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Questions on cardinal invariants of Boolean algebras

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Abstract

In the book Cardinal Invariants on Boolean Algebras by J. Donald Monk many such cardinal functions are defined and studied. Among them several are generalizations of well known cardinal characteristics of the continuum. Alongside a long list of open problems is given. Focusing on half a dozen of those cardinal invariants some of those problems are given an answer here, which in most of the cases is a definitive one. Most of them can be divided in two groups. The problems of the first group ask about the change on those cardinal functions when going from a given infinite Boolean algebra to its simple extensions, while in the second group the comparison is between a couple of given infinite Boolean algebras and their free product.

Keywords Cardinal invariants · Boolean algebras · Simple extensions · Free products

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1 Introduction

A way to study the structure of a given Boolean algebra is through the so called *cardinal invariants*, which can be described, somehow loosely, as bounds, either upper or lower, to the size of certain types of its substructures, be they algebraic, combinatorial or topological. They are mainly generalizations of cardinal invariants of topological spaces, as Boolean algebras always are, or of the characteristics of the continuum, which are usually defined on the Boolean algebra $P(\omega)/Fin$, i.e. the quotient of the power set of ω modulo the ideal of the finite subsets of ω. Of all these possibilities, in

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this text we will focus on a half dozen of them. The reader is assumed to have basic knowledge of Boolean algebras, as can be found in the first parts of [4].

Let $(A, +, \cdot, -, 0, 1)$, usually abbreviated as A, be an infinite Boolean algebra with $<$ as its order relation. The set of all non-zero elements of A will be denoted A^+. A subset $I \subseteq A$ is an *ideal* if $1 \notin I$, it is downward closed and $a+b \in I$, for all $a, b \in I$.

A *partition* of A is a non-empty subset $X \subseteq A^+$ such that for all $a, b \in X$, $a \cdot b = 0$, and such that for all $c \in A^+$ there exists $x \in X$ such that $c \cdot x \neq 0$. A *(decreasing) tower* is a non-empty subset $X \subseteq A^+$ well-ordered by $<^{-1}$ whose product is 0, i.e. whose only lower bound is 0. A *centered family* is a non-empty subset $X \subseteq A$ such that $\prod_{a \in F} a \neq 0$ for all $F \in [X]^{<\omega}$. If there exists $a \in A^+$ such that $a \leq x$, for all $x \in X$, it is said that a is a *pseudointersection* of X. The algebra A is said to be *atomless* if for all $a \in A^+$ there exists $b \in A^+$ such that $b < a$. If A is atomless, a *splitting* family of A is a non-empty subset $X \subseteq A$ such that for all $a \in A^+$ there exists $x \in X$ such that $a \cdot x \neq 0 \neq a \cdot -x$.

From these kinds of subfamilies of a given infinite Boolean algebra A one can define the following cardinal characteristics:

$$\mathfrak{a}(A) := \min\{|X| \mid X \subseteq A \text{ } is \text{ } an \text{ } infinite \text{ } partition\}$$
$$\mathfrak{t}(A) := \min\{|X| \mid X \subseteq A \text{ } is \text{ } a \text{ } tower\}$$
$$\mathfrak{p}(A) := \min\{|X| \mid X \subseteq A \text{ } is \text{ } a \text{ } centered \text{ } family \text{ } with \text{ } no \text{ } pseudointersection\}$$
$$\mathfrak{s}(A) := \min\{|X| \mid X \subseteq A \text{ } is \text{ } a \text{ } splitting \text{ } family\}.$$

For[1] the notions of centered families and pseudointersections there exist the dual notions of families with the *finite union property*, i.e. $X \subseteq A$ such that $\sum_{a \in F} a \neq 1$, for all $F \in [X]^{<\omega}$, and *pseudounion*, i.e. an upper bound not equal to 1. A $\mathfrak{p}$-family is a family with the finite union property with no pseudounion. Similarly one can define *increasing towers*. Thus the definitions of both $\mathfrak{p}(A)$ and $\mathfrak{t}(A)$ can be restated:

$$\mathfrak{t}(A) := \min\{|X| \mid X \subseteq A \text{ } is \text{ } an \text{ } increasing \text{ } tower\}$$
$$\mathfrak{p}(A) := \min\{|X| \mid X \subseteq A \text{ } is \text{ } a \text{ } \mathfrak{p}-family\}.$$

Since every infinite partition is a $\mathfrak{p}$-family and every decreasing tower is a centered family with no pseudointersection, it immediately follows that $\mathfrak{p}(A) \leq \mathfrak{a}(A), \mathfrak{t}(A)$. It is also easy to verify that every maximal centered subfamily of a splitting family is a centered family with no pseudointersection. So, $\mathfrak{p}(A) \leq \mathfrak{s}(A)$. Notice also that, when not ∞, $\mathfrak{t}(A)$ is a regular cardinal.

These small cardinal characteristics are generalizations of well-known cardinal characteristics of the continuum, when instead of A we have $P(\omega)/Fin$, simply denoted $\mathfrak{a}$, $\mathfrak{t}$, $\mathfrak{p}$ and $\mathfrak{s}$. For more information on these cardinals see [2].

Other larger cardinal invariants derive from the following concepts. A subset $X \subseteq A^+$ is *dense* if for all $a \in A^+$, there exists $x \in X$ such that $x \leq a$. A subset $X \subseteq A$ is

[1] If A is an infinite Boolean algebra, an infinite partition of A can be found through Axiom of Choice. If P is said partition, the set $\{-x \mid x \in P\}$ is a centered family with no pseudointersection. So $\mathfrak{a}(A)$ and $\mathfrak{p}(A)$ are well defined. If A is also atomless, then A^+ is a splitting family, and also $\mathfrak{s}(A)$ is well defined on atomless Boolean algebras. However, not every infinite Boolean algebra has a tower. In this case $\mathfrak{t}(A) = \infty$.

irredundant if x is not algebraically generated by $X \setminus \{x\}$, for all $x \in X$. If X is not irredundant, it will be called *redundant*. We can define

$$\pi\,(A) := \min\{|X| \mid X \subseteq A\ is\ dense\}$$
$$Irr\,(A) := \sup\{|X| \mid X \subseteq A\ is\ irredundant\}\,.$$

Of these concepts it is known that:

Theorem 1 *(McKenzie, [4], Proposition 4.23.) Every maximal irredundant family of a Boolean algebra generates a dense subalgebra.*

An immediate consequence of this proposition is that $\omega \leq \pi\,(A) \leq Irr\,(A)$ for all infinite Boolean algebra A.

In [5] there are several results on these cardinal functions, both concerning their relation to other functions and their behaviour on different kinds of Boolean algebras (although there it is written tow instead of $\mathfrak{t}$ and spl instead of $\mathfrak{s}$). In that book several questions are asked about these cardinal invariants. In this text an answer, be it partial or complete, is given to some of them. Here a list of these problems is provided, with the same enumeration as in [5].

- **Problem 7.** Does $A \leq_s B$ imply that $\mathfrak{a}(B) \leq \mathfrak{a}(A)$?
- **Problem 8.** Is it true that for all infinite BAs A, B one has $\mathfrak{a}(A \oplus B) = \min(\mathfrak{a}(A), \mathfrak{a}(B))$?
- **Problem 37.** Does $A \leq_s B$ imply that $tow_{spect}(A) \subseteq tow_{spect}(B)$ or $tow(B) \leq tow(A)$?
- **Problem 38.** Are there BAs A, B such that $A \leq_m B$ and $tow(B) < tow(A)$?
- **Problem 45.** Are there BAs A, B such that $A \leq_m B$ and $\mathfrak{p}(B) < \mathfrak{p}(A)$?
- **Problem 46.** Is it true that for all infinite BAs A, B we have $\mathfrak{p}(A \oplus B) = \min(\mathfrak{p}(A), \mathfrak{p}(B))$?
- **Problem 48.** Is

$$\mathfrak{p}(A) = \min\{|X| : X\ is\ a\ maximal\ ramification\ set\ in\ A\}?$$

- **Problem 52.** Is $spl(A \oplus B) = \min(spl(A), spl(B))$ for atomless A, B?
- **Problem 70.** Are there BAs A, B such that $A \leq_\sigma B$ and $\pi(A) > \pi(B)$?
- **Problem 71.** Are there BAs A, B such that $A \leq_s B$ and $\pi(A) > \pi(B)$?
- **Problem 86.** Can one have $Irr(A) < Irr(B)$ for $A \leq_s B$ or $A \leq_m B$?

