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A topological completeness theorem for transfinite provability logic

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Abstract

We prove a topological completeness theorem for the modal logic GLP containing operators $\{\langle\xi\rangle : \xi \in Ord\}$ intended to capture a wellordered sequence of consistency operators increasing in strength. More specifically, we prove that, given a tall-enough scattered space X, any sentence ϕ consistent with GLP can be satisfied on a polytopological space based on finitely many Icard topologies constructed over X and corresponding to the finitely many modalities that occur in ϕ.

Keywords GLP · Provability logic · Topological model

Mathematics Subject Classification 03F45, 03E10, 54G12

Contents

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1 Introduction

The purpose of this article is to prove a topological completeness theorem for the transfinite extension of Japaridze's logic GLP. GLP is a provability logic in a propositional language augmented with a possibly transfinite sequence of modal operators; our case of interest is that in which the sequence is wellfounded. Each of these operators can be interpreted arithmetically as asserting provability within a given theory, and the logic GLP relates these notions of provability to one another. For arithmetical interpretations of GLP, see Beklemishev [6], Fernández-Duque and Joosten [18], Cordón-Franco et al. [15], and others. As a modal logic, GLP has some unusual properties. For example, it is not complete with respect to any class of relational frames; a natural question is whether it is complete with respect to its neighborhood (i.e., topological) semantics. Beklemishev and Gabelaia [11] showed that GLP_ω, the restriction of GLP to ω-many modalities, is complete with respect to a natural topological space on the ordinal ε_0. Because sentences in the language of GLP contain instances of only finitely many modalities, the spaces constructed by Beklemishev and Gabelaia serve as models also for formulas in the language of transfinite GLP; hence it is also topologically complete. However, it is an open problem whether transfinite GLP is complete with respect to a single Beklemishev-Gabelaia space.[1] Another open problem is that of completeness with respect to what are known as the *canonical topological semantics* for GLP. The question of completeness with respect to these spaces has very interesting connections with stationary reflection and indescribable cardinals (see Bagaria [4], Bagaria-Magidor-Sakai [5] and Brickhill [14]). Completeness for the two-modality fragment with respect to these topologies was proved by Beklemishev [8]. It is not hard to see that GLP is not strongly complete with respect to its canonical topological semantics; we shall prove this below.

Completeness with respect to Beklemishev-Gabelaia spaces was proved by Fernández-Duque [16] for restrictions of GLP to any countable amount of modalities. It is not known whether this result can be extended to arbitrarily long sequences of modalities, but the results from [3] show that the techniques would need to be very different.

The topological completeness theorem we shall prove here goes in this direction. Roughly, given a sentence consistent with GLP, we construct a topological model for it with finitely many topologies. The new feature is that these topologies in a way correspond to the modalities appearing in the sentence; we call these $\vec{\vartheta}$-polytopologies, for $\vec{\vartheta}$ a finite sequence of ordinals. In particular, one can extend the space with intermediate increasing topologies corresponding to modalities not appearing in the sentence in such a way that each topology results in a model of the unimodal GL. Unfortunately, this extension (with the intermediate topologies) will not be a model of GLP, but we hope that a similar construction can yield completeness for Beklemishev-Gabelaia spaces with any amount of topologies. This hope is the main motivation for carrying out the work reported in this article.

[1] The anonymous reviewer has pointed out that if $\mathfrak{X} = (Ord, \{\mathcal{T}_\iota : \iota \in Ord\})$ is a Beklemishev-Gabelaia space, then GLP is complete with respect to the space $(Ord, \{\mathcal{T}_{\iota+1} : \iota \in Ord\})$ consisting of the successor topologies of $\mathfrak{X}$. (This can be proved by adapting the previous arguments for completeness.)

Our main tool is a technical "product lemma." Essentially, given two ordinals κ and λ, we find an ordinal Θ and natural projections $\pi_0 : \Theta \to \kappa$ and $\pi_1 : \Theta \to \lambda$ which preserve satisfiability of polymodal formulas, if the ordinals are equipped with the right topologies. This is a generalization of a technical lemma of Beklemishev-Gabelaia [10], which corresponds to the case in which the first element of the sequence $\vec{\vartheta}$ is 1. The proof is largely arithmetical and relies heavily on the theory of hyperexponentials and hyperlogarithms of Fernández-Duque and Joosten [17].

