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On undecidability of the propositional logic of an associative binary modality

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Abstract
It is shown that both classical and intuitionistic propositional logics of an associative binary modality are undecidable. The proof is based on the deduction theorem for these logics.

Keywords Logic of a binary modality · Deduction theorem · Undecidability · Relational semantics

Mathematics Subject Classification 03B45 · 03D35

1 Introduction

This paper deals with the classical and intuitionistic modal logics which are obtained from the respective propositional logics by extending their language with the binary modal connective $\bullet$ and adding the following axiom schemes and rules of inference.

The K-like axioms (left and right):

$$(\mathrm{K_L})\ \psi \bullet (\varphi \supset \chi) \supset (\psi \bullet \varphi \supset \psi \bullet \chi) \quad (\mathrm{K_R})\ (\varphi \supset \chi) \bullet \psi \supset (\varphi \bullet \psi \supset \chi \bullet \psi)^{1}$$

The associativity axioms:

$$(\mathrm{A_L})\ (\varphi \bullet \chi) \bullet \psi \supset \varphi \bullet (\chi \bullet \psi) \quad (\mathrm{A_R})\ \varphi \bullet (\chi \bullet \psi) \supset (\varphi \bullet \chi) \bullet \psi$$

The *necessitation* rules of inference (left and right):

[1] As usual, $\bullet$ binds more closely than the propositional connectives, but negation. We also omit the parentheses, when no ambiguity arises.

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$$(\mathrm{NEC_L})\ \frac{\varphi}{\chi \bullet \varphi} \quad (\mathrm{NEC_R})\ \frac{\varphi}{\varphi \bullet \chi}$$

These classical and intuitionistic logics will be denoted by $\boldsymbol{CL}^\bullet$ and $\boldsymbol{IL}^\bullet$, respectively. The former is tightly related to *relation algebras*, cf. [5–7], [3, Section 5.3], and [1, Section 5.2]. Namely, it is the logical counterpart of the fragment of a relational algebra consisting of a Boolean algebra augmented with a binary associative operator that is distributive under ordinary Boolean addition. In particular, the logical counterpart of that binary operator is the De Morgan dual of $\bullet$ defined in Sect. 4.1, and, naturally, the logical counterparts of the Bolean operators are the corresponding propositional connectives.

We assume that both $\boldsymbol{CL}^\bullet$ and $\boldsymbol{IL}^\bullet$ contain *falsity* $\bot$ and abbreviate $\neg\bot$ by $\top$. In what follows, $\boldsymbol{L}^\bullet$ may be any of $\boldsymbol{CL}^\bullet$ or $\boldsymbol{IL}^\bullet$ and the classical and intuitionistic propositional logics will be denoted by $\boldsymbol{CL}$ and $\boldsymbol{IL}$, respectively.

It is known from [10, 11] that $\boldsymbol{CL}^\bullet$ is undecidable. Namely, it is shown in [10, 11] that the equational theory of the Lindenbaum–Tarski algebra $\mathrm{LTA}_{\boldsymbol{CL}^\bullet}$ of $\boldsymbol{CL}^\bullet$ is undecidable, which is equivalent to undecidability of $\boldsymbol{CL}^\bullet$ itself. The proof in [10, 11] consists of first, restating the problem in an algebraic setting and then, translating the word problem of semigroups (that is undecidable) to equations in $\mathrm{LTA}_{\boldsymbol{CL}^\bullet}$, in a validity preserving way, see [11, Section 2] for details.

Such a detour obscures a logical nature of the problem: semigroups come from *concatenation* and the translation of the word problem of semigroups to equations in $\mathrm{LTA}_{\boldsymbol{CL}^\bullet}$ is, actually, the deduction theorem that (equivalently) converts the consequence relation to implication.

In our undecidability proof we reduce the undecidability of $\boldsymbol{L}^\bullet$ to the undecidability of the reachability problem for semi-Thue systems (i.e., string rewriting). This is done in two stages. In the first stage, we translate the reachability in semi-Thue systems to the consequence relation between implications of a suitable logic of concatenation (i.e., a logic of the free semigroup generated by the letters of the underlying alphabet) and, in the second stage, we translate the consequence relation in the logic of concatenation to the consequence relation in the logics of the binary connective. Then, the undecidability of the latter consequence relation gives rise to the undecidability of provability in these logics by encoding the unary modal operator $\Box$ with an S4-type deduction theorem into these logics.

Actually, our approach is "logically dual" to that in [10, 11], see Sect. 5, and, in particular, undecidability of the equational theory of $\mathrm{LTA}_{\boldsymbol{CL}^\bullet}$ established in [10, 11] follows from undecidability of $\boldsymbol{CL}^\bullet$. In addition to its transparency, an advantage of our proof is that it uniformly applies to both classical and intuitionistic logics, whereas the proofs in [10, 11] are for $\boldsymbol{CL}^\bullet$ only. Also, the deduction theorem, that converts assumption to implications' premises and is our main technical tool, seems to be of interest in its own right.

This paper is organized as follows. In the next section we prove the deduction theorem for $\boldsymbol{L}^\bullet$. In Sect. 3 we define semi-Thue systems and a logic of concatenation and prove their equivalence. Then, in Sect. 4, we embed the logic of concatenation into $\boldsymbol{L}^\bullet$, thus, proving its undecidability. We conclude the paper with a comparison of our approach to that in [11]. Finally, in the appendix, we show that the Lindenbaum–

Tarski algebra of the logic of of concatenation defined in Sect. 3.2 is the free semigroup generated by the set of all propositional variables.

2 Deduction theorem for $L^{\bullet}$

The deduction theorem for $\boldsymbol{L}^{\bullet}$ (Theorem 1 below) employs the following notation. For a formula φ, we denote the formula

$$\varphi \wedge \bot \bullet \varphi \wedge \varphi \bullet \bot \wedge \bot \bullet \varphi \bullet \bot^{2} \tag{1}$$

by $\Box\varphi$, cf. the "algebraically" dual unary term $c(x)$ in [10, 11]. In fact, Lemma 3 shows that (1) behaves like $\Box\varphi$ of modal logic S4.

Theorem 1 (Deduction theorem for $\boldsymbol{L}^{\bullet}$) *Let Θ, φ, and χ be a set of formulas and two formulas, respectively. Then $\Theta, \varphi \vdash_{\boldsymbol{L}^{\bullet}} \chi$ if and only if $\Theta \vdash_{\boldsymbol{L}^{\bullet}} \Box\varphi \supset \chi$.*

For the proof of Theorem 1 we need the following properties of $\boldsymbol{L}^{\bullet}$.

First, like in modal logic K, one can prove implications (2)–(5) below, see [4, Exercise 1-1(c), p. 21], say.

$$\varphi \bullet (\chi \wedge \psi) \supset (\varphi \bullet \chi \wedge \varphi \bullet \psi) \tag{2}$$
$$(\varphi \bullet \chi \wedge \varphi \bullet \psi) \supset \varphi \bullet (\chi \wedge \psi) \tag{3}$$
$$(\chi \wedge \psi) \bullet \varphi \supset (\chi \bullet \varphi \wedge \psi \bullet \varphi) \tag{4}$$

and

$$(\chi \bullet \varphi \wedge \psi \bullet \varphi) \supset (\chi \wedge \psi) \bullet \varphi \tag{5}$$

We shall also use the well-known derivable rules given by the following proposition.

Proposition 2 *Rules of monotonicity (6)–(9) are derivable in $\boldsymbol{L}^{\bullet}$:*

$$\frac{\varphi \supset \chi}{\psi \bullet \varphi \supset \psi \bullet \chi} \tag{6}$$
$$\frac{\varphi \supset \chi}{\varphi \bullet \psi \supset \chi \bullet \psi} \tag{7}$$
$$\frac{\varphi' \supset \chi' \quad \varphi'' \supset \chi''}{\varphi' \bullet \varphi'' \supset \chi' \bullet \chi''} \tag{8}$$

and

$$\frac{\varphi \supset \chi}{\psi \bullet \varphi \bullet \omega \supset \psi \bullet \chi \bullet \omega} \tag{9}$$

[2] Since $\bullet$ is associative, we may omit the parentheses.

Proof The derivation of (6) is

1. $\varphi \supset \chi$ assumption
2. $\psi \bullet (\varphi \supset \chi)$ follows from 1 by ($\mathrm{NEC_L}$)
3. $\psi \bullet (\varphi \supset \chi) \supset (\psi \bullet \varphi \supset \psi \bullet \chi)$ axiom ($\mathrm{K_L}$)
4. $\psi \bullet \varphi \supset \psi \bullet \chi$ follows from 2 and 3 by *modus ponens*

The derivation of (7) is symmetric to that of (6).