2 Simple and minimal extensions

If $A \subseteq B$ are two Boolean algebras with the same minimum and maximum elements, and if the operations of B restricted to A coincide with the operations of A (or, usually, if A is isomorphic to such a substructure of B), it is said that A is a *subalgebra* of B, or that B is an *extension* of A. This fact is generally denoted $A \leq B$.

If $A \leq B$, it is said that B is a *simple extension* of A, if there exists $x \in B$ such that $B = A\,(x)$, which means that B is the algebra generated by x and all the elements of

A. This fact will be denoted $A \leq_s B$. It is said that B is a *minimal extension* of A if any algebra C such that $A \leq C \leq B$ is either equal to A or equal to B. This fact is denoted $A \leq_m B$. If $A \leq_m B$, it follows easily $A \leq_s B$.

This section is about the changes, or the lack thereof, that can happen to the cardinal functions defined in the Introduction when they pass from a Boolean algebra to their simple and minimal extensions.

2.1 $\mathfrak{p}$-families, infinite partitions and towers

Considering the cardinal characteristics defined in the previous section and minimal extensions we have the following result.

Theorem 2 *Let A and B be infinite Boolean algebras. If* $A \leq_m B$*, then* $\mathfrak{p}(B) \leq \mathfrak{p}(A)$*,* $\mathfrak{t}(B) \leq \mathfrak{t}(A)$ *and* $\mathfrak{a}(B) \leq \mathfrak{a}(A)$.

This fact can be found in [5] (Propositions 3.34, 4.36, 4.54). In this book the author asks if this result can be extended to the case when B is a simple extension of A and also if the inequalities can be strict in the case of B being a minimal extension of A (Problems 7, 37, 38, 44, 45). For all these questions in this section affirmative answers are given. Firstly Theorem 2 will be generalized to simple extensions. Some ideals and their quotients will be important for this.

Definition 1 Let A and B be Boolean algebras such that $A \leq B$. If $x \in B$ define the ideal on A below x as follows:

$$A \upharpoonright x := \{a \in A \mid a \leq x\}.$$

Recall that for an ideal $I \subseteq A$ the *quotient* A/I is the Boolean algebra on the equivalence classes defined by the relation $a \sim_I b$ iff $a \triangle b = (a \cdot (-b)) + (b \cdot (-a)) \in I$, with the operations induced by those of A. This means that $[a]_I + [b]_I = [a+b]_I$, $[a]_I \cdot [b]_I = [a \cdot b]_I$, and $-[a]_I = [-a]_I$, where $[a]_I$ is the equivalence class of a, for all $a, b \in A$. Also recall that the *simple product* $A \times B$, whenever A and B are Boolean algebras, refers to the Boolean algebra on the set $A \times B$, with (1, 1) as maximum element, (0, 0) as minimum element, and the operations defined coordinatewise.

Lemma 1 *Let A be a Boolean algebra and suppose that* $A(x)$ *is a simple extension of A. If* $I_0 := A \upharpoonright x$ *and* $I_1 := A \upharpoonright -x$*, then* $A(x) \cong (A/I_0) \times (A/I_1)$.

On the face of Lemma 1, if we are working on simple extensions of Boolean algebras and some of their cardinal characteristics, it would be useful to know the behavior of said cardinal characteristics on simple products. Fortunately we have the following lemma.

Lemma 2 *Let A and B be two infinite Boolean algebras. Then*

- $\mathfrak{p}(A \times B) = \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$,
- $\mathfrak{t}(A \times B) = \min\{\mathfrak{t}(A), \mathfrak{t}(B)\}$ *and*
- $\mathfrak{a}(A \times B) = \min\{\mathfrak{a}(A), \mathfrak{a}(B)\}$.

Proofs of both lemmas can be found in [5], Propositions 2.28, 3.36, 4.37 and 4.55. If $b \in A$, the set

$$A \upharpoonright b := \{a \in A \mid a \leq b\}$$

with 0 as minimum element, b as maximum element, the product and sum of A, and the complement restricted to b is a Boolean algebra. We say that $X \subseteq A \upharpoonright b$ is a $\mathfrak{p}$-family (resp. infinite partition, resp. tower) below b if it is so in the Boolean algebra $A \upharpoonright b$. We now proceed to answer the questions.

Lemma 3 *Let A be an infinite Boolean algebra and $I \subseteq A$ be an ideal. Suppose that A/I is infinite and that $\mathfrak{p}(A) < \mathfrak{p}(A/I)$ (resp. $\mathfrak{a}(A) < \mathfrak{a}(A/I)$, resp. $\mathfrak{t}(A) < \mathfrak{t}(A/I)$). If $\{a_\alpha \mid \alpha < \kappa\}$ is a $\mathfrak{p}$-family in A (resp. infinite partition, resp. inc. tower) of minimum size, then there exist $b \in A^+$ and $E \in [\kappa]^\kappa$ such that $\{a_\alpha \cdot b \mid \alpha \in E\}$ is a $\mathfrak{p}$-family (resp. infinite partition, resp. tower) below b consisting of elements of I.*

Proof *Case 1*. Suppose that $\sum_{\alpha<\kappa} [a_\alpha]_I = [1]_I$.

Since A/I has no $\mathfrak{p}$-families of size κ, it follows that there exists $F \in [\kappa]^{<\omega}$ such that $\sum_{\alpha \in F} [a_\alpha]_I = [1]_I$. Hence $b := -\sum_{\alpha \in F} a_\alpha$ is an element of I. It follows immediately that $a_\alpha \cdot b$ is an element of I, for all $\alpha < \kappa$. Let E be the set of all $\alpha < \kappa$ such that $a_\alpha \cdot b \neq 0$. If E' were a finite subset of E such that $b = \sum_{\alpha \in E'} a_\alpha \cdot b$, it would follow that $1 = \sum_{\alpha \in E' \cup F} a_\alpha$, which is a contradiction.

We have that $b = \sum_{\alpha \in E} a_\alpha \cdot b$, since otherwise we would have a non-zero element $c \leq b$ such that $a_\alpha \cdot c = 0$ for all $\alpha \in E$, and hence for all $\alpha < \kappa$, contradicting the fact that $\{a_\alpha \mid \alpha < \kappa\}$ is a $\mathfrak{p}$-family of A. It follows that $\{a_\alpha \cdot b \mid \alpha \in E\}$ is a $\mathfrak{p}$-family below b. Finally, if we had $|E| < \kappa$, then $\{a_\alpha \cdot b \mid \alpha \in E\} \cup \{b\}$ would be a $\mathfrak{p}$-family of A of size less that $\mathfrak{p}(A)$. Therefore $|E| = \kappa$. (When dealing with infinite partitions there also exists $F \in [\kappa]^{<\omega}$ such that $\sum_{\alpha \in F} [a_\alpha]_I = [1]_I$. Define $b := -\sum_{\alpha \in F} a_\alpha$ and $E := \kappa \backslash F$. If $\{a_\alpha \mid \alpha < \kappa\}$ is a tower, there exists $\alpha < \kappa$ such that if $\alpha \leq \beta < \kappa$, then $\left[a_\beta\right]_I = [1]_I$. In this case $b := -a_\alpha$ and $E := \kappa \backslash \alpha$.)