2 Preliminaries

2.1 The polymodal logic of provability

For any ordinal number Λ we consider a language $\mathcal{L}_\Lambda$ consisting of a countable set of propositional variables $\mathbb{P}$ together with the constants $\top$, $\bot$; Boolean connectives $\wedge, \vee, \neg, \to$; and a modality $[\xi]$ for each ordinal $\xi < \Lambda$. As usual, we write $\langle\xi\rangle$ as a shorthand for $\neg[\xi]\neg$.

Definition 2.1 The logic GLP_Λ is then defined to be the least logic containing all propositional tautologies and the following axiom schemata:

(i) $[\xi](\varphi \to \psi) \to ([\xi]\varphi \to [\xi]\psi)$ for all $\xi < \Lambda$,
(ii) $[\xi]([\xi]\varphi \to \varphi) \to [\xi]\varphi$ for all $\xi < \Lambda$,
(iii) $[\xi]\varphi \to [\zeta]\varphi$ for all $\xi < \zeta < \Lambda$,
(iv) $\langle\xi\rangle\varphi \to [\zeta]\langle\xi\rangle\varphi$ for all $\xi < \zeta < \Lambda$,

and closed under the rules modus ponens and necessitation for each $[\xi]$:

$$\frac{\varphi \to \psi \qquad \varphi}{\psi}\ \text{MP} \qquad\qquad \frac{\varphi}{[\xi]\varphi}\ \text{nec}$$

We will often write simply GLP for GLP_Λ when we do not want to specify a Λ. Note that GLP when restricted to any one modality is simply the well-known logic GL. Modal logics are usually studied by means of relational semantics. A *Kripke* Λ*-frame* is a structure $(W, \{R_\xi\})_{\xi<\Lambda}$, where each R_ξ is a binary relation on W. We define a *valuation* $[\![\cdot]\!]$ to be a function assigning subsets of W to each $\mathcal{L}_\Lambda$-formula such that $[\![\cdot]\!]$ respects boolean connectives and such that

$$[\![\langle\xi\rangle\varphi]\!] = R_\xi^{-1}[\![\varphi]\!].$$

Proposition 2.2 (Segerberg [19]) *GLP_1 is complete with respect to the class of finite relational frames (W, R) that are conversely wellfounded trees.*

The preceding proposition provides a convenient way to study GLP_1. However, as is well known, GLP_Λ is incomplete with respect to any class of relational frames whenever $1 < \Lambda$. This motivates the search for topological models of the GLP.

Recall that x is a *limit point* of A if A intersects every punctured neighborhood of x. We call the set of limit points of A the *derived set* of A and denote it by dA. We may also denote it by $d_\tau A$ to emphasize the topology we are considering. The derived set operator is iterated transfinitely by setting

1. $d^0 A = A$,
2. $d^{\alpha+1} A = dd^\alpha A$, and
3. $d^\gamma A = \bigcap_{\alpha<\gamma} d^\alpha A$ for limit ordinals.

Since $d^\alpha X \supset d^\beta X$ whenever $\alpha < \beta$, there exists a minimal ordinal $ht(X)$—the *height* or *rank* of X—such that $d^{ht(X)} X = d^{ht(X)+1} X$. For any $x \in X$, we let $\rho_\tau x$, the *rank* of x, be the least ordinal ξ such that $x \notin d^{\xi+1} X$, if it exists.

Throughout this paper, we will speak about *rank-preserving* extensions of topologies. (A topology σ on a set X is a rank-preserving extension of a topology τ on X if $\tau \subset \sigma$ and $\rho_\sigma x = \rho_\tau x$ for all $x \in X$.)

Lemma 2.3 (Beklemishev-Gabelaia [10]). *A topology σ is a rank-preserving extension of a scattered topology τ if, and only if, $\rho_\tau[U]$ is an ordinal for each $U \in \sigma$.*

A point in A that is not a limit point is *isolated*. Thus a point is isolated if and only if it has rank 0. We denote by $iso(A)$ the set of isolated points in A. A topological space is *scattered* if $iso(A) \neq \varnothing$ for each $A \subset X$ (alternatively, if $d^{ht(X)} X = \varnothing$). Not all scattered spaces are T_1 (e.g., $X = \{0, 1\}$ with open sets $\varnothing$, $\{0\}$ and X), however, the examples in which we will focus are.