The derivation of (8) is

1. $\varphi' \supset \chi'$ assumption
2. $\varphi'' \supset \chi''$ assumption
3. $\varphi' \bullet \varphi'' \supset \chi' \bullet \varphi''$ follows from 1 by (7)
4. $\chi' \bullet \varphi'' \supset \chi' \bullet \chi''$ follows from 2 by (6)
5. $\varphi' \bullet \varphi'' \supset \chi' \bullet \chi''$ already derivable from 3 and 4 in $\boldsymbol{IL}$

Finally, the derivation of (9) is

1. $\varphi \supset \chi$ assumption
2. $\psi \bullet \varphi \supset \psi \bullet \chi$ follows from 1 by (6)
3. $\psi \bullet \varphi \bullet \omega \supset \psi \bullet \chi \bullet \omega$ follows from 2 by (7)

□

Finally, we shall need the following properties of $\Box$.

Lemma 3 *For all formulas φ and χ,*

(i) $\vdash_{\boldsymbol{L}^\bullet} \Box\varphi \supset \chi \bullet \varphi$ *and* $\vdash_{\boldsymbol{L}^\bullet} \Box\varphi \supset \varphi \bullet \chi$*;*
(ii) $\vdash_{\boldsymbol{L}^\bullet} \Box(\varphi \supset \chi) \supset (\Box\varphi \supset \Box\chi)$*;*
(iii) $\vdash_{\boldsymbol{L}^\bullet} \Box\varphi \supset \varphi$*;*
(iv) $\vdash_{\boldsymbol{L}^\bullet} \Box\varphi \supset \Box\Box\varphi$*; and*
(v) $\varphi \vdash_{\boldsymbol{L}^\bullet} \Box\varphi$.

Remark 4 Items (ii)–(v) of the lemma are an axiomatization of modal logic S4.

Proof of Lemma 3 (i) The proof of $\vdash_{\boldsymbol{L}^\bullet} \Box\varphi \supset \chi \bullet \varphi$ is presented below and the proof of $\vdash_{\boldsymbol{L}^\bullet} \Box\varphi \supset \varphi \bullet \chi$ is symmetric.

1. $\bot \supset \chi$ axiom
2. $\varphi \bullet \bot \supset \varphi \bullet \chi$ follows from 1 by (6)
3. $\Box\varphi \supset \varphi \bullet \bot$ already derivable in $\boldsymbol{IL}$ by the definition of $\Box\varphi$
4. $\Box\varphi \supset \varphi \bullet \chi$ already derivable from 3 and 2 in $\boldsymbol{IL}$

(ii) We have

(a) $(\varphi \supset \chi) \supset (\varphi \supset \chi)$,
(b) $\bot \bullet (\varphi \supset \chi) \supset (\bot \bullet \varphi \supset \bot \bullet \chi)$;
(c) $(\varphi \supset \chi) \bullet \bot \supset (\varphi \bullet \bot \supset \chi \bullet \bot)$; and
(d) $\bot \bullet (\varphi \supset \chi) \bullet \bot \supset (\bot \bullet \varphi \bullet \bot \supset \bot \bullet \chi \bullet \bot)$;

where (a) is already derivable in $\boldsymbol{IL}$, (b) and (c) are axioms ($\mathrm{K_L}$) and ($\mathrm{K_R}$), respectively (with ψ being $\bot$), and (d) follows from (c) by (6) and ($\mathrm{K_L}$). Obviously, (ii) is already derivable in $\boldsymbol{IL}$ from (a)–(d).

(iii) This is immediate, by the definition of $\Box$.
(iv) It follows from the definition and (2)–(5) that $\Box\Box\varphi$ is $\boldsymbol{IL}$-equivalent to the conjunction of

(1) φ,
(2) $\bot \bullet \varphi$,
(3) $\varphi \bullet \bot$,
(4) $\bot \bullet \varphi \bullet \bot$,
(5) $\bot \bullet \bot \bullet \varphi$,
(6) $\varphi \bullet \bot \bullet \bot$,
(7) $\bot \bullet \bot \bullet \varphi \bullet \bot$,
(8) $\bot \bullet \varphi \bullet \bot \bullet \bot$, and
(9) $\bot \bullet \bot \bullet \varphi \bullet \bot \bullet \bot$.

We shall denote the conjunct in item (i) by φ_i, $i = 1, \ldots, 9$. It suffices to show that

$$\vdash_{\boldsymbol{L}^\bullet} \Box\varphi \supset \varphi_i \tag{10}$$

$i = 1, \ldots, 9$.
The first four conjuncts are also conjuncts of $\Box\varphi$, implying (10) for $i = 1, 2, 3, 4$.
For $i = 5$, from the axiom $\bot \supset \bot \bullet \bot$, by (7), we obtain

$$\bot \bullet \varphi \supset (\bot \bullet \bot) \bullet \varphi$$

implying, by associativity of $\bullet$,

$$\bot \bullet \varphi \supset \bot \bullet \bot \bullet \varphi$$

Since $\bot \bullet \varphi$ is a conjunct of $\Box\varphi$, (10) follows.
The case of $i = 6$ is symmetric.
For $i = 7$, from the axiom $\bot \supset \bot \bullet \bot$, by (7), we obtain

$$\bot \bullet (\varphi \bullet \bot) \supset (\bot \bullet \bot) \bullet (\varphi \bullet \bot)$$

implying, by associativity of $\bullet$,

$$\bot \bullet \varphi \bullet \bot \supset \bot \bullet \bot \bullet \varphi \bullet \bot$$

Since $\bot \bullet \varphi \bullet \bot$ is a conjunct of $\Box\varphi$, (10) follows.
The case of $i = 8$ is symmetric.
Finally, for $i = 9$, from the axiom $\bot \supset \bot \bullet \bot$, by (6), we obtain

$$(\bot \bullet \bot \bullet \varphi) \bullet \bot \supset (\bot \bullet \bot \bullet \varphi) \bullet (\bot \bullet \bot)$$

implying, by associativity of $\bullet$,

$$\bot \bullet \bot \bullet \varphi \bullet \bot \supset \bot \bullet \bot \bullet \varphi \bullet \bot \bullet \bot$$

This, together with (10) for $i = 7$ implies (10) for $i = 9$.

(v) It suffices to show that each conjunct of $\Box\varphi$ is derivable from φ. Trivially, φ is derivable from itself and $\bot \bullet \varphi$ and $\varphi \bullet \bot$ are derivable from φ by ($\mathrm{NEC_L}$) and ($\mathrm{NEC_R}$), respectively. Finally, $\bot \bullet \varphi \bullet \bot$ is derivable from $\varphi \bullet \bot$ by ($\mathrm{NEC_L}$). □

By Remark 4, $\Box$ behaves like the ordinary S4 modality and, indeed, the proof of Theorem 1 is very similar to the proof of the deduction theorem for S4 in [16].

Proof of Theorem 1 The "if" part of the theorem follows from Lemma 3(v) by *modus ponens*.

The proof of the "only if" part is by induction on the length of the derivation of χ from Θ, φ.

For the basis, χ is an axiom, or belongs to Θ, or is φ itself. In the two former cases, $\Box\varphi \supset \chi$ is already derivable from χ in $\boldsymbol{IL}$ and the latter case is Lemma 3(iii).

For the induction step, χ is obtained from previously derived formulas by one of the rules of inference—*modus ponens*, necessitation ($\mathrm{NEC_L}$), or necessitation ($\mathrm{NEC_R}$).

Assume that χ is obtained by *modus ponens* from ψ and $\psi \supset \chi$:

$$\frac{\psi \quad \psi \supset \chi}{\chi}$$

By the induction hypothesis, $\Theta \vdash_{\boldsymbol{L}^\bullet} \Box\varphi \supset \psi$ and $\Theta \vdash_{\boldsymbol{L}^\bullet} \Box\varphi \supset (\psi \supset \chi)$ from which $\Theta \vdash_{\boldsymbol{L}^\bullet} \Box\varphi \supset \chi$ is already derivable in $\boldsymbol{IL}$, see, e.g., [13, pp. 28–29].[3]

Assume that χ is obtained by ($\mathrm{NEC_L}$) from χ':

$$\frac{\chi'}{\chi'' \bullet \chi'}$$

That is, χ is of the form $\chi'' \bullet \chi'$. By the induction hypothesis, $\Theta \vdash_{\boldsymbol{L}^\bullet} \Box\varphi \supset \chi'$ from which we proceed as follows.

