ j\}$$

It is clear that M' is a proper ideal. Suppose $(x_i)_i \notin M'$. Then $x_j \notin M_j$ for some j. Therefore $\neg x_j \in M_j$. It follows that $(\neg x_i)_i \in M'$ and so M' is a super maximal ideal of A'. But A is a subalgebra of A', so also A has a super maximal ideal. □

In general the class BP is not closed under MV-homomorphisms. In fact, every free MV-algebra is in BP, [0, 1] is not in BP (since it is not spectral) but [0, 1], like every MV-algebra, is a homomorphic image of a free MV-algebra.

4.1 Axiomatizing bipartite MV-algebras

We begin with an infinite axiomatization of bipartite MV-algebras in first order logic and then we show that the axiomatization can be simplified to a finite one.

Theorem 21 *An MV-algebra A is in BP if and only if for every k, n, A verifies the formula*

$$(B_{n,k}) \qquad NOT(x_1 \wedge \neg x_1) \vee \ldots \vee (x_k \wedge \neg x_k) \geq 1/n.$$

Remark: in this and the following proofs, AND, OR, NOT are intended as logical operators, whereas $\vee$, $\wedge$, $\neg$ are the MV-algebra operators.

Proof By [6, 7] we can suppose that A is embedded in a power of an ultrapower of $[0, 1]$, say $A \subseteq ([0, 1]^*)^I$.

$(BP \rightarrow B_{n,k})$ Suppose A is in *BP*. So A contains a maximal ideal M with $A/M = \{0, 1\}$.

For $x \in A$ let

$$A_{n,x} = \{i \in I \mid x^i \leq 1/n \; OR \; \neg x^i \leq 1/n\}.$$

Note $A_{n,x} = A_{n,\neg x}$.

Then the sets $A_{n,x}$ with n positive and $x \in A$ must enjoy the finite intersection property. In fact, suppose the finite intersection property is false.

Then $A_{n_1,x_1} \cap \ldots \cap A_{n_k,x_k} = \emptyset$. Up to exchange x with $\neg x$ we can suppose $x_1, \ldots, x_k \in M$, and up to take the maximum of $n_1, \ldots, n_k$ we can suppose $n_1 = \ldots = n_k = n$.

Now for every $i \in I$, if $x_1^i, \ldots, x_{k-1}^i \leq 1/n$ then $x_k^i \geq 1/n$. So $x_1 \vee \cdots \vee x_k \geq 1/n$, and multiplying by n we have $n(x_1 \vee \ldots \vee x_k) = 1$, and since M is an ideal we have $1 \in M$, contrary to the definition of maximal ideal. This proves the finite intersection property.

Now suppose $i \in A_{n+1,x_1} \cap \cdots \cap A_{n+1,x_k}$. Then

$$NOT(x_1^i \wedge \neg x_1^i) \vee \cdots \vee (x_k^i \wedge \neg x_k^i) \geq 1/n$$

and the left hand side is the i-th component of $(x_1 \wedge \neg x_1) \vee \cdots \vee (x_k \wedge \neg x_k)$, so the latter cannot be greater than $1/n$. This proves $B_{n,k}$.

$(B_{n,k} \rightarrow BP)$ Conversely, suppose A verifies $B_{n,k}$, that is

$$NOT(x_1 \wedge \neg x_1) \vee \cdots \vee (x_k \wedge \neg x_k) \geq 1/n.$$

Then, the sets $A_{n,x}$ with $x \in A$ and n arbitrary must enjoy the finite intersection property as above.

In fact, fix $n, x_1, \ldots, x_k$; there is an index i such that

$$NOT(x_1^i \wedge \neg x_1^i) \vee \cdots \vee (x_k^i \wedge \neg x_k^i) \geq 1/n,$$

in other words

$$(x_1^i \wedge \neg x_1^i) \vee \cdots \vee (x_k^i \wedge \neg x_k^i) < 1/n,$$

or by eliminating the supremum

$$(x_1^i \wedge \neg x_1^i) < 1/n \; AND \cdots AND (x_k^i \wedge \neg x_k^i) < 1/n,$$

so

$$i \in A_{n,x_1} \cap \cdots \cap A_{n,x_k}.$$

By the finite intersection property there is an ultrafilter U on I which contains $A_{n,x}$ for every n, x. By definition of ultrafilter, for every n, x, either U contains $\{i \mid x^i \leq 1/n\}$ or contains $\{i \mid \neg x^i \leq 1/n\}$. Now consider the set

$$M = \{x \in A \ \mid \forall^\infty n \ \{i \mid x^i \leq 1/n\} \in U\}$$

where $\forall^\infty n$ means that the property is true for every n except for a finite number.

