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The fixed point and the Craig interpolation properties for sublogics of IL

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Abstract

We study the fixed point property and the Craig interpolation property for sublogics of the interpretability logic **IL**. We provide a complete description of these sublogics concerning the uniqueness of fixed points, the fixed point property and the Craig interpolation property.

Keywords Interpretability logic · Fixed point property · Uniqueness of fixed points · Craig interpolation property

Mathematics Subject Classification 03B45 · 03F45

1 Introduction

De Jongh and Sambin's fixed point theorem [12] for the modal propositional logic **GL** is one of notable results of modal logical investigation of formalized provability. For any modal formula A, let $v(A)$ be the set of all propositional variables contained in A. A logic L is said to have the fixed point property (FPP) if for any modal formula $A(p)$ in which the propositional variable p appears only in the scope of $\Box$, there exists a modal formula B such that $v(B) \subseteq v(A) \setminus \{p\}$ and $L \vdash B \leftrightarrow A(B)$. De Jongh and

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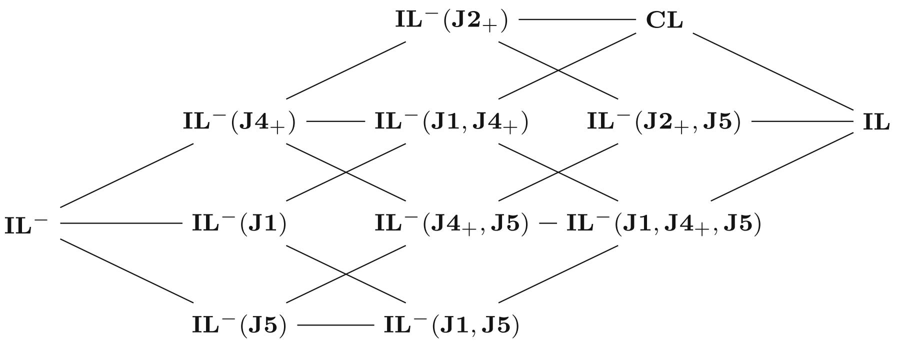

Fig. 1 Sublogics of **IL**

Sambin's theorem states that **GL** has FPP, and this is understood as a counterpart of the fixed point theorem in formal arithmetic (see [4]). Bernardi [2] also proved the uniqueness of fixed points (UFP) for **GL**.

A logic L is said to have the Craig interpolation property (CIP) if for any formulas A and B, if $L \vdash A \to B$, then there exists a formula C such that $v(C) \subseteq v(A) \cap v(B)$, $L \vdash A \to C$, and $L \vdash C \to B$. Smoryński [13] and Boolos [3] independently proved that **GL** has CIP. Smoryński also made an important observation that FPP for **GL** follows from CIP and UFP.

The interpretability logic **IL** is an extension of **GL** in the language of **GL** equipped with the binary modal operator $\rhd$, where the modal formula $A \rhd B$ is read as "$T + B$ is relatively interpretable in $T + A$". It is natural to ask whether **IL** also has the properties that hold for **GL**. Indeed, de Jongh and Visser [8] proved UFP for **IL** and that **IL** has FPP. Also Areces, Hoogland, and de Jongh [1] proved that **IL** has CIP.

Ignatiev [5] introduced the sublogic **CL** of **IL** as a base logic of the modal logical investigation of the notion of partial conservativity, and proved that **CL** is complete with respect to relational semantics (that is, usual Veltman semantics). Kurahashi and Okawa [9] also introduced several sublogics of **IL**, and showed the completeness and the incompleteness of these sublogics with respect to relational semantics.

In this paper, we investigate UFP, FPP, and CIP for sublogics of **IL** shown in Fig. 1.

Moreover, for technical reasons, we introduce and investigate the notions of ℓUFP and ℓFPP that are restricted versions of UFP and FPP with respect to some particular forms of formulas, respectively. Table 1 summarizes a complete description of these sublogics concerning ℓUFP, UFP, ℓFPP, FPP, and CIP.

The paper is organized as follows. In Sect. 3, we show that UFP holds for extensions of $\mathbf{IL}^-(\mathbf{J4}_+)$, and that UFP is not the case for sublogics of $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$. We also show that ℓUFP holds for extensions of $\mathbf{IL}^-$. In Sect. 4, we prove that the logic $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ has CIP by modifying a semantical proof of CIP for **IL** by Areces, Hoogland, and de Jongh. We also notice that CIP for **IL** easily follows from CIP for $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$. In Sect. 5, we observe that FPP for $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ immediately follows from our results in the previous sections. Also, we give a syntactical proof of FPP for $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$. Moreover, we prove that $\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5})$ has ℓFPP. In Sect. 6, we provide counter models of ℓFPP for **CL** and $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$ and a counter model of FPP for $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$.

Table 1 ℓUFP, UFP, ℓFPP, FPP, and CIP for sublogics of **IL**

	ℓUFP	UFP	ℓFPP	FPP	CIP
$\mathbf{IL}^-$	✓	×	×	×	×
$\mathbf{IL}^-(\mathbf{J1})$	✓	×	×	×	×
$\mathbf{IL}^-(\mathbf{J5})$	✓	×	×	×	×
$\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$	✓	×	×	×	×
$\mathbf{IL}^-(\mathbf{J4}_+)$	✓	✓	×	×	×
$\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+)$	✓	✓	×	×	×
$\mathbf{IL}^-(\mathbf{J2}_+)$	✓	✓	×	×	×
CL	✓	✓	×	×	×
$\mathbf{IL}^-(\mathbf{J4}_+, \mathbf{J5})$	✓	✓	✓	×	×
$\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$	✓	✓	✓	×	×
$\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$	✓	✓	✓	✓	✓
IL	✓	✓ [8]	✓	✓ [8]	✓ [1]

As a consequence, we also show that CIP is not the case for these sublogics except $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ and **IL**.

2 Preliminaries

2.1 IL and its sublogics

The interpretability logic **IL** is a base logic of modal logical investigations of the notion of relative interpretability (see [15, 16]). The language of **IL** consists of propositional variables $p, q, \ldots$, the propositional constant $\bot$, the logical connective $\to$, the unary modal operator $\Box$, and the binary modal operator $\rhd$. Other logical connectives, the propositional constant $\top$, and the modal operator $\Diamond$ are introduced as usual abbreviations. The formulas of **IL** are generated by the following grammar:

$$A ::= \bot \mid p \mid A \to A \mid \Box A \mid A \rhd A.$$

For each formula A, let $\boxdot A \equiv A \wedge \Box A$.

Definition 1 The axioms of the modal propositional logic **IL** are as follows:

L1 All tautologies in the language of **IL**;
L2 $\Box(A \to B) \to (\Box A \to \Box B)$;
L3 $\Box(\Box A \to A) \to \Box A$;
J1 $\Box(A \to B) \to A \rhd B$;
J2 $(A \rhd B) \wedge (B \rhd C) \to A \rhd C$;
J3 $(A \rhd C) \wedge (B \rhd C) \to (A \vee B) \rhd C$;
J4 $A \rhd B \to (\Diamond A \to \Diamond B)$;
J5 $\Diamond A \rhd A$.

The inference rules of **IL** are Modus Ponens $\dfrac{A \quad A \to B}{B}$ and Necessitation $\dfrac{A}{\Box A}$.

The conservativity logic **CL** is obtained from **IL** by removing the axiom scheme **J5**, that was introduced by Ignatiev [5] as a base logic of modal logical investigations of the notion of partial conservativity. Several other sublogics of **IL** were introduced in [9]. The basis for these newly introduced logics is the logic $\mathbf{IL}^-$.

Definition 2 The language of $\mathbf{IL}^-$ is that of **IL**, and the axioms of $\mathbf{IL}^-$ are **L1**, **L2**, **L3**, **J3**, and **J6**: $\Box A \leftrightarrow (\neg A) \rhd \bot$. The inference rules of $\mathbf{IL}^-$ are Modus Ponens, Necessitation, **R1** $\dfrac{A \to B}{C \rhd A \to C \rhd B}$, and **R2** $\dfrac{A \to B}{B \rhd C \to A \rhd C}$.

For schemata $\Sigma_1, \ldots, \Sigma_n$, let $\mathbf{IL}^-(\Sigma_1, \ldots, \Sigma_n)$ be the logic obtained by adding $\Sigma_1, \ldots, \Sigma_n$ as axiom schemata to $\mathbf{IL}^-$. The following schemata $\mathbf{J2}_+$ and $\mathbf{J4}_+$ were introduced in [9] and [15], respectively:

$\mathbf{J2}_+$ $(A \rhd (B \lor C)) \land (B \rhd C) \to A \rhd C$;
$\mathbf{J4}_+$ $\Box(A \to B) \to (C \rhd A \to C \rhd B)$.

In this paper, we mainly deal with logics consisting of some of the axiom schemata **J1**, $\mathbf{J2}_+$, $\mathbf{J4}_+$, and **J5** (see Fig. 1 in Sect. 1). Then, we have the following proposition.

Proposition 1 *Let A, B, and C be any formulas.*

1. $\mathbf{IL}^- \vdash \Box\neg A \to A \rhd B$.
2. $\mathbf{IL}^- \vdash \Box(A \to B) \to (B \rhd C \to A \rhd C)$.
3. $\mathbf{IL}^- \vdash (\neg A \land B) \rhd C \to (A \rhd C \to B \rhd C)$.
4. $\mathbf{IL}^-(\mathbf{J4}_+) \vdash \mathbf{J4}$.
5. $\mathbf{IL}^-(\mathbf{J2}_+) \vdash \mathbf{J2} \land \mathbf{J4}_+$.
6. $\mathbf{IL}^-(\mathbf{J2}_+) \vdash (A \rhd B) \land ((B \land \neg C) \rhd C) \to (A \rhd C)$.
7. $\mathbf{IL}^-(\mathbf{J1}) \vdash A \rhd A$.
8. **CL** *is deductively equivalent to* $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J2}_+)$.
9. **IL** *is deductively equivalent to* $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J2}_+, \mathbf{J5})$.

Proof Except clause 3, see [9]. For 3, by **J3**, $\mathbf{IL}^- \vdash ((\neg A \land B) \rhd C) \land (A \rhd C) \to ((\neg A \land B) \lor A) \rhd C$. Since $\mathbf{IL}^- \vdash B \to ((\neg A \land B) \lor A)$, we have $\mathbf{IL}^- \vdash ((\neg A \land B) \lor A) \rhd C \to B \rhd C$ by the rule **R2**. Thus, $\mathbf{IL}^- \vdash ((\neg A \land B) \rhd C) \land (A \rhd C) \to B \rhd C$. □

The following lemma (Lemma 1) plays an important role in our proofs of CIP and FPP for $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ in Sects. 4 and 5.

Fact 1 (See [17]) *For any formula A,*

$$\mathbf{IL}^- \vdash (A \lor \Diamond A) \leftrightarrow ((A \land \Box\neg A) \lor \Diamond(A \land \Box\neg A)).$$

Lemma 1 *Let A and C be any formulas.*

1. $\mathbf{IL}^-(\mathbf{J2}, \mathbf{J5}) \vdash ((A \land \Box\neg A) \rhd C) \leftrightarrow (A \rhd C)$.
2. $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5}) \vdash (C \rhd (A \land \Box\neg A)) \leftrightarrow (C \rhd A)$.

Proof In this proof, let $B \equiv (A \wedge \Box\neg A)$.

1. ($\leftarrow$): Since $\mathbf{IL}^- \vdash B \to A$, we have $\mathbf{IL}^- \vdash A \rhd C \to B \rhd C$ by **R2**.

($\rightarrow$): Since $\mathbf{IL}^-(\mathbf{J5}) \vdash \Diamond B \rhd B$, we have $\mathbf{IL}^-(\mathbf{J2}, \mathbf{J5}) \vdash B \rhd C \to \Diamond B \rhd C$. Hence, by **J3**,

$$\mathbf{IL}^-(\mathbf{J2}, \mathbf{J5}) \vdash B \rhd C \to (B \vee \Diamond B) \rhd C.$$

By Fact 1 and **R2**, we obtain

$$\mathbf{IL}^-(\mathbf{J2}, \mathbf{J5}) \vdash B \rhd C \to (A \vee \Diamond A) \rhd C.$$

Since $\mathbf{IL}^- \vdash A \to (A \vee \Diamond A)$, we obtain

$$\mathbf{IL}^-(\mathbf{J2}, \mathbf{J5}) \vdash B \rhd C \to A \rhd C$$

by **R2**.

2. ($\rightarrow$): This is immediate from $\mathbf{IL}^- \vdash B \to A$ and **R1**.

($\leftarrow$): Since $\mathbf{IL}^- \vdash A \to (A \vee \Diamond A)$, we obtain

$$\mathbf{IL}^- \vdash C \rhd A \to C \rhd (A \vee \Diamond A)$$

by **R1**. Then, by Fact 1 and **R1**,

$$\mathbf{IL}^- \vdash C \rhd A \to C \rhd (B \vee \Diamond B).$$

Since $\mathbf{IL}^-(\mathbf{J5}) \vdash \Diamond B \rhd B$, we obtain

$$\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5}) \vdash C \rhd A \to C \rhd B$$

because $(C \rhd (\Diamond B \vee B)) \wedge (\Diamond B \rhd B) \to C \rhd B$ is an instance of $\mathbf{J2}_+$. □

2.2 $\mathbf{IL}^-$-frames and models

Definition 3 We say that a system $\langle W, R, \{S_w\}_{w \in W}\rangle$ is an $\mathbf{IL}^-$-*frame* if it satisfies the following three conditions:

1. W is a non-empty set;
2. R is a transitive and conversely well-founded binary relation on W;
3. For each $w \in W$, S_w is a binary relation on W with

$$\forall x, y \in W (x S_w y \Rightarrow w R x).$$

A system $\langle W, R, \{S_w\}_{w \in W}, \Vdash\rangle$ is called an $\mathbf{IL}^-$-*model* if $\langle W, R, \{S_w\}_{w \in W}\rangle$ is an $\mathbf{IL}^-$-frame and $\Vdash$ is a usual satisfaction relation on the Kripke frame $\langle W, R\rangle$ with the following additional condition:

$$w \Vdash A \rhd B \iff \forall x \in W (w R x \;\&\; x \Vdash A \Rightarrow \exists y \in W (x S_w y \;\&\; y \Vdash B)).$$

A formula A is said to be *valid* in an $\mathbf{IL}^-$-frame $\langle W, R, \{S_w\}_{w \in W}\rangle$ if $w \Vdash A$ for any satisfaction relation $\Vdash$ on the frame and any $w \in W$.

For each $w \in W$, let $R[w] := \{x \in W : wRx\}$.

Proposition 2 (See [9] and [15]) *Let $\mathcal{F} = \langle W, R, \{S_w\}_{w \in W}\rangle$ be any $\mathbf{IL}^-$-frame.*

1. $\mathbf{J1}$ *is valid in $\mathcal{F}$ if and only if for any $w, x \in W$, if wRx, then $xS_w x$.*
2. $\mathbf{J2}_+$ *is valid in $\mathcal{F}$ if and only if $\mathbf{J4}_+$ is valid in $\mathcal{F}$ and for any $w \in W$, S_w is transitive.*
3. $\mathbf{J4}_+$ *is valid in $\mathcal{F}$ if and only if for any $w \in W$, S_w is a binary relation on $R[w]$.*
4. $\mathbf{J5}$ *is valid in $\mathcal{F}$ if and only if for any $w, x, y \in W$, wRx and xRy imply $xS_w y$.*

Theorem 1 (See [5], [7] and [9]) *Let L be one of logics shown in Fig. 1 in Sect. 1. Then, for any formula A, the following are equivalent:*

1. $L \vdash A$.
2. *A is valid in all (finite) $\mathbf{IL}^-$-frames in which all axioms of L are valid.*

2.3 The fixed point and the Craig interpolation properties

For each formula A, let $v(A)$ be the set of all propositional variables contained in A.