We study *polytopological spaces*—structures $(X, \{\mathcal{T}_\iota\}_{\iota<\Lambda})$, where X is a set and $\{\mathcal{T}_\iota\}_{\iota<\Lambda}$ is a sequence of topologies of length Λ. Topological semantics for modal logics may be defined by interpreting diamonds as topological derivatives.

Definition 2.4 (Topological semantics). Let $\mathfrak{X} = (X, \{\mathcal{T}_\iota\}_{\iota<\Lambda})$ be a polytopological space. A *valuation* is a function $[\![\cdot]\!] : \mathcal{L}_\Lambda \to \mathcal{P}(X)$ such that for any $\mathcal{L}_\Lambda$-formulae φ, ψ:

(i) $[\![\bot]\!] = \varnothing$;
(ii) $[\![\neg\varphi]\!] = X \backslash [\![\varphi]\!]$;
(iii) $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$;
(iv) $[\![\langle\xi\rangle\varphi]\!] = d_{\mathcal{T}_\xi} [\![\varphi]\!]$.

A *model* $\mathfrak{M} = (\mathfrak{X}, [\![\cdot]\!])$ is a polytopological space together with a valuation. We say that φ is *satisfied* in $\mathfrak{M}$ if $[\![\varphi]\!]$ is nonempty and we say φ is *valid* in a space $\mathfrak{X}$ and write $\mathfrak{X} \models \varphi$ if $[\![\varphi]\!] = X$ for any model based on $\mathfrak{X}$.

Observe that $x \in [\![[\xi]\varphi]\!]$ if, and only if, there is a $\mathcal{T}_\xi$-neighborhood of x all of whose points belong to $[\![\varphi]\!]$, except possibly for x. In order that a space validate the axioms of GLP, we need to impose some regularity conditions (see Beklemishev-Bezhanishvili-Icard [9]). A space $(X, \{\mathcal{T}_\iota\}_{\iota<\Lambda})$ is a GLP_Λ*-space* if $\{\mathcal{T}_\iota\}_{\iota<\Lambda}$ is non-decreasing, scattered, and

$$d_\xi A \in \mathcal{T}_\zeta \text{ for all } \xi < \zeta \text{ and all } A \in \mathcal{P}(X) \tag{2.1}$$

In the situation above, we refer to $\{\mathcal{T}_\iota\}_{\iota<\Lambda}$ as a GLP_Λ*-polytopology*. Clearly, we have:

Lemma 2.5 *Any GLP_Λ-space validates all theorems of GLP_Λ.*

A natural way of constructing GLP_Λ-polytopologies appears to be to start with any scattered topology and simply add all derived sets at each stage. This results in what has come to be known as the *canonical GLP-space generated by X*. In doing so, the topologies quickly become extremely fine. In fact, for the most natural examples, their non-discreteness becomes undecidable within ZFC after two or three iterations.

One way out of this, explored in Beklemishev-Gabelaia [10], is to extend the topology at each stage before adding derived sets. Extending the topology reduces the amount of derived sets attainable and makes subsequent topologies coarser. A different approach, introduced in Fernández-Duque [16], is to fix increasing topologies from the beginning and restrict the algebra of possible valuations. We will consider the first approach here.

Let us finish this section with the remark that GLP is not strongly complete with respect to its canonical semantics. By *strong completeness* (with respect to a class of models $\mathcal{X}$), we mean the following assertion: whenever Γ is a set of $\mathcal{L}_\Lambda$-sentences consistent with GLP_Λ, then there is some model $\mathfrak{X} \in \mathcal{X}$ where Γ is satisfied.

Proposition 2.6 *Suppose X is a scattered space in which every G_δ set is open. Then GL is not strongly complete with respect to X.*

Proof This is a generalization of the usual proof that GL is not strongly complete with respect to trees. Let

$$\Gamma = \{\Diamond p_0\} \cup \{\Box(p_i \to \Diamond p_{i+1}) : i < \omega\}.$$

Suppose Γ is satisfied at some x. Then, for each i, there is a neighborhood U_i of x such that every $y \in U_i$ with $x \neq y$, if y satisfies p_i, then y is a limit of points satisfying p_{i+1}. Since $U := \bigcap_{i<\omega} U_i$ is G_δ, it is open. Since x satisfies $\Diamond p_0$ and U is a neighborhood of x, U contains some $x_0 \neq x$ of some rank $\alpha_0 < \rho(x)$ satisfying p_0. Inductively, for each $i < \omega$, there is some $x_i \in U$ satisfying p_i with $x_i \neq x$, and, since $x_i \in U_i$, x_i is a limit of points satisfying p_{i+1}; in particular, there is one such point x_{i+1} in U, say of rank $\alpha_{i+1} < \alpha_i$ (so in particular $x_{i+1} \neq x$). This gives an infinite decreasing sequence of ordinals. □