We note that M is an ideal of A and also that for every $x \in A$ we have either $x \in M$ or $\neg x \in M$. It follows that M is a maximal ideal and $A/M = \{0, 1\}$. □

Now let us continue the investigation:

Lemma 7 *Let A be an MV-algebra embedded in a power of an ultrapower* $([0, 1]^*)^I$ *(such an embedding always exists by [6]).*

An element x of A has the form $y \wedge \neg y$ if and only if $x \leq 1/2$.

Proof Clearly $y \wedge \neg y \leq 1/2$ for every $y \in A$. Conversely, if $x \leq 1/2$, then $x \leq \neg x$, so $x = x \wedge \neg x$. □

By the previous lemma and the definition of $B_{n,k}$ we have:

Corollary 5 *An MV-algebra verifies $B_{n,k}$ if and only if for every $x_1, \ldots, x_k \leq 1/2$ one has $NOT(x_1 \vee \ldots \vee x_k \geq 1/n)$.*

So we have the following simplification:

Corollary 6 *For every n, k, $B_{n,k}$ is equivalent to $B_{n,1}$.*

Proof The negation of $B_{n,1}$ implies the negation of $B_{n,k}$. In fact, if there is $x \leq 1/2$ with $x \geq 1/n$ then we can take $x_1 = \ldots = x_k = x$.

Conversely, the negation of $B_{n,k}$ implies the negation of $B_{n,1}$. In fact, if for some $x_1, \ldots, x_k \leq 1/2$ one has $x_1 \vee \ldots x_k \geq 1/n$, then we can define $y = x_1 \vee \ldots \vee x_k$ and we have $y \leq 1/2$ and $y \geq 1/n$. □

Before the concluding theorem we need a technical lemma on McNaughton functions:

Lemma 8 *For every $n \geq 4$ there is a McNaughton function $f : [0, 1] \to [0, 1]$ sending the interval $[1/(n + 1), 1/2]$ to $[1/n, 1/2]$.*

Proof We use the theory of McNaughton functions (or Z-maps) developed in [17].

The set

$$T = \{1/(n + 1), 1/n, \ldots, 1/2\}$$

is a regular triangulation (or segmentation in our case) of the interval $I = [1/(n + 1), 1/2]$.

In fact, let $r \in I$ be a rational element different from $1/(n + 1), \ldots, 1/2$. Then $r = p/d$ where $p \geq 2$ and $(p, d) = 1$. For some q one has $1/(q + 1) < r < 1/q$, hence $p/p(q+1) < p/d < p/pq$, hence $d > pq$, $d \geq pq+1 \geq 2q+1 = q+q+1$. So the denominator of r is at least the sum of the denominators of $1/(q+1)$ and $1/q$. Since r is arbitrary, the segmentation is regular.

Hence by [17, Lemma 3.7.iii] there is a McNaughton function $\eta : [0, 1] \to [0, 1]$ such that:

1. η sends $1/(n+1)$ to $2/(n+1)$;
2. η fixes $1/n, \ldots, 1/2$;
3. η is linear on $[1/q+1, 1/q]$ for every q with $2 \leq q \leq n$.

Since η is linear in each piece of the segmentation, the range of η is included in the smallest interval J containing the images of the vertices of T, which are $2/(n+1)$, $1/n, \ldots, 1/2$. Since $1/n \leq 2/(n+1) \leq 1/2$, this interval is $J = [1/n, 1/2]$. □

Here is the final collapse result:

Theorem 22 *The $B_{n,1}$ hierarchy collapses.*

Proof As usual we can suppose that A is embedded in $([0, 1]^*)^I$. It is enough to show that for $n \geq 4$, if an MV-algebra A contains an element x such that $1/(n+1) \leq x \leq 1/2$, then it contains an element y such that $1/n \leq y \leq 1/2$.

Now suppose $1/(n+1) \leq x \leq 1/2$. All the components of x are comprised between $1/(n+1)$ and $1/2$. By the previous lemma, there is a unary McNaughton function f which sends the interval $[1/(n+1), 1/2]$ into the interval $[1/n, 1/2]$. Now it is enough to put $y = f(x)$. □

Since the $B_{n,k}$ hierarchy collapses we get:

Corollary 7 *The class* BP *is finitely axiomatizable in first order logic.*

Finally we can make a remark on abelian ℓ-groups.

We have seen that spectral MV-algebras are axiomatizable in first order logic, say via a formula β. So, the class of unital abelian ℓ-groups (G, u) such that $\Gamma(G, u)$ is a spectral MV-algebra is also finitely axiomatizable via a formula β', which is obtained from β by restricting quantifiers to the interval $[0, u]$. This follows from the fact that the MV-algebra $\Gamma(G, u)$ is definable, in first order logic, in the unital ℓ-group (G, u).

The same holds for abelian bipartite unital ℓ-groups.