Definition 4 We say that a formula A is *modalized* in a propositional variable p if every occurrence of p in A is in the scope of some modal operators $\Box$ or $\rhd$.

Definition 5 A logic L is said to have the *fixed point property* (FPP) if for any propositional variable p and any formula $A(p)$ which is modalized in p, there exists a formula F such that $v(F) \subseteq v(A) \setminus \{p\}$ and $L \vdash F \leftrightarrow A(F)$.

Definition 6 We say that the *uniqueness of fixed points* (UFP) holds for a logic L if for any propositional variables p, q and any formula $A(p)$ which is modalized in p and does not contain q,

$$L \vdash \boxdot(p \leftrightarrow A(p)) \land \boxdot(q \leftrightarrow A(q)) \to (p \leftrightarrow q).$$

Theorem 2 (De Jongh and Visser [8])

1. $\mathbf{IL}$ *has FPP.*
2. *UFP holds for* $\mathbf{IL}$.

In particular, de Jongh and Visser showed that a fixed point of a formula $A(p) \rhd B(p)$ is $A(\top) \rhd B(\Box\neg A(\top))$. Then, a fixed point of every formula $A(p)$ which is modalized in p is explicitly calculable by a usual argument.

Definition 7 A logic L is said to have the *Craig interpolation property* (CIP) if for any formulas A and B, if $L \vdash A \to B$, then there exists a formula C such that $v(C) \subseteq v(A) \cap v(B)$, $L \vdash A \to C$, and $L \vdash C \to B$.

Theorem 3 (Areces, Hoogland, and de Jongh [1]) $\mathbf{IL}$ *has CIP.*

3 Uniqueness of fixed points

In this section, we investigate the uniqueness of fixed points for sublogics. Firstly, we show that UFP holds for extensions of $\mathbf{IL}^-(\mathbf{J4}_+)$. Secondly, we prove that UFP is not the case for sublogics of $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$. Then, we investigate the newly introduced notion that a formula $A(p)$ is *left-modalized* in a propositional variable p. We prove that UFP with respect to formulas which are left-modalized in p (ℓUFP) holds for all extensions of $\mathbf{IL}^-$. Finally, we discuss Smoryński's implication "CIP + UFP ⇒ FPP" in our framework.

3.1 UFP

By adapting Smoryński's argument [14], de Jongh and Visser [8] showed that UFP holds for every logic closed under Modus Ponens and Necessitation, and containing **L1**, **L2**, **L3**, **E1**, and **E2**, where

E1 $\Box(A \leftrightarrow B) \to (A \rhd C \leftrightarrow B \rhd C)$;
E2 $\Box(A \leftrightarrow B) \to (C \rhd A \leftrightarrow C \rhd B)$.

Since **E1** and **E2** are easy consequences of Proposition 1.2 and $\mathbf{J4}_+$ respectively, we obtain the following theorem.

Theorem 4 (UFP for $\mathbf{IL}^-(\mathbf{J4}_+)$) *UFP holds for every extension of the logic* $\mathbf{IL}^-(\mathbf{J4}_+)$.

As shown in [8], in the proof of Theorem 4, the use of the following substitution principle is essential.

Proposition 3 (The Substitution Principle) *Let A, B, and $C(p)$ be any formulas.*

1. $\mathbf{IL}^-(\mathbf{J4}_+) \vdash \boxdot(A \leftrightarrow B) \to (C(A) \leftrightarrow C(B))$.
2. *If $C(p)$ is modalized in p, then* $\mathbf{IL}^-(\mathbf{J4}_+) \vdash \Box(A \leftrightarrow B) \to (C(A) \leftrightarrow C(B))$.

Proposition 3.2 shows that every extension L of $\mathbf{IL}^-(\mathbf{J4}_+)$ proves $\Box(A \leftrightarrow B) \to (C(A) \leftrightarrow C(B))$ for any formula $C(p)$ which is modalized in p. We show that the converse of this statement also holds.

Proposition 4 *Let L be any extension of $\mathbf{IL}^-$. Suppose that for any formula $C(p)$ which is modalized in p, $L \vdash \Box(A \leftrightarrow B) \to (C(A) \leftrightarrow C(B))$. Then, $L \vdash \mathbf{J4}_+$.*

Proof Let A, B, and C be any formulas and assume $p \notin v(C)$. Then, the formula $C \rhd p$ is modalized in p. By the supposition, we have

$$L \vdash \Box(A \leftrightarrow A \wedge B) \to (C \rhd A \leftrightarrow C \rhd (A \wedge B)).$$

Since $\mathbf{IL}^- \vdash \Box(A \to B) \to \Box(A \leftrightarrow A \wedge B)$ and $\mathbf{IL}^- \vdash C \rhd (A \wedge B) \to C \rhd B$, we obtain $L \vdash \Box(A \to B) \to (C \rhd A \to C \rhd B)$. □

On the other hand, we show that UFP does not hold for sublogics of $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$ in general.

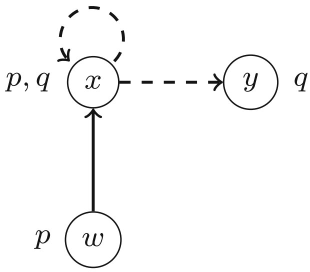

Fig. 2 A counter model of UFP for $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$

Proposition 5 *Let p, q be distinct propositional variables. Then,*

$$\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5}) \nvdash \boxdot(p \leftrightarrow (\top \rhd \neg p)) \land \boxdot(q \leftrightarrow (\top \rhd \neg q)) \to (p \leftrightarrow q).$$

Proof We define an $\mathbf{IL}^-$-frame $\mathcal{F} = \langle W, R, \{S_w\}_{w \in W} \rangle$ as follows:

- $W := \{w, x, y\}$;
- $R := \{\langle w, x \rangle\}$;
- $S_w := \{\langle x, x \rangle, \langle x, y \rangle\}$, $S_x := \emptyset$, $S_y := \emptyset$.

Obviously, by Proposition 2, $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$ is valid in $\mathcal{F}$. Let $\Vdash$ be a satisfaction relation on $\mathcal{F}$ satisfying the following conditions:

- $w \Vdash p$ and $w \nVdash q$;
- $x \Vdash p$ and $x \Vdash q$;
- $y \nVdash p$ and $y \Vdash q$.

We prove $w \Vdash \boxdot(p \leftrightarrow (\top \rhd \neg p)) \land \boxdot(q \leftrightarrow (\top \rhd \neg q)) \land \neg(p \leftrightarrow q)$. Since $w \Vdash p$ and $w \nVdash q$, $w \Vdash \neg(p \leftrightarrow q)$ is obvious. We show $w \Vdash (p \leftrightarrow (\top \rhd \neg p)) \land (q \leftrightarrow (\top \rhd \neg q))$. Since $w \Vdash p$ and $w \nVdash q$, it suffices to prove $w \Vdash \top \rhd \neg p$ and $w \Vdash \neg(\top \rhd \neg q)$.

- $w \Vdash \top \rhd \neg p$: Let $z \in W$ be any element with wRz. Then, $z = x$. Since xS_wy and $y \Vdash \neg p$, we obtain $w \Vdash \top \rhd \neg p$.
- $w \Vdash \neg(\top \rhd \neg q)$: Let $z \in W$ be any element with xS_wz. Then, $z = x$ or $z = y$. In either case, we obtain $z \Vdash q$. Since xRy, we conclude $w \Vdash \neg(\top \rhd \neg q)$.

At last, we show $w \Vdash \Box(p \leftrightarrow (\top \rhd \neg p)) \land \Box(q \leftrightarrow (\top \rhd \neg q))$. Let $z \in W$ be such that wRz. Then, $z = x$. Since there is no $z' \in W$ such that xRz', $x \Vdash (\top \rhd \neg p) \land (\top \rhd \neg q)$. Since $x \Vdash p$ and $x \Vdash q$, we have $x \Vdash (p \leftrightarrow (\top \rhd \neg p)) \land (q \leftrightarrow (\top \rhd \neg q))$. Hence, we obtain $w \Vdash \Box(p \leftrightarrow (\top \rhd \neg p)) \land \Box(q \leftrightarrow (\top \rhd \neg q))$.

Therefore, $w \Vdash \boxdot(p \leftrightarrow (\top \rhd \neg p)) \land \boxdot(q \leftrightarrow (\top \rhd \neg q)) \land \neg(p \leftrightarrow q)$. □

3.2 ℓUFP

Even for extensions of $\mathbf{IL}^-$, Proposition 1.2 suggests that the uniqueness of fixed points may hold with respect to formulas in some particular forms. From this perspective, we introduce the notion that formulas are left-modalized in p.

Definition 8 We say that a formula A is *left-modalized* in a propositional variable p if A is modalized in p and $p \notin v(C)$ for any subformula $B \rhd C$ of A.

Then, we obtain the following version of the substitution principle.

Proposition 6 *Let A, B, and $C(p)$ be any formulas such that $p \notin v(E)$ for any subformula $D \rhd E$ of C.*

1. $\mathbf{IL}^- \vdash \boxdot(A \leftrightarrow B) \to (C(A) \leftrightarrow C(B))$.
2. *If $C(p)$ is left-modalized in p, then* $\mathbf{IL}^- \vdash \Box(A \leftrightarrow B) \to (C(A) \leftrightarrow C(B))$.

Proof 1. This proposition is proved by induction on the construction of $C(p)$. We only prove the case $C(p) \equiv D(p) \rhd E$ (By our supposition, $p \notin v(E)$). For any subformula $D' \rhd E'$ of D, it is also a subformula of C, and hence $p \notin v(E')$. Then, by the induction hypothesis, we obtain

$$\mathbf{IL}^- \vdash \boxdot(A \leftrightarrow B) \to (D(A) \leftrightarrow D(B)).$$

Then, $\mathbf{IL}^- \vdash \Box(A \leftrightarrow B) \to \Box(D(A) \leftrightarrow D(B))$. Therefore, by Proposition 1.2,

$$\mathbf{IL}^- \vdash \Box(A \leftrightarrow B) \to (D(A) \rhd E \leftrightarrow D(B) \rhd E).$$

Since $p \notin v(E)$, we find $C(A) \equiv (D(A) \rhd E)$ and $C(B) \equiv (D(B) \rhd E)$. Therefore,

$$\mathbf{IL}^- \vdash \Box(A \leftrightarrow B) \to (C(A) \leftrightarrow C(B)).$$

2. This follows from our proof of clause 1. □

We introduce our restricted versions of UFP and FPP.

Definition 9 We say that ℓUFP holds for a logic L if for any formula $A(p)$ which is left-modalized in p, $L \vdash \boxdot(p \leftrightarrow A(p)) \land \boxdot(q \leftrightarrow A(q)) \to (p \leftrightarrow q)$.

Definition 10 We say that a logic L has ℓFPP if for any formula $A(p)$ which is left-modalized in p, there exists a formula F such that $v(F) \subseteq v(A) \setminus \{p\}$ and $L \vdash F \leftrightarrow A(F)$.

Then, ℓUFP holds for every our sublogic of $\mathbf{IL}$.

Theorem 5 (ℓUFP for $\mathbf{IL}^-$) *ℓUFP holds for all extensions of* $\mathbf{IL}^-$.

Proof Let $A(p)$ be any formula which is left-modalized in p. Then, by Proposition 6.2, $\mathbf{IL}^- \vdash \Box(p \leftrightarrow q) \to (A(p) \leftrightarrow A(q))$. Therefore,

$$\begin{aligned}
\mathbf{IL}^- \vdash \boxdot(p \leftrightarrow A(p)) \land \boxdot(q \leftrightarrow A(q)) &\to (\Box(p \leftrightarrow q) \to (A(p) \leftrightarrow A(q))) \\
&\to (\Box(p \leftrightarrow q) \to (p \leftrightarrow q)) \\
&\to (\Box(\Box(p \leftrightarrow q) \to (p \leftrightarrow q))) \\
&\to \Box(p \leftrightarrow q) \\
&\to (p \leftrightarrow q).
\end{aligned}$$

□

3.3 Applications of Smoryński's argument

We have shown that UFP and the substitution principle hold for extensions of $\mathbf{IL}^-(\mathbf{J4}_+)$ (Theorem 4 and Proposition 3). Then, by applying Smoryński's argument [13], we prove that for any appropriate extension of $\mathbf{IL}^-(\mathbf{J4}_+)$, CIP implies FPP.

Lemma 2 *Let L be any extension of $\mathbf{IL}^-(\mathbf{J4}_+)$ that is closed under substituting a formula for a propositional variable. If L has CIP, then L also has FPP.*

Proof Suppose $L \supseteq \mathbf{IL}^-(\mathbf{J4}_+)$ and L has CIP. Let $A(p)$ be any formula modalized in p. Then, by Theorem 4,

$$L \vdash \boxdot(p \leftrightarrow A(p)) \wedge \boxdot(q \leftrightarrow A(q)) \to (p \leftrightarrow q).$$

We have

$$L \vdash \boxdot(p \leftrightarrow A(p)) \wedge p \to (\boxdot(q \leftrightarrow A(q)) \to q).$$

Since L has CIP, there exists a formula F such that $v(F) \subseteq v(A)\setminus\{p\}$, $L \vdash \boxdot(p \leftrightarrow A(p)) \wedge p \to F$ and $L \vdash F \to (\boxdot(q \leftrightarrow A(q)) \to q)$. Since $q \notin v(F)$, we have $L \vdash F \to (\boxdot(p \leftrightarrow A(p)) \to p)$ by substituting p for q. Then,

$$L \vdash \boxdot(p \leftrightarrow A(p)) \to (F \leftrightarrow p).$$

By substituting $A(F)$ for p, we get

$$L \vdash \boxdot(A(F) \leftrightarrow A(A(F))) \to (F \leftrightarrow A(F)). \tag{1}$$

Then

$$L \vdash \Box(A(F) \leftrightarrow A(A(F))) \to \Box(F \leftrightarrow A(F)).$$

Since $A(p)$ is modalized in p, by Proposition 3.2,

$$L \vdash \Box(A(F) \leftrightarrow A(A(F))) \to (A(F) \leftrightarrow A(A(F))).$$

Then, by applying the axiom scheme **L3**, we obtain $L \vdash A(F) \leftrightarrow A(A(F))$. From this with (1), we conclude $L \vdash F \leftrightarrow A(F)$. Therefore, F is a fixed point of $A(p)$ in L. □

Also, we have shown that ℓUFP and the substitution principle with respect to left-modalized formulas hold for extensions of $\mathbf{IL}^-$ (Theorem 5 and Proposition 6). Thus, our proof of Lemma 2 also works for proving the following lemma.

Lemma 3 *Let L be any extension of $\mathbf{IL}^-$ that is closed under substituting a formula for a propositional variable. If L has CIP, then L also has ℓFPP.*

4 The Craig interpolation property

In this section, we prove the following theorem.

Theorem 6 (CIP for $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$) *The logic* $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ *has CIP.*

Our proof of Theorem 6 is based on a semantical proof of CIP for **IL** due to Areces, Hoogland, and de Jongh [1].