1. $\Box\varphi \supset \chi'$	induction hypothesis
2. $\Box(\Box\varphi \supset \chi')$	follows from 1 by Lemma 3 (v)
3. $\Box(\Box\varphi \supset \chi') \supset (\Box\Box\varphi \supset \Box\chi')$	Lemma 3 (ii)
4. $\Box\Box\varphi \supset \Box\chi'$	follows from 2 and 3 by *modus ponens*
5. $\Box\varphi \supset \Box\Box\varphi$	Lemma 3 (iv)
6. $\Box\varphi \supset \Box\chi'$	is already derivable from 5 and 4 in $\boldsymbol{IL}$
7. $\Box\chi' \supset \chi'' \bullet \chi'$	Lemma 3 (i)
8. $\Box\varphi \supset \chi'' \bullet \chi'$	is already derivable from 6 and 7 in $\boldsymbol{IL}$

The case of ($\mathrm{NEC_R}$) is symmetric to that of ($\mathrm{NEC_L}$). □

A routine inspection of the proof of Theorem 1 shows that it holds for any extension of $\boldsymbol{L}^\bullet$ with new connectives and axioms (but not rules of inference).

[3] The proof in [13] does not involve $\neg$.

3 Semi-Thue systems and a logic of concatenation

This section contains the definitions of semi-Thue systems and logic of concatenation $\boldsymbol{LC}$ and the proof of their equivalence. Namely, semi-Thue systems are defined in Sect. 3.1, $\boldsymbol{LC}$ is defined in Sect. 3.2, and the equivalence proof is presented in Sect. 3.3.

3.1 Semi-Thue systems

As we have already mentioned in the introduction, undecidability of $\boldsymbol{L}^{\bullet}$ is reduced to undecidability of the reachability problem in semi-Thue systems. This section contains the relevant definitions.

In what follows, Σ is a finite alphabet not containing $\to$ and, as usual, Σ^* and Σ^+ denote the sets of all words and of all nonempty words over Σ, respectively.

Definition 5 A *semi-Thue system* (over Σ) is a pair $T = (\Sigma, P)$, where P is a finite set of *productions*, which are expressions of the form $u \to v$, where $u, v \in \Sigma^*$.

A semi-Thue system (Σ, P) is a *Thue system*, if for each production $u \to v \in P$, the production $v \to u$ is also in P.

A semi-Thue system $T = (\Sigma, P)$ is *positive* if for all $u \to v \in P$, both u and v are nonempty.

A semi-Thue system T induces the following binary relation $\Rightarrow_T$ on Σ^*: $w \Rightarrow_T z$, if for some $u \to v \in P$ and some $w', w'' \in \Sigma^*$, $w = w'uw''$ and $z = w'vw''$.

We write $w' \Rightarrow_T^n w''$, $n = 0, 1, \ldots$, if there is the sequence of words $w_0, w_1, \ldots, w_n$ such that $w_0 = w'$, $w_n = w''$, and $w_i \Rightarrow_T w_{i+1}$, $i = 0, 1, \ldots, n-1$. Such a sequence is called a *derivation* of w'' from w'.

Also, we write $w' \Rightarrow_T^* w''$, if for some $n = 0, 1, \ldots$, $w' \Rightarrow_T^n w''$.[4]

Definition 6 The *reachability* problem for semi-Thue systems is whether for a semi-Thue system $T = (\Sigma, P)$ and $w, z \in \Sigma^*$, $w \Rightarrow_T^* z$?

Theorem 7 [15] *The reachability problem for semi-Thue systems over alphabets with more than one letter is undecidable.*[5]

3.2 Logic of concatenation $\boldsymbol{LC}$

In this section we define the *logic of concatenation* $\boldsymbol{LC}$ and, in Sect. 3.3, we show that this logic is "equivalent" to positive semi-Thue systems. This equivalence will be used in Sect. 4 for the proof of undecidability of $\boldsymbol{L}^{\bullet}$.

$\boldsymbol{LC}$ has a countably infinite set $\mathsf{Var}_{\boldsymbol{LC}}$ of propositional variables and a single binary connective $\cdot$ to extend $\mathsf{Var}_{\boldsymbol{LC}}$ to the set $\mathsf{Fm}_{\boldsymbol{LC}}$ of formulas of $\boldsymbol{LC}$. An $\boldsymbol{LC}$-*sequent* is an expression $\varphi \to \chi$, where φ and χ are $\boldsymbol{LC}$-formulas.

The axioms of $\boldsymbol{LC}$ are sequents of the form

$$\varphi \to \varphi \tag{11}$$

[4] That is, $\Rightarrow_T^*$ is the reflexive transitive closure of $\Rightarrow_T$.

[5] The semi-Thue system in [15] is positive.

$$(\varphi \cdot \chi) \cdot \psi \to \varphi \cdot (\chi \cdot \psi) \tag{12}$$

and

$$\varphi \cdot (\chi \cdot \psi) \to (\varphi \cdot \chi) \cdot \psi \tag{13}$$

and the rules of inference are

$$\frac{\varphi \to \psi \quad \psi \to \chi}{\varphi \to \chi} \tag{14}$$

and

$$\frac{\varphi' \to \chi' \quad \varphi'' \to \chi''}{\varphi' \cdot \varphi'' \to \chi' \cdot \chi''} \tag{15}$$

Indeed, as we shall see in the next section, $\boldsymbol{LC}$ can be thought of as a logic of concatenation of non-empty words over $\mathsf{Var}_{\boldsymbol{LC}}$.

Equivalently, $\boldsymbol{LC}$ can be thought of as a logic of the free semigroup generated by $\mathsf{Var}_{\boldsymbol{LC}}$. Namely, the Lindenbaum–Tarski algebra $\mathrm{LTA}_{\boldsymbol{LC}}$ of $\boldsymbol{LC}$ is isomorphic to that semigroup, see the appendix.

Propositions 8 and 9 below will be used in Sect. 4 for embedding $\boldsymbol{LC}$ into $\boldsymbol{CL}^{\bullet}$.

Proposition 8 *All propositional variables occurring in an $\boldsymbol{LC}$-derivation of $\varphi \to \chi$ from a set of assumption Θ either occur in both φ and χ or they occur in sequents of Θ.*

Proof The proof is by induction on the derivation length of $\varphi \to \chi$ from Θ.

The basis, i.e., the case of an $\boldsymbol{LC}$-axiom or an assumption from Θ is trivial.

For the induction step, if the derivation ends in an application of (14), then, by the induction hypothesis, all propositional variables in the derivation of the left premise are common to φ and ψ or belong to Θ, likewise all propositional variables in the derivation of the right premise are common to ψ and χ or belong to Θ.

Suppose a propositional variable in the derivation of $\varphi \to \chi$ from Θ does not belong to Θ. If it occurs in the derivation of the left premise, by the induction hypothesis, it is common to φ and ψ. As it occurs in ψ, by the induction hypothesis, it occurs in the derivation of the right premise as well, and, again, by the induction hypothesis, this propositional variable is common to ψ and χ. Hence, it is common to φ and χ.

Similarly, if the propositional variable occurs in the derivation of the right premise, it is, by the induction hypothesis, common to ψ and χ. As it occurs in ψ, it occurs in the derivation of the left premise as well, and, again, by the induction hypothesis, this propositional variable is common to φ and ψ. Hence, it is common to φ and χ.

If the derivation ends in an application of (15), by the induction hypothesis, a propositional variable in the derivation that does not belong to Θ is common in φ' and χ', or common in φ'' and χ'', hence, in any case, it is common in $\varphi' \cdot \varphi''$ and $\chi' \cdot \chi''$. □

To state Proposition 9 we need the following notation.

For a set of sequents Θ we define the set of sequents $\Theta^{\leftarrow}$ by

$$\Theta^{\leftarrow} = \{\chi \to \varphi : \varphi \to \chi \in \Theta\}$$

Proposition 9 *Let Θ and $\varphi \to \chi$ be a set of sequents and a sequent respectively. Then $\Theta \vdash_{\boldsymbol{LC}} \varphi \to \chi$ if and only if $\Theta^{\leftarrow} \vdash_{\boldsymbol{LC}} \chi \to \varphi$.*

Proof We prove the "only if" direction only. The proof of the "if" direction follows from $(\Theta^{\leftarrow})^{\leftarrow}$ being Θ.

The proof is by induction on the length of the derivation of $\varphi \to \chi$ from Θ.

For the basis, $\varphi \to \chi$ is either an axiom or belongs to Θ. If $\varphi \to \chi$ is axiom (11), then $\chi \to \varphi$ is also axiom (11), if $\varphi \to \chi$ is axiom (12) then $\chi \to \varphi$ is axiom (13), and, vice versa. If $\varphi \to \chi$ is an assumption from Θ, then, by the definition of $\Theta^{\leftarrow}$, $\chi \to \varphi$ is an assumption from $\Theta^{\leftarrow}$.

For the induction step, $\varphi \to \chi$ is obtained from previously derived sequents either by rule (14) or by rule (15).