4.2 Bipartite MV-algebras and Bayes' property

In this section, we give some remarks related to the relationship between bipartite MV-algebras and Bayes' property contained in Rybárik's paper [21]. As explained in the latter paper, the entropy of partitions on probabilistic spaces was introduced by Kolmogorov and Sinaj [16, 22] as a criterion to distinguish nonisomorphic dynamical systems. In this section we collect a few results on the entropy of partitions in MV-algebras.

For the sake of completeness we recall some definitions inspired by (but not exactly copied from) [21].

Notation 1 *Given two elements x, y of an MV-algebra A, we write $x \perp y$ if $x \odot y = 0$.*

Recall that when $x \perp y$ we write $x + y = x \oplus y$. So the operation $+$ is a partial binary operation on MV-algebras.

Definition 7 A state on an MV-algebra A is a map $s : A \to [0, 1]$ such that $s(1) = 1$ and $s(x \oplus y) = s(x) + s(y)$ when $x \perp y$.

Definition 8 A finite system $P = (x_1, x_2, \ldots, x_k)$ in A is said to be an $\oplus$-orthogonal system if

$$\left(\bigoplus_{i=1}^{l} x_i\right) \perp x_{l+1}$$

for every $l = 1, \ldots, k-1$. P is said to be a partition of A with respect to the state s if
(P1) P is an $\oplus$-orthogonal system.
(P2) $s(\bigoplus_{i=1}^{k} x_i) = 1$.

Definition 9 We define partition of unity in A any finite sequence $(x_1, \ldots, x_p)$ such that $x_1 + \cdots + x_p = 1$.

Obviously, each partition according to Definition 9 is also a partition with respect to an arbitrary state.

Definition 10 A state s has Bayes' property if it satisfies the following condition:
Let $(x_1, x_2, \ldots, x_l)$ be any partition corresponding to a state s and $y \in A$, then

$$s\left(\oplus_{j=1}^{l}(y \odot x_j)\right) = s(y)$$

Definition 11 Let $P = (x_1, x_2, \ldots, x_k)$ and $Q = (y_1, y_2, \ldots, y_l)$ be two partitions of A corresponding to a state s. Then the common refinement of these partitions will be defined as the system

$$P \cup Q = \{x_i \odot y_j : x_i \in P, y_j \in Q, i \in \{1, 2, \ldots, k\}; j \in \{1, 2, \ldots, l\}\}$$

The next lemma clarifies a condition that ensures $P \cup Q$ being a partition:

Lemma 9 *If the state s has Bayes' property, and P, Q are partitions with respect to s, then also the system $P \cup Q$ is a partition with respect to s.*

Proof See [21, Lemma 5]. □

Example 2 Consider a subset A of the interval $[0, 1]$ of real numbers such that $0 \in A$, and if $x, y \in A$, then

$$x \oplus y = \min(1, x + y) \in A, \ x \odot y = \max(0, x + y - 1) \in A, \ \neg x = 1 - x \in A$$

where symbols $+$ and $-$ denote the usual sum and difference of real numbers. Let $P = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $Q = (\frac{1}{2}, \frac{1}{2})$ be two partitions corresponding to the identity state. We have $\frac{1}{3} \odot \frac{1}{2} = 0$, hence $P \cup Q$ is not a partition any more.

We mention the following, apparently open problem:

Question 1 *Let $P = (x_1, x_2, \ldots, x_k)$ and $Q = (y_1, y_2, \ldots, y_l)$ be two partitions of A with respect to a state s. If $P \cup Q$ is a partition with respect to s for every P, Q, then does s have Bayes' property?*

The question, if answered positively, coupled with its converse Lemma 9, could help clarify the theory of Bayes states.

We will now investigate which MV-algebras possess a state m with Bayes' property and which ones do not. In Boolean algebras there are plenty of states with Bayes' property, because every homomorphism in $\{0, 1\}$ is a state. Other examples of MV-algebras which admit states with Bayes' property are contained in Example 3:

Definition 12 A is a *Bayes MV-algebra* if it admits a state with Bayes' property.

Example 3 Let h be an MV-homomorphism from A to $\{0, 1\}$, then h is a state with Bayes' property.

In particular,

(i) Boolean algebras are Bayes;
(ii) Let A be the MV-algebra of all sequences converging to either 0 or 1, and consider the state s which maps every sequence to its limit, then s has Bayes' property;
(iii) Let $A = \{0, 1\} \times [0, 1]$, then it is Bayes.

Now we give the following definition, which apparently generalizes idempotent MV-algebras (that is, Boolean algebras), but actually it collapses to them:

Definition 13 We will say that an MV-algebra is *weakly idempotent* if

$$x \odot x = 0 \implies x = 0$$

Obviously, Boolean algebras are weakly idempotent while MV-chains of length greater than 2 (and MV-algebras which contain them) are not.