4.1 Preparations for our proof of Theorem 6

In this subsection, we prepare several definitions and prove some lemmas that are used in our proof of Theorem 6. Only in this section, we write $\vdash A$ instead of $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5}) \vdash A$ if there is no confusion. Notice that by Proposition 1, $\vdash \mathbf{J2} \wedge \mathbf{J4} \wedge \mathbf{J4}_+$.

For a formula A, we define the formula $\sim A$ as follows:

$$\sim A :\equiv \begin{cases} B & \text{if } A \equiv \neg B \text{ for some formula } B, \\ \neg A & \text{otherwise.} \end{cases}$$

For a set X of formulas, by $\mathcal{L}_X$ we denote the set of all formulas built up from $\bot$ and propositional variables occurring in formulas in X. We simply write $\mathcal{L}_A$ instead of $\mathcal{L}_{\{A\}}$. For a finite set X of formulas, let $\bigwedge X$ be a conjunction of all elements of X. For the sake of simplicity, only in this section, $\vdash \bigwedge X \to A$ will be written as $\vdash X \to A$.

For a set Φ of formulas, we define

$$\Phi_{\rhd} := \{A : \text{there exists a formula } B \text{ such that } A \rhd B \in \Phi \text{ or } B \rhd A \in \Phi\}.$$

Definition 11 A set Φ of formulas is said to be *adequate* if it satisfies the following conditions:

1. Φ is closed under taking subformulas and the $\sim$-operation;
2. $\bot \in \Phi_{\rhd}$;
3. If $A, B \in \Phi_{\rhd}$, then $A \rhd B \in \Phi$;
4. If $A \in \Phi_{\rhd}$, then $\Box \sim A \in \Phi$.

Our notion of adequate sets is essentially the same as that introduced by de Jongh and Veltman [7].

Note that for any finite set X of formulas, there exists the smallest finite adequate set Φ containing X. We denote this set by Φ_X.

Definition 12 1. A pair (Γ_1, Γ_2) of finite sets of formulas is said to be *separable* if for some formula $I \in \mathcal{L}_{\Gamma_1} \cap \mathcal{L}_{\Gamma_2}$, $\vdash \Gamma_1 \to I$, and $\vdash \Gamma_2 \to \neg I$. A pair is said to be *inseparable* if it is not separable.
2. A pair (Γ_1, Γ_2) of finite sets of formulas is said to be *complete* if it is inseparable and

- For each $F \in \Phi_{\Gamma_1}$, either $F \in \Gamma_1$ or $\sim F \in \Gamma_1$;
- For each $F \in \Phi_{\Gamma_2}$, either $F \in \Gamma_2$ or $\sim F \in \Gamma_2$.

We say a finite set X of formulas is *consistent* if $\nvdash X \to \bot$. If a pair (Γ_1, Γ_2) is inseparable, then it can be shown that both of Γ_1 and Γ_2 are consistent.

In the rest of this subsection, we fix some sets X and Y of formulas. Put $\Phi^1 := \Phi_X$ (resp. $\Phi^2 := \Phi_Y$) and $\mathcal{L}_1 := \mathcal{L}_X$ (resp. $\mathcal{L}_2 := \mathcal{L}_Y$). Let $X' \subseteq \Phi^1$ and $Y' \subseteq \Phi^2$. It is easily proved that if (X', Y') is inseparable, then for any formula $A \in \Phi^1$, at least one of $(X' \cup \{A\}, Y')$ and $(X' \cup \{\sim A\}, Y')$ is inseparable. Also, a similar statement holds for Φ^2 and Y'. Then, we obtain the following proposition.

Proposition 7 *If (X, Y) is inseparable, then there exists some complete pair $\Gamma' = (\Gamma_1, \Gamma_2)$ such that $X \subseteq \Gamma_1 \subseteq \Phi^1$ and $Y \subseteq \Gamma_2 \subseteq \Phi^2$.*

Let $K(\Phi^1, \Phi^2)$ be the set of all complete pairs (Γ_1, Γ_2) satisfying $\Gamma_1 \subseteq \Phi^1$ and $\Gamma_2 \subseteq \Phi^2$. Note that the set $K(\Phi^1, \Phi^2)$ is finite. For each $\Gamma \in K(\Phi^1, \Phi^2)$, let Γ_1 and Γ_2 be the first and the second components of Γ, respectively.

Definition 13 We define a binary relation $\prec$ on $K(\Phi^1, \Phi^2)$ as follows: For $\Gamma, \Delta \in K(\Phi^1, \Phi^2)$,

$$\Gamma \prec \Delta :\Leftrightarrow \begin{array}{l}\text{For } i = \{1, 2\},\ \text{if } \Box A \in \Gamma_i,\ \text{then } \Box A, A \in \Delta_i,\ \text{and} \\ \text{there exists some } \Box B \text{ such that } \Box B \in \Delta_1 \cup \Delta_2 \text{ and } \Box B \notin \Gamma_1 \cup \Gamma_2.\end{array}$$

Then, $\prec$ is a transitive and conversely well-founded binary relation on $K(\Phi^1, \Phi^2)$.

Definition 14 Let $\Gamma, \Delta \in K(\Phi^1, \Phi^2)$ and $A \in \Phi^1_{\rhd} \cup \Phi^2_{\rhd}$. We say that Δ is an *A-critical successor* of Γ (write $\Gamma \prec_A \Delta$) if the following conditions are met:

1. $\Gamma \prec \Delta$;
2. If $A \in \Phi^1_{\rhd}$, then

$$\begin{aligned}\Gamma_1^A :=& \{\Box{\sim}B, {\sim}B : B \rhd A \in \Gamma_1\} \subseteq \Delta_1; \\ \Gamma_2^A :=& \{\Box{\sim}C, {\sim}C : C \in \Phi^2_{\rhd} \text{ and for some } I \in \mathcal{L}_1 \cap \mathcal{L}_2, \\ & \vdash \Gamma_1 \to (I \wedge \neg A) \rhd A \ \&\ \vdash \Gamma_2 \to C \rhd I\} \subseteq \Delta_2.\end{aligned}$$

3. If $A \in \Phi^2_{\rhd}$, then

$$\begin{aligned}\Gamma_1^A :=& \{\Box{\sim}B, {\sim}B : B \in \Phi^1_{\rhd} \text{ and for some } I \in \mathcal{L}_1 \cap \mathcal{L}_2, \\ & \vdash \Gamma_1 \to B \rhd I \ \&\ \vdash \Gamma_2 \to (I \wedge \neg A) \rhd A\} \subseteq \Delta_1; \\ \Gamma_2^A :=& \{\Box{\sim}C, {\sim}C : C \rhd A \in \Gamma_2\} \subseteq \Delta_2.\end{aligned}$$

The notion of A-critical successors is originally introduced by de Jongh and Veltman [7]. Our definition is based on a modification due to Areces, Hoogland, and de Jongh [1].

From the following claim, Definition 14 makes sense.

Proposition 8 *If $A \in \Phi^1_{\rhd} \cap \Phi^2_{\rhd}$, then the sets Γ^A_1 in clauses 2 and 3 of Definition 14 coincide. This is also the case for Γ^A_2.*

Proof We prove only for Γ^A_1. It suffices to show that for any formula B, the following are equivalent:

1. $B \rhd A \in \Gamma_1$.
2. $B \in \Phi^1_{\rhd}$ and for some $I \in \mathcal{L}_1 \cap \mathcal{L}_2$, $\vdash \Gamma_1 \to B \rhd I$, and $\vdash \Gamma_2 \to (I \wedge \neg A) \rhd A$.

(1 $\Rightarrow$ 2): Suppose $B \rhd A \in \Gamma_1$, then $B \in \Phi^1_{\rhd}$. By Proposition 1.1, we have $\mathbf{IL}^- \vdash (A \wedge \neg A) \rhd A$ because $\mathbf{IL}^- \vdash \Box\neg(A \wedge \neg A)$. Since $A \in \mathcal{L}_1 \cap \mathcal{L}_2$, the clause 2 holds by letting $I \equiv A$.

(2 $\Rightarrow$ 1): Assume that clause 2 holds. Then, $A \rhd B \in \Phi^1$ because $A, B \in \Phi^1_{\rhd}$. Suppose, towards a contradiction, that $\neg(B \rhd A) \in \Gamma_1$. By Proposition 1.6, $\vdash (B \rhd I) \wedge ((I \wedge \neg A) \rhd A) \to B \rhd A$. Then, we obtain $\vdash \Gamma_1 \to \neg((I \wedge \neg A) \rhd A)$. This contradicts the inseparability of Γ because $(I \wedge \neg A) \rhd A \in \mathcal{L}_1 \cap \mathcal{L}_2$. Hence, $\neg(B \rhd A) \notin \Gamma_1$. Since Γ is complete, $B \rhd A \in \Gamma_1$. □

Lemma 4 *For $\Gamma, \Delta \in K(\Phi^1, \Phi^2)$, if $\Gamma \prec \Delta$, then $\Gamma \prec_{\bot} \Delta$.*

Proof Notice that $\bot \in \Phi^1_{\rhd} \cap \Phi^2_{\rhd}$. By Proposition 8, it suffices to show that if $C \rhd \bot \in \Gamma_1$ (resp. Γ_2), then $\Box{\sim}C, {\sim}C \in \Delta_1$ (resp. Δ_2). Suppose $C \rhd \bot \in \Gamma_1$. Then, by (**J6**), $\vdash \Gamma_1 \to \Box{\sim}C$. Note that $\Box{\sim}C \in \Phi^1$, and hence $\Box{\sim}C \in \Gamma_1$. By $\Gamma \prec \Delta$, $\Box{\sim}C, {\sim}C \in \Delta_1$. The case $C \rhd \bot \in \Gamma_2$ is proved similarly. Therefore, $\Gamma \prec_{\bot} \Delta$. □

Lemma 5 *For $\Gamma, \Delta, \Theta \in K(\Phi^1, \Phi^2)$ and $A \in \Phi^1_{\rhd} \cup \Phi^2_{\rhd}$, if $\Gamma \prec_A \Delta$ and $\Delta \prec \Theta$, then $\Gamma \prec_A \Theta$.*

Proof We only prove the case $A \in \Phi^1_{\rhd}$. Let Γ^A_1 and Γ^A_2 be the sets as in Definition 14. If $\Box{\sim}B, {\sim}B \in \Gamma^A_1$, then $\Box{\sim}B \in \Delta_1$ because $\Gamma \prec_A \Delta$. Thus, $\Box{\sim}B, {\sim}B \in \Theta_1$ because $\Delta \prec \Theta$. Similarly, if $\Box{\sim}C, {\sim}C \in \Gamma^A_2$, then Θ_2 contains $\Box{\sim}C$ and ${\sim}C$. This means $\Gamma \prec_A \Theta$. □

In order to prove the Truth Lemma (Lemma 4.2), we show the following two lemmas.

Lemma 6 *Let $\Gamma \in K(\Phi^1, \Phi^2)$. If $\neg(G \rhd F) \in \Gamma_1 \cup \Gamma_2$, then there exists a pair $\Delta \in K(\Phi^1, \Phi^2)$ such that:*

1. *$\Gamma \prec_F \Delta$;*
2. *$G, \Box{\sim}F \in \Delta_1 \cup \Delta_2$.*

Proof Suppose $\neg(G \rhd F) \in \Gamma_1$. Let

$$X' := \boxdot\,\Gamma_1 \cup \{G, \Box{\sim}G, \Box{\sim}F\} \cup \{\Box{\sim}A, {\sim}A : A \rhd F \in \Gamma_1\};$$
$$Y' := \boxdot\,\Gamma_2 \cup \{\Box{\sim}B, {\sim}B : B \in \Phi^2_{\rhd} \text{ and for some } I \in \mathcal{L}_1 \cap \mathcal{L}_2,\ \vdash \Gamma_1 \to (I \wedge \neg F) \rhd F \ \&\ \vdash \Gamma_2 \to B \rhd I\},$$

where $\boxdot\Gamma_i$ $(i = 1, 2)$ denotes the set $\{\Box C, C : \Box C \in \Gamma_i\}$.

We claim $\Box{\sim}G \notin \Gamma_1 \cup \Gamma_2$. Assume $\Box\neg G \in \Gamma_1$. Then, $\vdash \Gamma_1 \rightarrow \Box\neg G$. By Proposition 1.1, $\vdash \Box\neg G \rightarrow G \rhd F$. Hence, $\vdash \Gamma_1 \rightarrow G \rhd F$. This implies that Γ_1 is inconsistent, a contradiction. Thus, $\Box{\sim}G \notin \Gamma_1$. Moreover, if $\Box{\sim}G \in \Gamma_2$, then $\Diamond G$ separates (Γ_1, Γ_2) because $\Diamond G \in \mathcal{L}_1 \cap \mathcal{L}_2$. This contradicts the inseparability of Γ. Hence, $\Box{\sim}G \notin \Gamma_2$.

We show that (X', Y') is inseparable. Suppose, for a contradiction, that $J \in \mathcal{L}_1 \cap \mathcal{L}_2$ separates (X', Y'). From $\vdash Y' \rightarrow \neg J$,

$$\vdash \boxdot\Gamma_2 \rightarrow \left(J \rightarrow \bigvee_{j\in\kappa} (\Diamond B_j \vee B_j)\right),$$

where κ is an appropriate index set for Y' such that for each $j \in \kappa$, $B_j \in \Phi^2_{\rhd}$ and there exists a formula $I_j \in \mathcal{L}_1 \cap \mathcal{L}_2$ such that

$$\vdash \Gamma_1 \rightarrow (I_j \wedge \neg F) \rhd F, \text{ and} \tag{2}$$

$$\vdash \Gamma_2 \rightarrow B_j \rhd I_j. \tag{3}$$

Then

$$\vdash \Gamma_2 \rightarrow \Box\left(J \rightarrow \bigvee_{j\in\kappa} (\Diamond B_j \vee B_j)\right).$$

By Proposition 1.2,

$$\vdash \Gamma_2 \rightarrow \left(\left(\bigvee_{j\in\kappa} (\Diamond B_j \vee B_j)\right) \rhd \bigvee_{j\in\kappa} I_j \rightarrow J \rhd \bigvee_{j\in\kappa} I_j\right).$$

By (3), **J2**, **J3**, and **J5**, we have $\vdash \Gamma_2 \rightarrow \left(\bigvee_{j\in\kappa} (\Diamond B_j \vee B_j)\right) \rhd \bigvee_{j\in\kappa} I_j$. Hence

$$\vdash \Gamma_2 \rightarrow J \rhd \bigvee_{j\in\kappa} I_j. \tag{4}$$

On the other hand, from $\vdash X' \rightarrow J$,

$$\vdash \boxdot\Gamma_1 \rightarrow \left(\neg J \wedge G \wedge \Box\neg G \rightarrow \bigvee_{A\rhd F\in\Gamma_1} (\Diamond A \vee A) \vee \Diamond F\right),$$

(By Proposition 1.2)

$$\vdash \Gamma_1 \rightarrow \Box\left(\neg J \wedge G \wedge \Box\neg G \rightarrow \bigvee_{A\rhd F\in\Gamma_1} (\Diamond A \vee A) \vee \Diamond F\right),$$

$$\vdash \Gamma_1 \to \left(\left(\bigvee_{A \rhd F \in \Gamma_1} (\Diamond A \vee A) \vee \Diamond F\right) \rhd F \to (\neg J \wedge G \wedge \Box\neg G) \rhd F\right).$$

By **J2**, **J3**, and **J5**, we have $\vdash \Gamma_1 \to \left(\bigvee_{A \rhd F \in \Gamma_1} (\Diamond A \vee A) \vee \Diamond F\right) \rhd F$. Hence, we obtain $\vdash \Gamma_1 \to (\neg J \wedge G \wedge \Box\neg G) \rhd F$. By Proposition 1.3, $\vdash \Gamma_1 \to (J \rhd F \to (G \wedge \Box\neg G) \rhd F)$. By Lemma 1.1, $\vdash \Gamma_1 \to (J \rhd F \to G \rhd F)$. Since $\vdash \Gamma_1 \to \neg(G \rhd F)$, we get $\vdash \Gamma_1 \to \neg(J \rhd F)$. From (2) and **J3**, we obtain $\vdash \Gamma_1 \to \left(\bigvee_{j \in \kappa} I_j \wedge \neg F\right) \rhd F$. By Proposition 1.6, $\vdash \Gamma_1 \to \left(J \rhd \bigvee_{j \in \kappa} I_j \to J \rhd F\right)$. Hence,

$$\vdash \Gamma_1 \to \neg\left(J \rhd \bigvee_{j \in \kappa} I_j\right).$$

From this and (4), we conclude that $\neg(J \rhd \bigvee_{j \in \kappa} I_j)$ separates (Γ_1, Γ_2), a contradiction. Therefore, (X', Y') is inseparable.