Case 2. Suppose that $\sum_{\alpha<\kappa} [a_\alpha]_I \neq [1]_I$.

There exists $b \in A \backslash I$ such that $[b]_I \cdot [a_\alpha]_I = [0]_I$, i.e. $b \cdot a_\alpha \in I$, for all $\alpha < \kappa$. Consider $E := \{\alpha < \kappa \mid a_\alpha \cdot b \neq 0\}$. If E' were a finite subset of E such that $b = \sum_{\alpha \in E'} b \cdot a_\alpha$, it would follow that $b \in I$, which is a contradiction. Therefore $\{a_\alpha \cdot b \mid \alpha \in E\}$ is a $\mathfrak{p}$-family below b and, as was observed in case 1, the size of E is κ (resp. infinite partition, resp. contains a tower). □

Theorem 3 *Let A and B be infinite Boolean algebras. If $A \leq_s B$, then $\mathfrak{p}(B) \leq \mathfrak{p}(A)$, $\mathfrak{a}(B) \leq \mathfrak{a}(A)$ and $\mathfrak{t}(B) \leq \mathfrak{t}(A)$.*

Proof From Lemma 1, it is known that there exist I_0 and I_1 ideals of A, such that $I_0 \cap I_1 = \{0\}$ and $B \cong (A/I_0) \times (A/I_1)$. We have two cases.

Case 1. The size of, say, A/I_0 is finite.

In this case $\mathfrak{p}(B) = \mathfrak{p}(A/I_1)$. For proving this take $\{c_\alpha \mid \alpha < \mathfrak{p}(A/I_1)\}$, a $\mathfrak{p}$-family in A/I_1. It follows that $\{([1]_{I_0}, c_\alpha) \mid \alpha < \mathfrak{p}(A/I_1)\}$ is a $\mathfrak{p}$-family in B. Therefore $\mathfrak{p}(B) \leq \mathfrak{p}(A/I_1)$.

Now take $\lambda < \mathfrak{p}(A/I_1)$ and $P = \{(d_\alpha, c_\alpha) \mid \alpha < \lambda\} \subseteq B$ such that

$$\sum_{\alpha \in F} (d_\alpha, c_\alpha) \neq ([1]_{I_0}, [1]_{I_1})$$

for all $F \in [\lambda]^{<\omega}$. Since A/I_0 is finite, there exists $F \in [\lambda]^{<\omega}$ such that for all $\alpha < \lambda$ there exists $\beta \in F$ such that $d_\alpha = d_\beta$. Therefore

$$\sum_{\alpha < \lambda} d_\alpha = \sum_{\beta \in F} d_\beta .$$

If $\sum_{\beta \in F} d_\beta \neq [1]_{I_0}$, then $(\sum_{\beta \in F} d_\beta, [1]_{I_1})$ is a pseudounion of P witnessing that that P is not a $\mathfrak{p}$-family in B. If $\sum_{\beta \in F} d_\beta = [1]_{I_0}$, it follows that $\sum_{\alpha \in F'} c_\alpha \neq [1]_{I_1}$, for all $F' \in [\lambda]^{<\omega}$. If $c \in A/I_1$ is a pseudounion of $\{c_\alpha \mid \alpha < \lambda\}$, then $([1]_{I_0}, c)$ witnesses that P is not a $\mathfrak{p}$-family in B. Therefore $\lambda < \mathfrak{p}(B)$. We conclude that $\mathfrak{p}(B) = \mathfrak{p}(A/I_1)$ (analogously for the other two functions).

Suppose that $\kappa = \mathfrak{p}(A) < \mathfrak{p}(A/I_1)$. Let $\{a_\alpha \mid \alpha < \kappa\}$ be a $\mathfrak{p}$-family in A, and let b and E be as given by Lemma 3 with $I = I_1$. Since A/I_0 is finite, there exist $\alpha < \beta$ in E such that $b \cdot a_\alpha \sim_{I_0} b \cdot a_\beta$, though $b \cdot a_\alpha \neq b \cdot a_\beta$. Hence $b \cdot a_\alpha \triangle b \cdot a_\beta \neq 0$ and lies both in I_0 and I_1, which is a contradiction. Therefore, $\mathfrak{p}(A) \geq \mathfrak{p}(A/I_1) = \mathfrak{p}(B)$. The proof is analogous for the other two cardinal functions.

Case 2. Both the size of A/I_0 and the size of A/I_1 are infinite.

In this case $\mathfrak{p}(B) = \min\{\mathfrak{p}(A/I_0), \mathfrak{p}(A/I_1)\}$, (resp. with the other two functions). Suppose that $\kappa = \mathfrak{p}(A) < \mathfrak{p}(A/I_0), \mathfrak{p}(A/I_1)$. Let $\{a_\alpha \mid \alpha < \kappa\}$ be a $\mathfrak{p}$-family in A (resp. infinite partition). If E and b are as given by Lemma 3 with $I_0 = I$, it follows that $\{a_\alpha \cdot b \mid \alpha \in E\} \cup \{-b\}$ is a $\mathfrak{p}$-family in A (resp. infinite partition) whose elements, but possibly one of them, lie in I_0. Applying Lemma 3 to this $\mathfrak{p}$-family (partition) and I_1, it follows that $I_0 \cap I_1 \neq \{0\}$, which is a contradiction. Therefore, $\mathfrak{p}(A) \geq \min\{\mathfrak{p}(A/I_0), \mathfrak{p}(A/I_1)\} = \mathfrak{p}(B)$ (resp. for the function $\mathfrak{a}$).

(When $\{a_\alpha \mid \alpha < \kappa\}$ is an increasing tower, we take b and E corresponding to I_0 and we are in one of the following two cases:

1. $\sum_{\alpha \in E} [b \cdot a_\alpha]_{I_1} = [b]_{I_1}$ or
2. $\sum_{\alpha \in E} [b \cdot a_\alpha]_{I_1} \neq [b]_{I_1}$.

Repeating the proof of Lemma 3 we get a similar contradiction.) □

Now an example of these inequalities being strict will be given. The following lemma tells us when a simple extension is a minimal one.

Lemma 4 *Let $B := A(x)$ be a simple extension of a Boolean algebra A. Then B is a minimal extension of A iff Smp_x^A, the ideal generated by $A \upharpoonright x$ and $A \upharpoonright -x$, is either equal to A or a maximal ideal of A.*

For a proof of this lemma see [5], Proposition 2.32. Recall that an ideal $I \subseteq A$ is *maximal* if for any ideal J such that $I \subseteq J \subseteq A$, it follows that $I = J$.

Theorem 4 *There exist A and B infinite Boolean algebras such that $A \leq_m B$ and $\mathfrak{t}(B) < \mathfrak{t}(A)$.*

Proof Let $\{X_n \mid n < \omega\}$ be a set of disjoint copies of $\beta\omega\backslash\omega$ and p be an element of X_0. Define $X := \bigcup_{n<\omega} X_n$,

$$U := \{a \in P(X) \mid (\forall n < \omega)(a \cap X_n \in clop(X_n)) \wedge (\exists n < \omega)(\forall m \geq n)(a \cap X_n = X_n) \wedge p \in a \cap X_0\}$$

and

$$I := \{a \in P(X) \mid (\forall n < \omega)(a \cap X_n \in clop(X_n)) \wedge (\exists n < \omega)(\forall m \geq n)(a \cap X_n = \emptyset) \wedge p \notin a \cap X_0\},$$

where $P(X)$ is the power set of X and $clop(X_n)$ is the family of clopen subsets of X_n.