Conversely:

Theorem 23 *Every weakly idempotent MV-algebra A is a Boolean algebra. So weakly idempotent MV-algebras and Boolean algebras coincide.*

Proof First, we claim that Chang distance d in $[0, 1]$ verifies

$$d(x, x \oplus x) \odot d(x, x \oplus x) = 0$$

To verify the claim, we distinguish the cases $x \leq 1/2$ and $x \geq 1/2$. If $x \leq 1/2$ then $d(x, x \oplus x) = x \leq 1/2$ so $d(x, x \oplus x) \odot d(x, x \oplus x) = 0$.. If $x \geq 1/2$ then $x \oplus x = 1$ so $d(x, x \oplus x) = d(x, 1) = 1 - x \leq 1/2$ and again $d(x, x \oplus x) \odot d(x, x \oplus x) = 0$.

Since $[0, 1]$ generates the variety of MV-algebras, the claim extends to every MV-algebra.

Suppose the MV-algebra A is weakly idempotent. Then we have $d(x, x \oplus x) = 0$ for every x. As d is a metric we get $x \oplus x = x$ for every x. By Theorem 1 the MV-algebra is a Boolean algebra. □

Definition 14 A state s on an MV-algebra A is called idempotent if $s(x \odot x) = s(x)$ for every $x \in A$.

Definition 15 Given a state s on an MV-algebra A, we denote by $N(s)$ the set of elements x of A such that $s(x) = 0$.

Proposition 24 *Let A be a MV-algebra. There is an idempotent state s on A if and only if there exists an MV-algebra homomorphism from A to $\{0, 1\}$.*

Proof Suppose there is an idempotent state s on A. $N(s)$ is an ideal of A, and since s is idempotent, $A/N(s)$ is a weakly idempotent MV-algebra. By Theorem 23 there is an MV-homomorphism from $A/N(s)$ to $\{0, 1\}$, hence there exists an MV-homomorphism from A to $\{0, 1\}$.

Vice versa suppose that there exists a homomorphism f from A to $\{0, 1\}$. It is easy to check that f itself is an idempotent state. □

Proposition 25 *Let A be an MV-algebra and s be a state on A with Bayes' property, then s is idempotent.*

Proof Suppose s is a state on A with Bayes' property. Let $x \in A$. By applying Bayes property to the partition $(x, \neg x)$ we obtain

$$s(x) = s(x \odot x) + s(x \odot \neg x) = s(x \odot x) + s(0) = s(x \odot x).$$

□

Moreover we have the following relation between a state s and its null set $N(s)$:

Proposition 26 *The state s is idempotent if and only if the quotient $A/N(s)$ is Boolean.*

Proof Suppose s is an idempotent state. Then $A/N(s)$ is weakly idempotent since $x \in N(s)$ if and only if $x \odot x \in N(s)$, and by Theorem 23, $A/N(s)$ is Boolean.

Conversely, suppose $A/N(s)$ is Boolean. Then for every $x \in A$, x and $x \odot x$ are congruent modulo $N(s)$, so $d(x, x \odot x) \in N(s)$ and $s(d(x, x \odot x)) = 0$. In particular $s(x \ominus (x \odot x)) = 0$, but $x \odot x \leq x$, so $s(x) = s(x \odot x)$, and s is idempotent. □

Proposition 27 *An MV-algebra A is Bayes if and only if it admits an idempotent state.*

Proof By Proposition 25 every state with Bayes' property is idempotent.

Vice versa, suppose that there is an idempotent state s on A. By Proposition 26 the MV-algebra $A/N(s)$ admits a state $\hat{m}$ with Bayes' property, since it is a Boolean algebra. Then the map m defined by elements $m(x) := \hat{m}(x \; mod \; N(s))$ is a state of A with Bayes' property.

In fact, clearly m is a state. We verify Bayes' property. Write $\hat{x} = x \; mod \; N(s)$. Let $P = (y_1, y_2, \ldots, y_l)$ be any partition corresponding to m. Then $\hat{P} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_l)$ is a partition corresponding to $\hat{m}$. Since $\hat{m}$ has Bayes' property, we have

$$\hat{m}(\hat{x}) = \hat{m}(\oplus_{j=1}^{l}(\hat{x} \odot \hat{y}_j))$$

whence

$$m(x) = \hat{m}(\hat{x}) = \hat{m}(\oplus_{j=1}^{l}(\hat{x} \odot \hat{y}_j)) = \hat{m}(\widehat{\oplus_{j=1}^{l}(x \odot y_j)}) = m(\oplus_{j=1}^{l}(x \odot y_j))$$

in other words m has Bayes' property. □

Theorem 28 *An MV-algebra A is Bayes if and only if it is bipartite.*

Proof It follows from Propositions 24 and 27. □

The theorem implies that Bayes' property is not stable under quotients, because we know that bipartite MV-algebras are not closed under quotients; other information is given in the following

Lemma 10 *1. Let A, F be two MV-algebras and let s be a state on A with Bayes' property, let $h : F \to A$ be an MV-homomorphism. Then $s \circ h$ is a state on F with Bayes' property.*
2. If A is Bayes, then for each MV-algebra F $A \times F$ is Bayes (in particular Bayes MV-algebras are closed under product).
3. Bayes MV-algebras are closed under subalgebras.
4. Bayes MV-algebras are closed under ultraproducts.