Now let $\Delta \in K(\Phi^1, \Phi^2)$ be a complete pair extending (X', Y'). We have $\Gamma \prec_F \Delta$ and $G, \Box{\sim}F \in \Delta_1$. The other case where $\neg(G \rhd F) \in \Gamma_2$ is proved in a similar way. □

Lemma 7 *Let $\Gamma, \Delta \in K(\Phi^1, \Phi^2)$. Suppose that $\Gamma \prec_A \Delta$, $G \rhd F \in \Gamma_1 \cup \Gamma_2$ and $G \in \Delta_1 \cup \Delta_2$. Then, there exists a pair $\Theta \in K(\Phi^1, \Phi^2)$ such that:*

- *$\Gamma \prec_A \Theta$;*
- *$F \in \Theta_1 \cup \Theta_2$;*
- *$\Box{\sim}A, {\sim}A \in \Theta_1 \cup \Theta_2$.*

Proof Suppose $G \rhd F \in \Gamma_1$. From $G \in \Delta_1 \cup \Delta_2$ and $G \in \Phi_1$, we obtain $G \in \Delta_1$ by the inseparability of Δ. We distinguish the following two cases:

(Case 1): Assume $A \in \Phi^1_{\rhd}$. Then, $G \rhd A \in \Phi^1$. If $G \rhd A \in \Gamma_1$, then ${\sim}G \in \Delta_1$ because $\Gamma \prec_A \Delta$. This contradicts the consistency of Δ_1. Therefore, $G \rhd A \notin \Gamma_1$. Since Γ is complete, we have $\neg(G \rhd A) \in \Gamma_1$.

Let:

$$\begin{aligned} X' &:= \boxdot\, \Gamma_1 \cup \{\Box{\sim}F, F, \Box{\sim}A, {\sim}A\} \cup \{\Box{\sim}B, {\sim}B : B \rhd A \in \Gamma_1\}; \\ Y' &:= \boxdot\, \Gamma_2 \cup \{\Box{\sim}C, {\sim}C : C \in \Phi^2_{\rhd} \text{ and for some } I \in \mathcal{L}_1 \cap \mathcal{L}_2, \\ &\qquad \vdash \Gamma_1 \to (I \wedge \neg A) \rhd A \ \&\ \vdash \Gamma_2 \to C \rhd I\}. \end{aligned}$$

We show $\Box{\sim}F \notin \Gamma_1 \cup \Gamma_2$. If $\Box{\sim}G \in \Gamma_1$, then ${\sim}G \in \Delta_1$ because $\Gamma \prec \Delta$. This contradicts the consistency of Δ_1. Hence, $\Box{\sim}G \notin \Gamma_1$. Since $\vdash \Gamma_1 \to (G \rhd F) \wedge \Diamond G$, we have $\vdash \Gamma_1 \to \Diamond F$ by **J4**. Therefore, $\Box{\sim}F \notin \Gamma_1$. Moreover, if $\Box{\sim}F \in \Gamma_2$, then $\Diamond F$ would separate (Γ_1, Γ_2), a contradiction. Thus, $\Box{\sim}F \notin \Gamma_2$.

We show that (X', Y') is inseparable. Suppose, for a contradiction, that for some $J \in \mathcal{L}_1 \cap \mathcal{L}_2$, $\vdash X' \to J$ and $\vdash Y' \to \neg J$.

From $\vdash Y' \to \neg J$,

$$\vdash \boxdot \Gamma_2 \to \left(J \to \bigvee_{j \in \kappa} (\Diamond C_j \vee C_j) \right),$$

where κ is an appropriate index set such that for each $j \in \kappa$, $C_j \in \Phi^2_{\rhd}$ and there exists a formula $I_j \in \mathcal{L}_1 \cap \mathcal{L}_2$ such that $\vdash \Gamma_1 \to (I_j \wedge \neg A) \rhd A$ and $\vdash \Gamma_2 \to C_j \rhd I_j$. Then,

$$\vdash \Gamma_2 \to \Box \left(J \to \bigvee_{j \in \kappa} (\Diamond C_j \vee C_j) \right).$$

Since $\vdash \Gamma_2 \to \left(\bigvee_{j \in \kappa} (\Diamond C_j \vee C_j) \right) \rhd \bigvee I_j$, by Proposition 1.2, we obtain

$$\vdash \Gamma_2 \to J \rhd \bigvee I_j. \tag{5}$$

On the other hand, from $\vdash X' \to J$,

$$\vdash \boxdot \Gamma_1 \to \left(\neg J \wedge \Box \neg F \wedge F \wedge \neg A \to \Diamond A \vee \bigvee_{B \rhd A \in \Gamma_1} (\Diamond B \vee B) \right),$$

$$\vdash \Gamma_1 \to \Box \left(\neg J \wedge \Box \neg F \wedge F \wedge \neg A \to \Diamond A \vee \bigvee_{B \rhd A \in \Gamma_1} (\Diamond B \vee B) \right).$$

Then, by Proposition 1.2, we obtain $\vdash \Gamma_1 \to (\neg J \wedge \Box \neg F \wedge F \wedge \neg A) \rhd A$ because $\vdash \Gamma_1 \to \left(\Diamond A \vee \bigvee_{B \rhd A \in \Gamma_1} (\Diamond B \vee B) \right) \rhd A$. By Proposition 1.3, $\vdash \Gamma_1 \to (J \rhd A \to (\Box \neg F \wedge F \wedge \neg A) \rhd A)$. By Lemma 1.2, we have $\vdash \Gamma_1 \to G \rhd (\Box \neg F \wedge F)$. Then, by Proposition 1.6, we obtain $\vdash \Gamma_1 \to ((\Box \neg F \wedge F \wedge \neg A) \rhd A \to G \rhd A)$. Thus, $\vdash \Gamma_1 \to (J \rhd A \to G \rhd A)$. Since $\neg (G \rhd A) \in \Gamma_1$, we get $\vdash \Gamma_1 \to \neg (J \rhd A)$. Since $\vdash \Gamma_1 \to \left(\bigvee_{j \in \kappa} I_j \wedge \neg A \right) \rhd A$, we have $\vdash \Gamma_1 \to \left(J \rhd \bigvee_{j \in \kappa} I_j \to J \rhd A \right)$ by Proposition 1.6. Therefore,

$$\vdash \Gamma_1 \to \neg \left(J \rhd \bigvee_{j \in \kappa} I_j \right).$$

From this and (5), we conclude that $\neg (J \rhd \bigvee_{j \in \kappa} I_j)$ separates (Γ_1, Γ_2), a contradiction.

(Case 2): Assume $A \in \Phi^2_{\rhd}$. Let:

$$X' := \boxdot \Gamma_1 \cup \{\Box {\sim} F, F\}$$

$$\cup \{\Box{\sim}B, {\sim}B : B \in \Phi^1_{\rhd} \text{ and for some } I \in \mathcal{L}_1 \cap \mathcal{L}_2, \vdash \Gamma_1 \to B \rhd I \ \& \vdash \Gamma_2 \to (I \wedge \neg A) \rhd A\};$$
$$Y' := \boxdot \Gamma_2 \cup \{\Box{\sim}A, {\sim}A\} \cup \{\Box{\sim}C, {\sim}C : C \rhd A \in \Gamma_2\}.$$

As in Case 1, it can be shown $\Box{\sim}F \notin \Gamma_1 \cup \Gamma_2$. We prove that (X', Y') is inseparable. Suppose, for a contradiction, that for some $J \in \mathcal{L}_1 \cap \mathcal{L}_2$, $\vdash X' \to J$ and $\vdash Y' \to \neg J$. From $\vdash X' \to J$,

$$\vdash \boxdot\Gamma_1 \to \left(\Box\neg F \wedge F \wedge \neg J \to \bigvee_{j \in \kappa} (\Diamond B_j \vee B_j)\right),$$

where κ is an appropriate index set such that for each $j \in \kappa$, $B_j \in \Phi^1_{\rhd}$ and there exists a formula $I_j \in \mathcal{L}_1 \cap \mathcal{L}_2$ such that $\vdash \Gamma_1 \to B_j \rhd I_j$ and $\vdash \Gamma_2 \to (I_j \wedge \neg A) \rhd A$. Then,

$$\vdash \Gamma_1 \to \Box\left(\Box\neg F \wedge F \wedge \neg J \to \bigvee_{j \in \kappa} (\Diamond B_j \vee B_j)\right).$$

Since $\vdash \Gamma_1 \to \left(\bigvee_{j \in \kappa} (\Diamond B_j \vee B_j)\right) \rhd \bigvee I_j$, we have

$$\vdash \Gamma_1 \to (\Box\neg F \wedge F \wedge \neg J) \rhd \bigvee_{j \in \kappa} I_j$$

by Proposition 1.2. Then,

$$\vdash \Gamma_1 \to \left(\Box\neg F \wedge F \wedge \bigwedge_{j \in \kappa} \neg I_j \wedge \neg J\right) \rhd \left(\bigvee_{j \in \kappa} I_j \vee J\right),$$
$$\vdash \Gamma_1 \to \left(\Box\neg F \wedge F \wedge \neg\left(\bigvee_{j \in \kappa} I_j \vee J\right)\right) \rhd \left(\bigvee_{j \in \kappa} I_j \vee J\right).$$

Since $G \rhd F \in \Gamma_1$, by Lemma 1.2, we obtain $\vdash \Gamma_1 \to G \rhd (\Box\neg F \wedge F)$. Therefore, by Proposition 1.6, we obtain

$$\vdash \Gamma_1 \to G \rhd \left(\bigvee_{j \in \kappa} I_j \vee J\right). \tag{6}$$

On the other hand, from $\vdash Y' \to \neg J$,

$$\vdash \boxdot \Gamma_2 \to \left(J \wedge \neg A \to \Diamond A \vee \bigvee_{C \rhd A \in \Gamma_2} (\Diamond C \vee C) \right),$$

$$\vdash \Gamma_2 \to \Box \left(J \wedge \neg A \to \Diamond A \vee \bigvee_{C \rhd A \in \Gamma_2} (\Diamond C \vee C) \right).$$

Since $\vdash \Gamma_2 \to \left(\Diamond A \vee \bigvee_{C \rhd A \in \Gamma_2} (\Diamond C \vee C)\right) \rhd A$, we obtain $\vdash \Gamma_2 \to (J \wedge \neg A) \rhd A$ by Proposition 1.2. Since $\vdash \Gamma_2 \to \left(\bigvee_{j \in \kappa} I_j \wedge \neg A\right) \rhd A$, we have

$$\vdash \Gamma_2 \to \left(\left(\bigvee_{j \in \kappa} I_j \vee J \right) \wedge \neg A \right) \rhd A.$$

From this and (6), we conclude $\sim G \in \Delta_1$ because $\Gamma \prec_A \Delta$. This contradicts the consistency of Δ_1.

In both cases, (X', Y') is inseparable, and hence we can obtain a complete pair $\Theta \in K(\Phi^1, \Phi^2)$ which extends (X', Y') and satisfies the desired conditions. $\square$

4.2 Proof of Theorem 6

We are ready to prove Theorem 6.

Proof of Theorem 6 Suppose that the implication $A_0 \to B_0$ has no interpolant, and we would like to show $\nvdash A_0 \to B_0$. It follows that $(\{A_0\}, \{\neg B_0\})$ is inseparable. Let Φ^1 (resp. Φ^2) be the smallest finite adequate set containing A_0 (resp. $\neg B_0$), and put $K := K(\Phi^1, \Phi^2)$. There exists $\Gamma' \in K(\Phi^1, \Phi^2)$ such that $A_0 \in \Gamma'_1$ and $\neg B_0 \in \Gamma'_2$. For $\Gamma \in K$, we define inductively the rank of Γ (write $\mathrm{rank}(\Gamma)$) as $\mathrm{rank}(\Gamma) := \sup\{\mathrm{rank}(\Delta) + 1 : \Gamma \prec \Delta\}$, where $\sup \emptyset = 0$. This notion is well-defined because $\prec$ is conversely well-founded.

For finite sequences τ and σ of formulas, let $\tau \subseteq \sigma$ mean that σ is an end-extension of τ. Let $\tau * \langle A \rangle$ be the sequence obtained from τ by adding A as the last element.

We define an $\mathbf{IL}^-$-model $M = \langle W, R, \{S_w\}_{w \in W}, \Vdash \rangle$ as follows:

$$W := \{\langle \Gamma, \tau \rangle : \Gamma \in K \text{ and } \tau \text{ is a finite sequence of elements of } \Phi^1_{\rhd} \cup \Phi^2_{\rhd} \text{ with } \mathrm{rank}(\Gamma) + |\tau| \leq \mathrm{rank}(\Gamma')\};$$

$$\langle \Gamma, \tau \rangle R \langle \Delta, \sigma \rangle :\Leftrightarrow \Gamma \prec \Delta \text{ and } \tau \subsetneq \sigma;$$

$$\langle \Delta, \sigma \rangle S_{\langle \Gamma, \tau \rangle} \langle \Theta, \rho \rangle :\Leftrightarrow \begin{cases} \langle \Gamma, \tau \rangle R \langle \Delta, \sigma \rangle, \langle \Gamma, \tau \rangle R \langle \Theta, \rho \rangle \text{ and} \\ \text{if } \tau * \langle A \rangle \subseteq \sigma, \Gamma \prec_A \Delta \text{ and } \Box{\sim}A \in \Delta_1 \cup \Delta_2, \\ \text{then } \tau * \langle A \rangle \subseteq \rho, \Gamma \prec_A \Theta \text{ and } \Box{\sim}A, {\sim}A \in \Theta_1 \cup \Theta_2; \end{cases}$$

$$\langle \Gamma, \tau \rangle \Vdash p :\Longleftrightarrow p \in \Gamma_1 \cup \Gamma_2.$$

Claim $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ is valid in the frame of M.

Proof It is clear that R is transitive and conversely well-founded.