If the last rule in the derivation of $\varphi \to \chi$ is (14):

$$\frac{\varphi \to \psi \qquad \psi \to \chi}{\varphi \to \chi}\ (14)$$

then, by the induction hypothesis, $\Theta^{\leftarrow} \vdash_{\boldsymbol{LC}} \psi \to \varphi$ and $\Theta^{\leftarrow} \vdash_{\boldsymbol{LC}} \chi \to \psi$, from which, by (14), we obtain $\chi \to \varphi$:

$$\frac{\chi \to \psi \qquad \psi \to \varphi}{\chi \to \varphi}\ (14)$$

If the last rule in the derivation of $\varphi \to \chi$ is (15):

$$\frac{\varphi' \to \chi' \qquad \varphi'' \to \chi''}{\varphi' \cdot \varphi'' \to \chi' \cdot \chi''}\ (15)$$

i.e., φ and χ are of the form $\varphi' \cdot \varphi''$ and $\chi' \cdot \chi''$ respectively, then, by the induction hypothesis, $\Theta^{\leftarrow} \vdash_{\boldsymbol{LC}} \chi' \to \varphi'$ and $\Theta^{\leftarrow} \vdash_{\boldsymbol{LC}} \chi'' \to \varphi''$, from which, by (15), we obtain $\chi' \cdot \chi'' \to \varphi' \cdot \varphi''$:

$$\frac{\chi' \to \varphi' \qquad \chi'' \to \varphi''}{\chi' \cdot \chi'' \to \varphi' \cdot \varphi''}\ (15)$$

□

Next, we recall the relational semantics of $\boldsymbol{LC}$.

An *interpretation* is a triple $\mathfrak{I} = \langle W, R, V\rangle$, where W is a non-empty set of (possible) worlds, R is a ternary (accessibility) relation on W, and V is a (valuation) function from W into sets of propositional variables (propositional interpretations).

The satisfiability relation $\models$ between worlds in W and $\boldsymbol{LC}$-formulas and sequents is defined as follows.

Let $u \in W$.

- If φ is a propositional variable, then $\Im, u \models \varphi$, if $\varphi \in V(u)$;
- $\Im, u \models \varphi \cdot \chi$, if for some $v, w \in W$ such that $R(u, v, w)$, $\Im, v \models \varphi$ and $\Im, w \models \chi$; and
- $\Im, u \models \varphi \to \chi$, if $\Im, u \models \varphi$ implies $\Im, u \models \chi$.

A sequent $\varphi \to \chi$ is *satisfiable*, if $\Im, u \models \varphi \to \chi$ for some interpretation $\Im = \langle W, R, V\rangle$ and some $u \in W$. Also, we say that $\Im$ satisfies a sequent $\varphi \to \chi$, denoted $\Im \models \varphi \to \chi$, if $\Im, u \models \varphi \to \chi$, for all $u \in W$ and we say that $\Im$ satisfies a set of sequents Θ, denoted $\Im \models \Theta$, if $\Im \models \varphi \to \chi$, for all $\varphi \to \chi \in \Theta$. Finally, a set of sequents Θ *semantically entails* a sequent $\varphi \to \chi$, denoted $\Theta \models \varphi \to \chi$, if for each interpretation $\Im$, $\Im \models \Theta$ implies $\Im \models \varphi \to \chi$.

Definition 10 Let Θ be a set of $\boldsymbol{LC}$-sequents. The Θ-*canonical* interpretation $\Im_\Theta = \langle W_\Theta, R_\Theta, V_\Theta\rangle$ is defined as follows.

- W_Θ is $\mathsf{Fm}_{\boldsymbol{LC}}$;
- $R_\Theta = \{(\varphi, \chi, \psi) \in W_\Theta^3 : \Theta \vdash_{\boldsymbol{LC}} \varphi \to \chi \cdot \psi\}$; and
- $V_\Theta(\varphi) = \{p \in \mathsf{Var}_{\boldsymbol{LC}} : \Theta \vdash_{\boldsymbol{LC}} \varphi \to p\}$.

Example 11 For all formulas χ, $\Im_\Theta, \chi \models \chi$. The proof is by a straightforward induction on the complexity of χ. The basis (in which χ is a propositional variable) is by definition; and for the induction step assume that χ is of the form $\chi' \cdot \chi''$. Then, by the induction hypothesis, $\Im_\Theta, \chi' \models \chi'$ and $\Im_\Theta, \chi'' \models \chi''$, from which $\Im_\Theta, \chi' \cdot \chi'' \models \chi' \cdot \chi''$ follows by the definition of $\models$ and the axiom $\chi' \cdot \chi'' \to \chi' \cdot \chi''$.

Proposition 12 *Let Θ be a set of $\boldsymbol{LC}$-sequents. Then, for $\boldsymbol{LC}$-formulas φ and χ, $\Im_\Theta, \varphi \models \chi$ if and only if $\Theta \vdash_{\boldsymbol{LC}} \varphi \to \chi$.*

Proof The proof of the "only if' direction of the proposition is by induction on the complexity of χ. The basis, i.e., the case in which χ is a propositional variable, immediately follows from the definition. The induction step is equally easy.

Let χ be of the form $\chi' \cdot \chi''$ and assume $\Im_\Theta, \varphi \models \chi$. That is, for some formulas φ' and φ'' such that $\Theta \vdash_{\boldsymbol{LC}} \varphi \to \varphi' \cdot \varphi''$, $\Im_\Theta, \varphi' \models \chi'$ and $\Im_\Theta, \varphi'' \models \chi''$. By the induction hypothesis, $\Theta \vdash_{\boldsymbol{LC}} \varphi' \to \chi'$ and $\Theta \vdash_{\boldsymbol{LC}} \varphi'' \to \chi''$, implying

$$\dfrac{\varphi \to \varphi' \cdot \varphi'' \qquad \dfrac{\varphi' \to \chi' \qquad \varphi'' \to \chi''}{\varphi' \cdot \varphi'' \to \chi'' \cdot \chi''}(15)}{\varphi \to \chi' \cdot \chi''}(14)$$

For the proof of the "if" direction of the proposition, assume $\Theta \vdash_{\boldsymbol{LC}} \varphi \to \chi' \cdot \chi''$. Then, $R_\Theta(\varphi, \chi', \chi'')$ and, by Example 11, $\Im_\Theta, \chi' \models \chi'$ and $\Im_\Theta, \chi'' \models \chi''$. Thus, by the definition of $\models$, $\Im_\Theta, \varphi \models \chi' \cdot \chi''$. $\square$

Definition 13 [2] A ternary relation R on a set W is *associative*, if for all $u, v, w, x \in W$ the following holds.

- There exists y such that $R(y, v, w)$ and $R(u, y, x)$ if and only if there exists z such that $R(z, w, x)$ and $R(u, v, z)$.

An interpretation $\Im = \langle W, R, V\rangle$ is associative, if R is associative.

Proposition 14 *Relation R_Θ is associative.*

Proof Assume $R_\Theta(\tau, \upsilon, \varphi)$ and $R_\Theta(\chi, \tau, \psi)$. We have to show that for some formula ω, $R_\Theta(\omega, \varphi, \psi)$ and $R_\Theta(\chi, \upsilon, \omega)$.

By the definitions of R_Θ, $\Theta \vdash_{\boldsymbol{LC}} \tau \to \upsilon \cdot \varphi$ and $\Theta \vdash_{\boldsymbol{LC}} \chi \to \tau \cdot \psi$, implying $\Theta \vdash_{\boldsymbol{LC}} \chi \to (\upsilon \cdot \varphi) \cdot \psi$. Thus, by (12) (and (14), of course), $\Theta \vdash_{\boldsymbol{LC}} \chi \to \upsilon \cdot (\varphi \cdot \psi)$. That is, the formula ω we are looking for is $\varphi \cdot \psi$.

The proof of the other direction of associativity is symmetric. □

Theorem 15 (Cf. [2, Proposition 1]) ***LC*** *is strongly sound and complete with respect to associative interpretations.*[6]

Proof For the proof of soundness, assume that $\Theta \vdash_{\boldsymbol{LC}} \varphi \to \chi$ and let $\mathfrak{I} = \langle W, R, V\rangle$ be an associative interpretation satisfying Θ. We shall prove by induction on the length of the derivation of $\varphi \to \chi$ from Θ that $\mathfrak{I}$ satisfies $\varphi \to \chi$ as well.

For the basis, $\varphi \to \chi$ is either an assumption, i.e., belongs to Θ, or is an axiom.

The case of an assumption follows from $\mathfrak{I} \models \Theta$, and the case of axiom (11) follows from the definition of $\models$.