Proof The proof of item 1 is straightforward.

The proof of 2 follows from 1 taking for h the projection from $E \times F$ to A.

3 holds because a restriction of a state with Bayes property is still a state with Bayes property.

Item 4 is true since Bayes MV-algebras coincide with bipartite MV-algebras, so they are first order definable. □

Example 4 In [0, 1] there is only one state s (like in every linearly ordered MV-algebra) and $s(x) = x$ for every x, so $s(\frac{1}{2}) = \frac{1}{2}$, and

$$s\left(\frac{1}{2} \odot \frac{1}{2} + \frac{1}{2} \odot \frac{1}{2}\right) = s(0) = 0,$$

hence s does not have Bayes' property and the MV-algebra [0, 1] is not Bayes.

More generally, recall that a Riesz MV-algebra is a MV-algebra together with an external product from $[0, 1] \times A$ to A satisfying a few axioms, see [11]. A state of a Riesz MV-algebra is defined as a state of its MV-algebra reduct.

Example 5 Riesz MV-algebras do not admit any states with Bayes' property. In fact, they contain an isomorphic copy of [0, 1] and the latter MV-algebra is not Bayes.

Since Riesz MV-algebras do not admit any states with Bayes' property, it's a natural question to find out a new definition of entropy in the framework of Riesz MV-algebras.

Now we focus our attention on finite MV-algebras.

Recall that

Proposition 29 *Any finite MV-algebra is isomorphic to a product* $\Pi_\alpha S_{n_\alpha}$ *for* α *in a finite set* A *and* $n_\alpha \in \mathbb{N}$.

Proof See [5, Corollary 3.5.4] or [2, Corollary 4.2.8] □

Proposition 30 *A finite MV-algebra* A *is Bayes if and only if at least one of its linear factors is* $\{0, 1\}$.

Proof The simple quotients of A coincide with its linear factors. □

We can apply this proposition also to infinite MV-algebras, for instance:

Example 6 Let A be the MV-algebra of all sequences in $[0, 1]^\omega$ converging to either 0 or $\frac{1}{2}$ or 1. Then A is not Bayes. In fact, A contains an isomorphic copy of $\{0, 1/2, 1\}$, which is not Bayes by the previous proposition.

5 The inverse spectrum problem

Informally, the problem we are referring to is to investigate the MV-algebras with a given prime spectrum. We will see that the prime spectrum is much more informative than the maximal spectrum: in fact, the MV-algebras with a given prime spectrum are (up to isomorphism) a set, whereas the MV-algebras with a given maximal spectrum are (up to isomorphism) a proper class.

Note that from the previous sections it follows:

Proposition 31 *Given two MV-algebras* A, B, *the following are equivalent:*

1. *the topological spaces* $Spec(A)$ *and* $Spec(B)$ *are homeomorphic;*
2. *the lattices* $Bell(A)$ *and* $Bell(B)$ *are isomorphic;*
3. *the lattices* $id_c(A)$ *and* $id_c(B)$ *are isomorphic.*

Definition 16 We say that MV-algebras have the same E-class whenever they are equivalent according to Proposition 31.

For the finite case we have:

Proposition 32 *Let* A *be an MV-algebra. Then the following are equivalent:*

1. $Bell(A)$ *is finite;*
2. $id_c(A)$ *is finite;*
3. $Spec(A)$ *is finite.*

Proof The first and second point are equivalent trivially since the lattices $Bell(A)$ and $id_c(A)$ are isomorphic for every MV-algebra A.

If $Spec(A)$ is finite, then $Bell(A) \cong K(Spec(A))$, where $K(Spec(A))$ is the lattice of compact open sets of $Spec(A)$; so $Bell(A)$ is finite, too.

Conversely, if $Bell(A)$ is finite, then $Spec(A) = Spec(Bell(A))$ is the spectrum of a finite lattice, so it is finite. □

By the previous proposition we can give the following definition:

Definition 17 An MV-algebra is of finite type if it satisfies any of the three equivalent properties of Proposition 32.

Note that every finite MV-algebra has finite type, but not the other way round (for instance [0, 1] has finite type).