- Suppose $\langle \Delta, \sigma\rangle S_{\langle \Gamma,\tau\rangle}\langle \Theta, \rho\rangle$. Then, we have $\langle \Gamma, \tau\rangle R\langle \Theta, \rho\rangle$ by the definition of $S_{\langle \Gamma,\tau\rangle}$. Therefore, $\mathbf{J4}_+$ is valid in the frame of M.
- Suppose $\langle \Delta, \sigma\rangle S_{\langle \Gamma,\tau\rangle}\langle \Theta, \rho\rangle S_{\langle \Gamma,\tau\rangle}\langle \Lambda, \pi\rangle$. Then, we have $\langle \Gamma, \tau\rangle R\langle \Delta, \sigma\rangle$ and $\langle \Gamma, \tau\rangle R\langle \Lambda, \pi\rangle$.
  Assume $\tau * \langle A\rangle \subseteq \sigma$, $\Gamma \prec_A \Delta$, and $\Box{\sim}A \in \Delta_1 \cup \Delta_2$. By $\langle \Delta, \sigma\rangle S_{\langle \Gamma,\tau\rangle}\langle \Theta, \rho\rangle$, we obtain $\tau * \langle A\rangle \subseteq \rho$, $\Gamma \prec_A \Theta$, and $\Box{\sim}A \in \Theta_1 \cup \Theta_2$. By $\langle \Theta, \rho\rangle S_{\langle \Gamma,\tau\rangle}\langle \Lambda, \pi\rangle$, we conclude $\tau * \langle A\rangle \subseteq \pi$, $\Gamma \prec_A \Lambda$, and $\Box{\sim}A, {\sim}A \in \Lambda_1 \cup \Lambda_2$.
  Thus, $\langle \Delta, \sigma\rangle S_{\langle \Gamma,\tau\rangle}\langle \Lambda, \pi\rangle$. We obtain that $\mathbf{J2}_+$ is valid in the frame of M.
- Suppose $\langle \Gamma, \tau\rangle R\langle \Delta, \sigma\rangle R\langle \Theta, \rho\rangle$. Then, $\langle \Gamma, \tau\rangle R\langle \Delta, \sigma\rangle$, and hence $\langle \Gamma, \tau\rangle R\langle \Theta, \rho\rangle$ by the transitivity of R.
  Assume $\tau * \langle A\rangle \subseteq \sigma$, $\Gamma \prec_A \Delta$, and $\Box{\sim}A \in \Delta_1 \cup \Delta_2$. Since $\sigma \subseteq \rho$, we have $\tau * \langle A\rangle \subseteq \rho$. Since $\Delta \prec \Theta$, we have $\Box{\sim}A, {\sim}A \in \Theta_1 \cup \Theta_2$. Also, by Lemma 5, $\Gamma \prec_A \Theta$.
  Thus, $\langle \Delta, \sigma\rangle S_{\langle \Gamma,\tau\rangle}\langle \Theta, \rho\rangle$. We conclude that $\mathbf{J5}$ is valid in the frame of M. □

Claim (The Truth Lemma) For $B \in \Phi^1 \cup \Phi^2$ and $\langle \Gamma, \tau\rangle \in W$, the following are equivalent:

1. $B \in \Gamma_1 \cup \Gamma_2$.
2. $\langle \Gamma, \tau\rangle \Vdash B$.

Proof The lemma is proved by induction on the construction of B. We only prove for $B \equiv G \rhd F$.

(1 $\Rightarrow$ 2): Assume $G \rhd F \in \Gamma_1 \cup \Gamma_2$. Let $\langle \Delta, \sigma\rangle \in W$ be any element such that $\langle \Gamma, \tau\rangle R\langle \Delta, \sigma\rangle$ and $\langle \Delta, \sigma\rangle \Vdash G$. By the induction hypothesis, $G \in \Delta_1 \cup \Delta_2$. We distinguish the following two cases:

(Case 1): Assume that $\tau * \langle A\rangle \subseteq \sigma$, $\Gamma \prec_A \Delta$, and $\Box{\sim}A \in \Delta_1 \cup \Delta_2$. By Lemma 7, there exists a pair $\Theta \in K$ such that $\Gamma \prec_A \Theta$, $F \in \Theta_1 \cup \Theta_2$, and $\Box{\sim}A, {\sim}A \in \Theta_1 \cup \Theta_2$.

Take $\rho := \tau * \langle A\rangle$. By $\Gamma \prec \Theta$, $\mathrm{rank}(\Theta) + 1 \leq \mathrm{rank}(\Gamma)$. We have

$$\mathrm{rank}(\Theta) + |\rho| = \mathrm{rank}(\Theta) + 1 + |\tau| \leq \mathrm{rank}(\Gamma) + |\tau| \leq \mathrm{rank}(\Gamma').$$

It follows that $\langle \Theta, \rho\rangle \in W$, and we have $\langle \Delta, \sigma\rangle S_{\langle \Gamma,\tau\rangle}\langle \Theta, \rho\rangle$. By the induction hypothesis, $\langle \Theta, \rho\rangle \Vdash F$. Therefore, $\langle \Gamma, \tau\rangle \Vdash G \rhd F$.

(Case 2): Otherwise, by Lemma 4, we have $\Gamma \prec_\bot \Delta$. By Lemma 7, there exists a pair $\Theta \in K$ such that $\Gamma \prec_\bot \Theta$ and $F \in \Theta_1 \cup \Theta_2$.

Take $\rho := \tau * \langle \bot\rangle$. Then, we have $\langle \Theta, \rho\rangle \in W$ by a similar argument as in Case 1. By the definition of $S_{\langle \Gamma,\tau\rangle}$ and induction hypothesis, $\langle \Delta, \sigma\rangle S_{\langle \Gamma,\tau\rangle}\langle \Theta, \rho\rangle$ and $\langle \Theta, \rho\rangle \Vdash F$. Therefore, $\langle \Gamma, \tau\rangle \Vdash G \rhd F$.

(2 $\Rightarrow$ 1): Assume $G \rhd F \notin \Gamma_1 \cup \Gamma_2$. Then, $\neg(G \rhd F) \in \Gamma_1 \cup \Gamma_2$ because Γ is complete. By Lemma 6, there exists a pair $\Delta \in K$ such that $\Gamma \prec_F \Delta$ and $G, \Box{\sim}F \in \Delta_1 \cup \Delta_2$. Let $\sigma := \tau * \langle F\rangle$. We have $\langle \Delta, \sigma\rangle \in W$. By the induction hypothesis, $\langle \Delta, \sigma\rangle \Vdash G$. It suffices to show that for any $\langle \Theta, \rho\rangle \in W$, if $\langle \Delta, \sigma\rangle S_{\langle \Gamma,\tau\rangle}\langle \Theta, \rho\rangle$, then $\langle \Theta, \rho\rangle \nVdash F$. Suppose $\langle \Delta, \sigma\rangle S_{\langle \Gamma,\tau\rangle}\langle \Theta, \rho\rangle$. Since $\tau * \langle F\rangle \subseteq \sigma$, $\Gamma \prec_F \Delta$, and

$\Box\sim F \in \Delta_1 \cup \Delta_2$, we have $\sim F \in \Theta_1 \cup \Theta_2$ (and hence $F \notin \Theta_1 \cup \Theta_2$). By the induction hypothesis, $\langle \Theta, \rho \rangle \nVdash F$. $\Box$

Let ϵ be the empty sequence, then $\langle \Gamma', \epsilon \rangle \in W$ because $\mathrm{rank}(\Gamma') + |\epsilon| \leq \mathrm{rank}(\Gamma')$. By the Truth Lemma (Lemma 4.2), $\langle \Gamma', \epsilon \rangle \Vdash A_0 \wedge \neg B_0$, and therefore $A_0 \to B_0$ is not valid in M. It follows that $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ does not prove $A_0 \to B_0$. $\Box$

4.3 Consequences of Theorem 6

In this subsection, we prove some consequences of Theorem 6 on interpolation properties. Firstly, we prove that $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ has a version of the $\rhd$-interpolation property. Secondly, we notice that CIP for $\mathbf{IL}$ easily follows from Theorem 6.

Before them, we show the so-called generated submodel lemma. Let $M = \langle W, R, \{S_w\}_{w \in W}, \Vdash \rangle$ be any $\mathbf{IL}^-$-model such that $\mathbf{J4}_+$ is valid in the frame of M. For each $r \in W$, we define an $\mathbf{IL}^-$-model $M^* = \langle W^*, R^*, \{S^*_w\}_{w \in W^*}, \Vdash^* \rangle$ as follows:

- $W^* := R[r] \cup \{r\}$;
- $x R^* y :\iff x R y$;
- $y S^*_x z :\iff y S_x z$;
- $x \Vdash^* p :\iff x \Vdash p$.

We call M^* the *submodel of M generated by r*. It is easy to show that if $\mathbf{J1}$ is valid in the frame of M, then it is also valid in the frame of M^*. This is also the case for $\mathbf{J2}_+$ and $\mathbf{J5}$. Also, the following lemma is easily obtained.

Lemma 8 (The Generated Submodel Lemma) *Suppose that $\mathbf{J4}_+$ is valid in the frame of an $\mathbf{IL}^-$-model $M = \langle W, R, \{S_w\}_{w \in W}, \Vdash \rangle$. For any $r \in W$, let $M^* = \langle W^*, R^*, \{S^*_w\}_{w \in W^*}, \Vdash^* \rangle$ be the submodel of M generated by r. Then, for any $x \in W^*$ and formula A, $x \Vdash A$ if and only if $x \Vdash^* A$.*

Proof This is proved by induction on the construction of A. We only prove the case $A \equiv (B \rhd C)$.

($\Rightarrow$): Suppose $x \Vdash B \rhd C$. Let $y \in W^*$ be any element such that $x R^* y$ and $y \Vdash^* B$. Then, $x R y$, and by the induction hypothesis, $y \Vdash B$. Hence, there exists $z \in W$ such that $y S_x z$ and $z \Vdash C$. Since $\mathbf{J4}_+$ is valid in the frame of M, $x R z$. Since $r R x$, we have $r R z$. Thus, $z \in W^*$. It follows $y S^*_x z$. By the induction hypothesis, $z \Vdash^* C$. Therefore, $x \Vdash^* B \rhd C$.

($\Leftarrow$): Suppose $x \Vdash^* B \rhd C$. Let $y \in W$ be any element with $x R y$ and $y \Vdash B$. Since $x \in W^*$, we have $y \in W^*$, and hence $x R^* y$. By the induction hypothesis, $y \Vdash^* B$. Then, for some $z \in W^*$, $y S^*_x z$ and $z \Vdash^* C$. We have $y S_x z$. By the induction hypothesis, $z \Vdash C$. Thus, we conclude $x \Vdash B \rhd C$. $\Box$

Proposition 9 *For any formulas A and B, the following are equivalent:*

1. $\vdash A \rhd B$.
2. $\vdash A \to \Diamond B$.

Proof (1 ⇒ 2): Suppose $\nvdash A \to \Diamond B$. Then, by Theorem 1, there exist an $\mathbf{IL}^-$-model $M = \langle W, R, \{S_w\}_{w \in W}, \Vdash \rangle$ and $r \in W$ such that $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ is valid in the frame of M and $r \Vdash A \wedge \Box \neg B$. By the Generated Submodel Lemma, we may assume that r is the root of M, that is, for all $w \in W \setminus \{r\}$, rRw.

We define a new $\mathbf{IL}^-$-model $M' = \langle W', R', \{S'_w\}_{w \in W'}, \Vdash' \rangle$ as follows:

- $W' := W \cup \{r_0\}$, where r_0 is a new element;
- $xR'y :\Longleftrightarrow \begin{cases} xRy & \text{if } x \neq r_0, \\ y \in W & \text{if } x = r_0; \end{cases}$
- $yS'_x z :\Longleftrightarrow \begin{cases} yS_x z & \text{if } x \neq r_0, \\ yRz & \text{if } x = r_0; \end{cases}$
- $x \Vdash' p :\Longleftrightarrow x \neq r_0$ and $x \Vdash p$.

Then, $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ is also valid in the frame of M'. Also, it is easily shown that for any $x \in W$ and any formula C, $x \Vdash C$ if and only if $x \Vdash' C$.

Then, $r \Vdash' A \wedge \Box \neg B$. Let $x \in W'$ be any element such that $rS'_{r_0}x$. Then, rRx, and hence $rR'x$. We have $x \nVdash' B$. Therefore, we obtain $r_0 \nVdash' A \rhd B$. It follows $\nvdash A \rhd B$.

(2 ⇒ 1): Suppose $\vdash A \to \Diamond B$, then $\vdash \Diamond B \rhd B \to A \rhd B$ by **R2**. Thus, $\vdash A \rhd B$. □

Areces, Hoogland, and de Jongh [1] proved that **IL** has the $\rhd$-interpolation property. Namely, for every formulas A and B with $\mathbf{IL} \vdash A \rhd B$, there exists a formula C such that $v(C) \subseteq v(A) \cap v(B)$, $\mathbf{IL} \vdash A \rhd C$, and $\mathbf{IL} \vdash C \rhd B$. We prove that $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ has a version of the $\rhd$-interpolation property.

Corollary 1 (A version of the $\rhd$-interpolation property) *Let A and B be any formulas. If $\vdash A \rhd B$, then there exists a formula C such that $v(C) \subseteq v(A) \cap v(B)$, $\vdash A \to C$, and $\vdash C \rhd B$.*

Proof Suppose $\vdash A \rhd B$. Then, by Proposition 9, $\vdash A \to \Diamond B$. By Theorem 6, there exists a formula C such that $v(C) \subseteq v(A) \cap v(B)$, $\vdash A \to C$, and $\vdash C \to \Diamond B$. By Proposition 9 again, we obtain $\vdash C \rhd B$. □

Problem 1 Does the logic $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ have the original version of the $\rhd$-interpolation property?

For each formula A, let $\mathrm{Sub}(A)$ be the set of all subformulas of A. Also, let $\mathrm{PSub}(A) := \mathrm{Sub}(A) \setminus \{A\}$. We prove that **IL** is embeddable into $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ in some sense.

Proposition 10 *For any formula A, the following are equivalent:*

1. $\mathbf{IL} \vdash A$.
2. *A is valid in all finite $\mathbf{IL}^-$-frames in which all axioms of* **IL** *are valid.*
3. $\vdash \boxdot \bigwedge \{B \rhd B : B \in \mathrm{PSub}(A)\} \to A$.

Proof (1 ⇒ 2) is obvious.

(3 ⇒ 1) follows from Proposition 1.7.

(2 $\Rightarrow$ 3): Suppose $L \nvdash \boxdot \bigwedge\{B \rhd B : B \in \mathrm{PSub}(A)\} \to A$. Then, by Theorem 1, there exist a finite $\mathbf{IL}^-$-model $M = \langle W, R, \{S_w\}_{w \in W}, \Vdash \rangle$ and $r \in W$ such that $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ is valid in the frame of M and $r \Vdash \boxdot \bigwedge\{B \rhd B : B \in \mathrm{PSub}(A)\} \wedge \neg A$. By the Generated Submodel Lemma, we may assume that r is the root of M.

We define an $\mathbf{IL}^-$-model $M' = \langle W', R', \{S'_w\}_{w \in W'}, \Vdash' \rangle$ as follows:

- $W' := W$;
- $xR'y :\iff xRy$;
- $yS'_x z :\iff yS_x z$ or (xRy and $z = y$);
- $x \Vdash' p :\iff x \Vdash p$.

Claim $\mathbf{IL}$ is valid in the frame of M'.

Proof By Proposition 1.8, it suffices to show that $\mathbf{J1}$, $\mathbf{J2}_+$, and $\mathbf{J5}$ are valid in the frame of M'.