For the case of axiom (12), let $u \in W$ and let $\mathfrak{I}, u \models (\varphi \cdot \chi) \cdot \psi$. That is, there are worlds x and y such that $R(u, y, x)$, $\mathfrak{I}, y \models \varphi \cdot \chi$, and $\mathfrak{I}, x \models \psi$; and there are worlds v and w, such that $R(y, v, w)$, $\mathfrak{I}, v \models \varphi$, and $\mathfrak{I}, w \models \chi$.

Since R is associative, there is a world z such that $R(z, w, x)$ and $R(u, v, z)$. Then, $\mathfrak{I}, z \models \chi \cdot \psi$, implying $\mathfrak{I}, u \models \varphi \cdot (\chi \cdot \psi)$.

The case of axiom (13) is symmetric and is omitted.

For the induction step, the case of rule (14) immediately follows from the definition of $\models$. The case of rule (15) is also straightforward:

Assume that the derivation ends in rule (15) and assume $\mathfrak{I}, u \models \varphi' \cdot \varphi''$. That is, there are worlds v and w such that $R(u, v, w)$, $\mathfrak{I}, v \models \varphi'$, and $\mathfrak{I}, w \models \varphi''$. Then, by the definition of $\models$ and the induction hypothesis, $\mathfrak{I}, v \models \chi'$ and $\mathfrak{I}, w \models \chi''$, implying $\mathfrak{I}, u \models \chi' \cdot \chi''$. That is, $\mathfrak{I}, u \models \varphi' \cdot \varphi'' \to \chi' \cdot \chi''$.

The proof of completeness is equally easy: if $\Theta \nvdash_{\boldsymbol{LC}} \varphi \to \chi$, Then, by Proposition 12, $\mathfrak{I}_\Theta, \varphi \not\models \chi$. Since, by Example 11, $\mathfrak{I}_\Theta, \varphi \models \varphi$, by the definition of satisfaction of sequents, $\mathfrak{I}_\Theta, \varphi \not\models \varphi \to \chi$. Thus, $\mathfrak{I}_\Theta \not\models \varphi \to \chi$ either. Note that, by Proposition 14, $\mathfrak{I}_\Theta$ is an associative interpretation. □

3.3 Embedding semi-Thue systems into *LC*

The proof of the undecidability theorem in Sect. 4 is based on embedding positive semi-Thue systems into $\boldsymbol{L}^\bullet$ via their embedding into ***LC*** and embedding ***LC*** into $\boldsymbol{L}^\bullet$.[7] In some sense, the translation theorems below (Theorems 16 and 17) and the passage from the word problem for semigroups to the Lindenbaum–Tarski algebra $\mathrm{LTA}_{\boldsymbol{LC}}$ in the algebraic original proof in [11] are related. Namely, the word problem of semigroups is an equation in $\mathrm{LTA}_{\boldsymbol{LC}}$ with suitable assumptions. We do not rely on this fact in our proof, but, for sake of completeness, present it in the appendix.

6 Actually, we need completeness only.

7 In fact, the embedding of ***LC*** into $\boldsymbol{L}^\bullet$ is dual to the translation of the word problem of semigroups to equations in $\mathrm{LTA}_{\boldsymbol{CL}^\bullet}$ in [11]. The logical counterpart of the construction in [11] is presented in Sect. 5.

Embedding positive semi-Thue systems into $\boldsymbol{LC}$ is based on the following translations of semi-Thue systems and $\boldsymbol{LC}$ to each other.

To translate $\boldsymbol{LC}$ to semi-Thue systems, with each formula φ of $\boldsymbol{LC}$ we associate the word over $\mathsf{Var}_{\boldsymbol{LC}}$, denoted $\overline{\varphi}$, that is defined by the following recursion.

- If φ is a propositional variable, then $\overline{\varphi}$ is φ itself; and
- $\overline{\varphi \cdot \chi}$ is the word concatenation $\overline{\varphi}\,\overline{\chi}$.

Then, the translation of a set of sequents Θ, denoted by $\overline{\Theta}$, is the set of productions

$$\overline{\Theta} = \{\overline{\varphi} \to \overline{\chi} : \varphi \to \chi \in \Theta\}$$

Theorem 16 *If $\Theta \vdash_{\boldsymbol{LC}} \varphi \to \chi$ and Θ is finite, then $\overline{\varphi} \Rightarrow^*_T \overline{\chi}$, where T is the semi-Thue system $(\Sigma, \overline{\Theta})$ and Σ is the set of the $\boldsymbol{LC}$ propositional variables occurring in $\Theta \cup \{\varphi \to \chi\}$.*

Proof The proof is by induction on the length of the $\boldsymbol{LC}$-derivation of $\varphi \to \chi$ from the set of assumptions Θ.

For the basis, i.e., derivations of length one, either $\varphi \to \chi$ is an instance of one of the axioms (11), (12), or (13), or is an assumption from Θ.

If $\varphi \to \chi$ is an axiom, then $\overline{\varphi} \Rightarrow^0_T \overline{\chi}$, because the case of axiom (11) is immediate and the cases of axioms (12) and (13) follow from associativity of (word) concatenation.

If $\varphi \to \chi$ is an an assumption from Θ, then $\overline{\varphi} \Rightarrow^1_T \overline{\chi}$ by the definition of T.

For the induction step, if the last rule in the derivation is (14):

$$\frac{\varphi \to \psi \qquad \psi \to \chi}{\varphi \to \chi}\ (14)$$

then, by Proposition 8, all propositional variables occurring in the "cut formula" ψ occur in $\Theta \cup \{\varphi \to \chi\}$. Thus, by the induction hypothesis, for some nonnegative integers i and j, $\overline{\varphi} \Rightarrow^i_T \overline{\psi}$ and $\overline{\psi} \Rightarrow^j_T \overline{\chi}$, implying

$$\overline{\varphi} \Rightarrow^i_T \overline{\psi} \Rightarrow^j_T \overline{\chi}$$

and, if the last rule in the derivation is (15):

$$\frac{\varphi' \to \chi' \qquad \varphi'' \to \chi''}{\varphi' \cdot \varphi'' \to \chi' \cdot \chi''}\ (15)$$

i.e., φ and χ are of the form $\varphi' \cdot \varphi''$ and $\chi' \cdot \chi''$ respectively, then, by the induction hypothesis, for some nonnegative integers i and j, $\overline{\varphi'} \Rightarrow^i_T \overline{\chi'}$ and $\overline{\varphi''} \Rightarrow^j_T \overline{\chi''}$, implying

$$\overline{\varphi'}\,\overline{\varphi''} \Rightarrow^i_T \overline{\chi'}\,\overline{\varphi''} \Rightarrow^j_T \overline{\chi'}\,\overline{\chi''}$$

$\square$

For the converse translation, renaming the symbols in Σ, if necessary, we may assume that Σ is a finite set of propositional variables. Then, for each non-empty word $u \in \Sigma^*$ there is a Σ-formula $\overline{u}$ such that $\overline{\overline{u}}$ is u. For example, $\overline{u}$ can be defined, recursively, as follows.

- If $u \in \Sigma$, then $\overline{u}$ is u itself; and
- for $u \in \Sigma^+$ and $p \in \Sigma$, $\overline{up}$ is $\overline{u} \cdot p$.

In fact, setting $\overline{u}$ to be any formula φ such that $\overline{\varphi}$ is u (or, equivalently, $\vdash_{\boldsymbol{LC}} \overline{u} \to \varphi$) does not affect the proof of Theorem 17 below. This is because the free semigroup generated by the propositional variables coincides with $\mathrm{LTA}_{\boldsymbol{LC}}$, see the appendix.

Next, the $\boldsymbol{LC}$-translation $\overline{P}$ of a set of productions P is defined by

$$\overline{P} = \{\overline{u} \to \overline{v} : u \to v \in P\}$$

Then $\overline{\overline{P}}$ is P.

Theorem 17 *For a positive semi-Thue system $T = (\Sigma, P)$, $w \Rightarrow_T^* z$ implies $\overline{P} \vdash_{\boldsymbol{LC}} \overline{w} \to \overline{z}$.*

Proof The proof is by induction on the length n of the derivation of

$$w = w_0 \Rightarrow w_1 \Rightarrow \cdots \Rightarrow w_n = z$$

The basis, $n = 0$, is immediate, because the $\boldsymbol{LC}$ counterpart of

$$w_0 \Rightarrow_T^0 w_0$$

is the corresponding instance of axiom (11).