Given two perfect chains A, B, if $Spec(A)$ and $Spec(B)$ are isomorphic, then A, B are not isomorphic in general: consider the MV-algebra C defined above and the variant

$$C_{\mathbb{R}} = \Gamma(\mathbb{Z}\ lex\ \mathbb{R}, (1, 0)).$$

Both $id_c(C)$ and $id_c(C_{\mathbb{R}})$ are distributive lattices with two elements, so they are isomorphic, whereas C is countable and $C_{\mathbb{R}}$ is uncountable, so C and $C_{\mathbb{R}}$ are not isomorphic.

Note that [6, Theorem 2] shows that every bounded linear order is isomorphic to the Belluce lattice of an MV-algebra (necessarily linear). In fact, it is known that:

Proposition 33 (see [6]) *For every bounded, linearly ordered lattice L there is a canonical perfect MV-chain $A(L)$ such that $Id_c(A(L)) \cong L$.*

Proof The MV-chain $A(L)$ is obtained by taking the group which is a lexicographic direct sum $\hat{\mathbb{Z}}_L$ of copies of $\mathbb{Z}$, one for every element of L, and then letting $A(L) = \Gamma(\hat{\mathbb{Z}}_L, 1_L)$, where 1_L is the sequence which has component 1 at the maximum element of L and 0 elsewhere. □

We put a definition coming from the theory of archimedean ℓ-groups:

Definition 18 (*commensurability*) Let x, y be elements of an ℓ-group. We say that x *dominates* y is $n \mid x \mid \geq \mid y \mid$ for some $n \in \mathbb{N}$. The elements x, y are *commensurable* if they dominate each other.

Let instead consider x, y elements of an MV-algebra. We say that x *dominates* y is $nx \geq y$ for some $n \in \mathbb{N}$. The elements x, y are *commensurable* if they dominate each other.

Moreover we have the following:

Proposition 34 *For every MV-chain A there is a perfect MV-chain B such that:*

1. *A, B have the same spectrum;*
2. *B has at most the same size as $max(\aleph_0, id_c(A))$;*
3. *in every principal ideal I of B, there is a generator g of I such that every element of I is infinitely close to a multiple of g.*

Proof By compactness we can construct an MV-chain U containing a subset S of pairwise incommensurable infinitesimals, such that S is order isomorphic to $id_c(A)$. Let B be the subalgebra of U generated by S.

Claim: $id_c(A) \cong id_c(B)$.

To prove the claim, first we can show that every infinitesimal element x of B is commensurable with an element of S. This can be shown by writing $x = p(s_1, \ldots, p_n)$

where p is an MV-polynomial and $s_1, \dots, s_n \in S$, and by proceeding by induction on p.

Moreover, different elements of S generate different principal ideals. So, there is an increasing bijection between $id_c(A)$ and S, and also one between $id_c(B)$ and S, and this proves the claim. □

So we have a map from the class of bounded linear orders to the class of MV-chains. Can this map be made functorial? This is an open problem to our knowledge.

5.1 Least and greatest chains

This subsection investigates smallest and largest chains with given spectrum.

We want to study the commensurability relation for MV-algebras defined above. However we note that the theory is much simpler if the MV-algebras are linearly ordered.

We stick to MV-chains for simplicity. In fact we have:

Lemma 11 *Let A be an MV-chain. There is an order preserving bijection between the commensurability classes of A and the principal ideals of A.*

Proof Let $x, y \in A$, then x, y generate the same principal ideal if and only if they belong to the same commensurability class. □

In an MV-algebra we can define $x + y = x \oplus y$ when $x \perp y$, and we can define $x - y = x \ominus y$ when $x \geq y$. So a partial sum and a partial subtraction are defined.

Definition 19 A subset G of an MV-algebra A is a *set of generators over* $\mathbb{Z}$ if every $x \in A$ can be uniquely written as $x = \Sigma_i n_i g_i$, where $g_i \in G$ and $n_i \in \mathbb{Z}$.

Lemma 12 *Let L be a bounded linear order. Let $A(L)$ be the MV-chain recalled in Proposition 33 and let B every MV-chain with spectrum homeomorphic to the one of $A(L)$. Then $A(L)$ embeds in B.*

Proof Let $\phi : id_c(A(L)) \to id_c(B)$ be an order isomorphism.

Recall the definition of $A(L)$ in the proof of Proposition 33. Let G be the set of sequences in $A(L)$ where one component is 1 and the others are 0. G is a pairwise incommensurable set of generators of $A(L)$.

Every element of $A(L)$ is a unique finite sum of elements of G. For every $g \in G$, let g' be a generator of $\phi(ideal(g))$.

This gives an injective embedding η of $A(L)$ into B. That is, we let $\eta(\Sigma_i n_i g_i) = \Sigma_i n_i g'_i$, where n_i are integers (possibly negative). □

In general, in an MV-algebra A there is no smallest subalgebra of A equispectral with A (smallest with respect to inclusion).