$\mathbf{J1}$: Suppose xRy. Then, $yS'_x y$ by the definition of S'_x. Thus, $\mathbf{J1}$ is valid.

$\mathbf{J4}_+$: Suppose $yS'_x z$. Then, $yS_x z$ or (xRy and $y = z$). If $yS_x z$, then xRz because $\mathbf{J4}_+$ is valid in the frame of M. If xRy and $y = z$, then xRz. Hence, in either case, we have xRz. Therefore, $\mathbf{J4}_+$ is valid.

$\mathbf{J2}_+$: Suppose $yS'_x z$ and $zS'_x u$. We distinguish the following four cases.

- (Case 1): $yS_x z$ and $zS_x u$. Since $\mathbf{J2}_+$ is valid in the frame of M, $yS_x u$.
- (Case 2): $yS_x z$, xRz and $z = u$. Then, $yS_x u$.
- (Case 3): xRy, $y = z$ and $zS_x u$. Then, $yS_x u$.
- (Case 4): xRy, $y = z$, xRz and $z = u$. Then, xRy and $y = u$.

In either case, we have $yS'_x u$. Since $\mathbf{J4}_+$ is valid, we obtain that $\mathbf{J2}_+$ is valid in the frame of M'.

$\mathbf{J5}$: Suppose $xR'y$ and $yR'z$. Then, xRy and yRz. Since $\mathbf{J5}$ is valid in the frame of M, $yS_x z$. Then, $yS'_x z$. Therefore, $\mathbf{J5}$ is valid. □

Claim For any $B \in \mathrm{Sub}(A)$ and $x \in W$, $x \Vdash B$ if and only if $x \Vdash' B$.

Proof We prove the claim by induction on the construction of B. We only give a proof of the case that B is $C \rhd D$.

($\Rightarrow$): Suppose $x \Vdash C \rhd D$. Let $y \in W$ be such that xRy and $y \Vdash' C$. By the induction hypothesis, $y \Vdash C$. Then, there exists $z \in W$ such that $yS_x z$ and $z \Vdash D$. Then, $yS'_x z$ and by the induction hypothesis, $z \Vdash' D$. Therefore, $x \Vdash' C \rhd D$.

($\Leftarrow$): Suppose $x \Vdash' C \rhd D$. Let $y \in W$ be such that xRy and $y \Vdash C$. By the induction hypothesis, $y \Vdash' C$. Hence, there exists $z \in W$ such that $yS'_x z$ and $z \Vdash' D$. By the induction hypothesis, $z \Vdash D$. By the definition of S'_x, we have either $yS_x z$ or (xRy and $y = z$). If $yS_x z$, then $x \Vdash C \rhd D$. If xRy and $y = z$, then xRy and $y \Vdash D$. Here either $x = r$ or rRw. Since $D \in \mathrm{PSub}(A)$, we obtain $x \Vdash D \rhd D$ because $r \Vdash \boxdot \bigwedge\{B \rhd B : B \in \mathrm{Sub}(A)\}$. Thus, for some $z' \in W$, $yS_x z'$ and $z' \Vdash D$. We conclude $x \Vdash C \rhd D$. □

Since $r \nVdash A$, we obtain $r \nVdash' A$ by the claim. Thus, A is not valid in some finite $\mathbf{IL}^-$-frame in which all axioms of $\mathbf{IL}$ are valid. □

Proof of Theorem 3 Suppose $\mathbf{IL} \vdash A \to B$. Then, by Proposition 10,

$$\vdash \boxdot \bigwedge\{C \rhd C : C \in \mathrm{PSub}(A \to B)\} \to (A \to B).$$

Since $\mathrm{PSub}(A \to B) = \mathrm{Sub}(A) \cup \mathrm{Sub}(B)$, we have

$$\vdash \boxdot \bigwedge\{C \rhd C : C \in \mathrm{Sub}(A)\} \wedge A \to \left(\boxdot \bigwedge\{C \rhd C : C \in \mathrm{Sub}(B)\} \to B\right).$$

By Theorem 6, there exists a formula D such that $v(D) \subseteq v(A) \cap v(B)$,

$$\vdash \boxdot \bigwedge\{C \rhd C : C \in \mathrm{Sub}(A)\} \wedge A \to D$$

and

$$\vdash D \to \left(\boxdot \bigwedge\{C \rhd C : C \in \mathrm{Sub}(B)\} \to B\right).$$

Then, by Proposition 1.7, we obtain $\mathbf{IL} \vdash A \to D$ and $\mathbf{IL} \vdash D \to B$. $\square$

5 The fixed point property

In this section, we investigate FPP and ℓFPP. First, we study FPP for the logic $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$. Then, we prove that $\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5})$ has ℓFPP.

5.1 FPP for $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$

From Theorem 6 and Lemma 2, we immediately obtain the following corollary.

Corollary 2 (FPP for $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$) $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ *has FPP.*

Moreover, we give a syntactical proof of FPP for $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ by modifying de Jongh and Visser's proof of FPP for $\mathbf{IL}$. Since the Substitution Principle (Proposition 3) holds for extensions of $\mathbf{IL}^-(\mathbf{J4}_+)$, as usual, it suffices to prove that every formula of the form $A(p) \rhd B(p)$ has a fixed point in $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$. As a consequence, we show that every formula $A(p)$ which is modalized in p has the same fixed point in $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ as given by de Jongh and Visser. That is,

Theorem 7 *For any formulas $A(p)$ and $B(p)$, $A(\top) \rhd B(\Box\neg A(\top))$ is a fixed point of $A(p) \rhd B(p)$ in $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$.*

Theorem 7 follows from the following five lemmas.

Lemma 9 *Let L be any extension of $\mathbf{IL}^-$. For any formulas A and B, if $L \vdash \Box\neg A \to (A \leftrightarrow B)$, then $L \vdash (A \wedge \Box\neg A) \leftrightarrow (B \wedge \Box\neg B)$.*

Proof Suppose $L \vdash \Box\neg A \to (A \leftrightarrow B)$. Then, $L \vdash \Box\neg A \to (\Box\neg A \leftrightarrow \Box\neg B)$ and hence $L \vdash \Box\neg A \to \Box\neg B$. By combining this with our supposition, we obtain

$$L \vdash (A \wedge \Box\neg A) \to (B \wedge \Box\neg B).$$

On the other hand, $L \vdash \neg B \to (\Box\neg A \to \neg A)$. Hence, by the axiom scheme **L3**, $L \vdash \Box\neg B \to \Box\neg A$. Therefore, by our supposition,

$$L \vdash (B \wedge \Box\neg B) \to (A \wedge \Box\neg A).$$

□

Lemma 10 *For any formulas A and C,*

$$\mathbf{IL}^-(\mathbf{J4}_+) \vdash (A(\top) \wedge \Box\neg A(\top)) \leftrightarrow (A(A(\top) \rhd C) \wedge \Box\neg A(A(\top) \rhd C)).$$

Proof By Proposition 1.1, $\mathbf{IL}^- \vdash \Box\neg A(\top) \to A(\top) \rhd C$. Therefore, we obtain $\mathbf{IL}^- \vdash \Box\neg A(\top) \to (\top \leftrightarrow (A(\top) \rhd C))$. Then, $\mathbf{IL}^- \vdash \Box\neg A(\top) \to \boxdot(\top \leftrightarrow (A(\top) \rhd C))$. Therefore, by Proposition 3.1, we obtain

$$\mathbf{IL}^-(\mathbf{J4}_+) \vdash \Box\neg A(\top) \to (A(\top) \leftrightarrow A(A(\top) \rhd C)).$$

The lemma directly follows from this and Lemma 9. □

Lemma 11 *For any formulas A, C, and D,*

$$\mathbf{IL}^-(\mathbf{J2}, \mathbf{J4}_+, \mathbf{J5}) \vdash (A(\top) \rhd D) \leftrightarrow (A(A(\top) \rhd C) \rhd D).$$

Proof By Lemma 10 and **R2**, we obtain

$$\mathbf{IL}^-(\mathbf{J4}_+) \vdash ((A(\top) \wedge \Box\neg A(\top)) \rhd D) \leftrightarrow ((A(A(\top) \rhd C) \wedge \Box\neg A(A(\top) \rhd C)) \rhd D).$$

Therefore, by Lemma 1.1, we obtain

$$\mathbf{IL}^-(\mathbf{J2}, \mathbf{J4}_+, \mathbf{J5}) \vdash (A(\top) \rhd D) \leftrightarrow (A(A(\top) \rhd C) \rhd D).$$

□

Lemma 12 *For any formulas B and C, $\mathbf{IL}^-(\mathbf{J4}_+)$ proves*

$$(B(\Box\neg C) \wedge \Box\neg B(\Box\neg C)) \leftrightarrow (B(C \rhd B(\Box\neg C)) \wedge \Box\neg B(C \rhd B(\Box\neg C))).$$

Proof Since $\mathbf{IL}^- \vdash \Box\neg B(\Box\neg C) \to \Box(\bot \leftrightarrow B(\Box\neg C))$,

$$\mathbf{IL}^-(\mathbf{J4}_+) \vdash \Box\neg B(\Box\neg C) \to (C \rhd \bot \leftrightarrow C \rhd B(\Box\neg C)).$$

Then, by **J6**, $\mathbf{IL}^-(\mathbf{J4}_+) \vdash \Box\neg B(\Box\neg C) \to (\Box\neg C \leftrightarrow C \rhd B(\Box\neg C))$ and hence $\mathbf{IL}^-(\mathbf{J4}_+) \vdash \Box\neg B(\Box\neg C) \to \boxdot(\Box\neg C \leftrightarrow C \rhd B(\Box\neg C))$. Therefore, by Proposition 3.1, we obtain

$$\mathbf{IL}^-(\mathbf{J4}_+) \vdash \Box\neg B(\Box\neg C) \to (B(\Box\neg C) \leftrightarrow B(C \rhd B(\Box\neg C))).$$

The lemma is a consequence of this with Lemma 9. □

Lemma 13 *For any formulas B, C, and D,*

$$\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5}) \vdash (D \rhd B(\Box\neg C)) \leftrightarrow (D \rhd B(C \rhd B(\Box\neg C))).$$

Proof By Lemma 12 and **R1**, $\mathbf{IL}^-(\mathbf{J4}_+)$ proves

$$D \rhd (B(\Box\neg C) \wedge \Box\neg B(\Box\neg C)) \leftrightarrow D \rhd (B(C \rhd B(\Box\neg C)) \wedge \Box\neg B(C \rhd B(\Box\neg C))).$$

Therefore, by Lemma 1.2,

$$\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5}) \vdash (D \rhd B(\Box\neg C)) \leftrightarrow (D \rhd B(C \rhd B(\Box\neg C))).$$

□

Proof of Theorem 7 Let $F \equiv A(\top) \rhd B(\Box\neg A(\top))$. By Lemma 11 for $C \equiv D \equiv B(\Box\neg A(\top))$, we obtain

$$\mathbf{IL}^-(\mathbf{J2}, \mathbf{J4}_+, \mathbf{J5}) \vdash F \leftrightarrow (A(F) \rhd B(\Box\neg A(\top))).$$

Furthermore, by Lemma 13 for $C \equiv A(\top)$ and $D \equiv F$,

$$\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5}) \vdash (A(F) \rhd B(\Box\neg A(\top))) \leftrightarrow (A(F) \rhd B(F)).$$

We conclude

$$\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5}) \vdash F \leftrightarrow A(F) \rhd B(F).$$

□

5.2 ℓFPP for $\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5})$

From Lemma 11, we immediately obtain the following corollary.

Corollary 3 *For any formulas $A(p)$ and B, if $p \notin v(B)$, then $A(\top) \rhd B$ is a fixed point of $A(p) \rhd B$ in $\mathbf{IL}^-(\mathbf{J2}, \mathbf{J4}_+, \mathbf{J5})$.*

Therefore, $\mathbf{IL}^-(\mathbf{J2}, \mathbf{J4}_+, \mathbf{J5})$ has ℓFPP. Moreover, we prove the following theorem.

Theorem 8 (ℓFPP for $\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5})$) *For any formulas $A(p)$ and B, if the formula $A(p) \rhd B$ is left-modalized in p, then $A(\Box\neg A(\top)) \rhd B$ is a fixed point of $A(p) \rhd B$ in $\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5})$. Therefore, $\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5})$ has ℓFPP.*

Before proving Theorem 8, we prepare two lemmas.

Lemma 14 *For any formula $A(p)$ such that $\Box A(p)$ is left-modalized in p,*

$$\mathbf{IL}^- \vdash \Box A(\top) \leftrightarrow \Box A(\Box A(\top)).$$

Proof This is proved in a usual way by using Proposition 6. □

Lemma 15 *Let $A(p)$ and B be any formulas such that $p \notin v(E)$ for any subformula $D \rhd E$ of $A(p)$. Then,*

$$\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5}) \vdash (A(\Box\neg A(p)) \rhd B) \leftrightarrow (A(A(p) \rhd B) \rhd B).$$

Proof By Proposition 1.1, $\mathbf{IL}^- \vdash \Box\neg A(p) \to A(p) \rhd B$. On the other hand, since $\mathbf{IL}^-(\mathbf{J4}) \vdash A(p) \rhd B \to (\Diamond A(p) \to \Diamond B)$, we have $\mathbf{IL}^-(\mathbf{J4}) \vdash \Box\neg B \to (A(p) \rhd B \to \Box\neg A(p))$. Hence, $\mathbf{IL}^-(\mathbf{J4}) \vdash \Box\neg B \to (\Box\neg A(p) \leftrightarrow A(p) \rhd B)$. Then,

$$\mathbf{IL}^-(\mathbf{J4}) \vdash \Box\neg B \to \boxdot(\Box\neg A(p) \leftrightarrow A(p) \rhd B).$$

By Proposition 6.1, we obtain

$$\mathbf{IL}^-(\mathbf{J4}) \vdash \Box\neg B \to (A(\Box\neg A(p)) \leftrightarrow A(A(p) \rhd B)).$$

Thus,

$$\mathbf{IL}^-(\mathbf{J4}) \vdash (A(\Box\neg A(p)) \vee \Diamond B) \leftrightarrow (A(A(p) \rhd B) \vee \Diamond B).$$

By **R2**, we obtain

$$\mathbf{IL}^-(\mathbf{J4}) \vdash ((A(\Box\neg A(p)) \vee \Diamond B) \rhd B) \leftrightarrow ((A(A(p) \rhd B) \vee \Diamond B) \rhd B).$$

Therefore, we conclude

$$\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5}) \vdash (A(\Box\neg A(p)) \rhd B) \leftrightarrow (A(A(p) \rhd B) \rhd B).$$

□

Proof of Theorem 8 Let $F :\equiv \Box\neg A(\top)$. Since $\Box\neg A(p)$ is left-modalized in p, $\mathbf{IL}^- \vdash F \leftrightarrow \Box\neg A(F)$ by Lemma 14. Since $\mathbf{IL}^-(\mathbf{J4}) \vdash \Box(F \leftrightarrow \Box\neg A(F))$, by Proposition 6.2, we have

$$\mathbf{IL}^-(\mathbf{J4}) \vdash (A(F) \rhd B) \leftrightarrow (A(\Box\neg A(F)) \rhd B).$$

By Lemma 15,

$$\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5}) \vdash (A(\Box\neg A(F)) \rhd B) \leftrightarrow (A(A(F) \rhd B) \rhd B).$$

Therefore,

$$\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5}) \vdash (A(F) \rhd B) \leftrightarrow (A(A(F) \rhd B) \rhd B).$$

□

6 Failure of ℓFPP, FPP, and CIP

In this section, we provide counter models of ℓFPP for **CL** and $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$, and also provide a counter model of FPP for $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$. We also show that CIP is not the case for our sublogics except $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$ and **IL**. Let ω be the set $\{0, 1, 2, \ldots\}$ of all natural numbers.