Take $A := I \cup U$ as a set algebra, i.e. as a subalgebra of $P(X)$. To verify that A is indeed a Boolean algebra take $a, b \in A$. If $b \in U$, then $a \cup b \in U$. If both $a, b \in I$, then $a \cup b \in I$. Similarly it can be verified that A is closed under intersections. Besides $a \in I$ iff $X \setminus a \in U$. By its definition I is a maximal ideal of A.

Claim $\omega < \mathfrak{t}(A)$.

Proof Let $C := \{a_n \mid n < \omega\}$ be a strictly increasing family of elements from A. If $C \subseteq I$, then $p \notin \bigcup_{n<\omega} X_0 \cap a_n$. Since $clop(\beta\omega \setminus \omega)$ has no countable towers, there exist $a, b \in clop(X_0)$ such that

1. $p \in b$
2. $\bigcup_{n<\omega} X_0 \cap a_n \subset a$,
3. $b \cap a = \emptyset$ and
4. $a \cup b \neq X_0$.

Then $a \cup b \cup \left(\bigcup_{0<n<\omega} X_n\right)$ is an element of A not equal to X which contains each element of C, and so this last set does not form a tower.

If C is not subset of I, without loss of generality, it can be supposed that $C \subseteq U$. Then there exists $m < \omega$ such that for all $n, k < \omega$, if $k \geq m$, then $a_n \cap X_k = X_k$. Therefore, there exists $k < m$ such that for all $n < \omega$, $a_n \cap X_k \subsetneq X_k$, i.e. $\{a_n \cap X_k \mid n < \omega\}$ is a strictly increasing family in $clop(X_k)$. Since this is an algebra with no countable towers, there exists $a \in clop(X_k)$ such that $a \cup \bigcup_{n\in\omega\setminus\{k\}} X_n$ witnesses that C is not a tower and the claim has been proved. □

Let $x := \bigcup_{n\in\omega\setminus\{0\}} X_n$ and $B := A(x)$. Since x is not element of A, B is a simple extension of A, not equal to A. Furthermore, it is a minimal extension of A. Indeed, both $A \upharpoonright x$ and $A \upharpoonright -x$ are respectively

$$\{y \in I \mid y \cap X_0 = \emptyset\}$$

and

$$\{y \in I \mid y \subset X_0\}.$$

It is easy to verify that $Smp_x^A = I$. Since I is a maximal ideal of A, it follows that $A \leq_m B$.

Trivially

$$\left\{ \bigcup_{k \leq n} X_k \mid n < \omega \right\}$$

is a countable tower of B. Then $\omega = \mathfrak{t}(B) < \mathfrak{t}(A)$. □

Lemma 5 *For any infinite Boolean algebra A, $\mathfrak{t}(A) = \omega$ iff $\mathfrak{a}(A) = \omega$ iff $\mathfrak{p}(A) = \omega$.*

Proof Observe that if $\{a_n \mid n < \omega\}$ is a $\mathfrak{p}$-family in some Boolean algebra A, then $\{b_n \mid n < \omega\}$, where $b_n := \sum_{i \leq n} a_i$ for each $n < \omega$, is a tower in A; if $\{b_n \mid n < \omega\}$ is an increasing tower in A, then $\{c_n \mid n < \omega\}$, where $c_n := b_n \cdot (-\sum_{i<n} b_i)$ for each $n < \omega$, is an infinite partition of A; and if $\{c_n \mid n < \omega\}$ is an infinite partition of A, it is a $\mathfrak{p}$-family in A. □

From this lemma and the previous theorem this corollary follows.

Corollary 1 *There exist A and B Boolean algebras, $A \leq_m B$ such that $\mathfrak{p}(B) < \mathfrak{p}(A)$ and $\mathfrak{a}(B) < \mathfrak{a}(A)$.*

2.2 Dense and irredundant families

Opposite to what was concluded at the end of the previous subsection, the cardinal function π is fixed on minimal extensions.

Theorem 5 *Let A and B be infinite Boolean algebras. If $A \leq_m B$, then $\pi(A) = \pi(B)$.*

See [5], Proposition 6.2. On the other hand there are cases where $A \leq_s B$, and $\pi(A) < \pi(B)$. Take for example $A = P(\omega)$ and $B = P(\omega)/Fin \times P(\omega)/\{\emptyset\}$. Trivially we have $\pi(A) = \omega$, and taking as witness any almost disjoint family of $P(\omega)/Fin$ of size $\mathfrak{c}$ it is verified that $\pi(B) = \mathfrak{c}$. Now, answering negatively to Problem 71, it will be proved impossible to have the opposite inequality.

Definition 2 If $I \subseteq A$ is an ideal, define $\pi(I)$ as the minimum size of a *dense* subset of I, i.e. a set $D \subset I \backslash \{0\}$ such that for all $x \in I \backslash \{0\}$ there exists $y \in D$ such that $y \leq x$.

Lemma 6 *If $I, J \subseteq A$ are ideals such that $I \cap J = \{0\}$, then $\pi(A/I) \geq \pi(J)$.*

Proof Suppose that $D := \{x_\alpha \mid \alpha < \pi(A/I)\} \subseteq A \backslash I$, is a set of representatives of a dense family of A/I. Define

$$D' := \{x_\alpha \in D \mid \exists y_\alpha \in J \text{ such that } [x_\alpha]_I \leq [y_\alpha]_I\}.$$

For each $x_\alpha \in D'$, fix y_α and define $z_\alpha := x_\alpha \cdot y_\alpha$. Since $x_\alpha \notin I$, while $x_\alpha \cdot (-y_\alpha) \in I$, for any $x_\alpha \in D'$, it follows that $z_\alpha \notin I$. In particular $z_\alpha \neq 0$, for all $x_\alpha \in D'$. The set of such z_α is dense on J. Indeed, if $y \in J\setminus\{0\}$, there exists $x_\alpha \in D$, such that $[x_\alpha]_I \leq [y]_I$. It follows that $x_\alpha \in D'$. Suppose that $z_\alpha \nleq y$. Then $0 \neq z_\alpha \cdot (-y) \leq x_\alpha \cdot (-y) \in I$. But $z_\alpha \in J$, and thus $0 \neq z_\alpha \cdot (-y) \in I \cap J$ which is a contradiction. Therefore, $z_\alpha \leq y$.

Since we got a dense subset of J of at most $\pi(A/I)$ many elements, it follows that $\pi(A/I) \geq \pi(J)$. □

Theorem 6 *Let A and B be infinite Boolean algebras. If $A \leq_s B$, then $\pi(A) \leq \pi(B)$.*

Proof Remember that if $A \leq_s B$, then there exist I_0 and I_1 ideals of A such that $I_0 \cap I_1 = \{0\}$ and $B \cong (A/I_0) \times (A/I_1)$. It follows easily that $\pi(B) = \max\{\pi(A/I_0), \pi(A/I_1)\}$.

Applying Lemma 6 to I_0 and I_1, we get that $\pi(A/I_0) \geq \pi(I_1)$ and that $\pi(A/I_1) \geq \pi(I_0)$. We can also apply it to

$$I' := \{a \in A \mid \forall x \in I_0\ \forall y \in I_1\ a \cdot x = 0 = a \cdot y\},$$

which is an ideal such that $I' \cap I_0 = \{0\} = I' \cap I_1$, to conclude that $\pi(I') \leq \pi(A/I_0), \pi(A/I_1)$. Since at least one of these last two cardinals is infinite, we conclude that $\pi(I_0) + \pi(I_1) + \pi(I') \leq \pi(A/I_0) + \pi(A/I_1) = \pi(B)$.

Now we prove that the left side of the last inequality is $\geq \pi(A)$. Take $D_0 \subseteq I_0$, $D_1 \subseteq I_1$ and $D' \subseteq I'$, dense subsets of minimum size of the respective ideals. Take $x \in A^+$. If there exists, for some $i < 2$, some non-zero $y_i \in I_i$ such that $y_i \cdot x \neq 0$, then there exists $z \in D_i$ such that $z \leq x$. If this is not the case, then $x \in I'$ and there exists $z \in D'$ such that $z \leq x$. So $D_0 \cup D_1 \cup D'$ is a dense subset of A.