For the induction step, assume

$$w_0 \Rightarrow_T \cdots \Rightarrow_T w_n \Rightarrow_T w_{n+1}$$

That is, w_n is of the form $w'uw''$, w_{n+1} is of the form $w'vw''$, and $u \to v \in P$. By the induction hypothesis, $\overline{P} \vdash_{\boldsymbol{LC}} \overline{w_0} \to \overline{w'uw''}$. Then,[8]

1. $\overline{w'} \to \overline{w'}$	axiom (11)
2. $\overline{u} \to \overline{v}$	assumption from $\overline{P}$
3. $\overline{w'} \cdot \overline{u} \to \overline{w'} \cdot \overline{v}$	follows from 1 and 2 by (15)
4. $\overline{w''} \to \overline{w''}$	axiom (11)
5. $(\overline{w'} \cdot \overline{u}) \cdot \overline{w''} \to (\overline{w'} \cdot \overline{v}) \cdot \overline{w''}$	follows from 3 and 4 by (15)
6. $\overline{w_0} \to \overline{w'uw''}$	induction hypothesis
7. $\overline{w'uw''} \to \overline{w'vw''}$	follows from 5 by (12), (13), and (14)
8. $\overline{w_0} \to \overline{w'vw''}$	follows from 6 and 7 by (14)

$\square$

[8] In the derivation below, we assume that both w' and w'' are nonempty. In the emptiness case, we just skip the corresponding derivation steps.

The corollary below immediately follows from Theorems 7, 16, and 17.

Corollary 18 *The consequence relation $\vdash_{\boldsymbol{LC}}$ is undecidable.*

Proof By Theorems 16 and 17, for a semi-Thue system (Σ, P) and words w and z over Σ, $w \Rightarrow^*_{(\Sigma,P)} z$ if and only if $\overline{P} \vdash_{\boldsymbol{LC}} \overline{w} \rightarrow \overline{z}$[9]; and, by Theorem 7, the reachability problem for semi-Thue systems over alphabets with more than one letter is undecidable. □

It follows from Theorem 15 and [2, Proposition 1] that the associative Lambek calculus $\boldsymbol{L}$ is a (strong) conservative extension of $\boldsymbol{LC}$. Since $\boldsymbol{L}$ is decidable (see [12, Section 8]), sequent derivability in the "pure" $\boldsymbol{LC}$, i.e., the $\boldsymbol{LC}$-derivability from the axioms only, is also decidable.

4 Embedding LC into $L^\bullet$ and its undecidability

The desired undecidability result is based on an embedding of (undecidable) $\boldsymbol{LC}$ into $\boldsymbol{L}^\bullet$. To embed $\boldsymbol{LC}$ into $\boldsymbol{L}^\bullet$, we translate $\boldsymbol{LC}$-sequents to $\boldsymbol{L}^\bullet$-formulas by replacing $\cdot$ by $\bullet$ and $\rightarrow$ by $\supset$. Namely, an $\boldsymbol{LC}$-formula φ is translated to the $\boldsymbol{L}^\bullet$-formula $\varphi^\bullet$, recursively, as follows.

- If φ is a propositional variable, then $\varphi^\bullet$ is φ; and
- $(\varphi \cdot \chi)^\bullet$ is $\varphi^\bullet \bullet \chi^\bullet$.

Then a set Θ of $\boldsymbol{LC}$-sequents is translated to the set of $\boldsymbol{L}^\bullet$-formulas $\Theta^\bullet$ defined by

$$\Theta^\bullet = \{\varphi^\bullet \supset \chi^\bullet : \varphi \rightarrow \chi \in \Theta\}$$

Theorem 19 *Let Θ and $\varphi \rightarrow \chi$ be a set of $\boldsymbol{LC}$-sequents and an $\boldsymbol{LC}$-sequent, respectively. Then $\Theta \vdash_{\boldsymbol{LC}} \varphi \rightarrow \chi$ if and only if $\Theta^\bullet \vdash_{\boldsymbol{L}^\bullet} \varphi^\bullet \supset \chi^\bullet$.*

We postpone the proof of Theorem 19 to the end of this section, because the proof involves the De Morgan dual connective of $\bullet$ and the relational semantics of $\boldsymbol{CL}^\bullet$. These are presented in Sects. 4.1 and 4.2, respectively.

Undecidability of $\boldsymbol{L}^\bullet$ follows from undecidability of $\boldsymbol{LC}$, its embedding into $\boldsymbol{L}^\bullet$, and the deduction theorem. We summarize these arguments in the proof of Theorem 20 below.

Theorem 20 *Both $\boldsymbol{CL}^\bullet$ and $\boldsymbol{IL}^\bullet$ are undecidable.*

Proof By Corollary 18, the consequence relation in $\boldsymbol{LC}$ is undecidable. This, in turn, implies, by Theorem 19 that the consequence relations in $\boldsymbol{L}^\bullet$ are undecidable. Namely, for $\boldsymbol{L}^\bullet$-formulas $\varphi_1, \varphi_2, \ldots, \varphi_n$, and φ (which are the $\bullet$-translations of $\boldsymbol{LC}$-sequents) it is undecidable whether

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vdash_{\boldsymbol{L}^\bullet} \varphi \tag{16}$$

9 Recall that Σ is a set of propositional variables.

By the deduction theorem (Theorem 1), (16) is equivalent to

$$\vdash_{\boldsymbol{L}^\bullet} \bigwedge_{i=1}^{n} \Box\varphi_i \supset \varphi$$

Therefore, derivability in $\boldsymbol{L}^\bullet$ is undecidable either. $\Box$

4.1 The dual of $\bullet$

We denote by $\circ$ the De Morgan dual connective of $\bullet$. That is, in $\boldsymbol{CL}^\bullet$, $\varphi \circ \chi$ is an abbreviation for $\neg(\neg\varphi \bullet \neg\chi)$, if $\bullet$ is the primary connective or, alternatively, $\varphi \bullet \chi$ is an abbreviation for $\neg(\neg\varphi \circ \neg\chi)$, if $\circ$ is the primary connective. In addition of being a technical tool in the proof of Theorem 19, the algebraic counterpart of $\circ$ (that is semigroup multiplication) plays the major role in the translation of the word problem of semigroups to equations in $\mathrm{LTA}_{\boldsymbol{CL}^\bullet}$ in [11], sketched in Sect. 5.

In the case of the primary connective $\circ$, $\boldsymbol{CL}^\bullet$ may be axiomatized by the following axioms and rules of inference (which are in addition to $\boldsymbol{CL}$).

Axioms:

$(\vee_\mathrm{L})$ $\psi \circ (\varphi \vee \chi) \supset (\psi \circ \varphi \vee \psi \circ \chi)$	$(\vee_\mathrm{R})$ $(\varphi \vee \chi) \circ \psi \supset (\varphi \circ \psi \vee \chi \circ \psi)$
$(\bot_\mathrm{L})$ $\varphi \circ \bot \supset \bot$	$(\bot_\mathrm{R})$ $\bot \circ \varphi \supset \bot$
$(\mathrm{A}^\circ_\mathrm{L})$ $(\varphi \circ \chi) \circ \psi \supset \varphi \circ (\chi \circ \psi)$	$(\mathrm{A}^\circ_\mathrm{R})$ $\varphi \circ (\chi \circ \psi) \supset (\varphi \circ \chi) \circ \psi$

The $\circ$-*monotonicity* rule of inference:

$$(\mathrm{MON})\ \frac{\varphi' \supset \chi' \qquad \varphi'' \supset \chi''}{\varphi' \circ \varphi'' \supset \chi' \circ \chi''}$$

cf. (8).

Proposition 21 *In* $\boldsymbol{CL}^\bullet$, *axioms* $(\mathrm{K_L})$, $(\mathrm{K_R})$, $(\mathrm{A_L})$, $(\mathrm{A_R})$ *and rules* $(\mathrm{NEC_L})$ *and* $(\mathrm{NEC_R})$ *are derivable from axioms* $(\mathrm{A}^\circ_\mathrm{L})$, $(\mathrm{A}^\circ_\mathrm{R})$, $(\vee_\mathrm{L})$, $(\vee_\mathrm{R})$, $(\bot_\mathrm{L})$, $(\bot_\mathrm{R})$, *and rule* (MON) *and vice versa, axioms* $(\vee_\mathrm{L})$, $(\vee_\mathrm{R})$, $(\bot_\mathrm{L})$, $(\bot_\mathrm{R})$, $(\mathrm{A}^\circ_\mathrm{L})$, $(\mathrm{A}^\circ_\mathrm{R})$ *and rule* (MON) *are derivable from axioms* $(\mathrm{K_L})$, $(\mathrm{K_R})$ $(\mathrm{A_L})$, $(\mathrm{A_R})$ *and rules* $(\mathrm{NEC_L})$ *and* $(\mathrm{NEC_R})$.

Proof We start with the first part of the proposition.

Axiom $(\mathrm{K_L})$ equivalently translates to

$$\neg(\neg\psi \circ \neg(\varphi \supset \chi)) \supset (\neg(\neg\psi \circ \neg\varphi) \supset \neg(\neg\psi \circ \neg\chi))$$

that is $\boldsymbol{CL}$ equivalent to

$$\neg\psi \circ \neg\chi \supset (\neg\psi \circ \neg\varphi \vee \neg\psi \circ \neg(\varphi \supset \chi))$$

and the latter can be derived as follows.