6.1 A counter model of ℓFPP for CL

In this subsection, we prove that $\mathbf{IL}^-$, $\mathbf{IL}^-(\mathbf{J1})$, $\mathbf{IL}^-(\mathbf{J4}_+)$, $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+)$, $\mathbf{IL}^-(\mathbf{J2}_+)$, and **CL** have neither ℓFPP nor CIP.

Theorem 9 *The formula $p \rhd q$ which is left-modalized in p has no fixed points in* **CL**. *That is, for any formula A which satisfies $v(A) \subseteq \{q\}$,*

$$\mathbf{CL} \nvdash A \leftrightarrow A \rhd q.$$

Proof We define an $\mathbf{IL}^-$-frame $\mathcal{F} = \langle W, R, \{S_w\}_{w\in W}\rangle$ as follows:

- $W := \{x_i, y_i : i \in \omega\}$;
- $R := \{\langle x_i, x_j\rangle, \langle x_i, y_j\rangle, \langle y_i, x_j\rangle, \langle y_i, y_j\rangle \in W^2 : i > j\}$;
- For each $i \in \omega$ and $w_i \in \{x_i, y_i\}$, $S_{w_i} := \{\langle a, a\rangle : w_i R a\} \cup \{\langle a, b\rangle :$ there exists an even number $k < i - 1$ such that $((a = x_k$ or $a = y_k)$ and $b = x_{k+1})\}$.

For example, S_{x_3}, S_{y_3}, S_{x_4}, and S_{y_4} are shown in the following figure (Fig. 3). In the figure, R relations and S relations are drawn by solid lines and broken lines, respectively. Since R is transitive, we draw only solid lines connecting the immediately preceding and succeeding elements.

It is easy to show that **J1** and $\mathbf{J2}_+$ are valid in $\mathcal{F}$. Thus, **CL** is valid in $\mathcal{F}$ by Proposition 1.9. Let $\Vdash$ be a satisfaction relation on $\mathcal{F}$ such that for any $i \in \omega$, we have $x_i \Vdash q$ and $y_i \nVdash q$. For each $w \in W$, we say that $i \in \omega$ is an *index* of w if either $w = x_i$ or $w = y_i$.

Claim 1 For any formula A with $v(A) \subseteq \{q\}$, there exists an $n_A \in \omega$ satisfying the following two conditions:

1. Either $\forall m \geq n_A\, (x_m \Vdash A)$ or $\forall m \geq n_A\, (x_m \nVdash A)$;

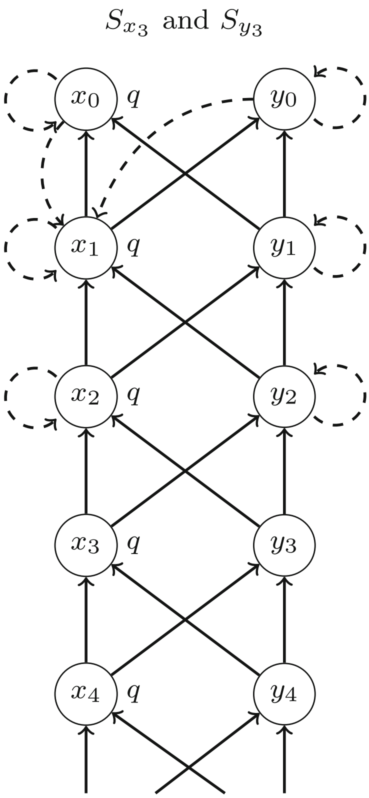

Fig. 3 A counter model of ℓFPP for **CL**

2. Either $\forall m \geq n_A$ ($y_m \Vdash A$) or $\forall m \geq n_A$ ($y_m \nVdash A$).

Proof We prove the claim by induction on the construction of A.
$A \equiv \bot$: Then, $n_A = 0$ and $\forall m \geq n_A$ ($x_m \nVdash A$ and $y_m \nVdash A$).
$A \equiv q$: Then, $n_A = 0$ and $\forall m \geq n_A$ ($x_m \Vdash q$ and $y_m \nVdash q$).
$A \equiv B \to C$: By the induction hypothesis, there exist $n_B, n_C \in W$ satisfying the statement of the claim for B and C, respectively. Let $n_A = \max\{n_B, n_C\}$. We distinguish the following three cases.

- $\forall m \geq n_A$ ($x_m \nVdash B$): Then, $\forall m \geq n_A$ ($x_m \Vdash B \to C$).
- $\forall m \geq n_A$ ($x_m \Vdash C$): Then, $\forall m \geq n_A$ ($x_m \Vdash B \to C$).
- $\forall m \geq n_A$ ($x_m \Vdash B$) and $\forall m \geq n_A$ ($x_m \nVdash C$): Then, $\forall m \geq n_A$ ($x_m \nVdash B \to C$).

In a similar way, it is proved that either $\forall m \geq n_A$ ($y_m \Vdash B \to C$) or $\forall m \geq n_A$ ($y_m \nVdash B \to C$).
$A \equiv \Box B$: We distinguish the following two cases.

- There exists an $n \in W$ such that either $x_n \nVdash B$ or $y_n \nVdash B$: Then, $n_A = n + 1$ and $\forall m \geq n_A$ ($x_m \nVdash \Box B$ and $y_m \nVdash \Box B$).
- For all $n \in W$, $x_n \Vdash B$ and $y_n \Vdash B$: Then, $n_A = 0$ and $\forall m \geq n_A$ ($x_m \Vdash \Box B$ and $y_m \Vdash \Box B$).

$A \equiv B \rhd C$: We distinguish the following five cases.

- (Case 1): There exists an even number k such that $x_k \Vdash B$, $x_k \nVdash C$, and $x_{k+1} \nVdash C$. Let $n_A = k + 2$ and $m \geq n_A$. Then, $x_m R x_k$ and $x_k \Vdash B$. For any $v \in W$ with $x_k S_{x_m} v$, either $v = x_k$ or $v = x_{k+1}$ by the definition of S_{x_m}. Thus, $v \nVdash C$. Therefore, we obtain $x_m \nVdash B \rhd C$. Since $y_m R x_k$, we also obtain $y_m \nVdash B \rhd C$ in a similar way.
- (Case 2): There exists an even number k such that $y_k \Vdash B$, $y_k \nVdash C$, and $x_{k+1} \nVdash C$. It is proved that $n_A = k + 2$ witnesses the claim as in Case 1.
- (Case 3): There exists an odd number k such that $x_k \Vdash B$ and $x_k \nVdash C$. Let $n_A = k + 1$ and $m \geq n_A$. Then, $x_m R x_k$ and $x_k \Vdash B$. For any $v \in W$ with $x_k S_{x_m} v$, we have $v = x_k$ by the definition of S_{x_m}. Thus, $v \nVdash C$. Therefore, we obtain $x_m \nVdash B \rhd C$. Since $y_m R x_k$, $y_m \nVdash B \rhd C$ is also proved.
- (Case 4): There exists an odd number k such that $y_k \Vdash B$ and $y_k \nVdash C$. It is proved that $n_A = k + 1$ witnesses the claim as in Case 3.
- (Case 5): Otherwise, all of the following conditions are satisfied.

  (I) For any even number k, if $x_k \Vdash B$, then either $x_k \Vdash C$ or $x_{k+1} \Vdash C$.
  (II) For any even number k, if $y_k \Vdash B$, then either $y_k \Vdash C$ or $x_{k+1} \Vdash C$.
  (III) For any odd number k, if $x_k \Vdash B$, then $x_k \Vdash C$.
  (IV) For any odd number k, if $y_k \Vdash B$, then $y_k \Vdash C$.

  By the induction hypothesis, there exists an $n_B \in \omega$ which is a witness of the statement of the claim for B. Now, we prove that there exists a natural number $n_A \geq 1$ such that for each $i \geq n_A - 1$, we have $x_i \Vdash \neg B \vee C$ and $y_i \Vdash \neg B \vee C$. We distinguish the following four cases.

  - $\forall m \geq n_B$ ($x_m \Vdash B$ and $y_m \Vdash B$): Then, by (III) and (IV), there are infinitely many odd numbers k such that $x_k \Vdash C$ and $y_k \Vdash C$. Thus, by the induction hypothesis, there exists an $n_C \in \omega$ such that $\forall m \geq n_C$ ($x_m \Vdash C$ and $y_m \Vdash C$). Then, we define $n_A := n_C + 1$.
  - $\forall m \geq n_B$ ($x_m \Vdash B$ and $y_m \nVdash B$): Then, by (III), there are infinitely many odd numbers k such that $x_k \Vdash C$. Thus, by the induction hypothesis, there exists an $n_C \in \omega$ such that $\forall m \geq n_C$ ($x_m \Vdash C$). Then, we define $n_A := \max\{n_B, n_C\} + 1$.
  - $\forall m \geq n_B$ ($x_m \nVdash B$ and $y_m \Vdash B$): Then, by (IV), there are infinitely many odd numbers k such that $y_k \Vdash C$. Thus, by the induction hypothesis, there exists an $n_C \in \omega$ such that $\forall m \geq n_C$ ($y_m \Vdash C$). Then, we define $n_A := \max\{n_B, n_C\} + 1$.
  - $\forall m \geq n_B$ ($x_m \nVdash B$ and $y_m \nVdash B$): We define $n_A := n_B + 1$.

  We prove that n_A witnesses the claim. Let $m \geq n_A$ and $z \in W$ be such that $x_m R z$ and $z \Vdash B$. We show that there exists a $v \in W$ such that $z S_{x_m} v$ and $v \Vdash C$. Let i be an index of z. If i is odd, then $z S_{x_m} z$ and $z \Vdash C$ by (III) and (IV). Assume that i is even. We distinguish the following two cases.

  - $n_A - 1 \leq i < m$: We obtain $z \Vdash \neg B \vee C$ by the choice of n_A. Since $z \Vdash B$, we have $z \Vdash C$. By the definition of S_{x_m}, we find $z S_{x_m} z$.

- $i < n_A - 1$: Then, $i < m - 1$. Therefore, $zS_{x_m}z$ and $zS_{x_m}x_{i+1}$. Furthermore, by (I) and (II), we obtain $z \Vdash C$ or $x_{i+1} \Vdash C$.

In any case, there exists $v \in W$ such that $zS_{x_m}v$ and $v \Vdash C$. Therefore, we obtain $x_m \Vdash B \rhd C$. Similarly, we have $y_m \Vdash B \rhd C$. □

We suppose, towards a contradiction, that there exists a formula A such that $v(A) \subseteq \{q\}$ and $\mathbf{CL} \vdash A \leftrightarrow A \rhd q$. Since $\mathbf{CL}$ is valid in $\mathcal{F}$, $A \leftrightarrow A \rhd q$ is valid in $\mathcal{F}$. Moreover, the following claim holds.

Claim 2 For any $w \in W$ whose index is n, n is even if and only if $w \Vdash A$.

Proof We prove the claim by induction on n. Let $w \in W$ be any element whose index is n.

For $n = 0$, since there is no $w' \in W$ such that wRw', we obtain $w \Vdash A \rhd q$ and hence, $w \Vdash A$. Suppose $n > 0$ and that the claim holds for any natural number less than n.

($\Leftarrow$): Assume that n is an odd number. Then, wRy_{n-1}. Since $n-1$ is even, $y_{n-1} \Vdash A$ by the induction hypothesis. Let v be any element in W satisfying $y_{n-1}S_w v$. By the definitions of S_w and $\Vdash$, we obtain $v = y_{n-1}$ and $v \nVdash q$. Therefore, $w \nVdash A \rhd q$ and hence $w \nVdash A$.

($\Rightarrow$): Assume that n is an even number. Let v be any element in W with wRv and $v \Vdash A$. Let m be the index of v. Since $m < n$ and $v \Vdash A$, m is even by the induction hypothesis. Since n is also even, $m < n - 1$ and hence $vS_w x_{m+1}$. Furthermore, $x_{m+1} \Vdash q$ by the definition of $\Vdash$. Therefore, we obtain $w \Vdash A \rhd q$ and hence, $w \Vdash A$. □

This contradicts Claim 1. Therefore, for any formula A with $v(A) \subseteq \{q\}$, we obtain $\mathbf{CL} \nvdash A \leftrightarrow A \rhd q$. □

Corollary 4 *Let L be any logic that is closed under substituting a formula for a propositional variable and satisfies $\mathbf{IL}^- \subseteq L \subseteq \mathbf{CL}$. Then, L has neither ℓFPP nor CIP.*

Proof By Theorem 9, every sublogic of $\mathbf{CL}$ does not have ℓFPP. By Lemma 3, every logic L such that $\mathbf{IL}^- \subseteq L \subseteq \mathbf{CL}$ does not have CIP. □

6.2 A counter model of ℓFPP for $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$

In this subsection, we prove that $\mathbf{IL}^-(\mathbf{J5})$ and $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$ have neither ℓFPP nor CIP.

Theorem 10 *The formula $p \rhd q$ which is left-modalized in p has no fixed point in $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$. That is, for any formula A which satisfies $v(A) \subseteq \{q\}$,*

$$\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5}) \nvdash A \leftrightarrow A \rhd q.$$

Proof We define an $\mathbf{IL}^-$-frame $\mathcal{F} = \langle W, R, \{S_w\}_{w \in W} \rangle$ as follows:

- $W := \omega \cup \{v\}$;

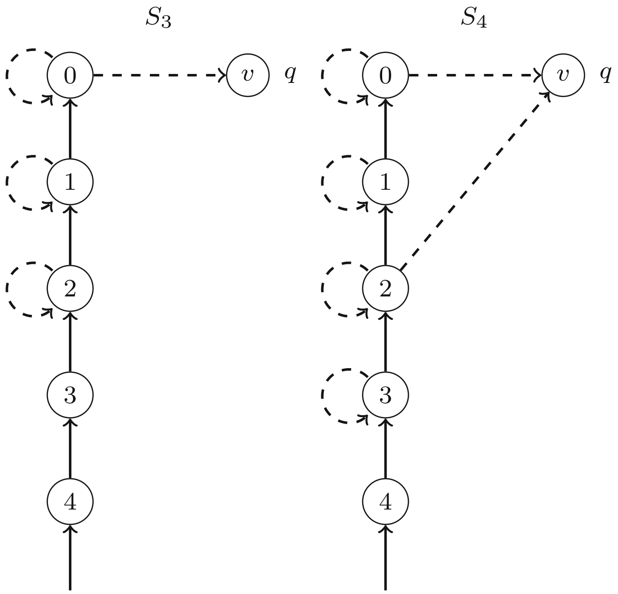

Fig. 4 A counter model of ℓFPP for $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$

- $R := \{\langle x, y\rangle \in W^2 : x, y \in \omega$ and $x > y\}$;
- $S_v := \emptyset$ and for each $n \in \omega$, $S_n := \{\langle x, y\rangle \in W^2 : nRx$ and $(y = x$ or xRy or $(x$ is even, $x < n - 1$ and $y = v))\}$.

For instance, the relations S_3 and S_4 are shown in the following figure (Fig. 4). In the case of xRy for $x, y < n$, xS_ny holds, and the corresponding broken lines are omitted in the figure.