Then $\pi(I_0) + \pi(I_1) + \pi\left(I'\right) \geq \pi(A)$. Finally we conclude that $\pi(A) \leq \pi(B)$. □

Finally, we state that Irr is not moved by simple extensions, giving an answer to problem 86. In order to prove this we need the following theorem (see [5], Theorem 8.4.).

Theorem 7 $Irr(A \times B) = \max\{Irr(A), Irr(B)\}$ *for all infinite Boolean algebras A and B.*

Theorem 8 *Let A and B be infinite Boolean algebras. If $A \leq_s B$, then $Irr(A) = Irr(B)$.*

Proof Since any irredundant subset of A is an irredundant set of B it follows that $Irr(A) \leq Irr(B)$. We also know that there exist two ideals I_0 and I_1 of A such that $I_0 \cap I_1 = \{0\}$ and $B \cong (A/I_0 \times A/I_1)$.

Theorem 7 tells us that $Irr(B) = \max\{Irr(A/I_0), Irr(A/I_1)\}$. Wlog $Irr(B) = Irr(A/I_0)$. We will consider two cases.

Case 1. $Irr(A/I_0)$ is a successor cardinal. Suppose that $Irr(A/I_0) = \kappa^+$ and take $\left\{[a_\alpha]_{I_0} \mid \alpha < \kappa^+\right\}$ an irredundant family of A/I_0. If $\left\{a_\alpha \mid \alpha < \kappa^+\right\}$ were redundant, we would have $\alpha < \kappa^+$ such that a_α is generated by $\left\{a_\beta \mid \alpha \neq \beta \in \kappa^+\right\}$, while $[a_\alpha]_{I_0}$ is not generated by $\left\{\left[a_\beta\right]_{I_0} \mid \alpha \neq \beta \in \kappa^+\right\}$, which is clearly a contradiction. So $\kappa^+ \leq Irr(A)$.

Case 2. $Irr(A/I_0)$ is a limit cardinal. Suppose that $Irr(A/I_0) = \lambda$ and take $\kappa < \lambda$. Let $\left\{[a_\alpha]_{I_0} \mid \alpha < \kappa\right\}$ be an irredundant family of A/I_0. As in the previous case $\{a_\alpha \mid \alpha < \kappa\}$ is irredundant, and it follows that $\kappa \leq Irr(A)$. Since λ is limit, we conclude that $\lambda \leq Irr(A)$.

In either case $Irr(B) \leq Irr(A)$ and the equality is verified. □

3 Free products

Definition 3 (Free product) If A and B are two Boolean algebras, their free product, denoted $A \oplus B$, is an algebra C such that there exist $A', B' \leq C$, such that $A \cong A'$, $B \cong B'$,

$$C = \langle A' \cup B' \rangle := \left\{ \sum_{i<n} a_i \cdot b_i \mid n < \omega, a_i \in A', b_i \in B' \right\}$$

and for all $a \in A' \setminus \{0\}$ and all $b \in B' \setminus \{0\}$, $a \cdot b \neq 0$.

Given two Boolean algebras A and B, this algebra exists and is unique up to isomorphisms.

In topological duality we have that if $A \cong clop\,(X)$ and $B \cong clop\,(Y)$, for some zero-dimensional compact Hausdorff spaces A and B, then $A \oplus B \cong clop\,(X \times Y)$. This algebra consists of sets of the form $\bigcup_{i<n} a_i \times b_i$, for some $n < \omega$, where $a_i \in clop(X)$ and $b_i \in clop(Y)$, for all $i < n$. It is worth observing that

$$\bigcup_{i<n} a_i \times b_i = \bigcup_{\emptyset \neq J \subseteq n} (\bigcap_{i \in J} a_i \setminus \bigcup_{j \notin J} a_j) \times (\bigcup_{i \in J} b_i).$$

Therefore we can always assume without loss of generality either that the a_i's are disjoint, or that the b_i's are disjoint. More basic information on free products can be found in [4], volume 1, chapter 4.

From now on A and B will be two infinite Boolean algebras isomorphic to (and often interchanged with) the algebra of clopen sets of two zero-dimensional compact Hausdorff spaces X and Y.

Theorem 9 *Let A and B be two infinite Boolean algebras. Then*

- $\mathfrak{p}(A \oplus B) \leq \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$,
- $\mathfrak{t}(A \oplus B) \leq \min\{\mathfrak{t}(A), \mathfrak{t}(B)\}$,
- $\mathfrak{a}(A \oplus B) \leq \min\{\mathfrak{a}(A), \mathfrak{a}(B)\}$ *and, if A and B are atomless,*
- $\mathfrak{s}(A \oplus B) \leq \min\{\mathfrak{s}(A), \mathfrak{s}(B)\}$.

Proof Take $\{a_\alpha \mid \alpha < \mathfrak{p}(A)\} \subseteq A$ a centered family with no pseudointersection. It follows that $\{a_\alpha \times Y \mid \alpha < \mathfrak{p}(A)\}$ is a centered family of $A \oplus B$. If it had a pseudointersection, wlog of the form $a \times b$ with $a \in A^+$ and $b \in B^+$, then a would be a pseudointersection of the a_α's, which is a contradiction. Therefore $\{a_\alpha \times Y \mid \alpha < \mathfrak{p}(A)\}$ has no pseudointersection and $\mathfrak{p}(A \oplus B) \leq \mathfrak{p}(A)$. Analogously $\mathfrak{p}(A \oplus B) \leq \mathfrak{p}(B)$.

For the other cardinal functions the proofs are analogous. □

Considering these simple inequalities a natural question to arise is whether equality holds in any of the four cases or any of them can be strict. Monk asks these questions (Problems 8, 46, 52) for $\mathfrak{p}$, $\mathfrak{a}$ and $\mathfrak{s}$, and claims to have an affirmative answer for $\mathfrak{t}$. Here we give an affirmative answer for $\mathfrak{p}$ and $\mathfrak{s}$, and provide other, perhaps better,[2] proof for $\mathfrak{t}$. For $\mathfrak{a}$ we give a lower bound.

Theorem 10 $\mathfrak{s}(A \oplus B) = \min\{\mathfrak{s}(A), \mathfrak{s}(B)\}$, *for atomless infinite Boolean algebras A and B.*

Proof Suppose that $\kappa < \min\{\mathfrak{s}(A), \mathfrak{s}(B)\}$ and that

$$C := \left\{ c_\alpha := \bigcup_{i<n_\alpha} a_i^\alpha \times b_i^\alpha \mid \alpha < \kappa \right\}$$

is a subset of $A \oplus B$. It follows that neither $\left\{a_i^\alpha \mid \alpha < \kappa, i < n_\alpha\right\}$ is a splitting family of A, nor $\left\{b_i^\alpha \mid \alpha < \kappa, i < n_\alpha\right\}$ is a splitting family of B. Let a and b non-empty witnesses of this fact, which means that:

1. $\forall\alpha < \kappa\ \forall i < n_\alpha$, either $a \cap a_i^\alpha = \emptyset$ or $a \subset a_i^\alpha$, and
2. $\forall\alpha < \kappa\ \forall i < n_\alpha$, either $b \cap b_i^\alpha = \emptyset$ or $b \subset b_i^\alpha$.