Example 7 Consider the perfect MV-chain $A(L)$ where L is the rational interval in $[0, 1]$. Let Q' be a proper subset of L isomorphic to L and containing 0 and 1. Let B be the subalgebra of $A(L)$ generated by the principal generators of the elements of Q'. Then $id_c(B)$ is order isomorphic to the rationals, so B is a proper subalgebra of $A(L)$ and $id_c(B)$ is isomorphic to $id_c(A(L))$.

We can also obtain a *greatest* perfect MV-chain with a given spectrum. To this aim we first recall the classical Hahn embedding theorem for totally ordered abelian groups.

Let Ω be a totally ordered set and let G be a totally ordered abelian group. We will say that the *Hahn group* $H(G, \Omega)$ is the group of all functions from Ω to G which are zero outside a well ordered set. Addition is defined componentwise. An element of $H(G, \Omega)$ is defined to be positive if the smallest nonzero component is positive. The theorem reads as follows:

Theorem 35 [13] (Hahn embedding theorem) *Let G be a totally ordered abelian group with a linearly ordered set Ω of archimedean classes. Then G embeds in the Hahn group $H(G, \Omega)$. Moreover the ordered set of archimedean classes of $H(G, \Omega)$ is isomorphic to Ω.*

Now we can derive from Hahn theorem the following result:

Proposition 36 *For every perfect MV-chain A there is a perfect MV-chain B which is maximal in the E-class of A.*

Proof Let us call Δ the equivalence of [10] between abelian ℓ-groups and perfect MV-algebras. Δ restricts to an equivalence between totally ordered abelian groups and perfect MV-chains. Let A be a perfect MV-chain and let $\Omega = id_c(A)$. Let $G = \Delta^{-1}(A)$. Let $B = \Delta(H(G, \Omega))$. From Theorem 35 it follows that G embeds in $H(G, \Omega)$ so A embeds in B. □

If we vary the principal ideal lattice Ω, the category of MV-algebras or abelian ℓ-groups with $id_c = \Omega$ may have different behavior. An example is the Cantor-Bernstein property.

We will say that a category verifies the *Cantor-Bernstein property* if two pairwise monomorphic objects are isomorphic. This property is notoriously false for MV-algebras in general. Note that there are "small" types for which the property is true and "large" types where the property is false. Here are two examples.

Proposition 37 *For the category of the totally ordered abelian groups G with $id_c(G) = \mathbb{Q}$ the Cantor Bernstein property is false.*

Proof Consider $G_1 = H(\mathbb{Z}, \mathbb{Q}) lex\ H(\mathbb{R}, \mathbb{Q})$ and $G_2 = H(\mathbb{R}, \mathbb{Q})$, where lex denotes the lexicographic product of totally ordered abelian groups. Then G_1 and G_2 embed into each other and have isomorphic lattices of principal ideals, that is, both lattices are isomorphic to $\mathbb{Q}$. However G_1 and G_2 are not isomorphic, since G_2 is divisible and G_1 is not. □

Proposition 38 *For the category of MV-chains A with $\mid id_c(A) \mid = 2$ (that is simple MV-algebras) the Cantor Bernstein property is true.*

Proof It follows from the known fact that no simple MV-algebra has a proper subalgebra isomorphic to itself. □

Conjecture 1 *Let n be a natural number. For the category of MV-chains A with $\mid id_c(A) \mid = n$ the Cantor Bernstein property is true.*

5.2 Specializing classical equivalences

Let Ω be any E-class. The two classical dualities of [18] and [10] preserve the archimedean type, so we have:

Proposition 39 *The equivalence Γ specializes to an equivalence between the category of MV-algebras of E-class Ω and the category of unital abelian ℓ-groups of E- class Ω.*

Corollary 8 *The equivalence Γ specializes to one between the category of MV-chains of E-class Ω and the category of unital totally ordered abelian groups of E-class Ω.*

Proposition 40 *The equivalence Δ specializes to an equivalence between the category of perfect MV-algebras of E-class Ω and the category of abelian ℓ-groups of E-class Ω.*

Corollary 9 *The equivalence Δ specializes to one between the category of perfect MV-chains of E-class Ω and the category of totally ordered abelian groups of E-class Ω.*

The last equivalence implies the existence of an interesting endofunctor:

Proposition 41 *For every E-class Ω, the perfect part functor $Perf$ is an endofunctor from the category C of MV-chains of E-class Ω and the category D of perfect MV-chains of E-class Ω. Moreover $Perf$ is a retraction.*

Proof $Perf$ is a retraction because D embeds in C via the subset functor j, and $j(Perf(d)) = d$ for every $d \in D$. □

Note that we do not expect that the previous proposition extends to perfect MV-*algebras*, because the E-class of an MV-algebra is different from the one of its perfect part.