1. $\neg\psi \supset \neg\psi$ tautology
2. $\neg\chi \supset (\neg\varphi \vee \neg(\varphi \supset \chi))$ derivable in $\boldsymbol{CL}$
3. $\neg\psi \circ \neg\chi \supset$
   $\neg\psi \circ (\neg\varphi \vee \neg(\varphi \supset \chi))$ follows from 1 and 2 by (MON)
4. $\neg\psi \circ (\neg\varphi \vee \neg(\varphi \supset \chi)) \supset$
   $(\neg\psi \circ \neg\varphi \vee \neg\psi \circ \neg(\varphi \supset \chi))$ axiom ($\vee_L$)
5. $\neg\psi \circ \neg\chi \supset$
   $(\neg\psi \circ \neg\varphi \vee \neg\psi \circ \neg(\varphi \supset \chi))$ already derivable from 3 and 4 in $\boldsymbol{CL}$

Axiom (A_L) equivalently translates to axiom (A_R°)

$$\neg\varphi \circ (\neg\chi \circ \neg\psi) \supset (\neg\varphi \circ \neg\chi) \circ \neg\psi$$

Rule (NEC_L) equivalently translates to

$$\frac{\varphi}{\neg\chi \circ \neg\varphi \supset \bot}$$

that can be derived as follows.

1. φ assumption
2. $\neg\varphi \supset \bot$ already derivable from 1 in$\boldsymbol{CL}$
3. $\neg\chi \supset \neg\chi$ tautology
4. $\neg\chi \circ \neg\varphi \supset \neg\chi \circ \bot$ follows from 2 and 3 by (MON)
5. $\neg\chi \circ \bot \supset \bot$ axiom ($\bot_L$)
6. $\neg\chi \circ \neg\varphi \supset \bot$ already derivable from 4 and 5 in$\boldsymbol{CL}$

The derivations of axioms (A_R), (K_R), and rule (NEC_R) are symmetric.

For the second part of the proposition, axiom ($\vee_L$) translates to

$$\neg(\neg\psi \bullet \neg(\varphi \vee \chi)) \supset (\neg(\neg\psi \bullet \neg\varphi) \vee \neg(\neg\psi \bullet \neg\chi))$$

that is equivalent to the instance

$$(\neg\psi \bullet \neg\varphi \wedge \neg\psi \bullet \neg\chi) \supset \neg\psi \bullet (\neg\varphi \wedge \neg\chi)$$

of (3).

Axiom ($\bot_L$) translates to

$$\neg(\neg\varphi \bullet \neg\bot) \supset \bot$$

that is equivalent to $\neg\varphi \bullet \top$ and the latter is derivable from $\top$ by (NEC_L).

Axiom (A_L°) equivalently translates to axiom (A_R)

$$\neg\varphi \bullet (\neg\chi \bullet \neg\psi) \supset (\neg\varphi \bullet \neg\chi) \bullet \neg\psi$$

The derivations of axioms ($\vee_R$), ($\bot_R$), and (A_R°), are symmetric.

Finally, rule (MON) translates to

$$\frac{\varphi' \supset \chi' \qquad \varphi'' \supset \chi''}{\neg(\neg\varphi' \bullet \neg\varphi'') \supset \neg(\neg\chi' \bullet \neg\chi'')}$$

or, equivalently to

$$\frac{\neg\chi' \supset \neg\varphi' \qquad \neg\chi'' \supset \neg\varphi''}{\neg\chi' \bullet \neg\chi'' \supset \neg\varphi' \bullet \neg\varphi''}$$

that is an instance of (8). □

4.2 The relational semantics of $\boldsymbol{CL}^\bullet$

In this section we recall the (ternary) relational semantics of $\boldsymbol{CL}^\bullet$.

For an interpretation $\mathfrak{I} = \langle W, R, V\rangle$, the satisfiability relation $\models$ between worlds in W and $\boldsymbol{CL}^\bullet$-formulas is defined as follows.

Let $u \in W$.

- If φ is a propositional variable, then $\mathfrak{I}, u \models \varphi$, if $\varphi \in V(u)$;
- $\mathfrak{I}, u \not\models \bot$;
- $\mathfrak{I}, u \models \varphi \wedge \chi$, if $\mathfrak{I}, u \models \varphi$ and $\mathfrak{I}, u \models \chi$;
- $\mathfrak{I}, u \models \varphi \vee \chi$, if $\mathfrak{I}, u \models \varphi$ or $\mathfrak{I}, u \models \chi$;
- $\mathfrak{I}, u \models \varphi \supset \chi$, if $\mathfrak{I}, u \not\models \varphi$ or $\mathfrak{I}, u \models \chi$;
- $\mathfrak{I}, u \models \neg\varphi$, if $\mathfrak{I}, u \not\models \varphi$;
- $\mathfrak{I}, u \models \varphi \circ \chi$, if there are $v, w \in W$ such that $R(u, v, w)$, $\mathfrak{I}, v \models \varphi$ and $\mathfrak{I}, w \models \chi$; and
- $\mathfrak{I}, u \models \varphi \bullet \chi$, if for all $v, w \in W$ such that $R(u, v, w)$, $\mathfrak{I}, v \models \varphi$ or $\mathfrak{I}, w \models \chi$.

The definitions of satisfiability by an interpretation and of semantical entailment are similar to the corresponding definitions in Sect. 3.2.

Theorem 22 *Associative interpretations*[10] *are strongly sound and complete for* $\boldsymbol{CL}^\bullet$.

Soundness can be proved by a straightforward induction on the length of the $\boldsymbol{CL}^\bullet$ derivation. In this paper, we do not use completeness and refer the reader to [8, Section 4], say, for the proof.

4.3 Proof of Theorem 19

We shall use one more translation of $\boldsymbol{LC}$-formulas to the $\boldsymbol{CL}^\bullet$ ones. This formula translation is similar to the $\bullet$-translation defined in the beginning of Sect. 4 and results in replacing $\cdot$ with $\circ$. That is, the $\circ$-translation of an $\boldsymbol{LC}$-formula φ, denoted φ°, is defined, recursively, as follows.

- If φ is a propositional variable, then φ° is φ; and
- $(\varphi \cdot \chi)^\circ$ is $\varphi^\circ \circ \chi^\circ$.

Then, similarly to the case of the $\bullet$-translation, for a set of $\boldsymbol{LC}$-sequents Θ we define the set of $\boldsymbol{CL}^\bullet$-formulas Θ° by

$$\Theta^\circ = \{\varphi^\circ \supset \chi^\circ : \varphi \to \chi \in \Theta\}$$

[10] See Definition 13.

Remark 23 It follows from the definition of the $\circ$-translation, by a straightforward induction on the length of an $\boldsymbol{LC}$-formula $p_1 \cdot p_2 \cdot \cdots \cdot p_\ell$, where $p_1, p_2, \ldots, p_\ell$ are propositional variables, that $(p_1 \cdot p_2 \cdot \cdots \cdot p_\ell)^\circ$ is $\boldsymbol{CL}^\bullet$-equivalent to $\neg(\neg p_1 \bullet \neg p_2 \bullet \cdots \bullet \neg p_\ell)$.

Proposition 24 *Let Θ and $\varphi \to \chi$ be a set of $\boldsymbol{LC}$-sequents and an $\boldsymbol{LC}$-sequent, respectively. Then $\Theta^\circ \vdash_{\boldsymbol{CL}^\bullet} \varphi^\circ \supset \chi^\circ$ implies $\Theta \vdash_{\boldsymbol{LC}} \varphi \to \chi$.*

Proof Let $\Theta^\circ \vdash_{\boldsymbol{CL}^\bullet} \varphi^\circ \supset \chi^\circ$ and assume to the contrary that $\Theta \nvdash_{\boldsymbol{LC}} \varphi \to \chi$. By completeness of the relational semantics of $\boldsymbol{LC}$, there is an an associative interpretation $\mathfrak{I} = \langle W, R, V \rangle$ satisfying Θ such that for some world $u \in W$, $\mathfrak{I}, u \not\models \varphi \to \chi$. Then also $\mathfrak{I} \models \Theta^\circ$, but $\mathfrak{I}, u \not\models \varphi^\circ \supset \chi^\circ$, because the interpretations of $\cdot$ and $\circ$ are the same. This, however, contradicts soundness of the relational semantics of $\boldsymbol{CL}^\bullet$. $\square$

Now we are ready for the proof of Theorem 19.