The set $a \times b$ witnesses that C is not a splitting family of $A \oplus B$. Indeed, take $\alpha < \kappa$ and suppose that $(a \times b) \cap c_\alpha \neq \emptyset$. Then there exists $i < n_\alpha$ such that $(a \times b) \cap \left(a_i^\alpha \times b_i^\alpha\right) \neq \emptyset$. But thus $a \cap a_i^\alpha \neq \emptyset$ and $b \cap b_i^\alpha \neq \emptyset$, from whence follows that $a \subset a_i^\alpha$ and $b \subset b_i^\alpha$ and hence that $(a \times b) \subset c_\alpha$. Therefore for all $\alpha < \kappa$ either $a \times b$ and c_α are disjoint or $a \times b$ is subset of c_α. We finally conclude that $\kappa < \mathfrak{s}(a \oplus b)$. □

Recall that a maximal centered family $U \subseteq A$ is called an *ultrafilter* and that if U is an ultrafilter of A and $\bigcup_{i<n} a_i \in U$, then there exists $i < n$ such that $a_i \in U$.

Theorem 11 $\mathfrak{p}(A \oplus B) = \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$.

Proof Suppose that $\kappa < \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$ and that

$$C := \left\{ c_\alpha := \bigcup_{i<n_\alpha} a_i^\alpha \times b_i^\alpha \mid \alpha < \kappa \right\}$$

is a centered family in $A \oplus B$. Extend C to an ultrafilter U of $A \oplus B$. So for all $\alpha < \kappa$ there is $i_\alpha < n_\alpha$ such that $a_{i_\alpha}^\alpha \times b_{i_\alpha}^\alpha$ is an element of U. Hence $\left\{a_{i_\alpha}^\alpha \times b_{i_\alpha}^\alpha \mid \alpha < \kappa\right\}$ is a centered family and both $\left\{a_{i_\alpha}^\alpha \mid \alpha < \kappa\right\}$ and $\left\{b_{i_\alpha}^\alpha \mid \alpha < \kappa\right\}$ are centered families in A and B respectively. Let a and b be some pseudointersections of each family. So $a \times b$ is a pseudointersection of $\left\{a_{i_\alpha}^\alpha \times b_{i_\alpha}^\alpha \mid \alpha < \kappa\right\}$ and therefore of C. It follows that $\kappa < \mathfrak{p}(A \oplus B)$. □

[2] The proof provided here is shorter and uses set-theoretic notation which seems to the author more intuitive when dealing with free products of Boolean algebras.

Implicit in the last proofs is the fact that if we have in $A \oplus B$ a centered family with no pseudointersection (resp. splitting) is because its projection on one coordinate is also a centered family with no pesudointersection (resp. is also splitting). So these structures on $A \oplus B$ strongly "inherit" the behavior they have on A and B.

On the other hand, if on $A \oplus B$ we have $c_\alpha := \bigcup_{i<n_\alpha} a_i^\alpha \times b_i^\alpha$, for $\alpha < \kappa$, which form a strictly decreasing family, and choose, through and ultrafilter, some $i_\alpha < n_\alpha$ for each α, as in the last proof, we can be sure only of taking on each coordinate a centered family, not precisely a decreasing family, and hence any hypothesis regarding $\mathfrak{t}(A)$ or $\mathfrak{t}(B)$ cannot be immediately used. For this reason, even when next result is analogous to both the previous ones, its proof is considerably less trivial.

Theorem 12 [3] $\mathfrak{t}(A \oplus B) = \min\{\mathfrak{t}(A), \mathfrak{t}(B)\}$.

Proof Let κ be a regular cardinal less than $\min\{\mathfrak{t}(A), \mathfrak{t}(B)\}$ and

$$C := \left\{ e_\alpha := \bigcup_{i<n_\alpha} a_i^\alpha \times b_i^\alpha \mid \alpha < \kappa \right\}$$

be a strictly decreasing family of $A \oplus B$. It will be proved that C has a pseudointersection.

First suppose that $\kappa = \omega$. Since $\omega < \mathfrak{t}(A)$ and $\omega < \mathfrak{t}(B)$, from Lemma 5 it follows that $\omega < \mathfrak{p}(A)$ and $\omega < \mathfrak{p}(B)$. Therefore, from Theorem 11, we conclude that $\omega = \kappa < \mathfrak{p}(A \oplus B) \leq \mathfrak{t}(A \oplus B)$ and that C has a pseudointersection.

Now suppose that $\omega < \kappa$. Suppose also that for all $\alpha < \kappa$ and $i < j < n_\alpha$ we have that $b_i^\alpha \cap b_j^\alpha = \emptyset$. Since κ is a regular uncountable cardinal, we may also suppose that there exists $n < \omega$ such that $n = n_\alpha$, for all $\alpha < \kappa$. It will be proved inductively that for all $n < \omega$ such a strictly decreasing family has a pseudointersection.

The basic step, when $n = 1$, is trivial. Suppose that n is greater than 1 and that the claim is already proved for all $m \in n \setminus \{0\}$. If $\alpha < \kappa$ and I is a nonempty subset of n, define

$$c_I^\alpha := \bigcap_{i \in I} a_i^\alpha \setminus \bigcup_{i \in n \setminus I} a_i^\alpha$$

and

$$d_I^\alpha := \bigcup_{i \in I} b_i^\alpha.$$

Clearly for all $\alpha < \kappa$

$$\bigcup_{I \in P(n) \setminus \{\emptyset\}} c_I^\alpha \times d_I^\alpha = \bigcup_{i<n} a_i^\alpha \times b_i^\alpha = e_\alpha.$$

[3] Theorem 4.40 in [5], states this fact, although it mainly refers to the spectrum of towers of the Boolean algebras in question, i.e. the set of regular cardinals κ such that there is a tower of size κ in the respective Boolean algebra. Notice that the following proof also serves for proving said theorem. Instead of beginning with $\kappa < \min\{\mathfrak{t}(A), \mathfrak{t}(B)\}$ begin with $\kappa \notin (\mathfrak{t}_{spec}(A) \cup \mathfrak{t}_{spec}(B))$ and everything else follows.

Observe also that for all $\alpha < \beta < \kappa$ and $J, I \in P(n)\backslash\{\emptyset\}$, if $c_J^\beta \cap c_I^\alpha \neq \emptyset$, it follows that $d_J^\beta \subseteq d_I^\alpha$. Indeed, if there exist $x \in c_J^\beta \cap c_I^\alpha$ and $y \in d_J^\beta \backslash d_I^\alpha$, we have that $(x, y) \in e_\beta$. On the other hand $(x, y) \notin c_I^\alpha \times d_I^\alpha$, because $y \notin d_I^\alpha$, and $(x, y) \notin c_{I'}^\alpha \times d_{I'}^\alpha$ for $I' \in P(n)\backslash\{\emptyset, I\}$, because $x \in c_I^\alpha$ and $c_I^\alpha \cap c_{I'}^\alpha = \emptyset$. Therefore $(x, y) \notin e_\alpha$. But this means that $e_\beta \nsubseteq e_\alpha$, which is a contradiction.[4]

Since the sets of the form

$$\bigcup_{I \in P(n)\backslash\{\emptyset\}} c_I^\alpha,$$

where $\alpha < \kappa$, form a decreasing family of non-empty sets, and therefore a centered family extendable to an ultrafilter, for all $\alpha < \kappa$ there exists $I_\alpha \in P(n) \setminus \{\emptyset\}$ such that $\left\{c_{I_\alpha}^\alpha \mid \alpha < \kappa\right\}$ is a centered family. So $\left\{d_{I_\alpha}^\alpha \mid \alpha < \kappa\right\}$ is a decreasing family. It follows that

$$C' := \left\{\bigcup_{i \in I_\alpha} a_i^\alpha \times b_i^\alpha \mid \alpha < \kappa\right\}$$

is a decreasing family of non-empty sets. Wlog we can suppose that there exists a nonempty $I \in P(n)$ such that $I_\alpha = I$ for all $\alpha < \kappa$. Notice that $\{\bigcap_{i \in I} a_i^\alpha \mid \alpha < \kappa\}$ is a centered family.