5.3 The category of MV-algebras of a given E-class

In this subsection we prove that the category of MV-algebras of a given E-class is small like the category of abelian ℓ-groups of a given E-class.

Theorem 42 *The category of MV-chains with a given E-class is small in the sense of category theory (that is the isomorphism classes of objects form a set) and has a greatest element.*

Proof By Hahn embedding theorem, the category of (unital) totally ordered abelian groups with given E-class is small and has a greatest element. Then, the result follows by Mundici equivalence. □

Theorem 43 *The category of MV-algebras of a given E-class is small.*

Proof Let A be an MV-algebra. We have a subdirect embedding

$$A \subseteq \Pi_{P \in Spec(A)} A/P.$$

The MV-algebras A/P are MV-chains with prime spectrum embeddable in the one of A, so they range over a set (depending only on $Spec(A)$) by the previous theorem. □

In the same vein we have:

Theorem 44 *The category of abelian ℓ-groups of a given E-class is small.*

Proof By the previous section, it is enough to prove that the category of perfect MV-algebras of a given E-class is small. But this follows from the previous theorem. □

5.4 The maximal case

In this subsection we show that Theorem 43 fails for the *maximal* spectrum. To prove this we need the notion of free product. By [19] any two MV-algebras A, B have a free product $A \coprod B$, which is an MV-algebra such that:

Lemma 13 *Any two MV-algebras A and B embed in* $A \coprod B$*. Moreover the topological spaces* $Max(A \coprod B)$ *and* $Max(A) \times Max(B)$ *are homeomorphic.*

Proof (i) The first claim follows from the definition of free product.

(ii) The maximal spectrum of any MV-algebra M is homeomorphic to the set of homomorphisms from M to $[0, 1]$. So it is enough to prove

$$Hom(A \coprod B, [0, 1]) \cong Hom(A, [0, 1]) \times Hom(B, [0, 1]).$$

Actually we exhibit a homeomorphism ϕ. Let $i_A : A \to A \coprod B$, $i_B : B \to A \coprod B$ the canonical embeddings. Given $h : A \coprod B \to [0, 1]$, define

$$\phi(h) = (h \circ i_A, h \circ i_B).$$

By definition of free product, ϕ is a bijection.

Moreover, let us prove that ϕ is continuous. In fact, both the maps $h \to h \circ i_A$ and $h \to h \circ i_B$ are continuous: given any basic open $A_x = \{k : A \to [0, 1] \mid k(x) = 0\}$, where $x \in A$, we have $h \circ i_A \in A_x$ if and only if $h \circ i_A(x) = 0$, that is $h(i_A(x)) = 0$, that is h is in $\{k : A \coprod B \to [0, 1] \mid k(i_A(x)) = 0\}$ which is open in $Hom(A \coprod B, [0, 1])$. So the preimage under $h \to h \circ i_A$ of any basic open is open, and the map $h \to h \circ i_A$ is continuous. Similarly the map $h \to h \circ i_B$ is continuous. Since $Max(A \coprod B)$ and $Max(A) \times Max(B)$ are compact Hausdorff spaces and there is a bijective continuous function between them, this function is a homeomorphism. □

We note that Theorem 43 fails for the *maximal* spectrum in the following strong sense:

Theorem 45 *For every compact Hausdorff topological space S, the MV-algebras whose maximal spectrum is S are (up to isomorphism) a proper class.*

Proof We know that S is a closed subset of some hypercube $[0, 1]^X$ for some set X, and S is the maximal space of the MV-algebra $C(S)$ of McNaughton functions on $[0, 1]^X$ restricted to S.
If B is any MV-chain then $Max(B)$ has one point, so by the previous lemma, $Max(A \coprod B) \cong Max(A)$. Taking $A = C(S)$ we have $Max(C(S) \coprod B) \cong S$. Since B embeds in $C(S) \coprod B$, and MV-chains are a proper class by Löwenheim-Skolem theorem, the MV-algebras of the form $C(S) \coprod B$ where B is an MV-chain are themselves a proper class, and all of them have maximal spectrum S. □

6 Conclusions

In this paper we give another contribution to understanding the relation between an MV-algebra and its prime spectrum and, more generally, with its ideals. The results may be generalized in several ways. For instance, bipartite MV-algebras can be generalized to k-partite MV-algebras, where there is a homomorphism from the MV-algebra into the MV-chain with k elements. Moreover, the study of equispectrality classes, especially their categorical properties, seems promising. The Cantor Bernstein property is only an example. So there is room for further investigations in future papers.

Funding Open access funding provided by Università degli Studi di Salerno within the CRUI-CARE Agreement.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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