Proof of Theorem 19 The "only if" direction is immediate, because the $\bullet$-translation of axiom (11) is $\boldsymbol{L}^\bullet$-derivable, the $\bullet$-translations of axioms (12) and (13) are $\boldsymbol{L}^\bullet$-axioms, rule (14) is $\boldsymbol{L}^\bullet$-derivable, and the $\bullet$-translation of rule (15) is rule (8).

For the proof of the "if" direction, we first consider the case of $\boldsymbol{CL}^\bullet$.

Let $\Theta^\bullet \vdash_{\boldsymbol{CL}^\bullet} \varphi^\bullet \supset \chi^\bullet$ and assume to the contrary that $\Theta \nvdash_{\boldsymbol{LC}} \varphi \to \chi$. Then, by (the contraposition of) Proposition 9, $\Theta^\leftarrow \nvdash_{\boldsymbol{LC}} \chi \to \varphi$ either, implying, by (the contraposition of) Proposition 24, $\Theta^{\leftarrow\circ} \nvdash_{\boldsymbol{CL}^\bullet} \chi^\circ \supset \varphi^\circ$.

Let φ and χ be $q_1 \cdot q_2 \cdot \cdots \cdot q_n$ and $r_1 \cdot r_2 \cdot \cdots \cdot r_m$, respectively. Then, by Remark 23,

$$\neg(\neg r_1 \bullet \neg r_2 \bullet \cdots \bullet \neg r_m) \supset \neg(\neg q_1 \bullet \neg q_2 \bullet \cdots \bullet \neg q_n)$$

is not $\boldsymbol{CL}^\bullet$-derivable from

$$\big\{\neg(\neg p''_1 \bullet \neg p''_2 \bullet \cdots \bullet \neg p''_{\ell''}) \supset \neg(\neg p'_1 \bullet \neg p'_2 \bullet \cdots \bullet \neg p'_{\ell'}) : \\ p'_1 \cdot p'_2 \cdot \cdots \cdot p'_{\ell'} \to p''_1 \cdot p''_2 \cdot \cdots \cdot p''_{\ell''} \in \Theta\big\}$$

or, equivalently,

$$\neg q_1 \bullet \neg q_2 \bullet \cdots \bullet \neg q_n \supset \neg r_1 \bullet \neg r_2 \bullet \cdots \bullet \neg r_m \tag{17}$$

is not $\boldsymbol{CL}^\bullet$-derivable from

$$\{\neg p'_1 \bullet \neg p'_2 \bullet \cdots \bullet \neg p'_{\ell'} \supset \neg p''_1 \bullet \neg p''_2 \bullet \cdots \bullet \neg p''_{\ell''} : \\ p'_1 \cdot p'_2 \cdot \cdots \cdot p'_{\ell'} \to p''_1 \cdot p''_2 \cdot \cdots \cdot p''_{\ell''} \in \Theta\} \tag{18}$$

However, replacing all propositional variables with their negations in the $\boldsymbol{CL}^\bullet$-derivation of $\varphi^\bullet \supset \chi^\bullet$ from $\Theta^\bullet$ we obtain a $\boldsymbol{CL}^\bullet$-derivation of (17) from (18). That is, we have arrived at a contradiction and the proof of the "if" direction is complete for the case of $\boldsymbol{CL}^\bullet$.

Finally, the "if" direction of the case of $\boldsymbol{IL}^\bullet$ follows from the case of $\boldsymbol{CL}^\bullet$, because $\Theta^\bullet \vdash_{\boldsymbol{IL}^\bullet} \varphi^\bullet \supset \chi^\bullet$ implies $\Theta^\bullet \vdash_{\boldsymbol{CL}^\bullet} \varphi^\bullet \supset \chi^\bullet$. $\square$

It follows from the proof of Theorem 19 and the completeness of the relational semantics of $\boldsymbol{CL}^\bullet$ that the binary connective $\cdot$ of $\boldsymbol{LC}$ is self dual. That is, the interpretation of $\cdot$ as

- $\mathfrak{I}, u \models \varphi \cdot \chi$, if for all $v, w \in W$ such that $R(u, v, w)$, $\mathfrak{I}, v \models \varphi$ or $\mathfrak{I}, w \models \chi$.

is also strongly sound and complete for $\boldsymbol{LC}$, cf. [14].

5 Back to the algebraic approach of [11]

In contrast to Theorem 20, the intuitionistic logic of an associative binary modality based on $\circ$ and denoted by $\boldsymbol{IL}^\circ$ is decidable, see [9, Proposition 64].[11]

As we have mentioned in the beginning of Sect. 2, the translation of the word problem of semigroups to equations in $\mathrm{LTA}_{\boldsymbol{CL}^\bullet}$ in [11] involves the unary term $c(\cdot)$ that is the algebraic counterpart of the dual of $\Box$, denoted (as usual) by $\Diamond$. That is, $\Diamond\varphi$ is

$$\varphi \vee \top \circ \varphi \vee \varphi \circ \top \vee \top \circ \varphi \circ \top$$

cf. (1).

Then, for a semigroup generated by a set of propositional variables with presentation $\{u_i = v_i\}_{i=1,\ldots,m}$, the (direct) logical counterpart of the "algebraic" translation of the word problem $u_0 = v_0$ in [11] is

$$\overline{u_0} \oplus \overline{v_0} \supset \Diamond \bigvee_{i=1}^{m} \overline{u_i} \oplus \overline{v_i} \tag{19}$$

where $\oplus$ is symmetric difference. Since (19) is an $\boldsymbol{IL}^\circ$-formula, the approach in [11] does not apply in the intuitionistic case.

Note that (19) is the (classical) contraposition of the translation

$$\Box \bigwedge_{i=1}^{m} \overline{u_i} \equiv \overline{v_i} \supset (\overline{u_0} \equiv \overline{v_0})$$

in Sect. 4.

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[11] By Proposition 21, this logic is classically equivalent to undecidable $\boldsymbol{IL}^\bullet$ that is based on $\bullet$.

Appendix: The Lindenbaum–Tarski algebra of $\boldsymbol{LC}$

The Lindenbaum–Tarski algebra $\mathrm{LTA}_{\boldsymbol{LC}}$ of $\boldsymbol{LC}$ is a free semigroup that is tightly related to the embedding semi-Thue systems into $\boldsymbol{LC}$ in Sect. 3.3 and to the undecidability proof in [11]. We precede its description with the following definition and proposition.

Definition 25 (Cf. Definition 5). A set Θ of $\boldsymbol{LC}$-sequents is called a *Thue set* if for each sequent $\varphi \to \chi \in \Theta$, the sequent $\chi \to \varphi$ is also in Θ.

Proposition 26 *Let Θ and $\varphi \to \chi$ be a Thue set of $\boldsymbol{LC}$-sequents and an $\boldsymbol{LC}$-sequent, respectively. Then $\Theta \vdash_{\boldsymbol{LC}} \varphi \to \chi$ if and only if $\Theta \vdash_{\boldsymbol{LC}} \chi \to \varphi$.*

Proof By the definition of the Thue set, $\Theta^{\leftarrow} = \Theta$. Thus, the proposition follows from Proposition 9. □

Next, for a Thue set of $\boldsymbol{LC}$-sequents Θ, we define a binary relation $\sim_\Theta$ on $\mathsf{Fm}_{\boldsymbol{LC}}$ by

$$\varphi \sim_\Theta \chi \text{ if and only if } \Theta \vdash_{\boldsymbol{LC}} \varphi \to \chi.$$

It follows from axiom (11), Proposition 26, and rule (14) that $\sim_\Theta$ is an equivalence relation. Moreover, by rule (15), $\sim_\Theta$ is a congruence and we define multiplication on $\mathsf{Fm}_{\boldsymbol{LC}}/{\sim_\Theta}$, also denoted by $\cdot$, by

$$[\varphi]_{\sim_\Theta} \cdot [\chi]_{\sim_\Theta} = [\varphi \cdot \chi]_{\sim_\Theta}$$

where, as usual, $[\varphi]_{\sim_\Theta}$ is the $\sim_\Theta$-congruence class of an $\boldsymbol{LC}$-formula φ.

By axioms (12) and (13), $\mathrm{LTA}_{\boldsymbol{LC}} = (\mathsf{Fm}_{\boldsymbol{LC}}/{\sim_\Theta}, \cdot)$ is a semigroup. It can be readily seen that this semigroup is generated by $\{[p]_{\sim_\Theta} : p \in \mathsf{Var}_{\boldsymbol{LC}}\}$ with the presentation $\{[\varphi]_{\sim_\Theta} = [\chi]_{\sim_\Theta} : \varphi \to \chi \in \Theta\}$, whereas the latter semigroup is isomorphic to the semigroup generated by $\mathsf{Var}_{\boldsymbol{LC}}$ with the presentation $\{\varphi = \chi : \varphi \to \chi \in \Theta\}$.

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