If $I \neq n$, there exists $m < n$ such that $|I| = m$. So, by induction hypothesis, C' has a pseudointersection, and hence C also has a pseudointersection. If $I = n$, we have two cases:

Case 1. There exists $\alpha < \kappa$ such that

$$\bigcap_{i<n} a_i^{\beta_1} \subseteq \bigcap_{i<n} a_i^{\beta_0}$$

for all $\alpha < \beta_0 < \beta_1 < \kappa$. It is clear that there exists $a \in A$ such that $a \subseteq \bigcap_{i<n} a_i^\beta$ for all $\alpha < \beta < \kappa$. If $b \in B$ is a pseudointersection of $\left\{\bigcup_{i<n} b_i^\alpha \mid \alpha < \kappa\right\}$, it follows that $a \times b$ is a pseudointersection for C.

Case 2. For all $\alpha < \kappa$ there exist $\alpha < \beta_0 < \beta_1 < \kappa$ such that

$$\bigcap_{i<n} a_i^{\beta_1} \nsubseteq \bigcap_{i<n} a_i^{\beta_0}.$$

Observe that this means that there exists $j < n$ such that $b_j^{\beta_0} \cap b_i^{\beta_1} = \emptyset$ for all $i < n$. Indeed, suppose that for all $j < n$ there exists $i < n$ such that $b_j^{\beta_0} \cap b_i^{\beta_1} \neq \emptyset$ and hence such that $a_i^{\beta_1} \subseteq a_j^{\beta_0}$ (as it was observed when dealing with the d's and c's). Take $x \in \bigcap_{i<n} a_i^{\beta_1}$ and $j < n$. Since $a_i^{\beta_1} \subseteq a_j^{\beta_0}$ for some $i < n$, it follows that $x \in a_j^{\beta_0}$. We

[4] As witnessed by this observation, observations (5), (6) and (7) in the proof of Theorem 4.40 in [5] heavily influenced the present proof.

conclude that $\bigcap_{i<n} a_i^{\beta_1} \subseteq \bigcap_{i<n} a_i^{\beta_0}$, which is contradiction. From this observation it follows that $\bigcup_{i<n} b_i^{\beta} \subseteq \bigcup_{i\in n\setminus\{j\}} b_i^{\beta_0}$ for all $\beta_1 \leq \beta < \kappa$.

With this idea it is easy to inductively construct a cofinal set $\{\beta_\alpha \mid \alpha < \kappa\} \subseteq \kappa$ and a sequence $\{j_\alpha \mid \alpha < \kappa\}$ such that

$$\left\{ \bigcup_{i\in n\setminus\{j_\alpha\}} a_i^{\beta_\alpha} \times b_i^{\beta_\alpha} \mid \alpha < \kappa \right\}$$

is a decreasing family. Applying the inductive hypothesis to this family we get a pseudointersection of C. □

Now take two disjoint elements of $(A \oplus B)^+$, say $a \times b$ and $c \times d$. There is nothing preventing, say, a and c from not being disjoint, being enough that b and d are disjoint. Hence it can be observed that given some infinite disjoint family of $A \oplus B$ its projection to either coordinate must not be, not even something close to, a disjoint family. Nevertheless, it can be noticed also that if we restrict the same disjoint family to some subfamily such that, say, projecting to the second coordinate give us a centered family, then necessarily in the first coordinate we get a disjoint family. With this thoughts the following result was obtained.

Theorem 13 $\min\{\min\{\mathfrak{a}(A), \mathfrak{a}(B)\}, \max\{\mathfrak{p}(A), \mathfrak{p}(B)\}\} \leq \mathfrak{a}(A \oplus B)$.

Proof Take $\omega \leq \kappa < \min\{\min\{\mathfrak{a}(A), \mathfrak{a}(B)\}, \max\{\mathfrak{p}(A), \mathfrak{p}(B)\}\}$, so wlog we can assume that $\kappa < \mathfrak{a}(A), \mathfrak{p}(B)$. Let $P := \{c_\alpha \mid \alpha < \kappa\}$ be a disjoint family of $A \oplus B$. Since each c_α can be replaced by the disjoint union of finitely many sets of the form $a_\alpha^i \times b_\alpha^i$, as was observed at the beginning of this section, and κ is an infinite cardinal, we may suppose that each $c_\alpha := a_\alpha \times b_\alpha$. We will prove that P is not a partition of $A \oplus B$. We consider two cases.

Case 1. For all $E \in [\kappa]^{\geq\omega}$ there exists $F \in [E]^{<\omega}$ such that $b_\alpha \subseteq \bigcup_{\beta\in F} b_\beta$ for all $\alpha \in E$.

Let F_0 be such a finite subset for $\kappa = E$. Suppose that for some $n < \omega$ we have already defined F_i, for all $i < n$. Let F_n be such a finite subset for $\kappa \setminus \bigcup_{i<n} F_i$. So recursively define a sequence $\{F_n \mid n < \omega\}$ of finite, pairwise disjoint subsets of κ such that if $m < n < \omega$, then

$$\bigcup_{\alpha\in F_n} b_\alpha \subseteq \bigcup_{\alpha\in F_m} b_\alpha.$$

Extending the centered family

$$\left\{ \bigcup_{\alpha\in F_n} b_\alpha \mid n < \omega \right\},$$

to an ultrafilter, it follows easily that we can choose for each $n < \omega$ some $\alpha_n \in F_n$ such that $\left\{b_{\alpha_n} \mid n < \omega\right\}$ is a centered family.

Extend the set $\{\alpha_n \mid n < \omega\}$ to D, a maximal subset of κ such that $\{b_\alpha \mid \alpha \in D\}$ is a centered family. Since for all $\alpha < \beta$ elements of D, c_α and c_β are disjoint, it follows that $\{a_\alpha \mid \alpha \in D\}$ is an infinite set of disjoint elements of A.

Since $\kappa < \mathfrak{a}(A)$, there exists $a \in A^+$ witnessing that $\{a_\alpha \mid \alpha \in D\}$ is not a partition. Also, $\kappa < \mathfrak{p}(B)$, which means that there exists $b \in B^+$, such that $b \subseteq b_\alpha$, for all $\alpha \in D$. Hence $a \times b$ witnesses that P is not a partition of $A \oplus B$.

Case 2. There exists $E \in [\kappa]^{\geq\omega}$ such that for all $F \in [E]^{<\omega}$ there exists $\alpha \in E$ such that $b_\alpha \nsubseteq \bigcup_{\beta\in F} b_\beta$.

Suppose that E is a maximal subset of κ with this property. It follows that $\{b_\alpha \mid \alpha \in \kappa\backslash E\}$ is a centered family in B. To prove it take $\alpha_0, ..., \alpha_n \in \kappa \setminus E$. Because of the maximality of E, if $i \leq n$, there exists F_i, a finite subset of E, such that

$$\bigcup_{\alpha\in E} b_\alpha \subseteq \bigcup_{\beta\in F_i} b_\beta \cup b_{\alpha_i}.$$

If $F := \bigcup_{i\leq n} F_i$, it follows that for all $i \leq n$

$$\bigcup_{\alpha\in E} b_\alpha \subseteq \bigcup_{\beta\in F} b_\beta \cup b_{\alpha_i}.$$

Also, by hypothesis, we know that there exists $\alpha' \in E$ such that $b_{\alpha'} \nsubseteq \bigcup_{\beta\in F} b_\beta$. Hence $\emptyset \neq b_{\alpha'}\backslash \bigcup_{\beta\in F} b_\beta \subseteq b_{\alpha_i}$ for all $i \leq n$. It follows that $\{b_\alpha \mid \alpha \in \kappa\backslash E\}$ is a centered family. If $\kappa \setminus E$ is infinite, as in the previous case, it can be proved that P is not a partition of $A \oplus B$.