Then, $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$ is valid in $\mathcal{F}$. Let $\Vdash$ be a satisfaction relation on $\mathcal{F}$ such that $v \Vdash q$ and for each $n \in \omega$, $n \nVdash q$.

Claim 1 For any formula A with $v(A) \subseteq \{q\}$, there exists an $n_A \in \omega$ such that

$$\forall m \geq n_A\, (m \Vdash A) \text{ or } \forall m \geq n_A\, (m \nVdash A).$$

Proof We prove the claim by induction on the construction of A. We only prove the case of $A \equiv B \rhd C$. We distinguish the following three cases.

- Case 1: There exists an even number k such that $k \Vdash B$, for all $j \leq k$, $j \nVdash C$ and $v \nVdash C$: Let $n_A = k + 1$ and $m \geq n_A$. Then, mRk and $k \Vdash B$. For any $w \in W$ with kS_mw, since either $w \leq k$ or $w = v$, we obtain $w \nVdash C$. Therefore, $m \nVdash B \rhd C$.
- Case 2: There exists an odd number k such that $k \Vdash B$ and for all $j \leq k$, $j \nVdash C$: Let $n_A = k + 1$ and $m \geq n_A$. Then, mRk and $k \Vdash B$. For any $w \in W$ with kS_mw, we have $w \nVdash C$ because $w \leq k$. Therefore, $m \nVdash B \rhd C$.
- Case 3: Otherwise: Then, the following conditions (I) and (II) are fulfilled.

  (I) For any even number k, if $k \Vdash B$, then there exists a $j \leq k$ such that $j \Vdash C$ or $v \Vdash C$.

(II) For any odd number k, if $k \Vdash B$, then there exists a $j \leq k$ such that $j \Vdash C$.

By the induction hypothesis, there exists an $n_B \in \omega$ such that $\forall m \geq n_B\ (m \Vdash B)$ or $\forall m \geq n_B\ (m \nVdash B)$. We may assume that n_B is an odd number. We distinguish the following two cases.

- $\forall m \geq n_B\ (m \Vdash B)$: Let $n_A = n_B + 1$ and $m \geq n_A$. Let k be any element in W satisfying mRk and $k \Vdash B$. Since n_B is odd and $n_B \Vdash B$, there exists a $j_0 \leq n_B$ such that $j_0 \Vdash C$ by (II). We distinguish the following three cases.
  - k is odd: By (II), there exists a $j_1 \leq k$ such that $j_1 \Vdash C$. Then, $kS_m j_1$ and $j_1 \Vdash C$.
  - k is even and $k \geq n_B$: Since $k \geq j_0$, we have $kS_m j_0$ and $j_0 \Vdash C$.
  - k is even and $k < n_B$: By (I), there exists a $j_1 \leq k$ such that $j_1 \Vdash C$ or $v \Vdash C$. Since $k < n_B \leq m - 1$, we obtain $k < m - 1$. Hence, $kS_m j_1$ and $kS_m v$.

  In any case, there exists a $w \in W$ such that $kS_m w$ and $w \Vdash C$. Therefore, $m \Vdash B \rhd C$.
- $\forall m \geq n_B\ (m \nVdash B)$: Let $n_A = n_B + 1$ and $m \geq n_A$. Let k be any element in W satisfying mRk and $k \Vdash B$. Then, $k < n_B$ because $k \Vdash B$. We distinguish the following two cases.
  - k is odd: Since there exists a $j \leq k$ such that $j \Vdash C$ by (II). Then, $kS_m j$.
  - k is even: By (I), there exists a $j \leq k$ such that $j \Vdash C$ or $v \Vdash C$. Since $k < n_B \leq m - 1$, we obtain $k < m - 1$ and hence $kS_m j$ and $kS_m v$.

In any case, there exists a $w \in W$ such that $kS_m w$ and $w \Vdash C$. Therefore, $m \Vdash B \rhd C$. □

We suppose, towards a contradiction, that there exists a formula A such that $v(A) \subseteq \{q\}$ and $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5}) \vdash A \leftrightarrow A \rhd q$. Since $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$ is valid in $\mathcal{F}$, we have that $A \leftrightarrow A \rhd q$ is also valid in $\mathcal{F}$. Then, the following claim holds.

Claim 2 For any $n \in \omega$, n is even if and only if $n \Vdash A$.

Proof We prove the claim by induction on n.

For $n = 0$, since obviously $0 \Vdash A \rhd q$, we have $0 \Vdash A$. Suppose $n > 0$ and the claim holds for any natural number less than n.

($\Leftarrow$): Assume that n is odd. Then, $nRn - 1$ and since $n - 1$ is even, $n - 1 \Vdash A$ by the induction hypothesis. Let w be the any element in W which satisfies $n - 1 S_n w$. By the definition of S_n, we find $w \leq n - 1$ and hence $w \nVdash q$. Therefore, $n \nVdash A \rhd q$, and thus $n \nVdash A$.

($\Rightarrow$): Assume that n is even. Let m be the any element in W which satisfies nRm and $m \Vdash A$. By the induction hypothesis, m is even and hence $m < n - 1$. Then, $mS_n v$ and $v \Vdash q$. Therefore, $n \Vdash A \rhd q$ and hence, $n \Vdash A$. □

This contradicts Claim 1. Threfore, for any formula A with $v(A) \subseteq \{q\}$, we obtain $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5}) \nvdash A \leftrightarrow A \rhd q$. □

As in Corollary 4, we obtain the following corollary.

Corollary 5 *Let L be any logic that is closed under substituting a formula for a propositional variable and satisfies $\mathbf{IL}^- \subseteq L \subseteq \mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$. Then, L has neither ℓFPP nor CIP.*

6.3 A counter model of FPP for $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$

In Theorems 9 and 10, we proved that the logics **CL** and $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J5})$ do not have ℓFPP. On the other hand, we proved in Theorem 8 that $\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5})$ has ℓFPP. Thus, we cannot provide a counter model of ℓFPP for extensions of $\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5})$. In this subsection, we prove that the logics $\mathbf{IL}^-(\mathbf{J4}_+, \mathbf{J5})$ and $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$ have neither FPP nor CIP.

Theorem 11 *The formula $\top \rhd \neg p$ has no fixed point in $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$. That is, for any formula A with $v(A) = \emptyset$,*

$$\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5}) \nvdash A \leftrightarrow \top \rhd \neg A.$$

Proof We define an $\mathbf{IL}^-$-frame $\mathcal{F} = \langle W, R, \{S_w\}_{w \in W}\rangle$ as follows:

- $W := \omega$;
- $xRy :\iff x > y$;
- For each $n \in W$, $S_n := \{\langle x, y\rangle \in W^2 : x, y < n$ and $(x \geq y$ or $(x = 0$ and $(y$ is even or $y = n - 1)))\}$.

We draw the relations S_3 and S_4. As in the proof of Theorem 10, in the case of xRy for $x, y < n$, xS_ny holds, and the corresponding broken lines are omitted in the figure (Fig. 5).

Then, $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$ is valid in $\mathcal{F}$. Let $\Vdash$ be an arbitrary satisfaction relation on $\mathcal{F}$.

Claim 1 For any formula A with $v(A) = \emptyset$, there exists an $n_A \in W$ such that

$$\forall m \geq n_A\,(m \Vdash A) \text{ or } \forall m \geq n_A\,(m \nVdash A).$$

Proof This is proved by induction on the construction of A. We prove only the case of $A \equiv B \rhd C$. We distinguish the following three cases.

- Case 1: There exists an $n > 0$ such that $n \Vdash B$ and for all $k \leq n$, $k \nVdash C$. Let $n_A = n + 1$ and $m \geq n_A$. Then, mRn and $n \Vdash B$. Also, for any $k \in W$, if nS_mk, then $k \leq n$ because $n \neq 0$. Therefore, $k \nVdash C$. Thus, $m \nVdash B \rhd C$.
- Case 2: $0 \Vdash B$ and for all even numbers k, $k \nVdash C$. By the induction hypothesis, there exists an $n_C \in W$ such that $\forall m \geq n_C\,(m \Vdash C)$ or $\forall m \geq n_C\,(m \nVdash C)$. Since there are infinitely many even numbers $k \in W$ such that $k \nVdash C$, we obtain $\forall m \geq n_C\,(m \nVdash C)$. Let $n_A = n_C + 1$. Then, for any $m \geq n_A$, we have $mR0$ and $0 \Vdash B$. Let $k \in W$ be such that $0S_mk$. Then, k is even or $k = m - 1$ by the definition of S_m. By our supposition, if k is even, then $k \nVdash C$. If $k = m - 1$, then $m - 1 \nVdash C$ because $m - 1 \geq n_C$. Therefore, in either case, $k \nVdash C$. Thus, $m \nVdash B \rhd C$.

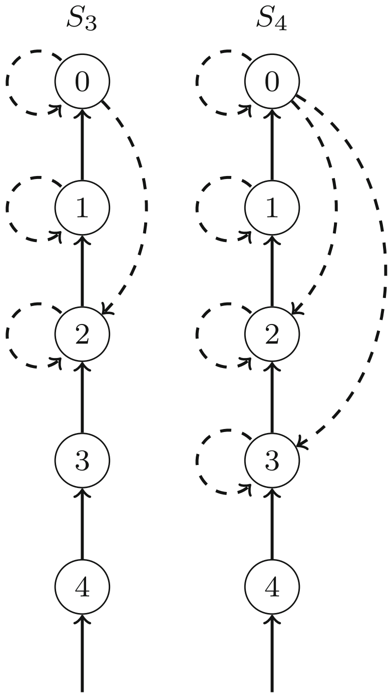

Fig. 5 A counter model of FPP for $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$

- Case 3: Otherwise: Then, the following conditions (I) and (II) are fulfilled.

  (I) For any $n > 0$, if $n \Vdash B$, then there exists a $k \in W$ such that $k \le n$ and $k \Vdash C$.
  (II) If $0 \Vdash B$, then there exists an even number $k \in W$ such that $k \Vdash C$.

  We distinguish the following two cases.

  - $0 \nVdash B$: Let $n_A = 0$ and $m \ge n_A$. For any $n \in W$ satisfying mRn and $n \Vdash B$, since $n \neq 0$, there exists a $k \le n$ such that $k \Vdash C$ by the condition (I). Since $nS_m k$, we obtain $m \Vdash B \rhd C$.
  - $0 \Vdash B$: By the condition (II), there exists an even number k such that $k \Vdash C$. Let $n_A = k + 1$ and $m \ge n_A$. Let $n \in W$ be such that mRn and $n \Vdash B$. If $n \neq 0$, then there exists a $k' \le n$ such that $k' \Vdash C$ and $nS_m k'$ by the condition (I). If $n = 0$, then since k is even and $k < m$, we obtain $nS_m k$ and $k \Vdash C$. Therefore, $m \Vdash B \rhd C$. □

We suppose, towards a contradiction, that there exists a formula A such that $v(A) = \emptyset$ and $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5}) \vdash A \leftrightarrow \top \rhd \neg A$. Then, $A \leftrightarrow \top \rhd \neg A$ is valid in $\mathcal{F}$ because so is $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$. Then, the following claim holds.

Claim 2 For any $n \in W$, n even if and only if $n \Vdash A$.

Proof We prove the by induction on n. For $n = 0$, obviously $0 \Vdash A$. Suppose $n > 0$ and the claim holds for any natural number less than n.

($\Leftarrow$): Assume that n is odd. Then, $nR0$. For any $k \in W$ which satisfies $0S_n k$, since n is odd, k is even and $k < n$. By the induction hypothesis, $k \Vdash A$. Thus, we obtain $n \nVdash \top \rhd \neg A$ and hence, $n \nVdash A$.

($\Rightarrow$): Assume that n is even. Let $m \in W$ be such that nRm. We distinguish the following three cases.

- $m = 0$: Then, $0S_n n-1$. Since $n-1$ is odd, $n-1 \Vdash \neg A$ by the induction hypothesis.
- m is even and $m \neq 0$: Then, $mS_n m-1$. Since $m-1$ is odd, $m-1 \Vdash \neg A$ by the induction hypothesis.
- m is odd: Then, $mS_n m$. Since m is odd, $m \Vdash \neg A$ by the induction hypothesis.

In any case, there exists a $w \in W$ such that $mS_n w$ and $w \Vdash \neg A$. Therefore, we obtain $n \Vdash \top \rhd \neg A$ and hence, $n \Vdash A$. □

This contradictions Claim 1. Therefore, there is no formula A such that $v(A) = \emptyset$ and $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5}) \nvdash A \leftrightarrow \top \rhd \neg A$. □

Corollary 6 *Every sublogic of* $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$ *does not have FPP. Furthermore, if* L *is closed under substituting a formula for a propositional variable and satisfies* $\mathbf{IL}^-(\mathbf{J4}_+) \subseteq L \subseteq \mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$*, then* L *does not have CIP.*

Proof By Theorem 11, every sublogic of $\mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$ does not have FPP. By Lemma 2, every logic L that is closed under substituting a formula for a propositional variable and satisfies $\mathbf{IL}^-(\mathbf{J4}_+) \subseteq L \subseteq \mathbf{IL}^-(\mathbf{J1}, \mathbf{J4}_+, \mathbf{J5})$ does not have CIP. □

7 Concluding remarks

In this paper, we provided a complete description of twelve sublogics of **IL** concerning UFP, FPP, and CIP. In particular, for these sublogics L, we proved that L has FPP if and only if L contains $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5})$. On the other hand, there are many other logics between $\mathbf{IL}^-$ and **IL**. For instance, Kurahashi and Okawa [9] introduced eight sublogics such as $\mathbf{IL}^-(\mathbf{J2}, \mathbf{J4}_+, \mathbf{J5})$ that are not in Fig. 1, and proved that these eight logics are not complete with respect to usual Veltman semantics but complete with respect to generalized Veltman semantics. Then, it is natural to investigate a sharper threshold for FPP in a larger class of sublogics. Then, for example, we propose a question if $\mathbf{J2}_+$ can be weakened by **J2** in the statement of Corollary 2.

Problem 2 Does the logic $\mathbf{IL}^-(\mathbf{J2}, \mathbf{J4}_+, \mathbf{J5})$ have FPP?

In our proofs of Theorems 6, 7, and 8, the use of the axiom scheme **J5** seems inevitable. In fact, **CL** ($= \mathbf{IL}^-(\mathbf{J1}, \mathbf{J2}_+)$) fails to have ℓFPP. From this observation, in an earlier version of the present paper, we had proposed the question whether **J5** is necessary or not for ℓFPP and FPP. Later on, the third author of the present paper settled this question. For each $n \geq 1$, let $\mathbf{J5}^n$ be the following axiom scheme:

$\mathbf{J5}^n$ $\Diamond^n A \rhd A$.

Concerning $\mathbf{J5}^n$, the following result is established.

Theorem 12 (Okawa [11]) *Let* $n \geq 1$.

1. $\mathbf{IL}^-(\mathbf{J5}^n) \vdash \mathbf{J5}^{n+1}$ *and* $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5}^{n+1}) \nvdash \mathbf{J5}^n$.
2. $\mathbf{IL}^-(\mathbf{J2}_+, \mathbf{J5}^n)$ *and* $\mathbf{IL}^-(\mathbf{J4}, \mathbf{J5}^n)$ *have FPP and* ℓ*FPP, respectively.*

Furthermore, the authors have already developed several studies related to the present paper (cf. [6, 10]).

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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