Suppose that $|\kappa \setminus E| < \omega$. By hypothesis we know that for all $F \in [E]^{<\omega}$, $Y \neq \bigcup_{\beta\in F} b_\beta$. Since $\kappa < \mathfrak{p}(B)$, it follows that there is $b \in B^+$ such that $b \cap b_\alpha = \emptyset$ for all $\alpha \in E$. Clearly if $\bigcup_{\alpha\in\kappa\backslash E} a_\alpha \neq X$, then the set $(X\backslash \bigcup_{\alpha\in\kappa\backslash E} a_\alpha) \times b$ would witness that P is not a partition of $A \oplus B$. Suppose that this is not the case, so $\bigcup_{\alpha\in\kappa\backslash E} a_\alpha = X$. Since $\kappa \setminus E$ is finite, we may suppose that it is equal to $\{\alpha_i \mid i \leq n\}$ for some $n < \omega$. As it was observed when proving that $\{b_\alpha \mid \alpha \in \kappa\backslash E\}$ is a centered family, there exist $\alpha' \in E$ and $F \in [E]^{<\omega}$ such that $\emptyset \neq b_{\alpha'}\backslash \bigcup_{\beta\in F} b_\beta \subseteq b_{\alpha_i}$ for all $i \leq n$. In particular $b_{\alpha'} \cap b_{\alpha_i} \neq \emptyset$ for all $i \leq n$. Also $a_{\alpha'} \cap a_{\alpha_j} \neq \emptyset$ for some $j \leq n$, which would imply that $c_{\alpha'} \cap c_{\alpha_j} \neq \emptyset$. This is a contradiction, since P is supposed to be a disjoint family. So $\bigcup_{\alpha\in\kappa\backslash E} a_\alpha \neq X$ and we conclude that P is not a partition.

All cases considered, it follows that $\kappa < \mathfrak{a}(A \oplus B)$. □

The complicated statement of the previous theorem can be translated as follows: If there exists a disjoint family of $A \oplus B$ which disproves the equality for $\mathfrak{a}$ in Theorem 9, its size must be at least as big as either $\mathfrak{p}(A)$ or $\mathfrak{p}(B)$.

We conclude this section pointing at a possibility for answering Problem 8. Simplifying to the case where $A = B$, it follows from Theorem 13 that if we wanted to get an example of $\mathfrak{a}(A \oplus A) < \mathfrak{a}(A)$, it would be necessary that $\mathfrak{p}(A) < \mathfrak{a}(A)$. Now take the better known case where $A = P(\omega)/Fin$. It is known to be consistent that $\mathfrak{p} < \mathfrak{a}$ (see [1], [3]). Hinting at a possible negative answer to Problem 8, the following question arises:

Question 1 *Is it consistent that* $\mathfrak{p} = \mathfrak{a}(P(\omega)/Fin \oplus P(\omega)/Fin) < \mathfrak{a}$*?*

4 Other questions

In this last section, as a kind of a appendix, we will briefly deal with a couple of questions that, though related to the cardinal invariants defined in the first section, remain somehow detached from those dealt with in sections 2 and 3. The first asks about the possibility of an alternative definition of $\mathfrak{p}$. That possible definition is related to the so called ramification sets.

Definition 4 Let A be a Boolean algebra. $X \subset A$ is said to be a *ramification* set if for all $a, b \in X$ either $a \leq b$, $b \leq a$, or $a \cdot b = 0$.

Problem 48 asks: Is

$$\mathfrak{p}\,(A) = \min\{|X| \mid X\ is\ maximal\ ramification\ set\ of\ A\}?$$

Here this question is answered negatively. The algebra which proves this is an interval algebra.

Definition 5 Let L be a linearly ordered set. The *interval* algebra of L, denoted $Intalg\,(L)$, is the set algebra on L generated by the intervals of the form $[a, b)$, where $a \in L \cup \{-\infty\}$ and $b \in L \cup \{\infty\}$.

It is easy to verify that for all linearly ordered set L, its interval algebra is the set of all $\bigcup_{i<n}[a_i, b_i)$, where $n < \omega$, $a_i \in L \cup \{-\infty\}$, $b_i \in L \cup \{\infty\}$, $a_i < b_i$ and $b_i < a_j$ for all $i < j < n$.

Take $A := Intalg\,(\omega_1)$. The set

$$\{[n, n+1) \mid n < \omega\} \cup \{[\omega, \infty)\}$$

witnesses that $\mathfrak{a}\,(A)$, and hence that $\mathfrak{p}\,(A)$, is equal to ω.

Now suppose that $R := \{c_n \mid n < \omega\}$ is ramification set of A. Each c_n is of the form $\bigcup_{i \leq m_n}[\alpha_i^n, \beta_i^n)$. Suppose that $\beta_{m_n}^n < \infty$ for all $n < \omega$. Take

$$\beta := \sup\{\beta_{m_n}^n \mid n < \omega\}.$$

Clearly the interval $[\beta, \infty)$, being disjoint to every element of R, witnesses that R is not a maximal ramification set.

On the other hand, suppose that there exists $n_0 < \omega$ such that $\beta_{m_{n_0}}^{n_0} = \infty$. Then take

$$\alpha := \sup(\{\alpha_{m_n}^n \mid n < \omega\} \cup \{\beta_{m_n}^n \mid n < \omega \wedge \beta_{m_n}^n < \infty\}).$$

Take $n < \omega$. If $\beta_{m_n}^n = \infty$, then $[\alpha, \infty)$ is subset of $[\alpha_{m_n}^n, \beta_{m_n}^n) \subseteq c_n$. If $\beta_{m_n}^n < \infty$, then $[\alpha_{m_n}^n, \beta_{m_n}^n) < [\alpha, \infty)$, and hence $[\alpha, \infty)$ is disjoint to c_n. Therefore $[\alpha, \infty)$ witnesses that R is not a maximal ramification set. We conclude that the minimum size of a maximal ramification set defines a cardinal function other than $\mathfrak{p}$.

The following question introduces a kind of extension other than those of the Sect. 2.

Definition 6 Let A and B two Boolean algebras such that $A \leq B$. It is said that A is σ-embedded in B, $A \leq_\sigma B$, if for all $b \in B$ the ideal $A \restriction b$ is σ-generated.

Recall that an ideal I of some Boolean algebra A is σ-generated if there exists $X \in [I]^\omega$ such that for all $a \in I$ there exists $x \in X$ such that $a \leq x$. The following theorem answers negatively to Problem 70: Are there BAs A, B such that $A \leq_\sigma B$ and $\pi(A) > \pi(B)$?

Theorem 14 *Let A and B two infinite Boolean algebras. If $A \leq_\sigma B$, then $\pi(A) \leq \pi(B)$.*

Proof A family $X \subseteq B \setminus \{1\}$ will be said to be *cofinal* in B if for all $b \in B \setminus \{1\}$ there exists $x \in X$ such that $b \leq x$. It is clear that $X \subseteq B \setminus \{1\}$ is cofinal in B iff $\{-x \mid x \in B\}$ is dense in B. Therefore there exists X a cofinal family in B of size $\pi(B)$. For $x \in X$ take $C_x \in [A]^\omega$, a set witnessing that $A \restriction x$ is σ-generated.

Take $a \in A \setminus \{1\}$. Since X is cofinal in B, there exists $x \in X$ such that $a \leq x$. It follows that a is an element of $A \restriction x$. Hence there exists $y \in C_x$ such that $a \leq y$. Therefore

$$Y := \bigcup_{x \in X} C_x$$

is cofinal in A. Since $\{-y \mid y \in Y\}$ is a dense set of A of size $\pi(B) \cdot \omega = \pi(B)$, it follows that $\pi(A) \leq \pi(B)$. $\square$

Declarations

Conflict of interest The author declares that they have no conflict of interest.

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