Recall that if κ is an ordinal of uncountable cofinality, then the intersection of countably many sets which are closed and cofinal in κ is also closed and cofinal in κ. Hence, Proposition 2.6 applies to the closed-unbounded topology from Blass [13]. More generally:

Corollary 2.7 *GL is not strongly complete with respect to topologies on ordinals given by countably complete filters, such as the closed-unbounded topology.*

The spaces we will consider will instead be built around the *Generalized Icard topologies*.

Definition 2.8 (Generalized Icard Topologies). Let (X, τ) be a scattered space of rank Θ. We define a topology $\tau_{\uparrow 1}$ generated by τ and all sets of the form

$$(\alpha, \beta)^\tau := \{x \in X : \alpha < \rho_\tau x < \beta\},$$

for ordinals $\alpha < \beta \leq \Theta + 1$. We iterate this construction by setting

- $\tau_{\uparrow(\iota+1)} = (\tau_{\uparrow\iota})_{\uparrow 1}$, and
- $\tau_{\uparrow\lambda} = \bigcup_{\xi<\lambda} \tau_{\uparrow\xi}$ at limit stages.

These are called the *generalized Icard topologies*.

These topologies were defined differently in [3]. By Lemma 2.20.2.20 below, both definitions coincide. Another equivalent formulation is as follows: $\tau_{\uparrow 1}$ is generated by τ and the family

$$\{d^{\xi+1}X : \xi \in Ord\}. \tag{2.2}$$

2.2 Arithmetic, I

Definition 2.9 We fix some notation related to ordinal arithmetic.

1. Whenever $\alpha < \beta$, we denote by $-\alpha + \beta$ the unique ordinal γ such that $\alpha + \gamma = \beta$.
2. Whenever A is a set of ordinals, we denote by $\alpha + A$ the set $\{\alpha + \beta : \beta \in A\}$. Expressions such as $-\alpha + A$ are defined analogously, if they make sense.
3. For all nonzero ξ, there exist ordinals α and β such that $\xi = \alpha + \omega^\beta$. Such a β is unique. We denote it by $\ell\xi$ and call it the *end-logarithm* of ξ.
4. For all nonzero ξ, there exists a unique ordinal η such that ξ can be written as $\omega^\eta + \gamma$, with $\gamma < \xi$. We denote this ordinal by $L\xi$ and call it the *initial logarithm* of ξ.

The operations ℓ and L should be regarded as functions on (a sufficiently large subset of) Ord. Nonetheless, in its use and in general whenever we deem it convenient, we will omit the symbol '$\circ$' for function composition, as well as perhaps parentheses.

Our completeness proof will rely heavily upon an analysis of generalized Icard topologies and their structure induced by the arithmetical properties of ordinals. Hence, developing a thorough intuition about them will be crucial. A most useful remark in this direction is the fact that they are to arbitrary topological spaces as the usual order topology is to ordinal numbers. Indeed, define the *initial segment topology* $\mathcal{I}_0$ on an ordinal Θ (or on Ord) to be generated by all initial segments $[0, \alpha]$, for $\alpha < \Theta$. Then $(\Theta, \mathcal{I}_0)$ is a scattered space: a rather trivial scattered space—it carries no further information than the usual ordering on Ord. For instance, we have $\rho_{\mathcal{I}_0}\alpha = \alpha$ for all α and $ht(\Theta, \mathcal{I}_0) = \Theta$.

Lemma 2.10 *$\mathcal{I}_1 := \mathcal{I}_{0\uparrow 1}$ is the order topology. We have $\rho_{\mathcal{I}_1}\alpha = \ell\alpha$ for all α, so in particular isolated points are exactly the successor ordinals. Moreover, $ht(\Theta + 1, \mathcal{I}_1) = L(\Theta) + 1$.*

Proof It is not hard to see that $\mathcal{I}_1$ is the order topology, and that the rank function is ℓ is established by a simple induction. Finally, let $\mathbb{H}$ be the class of additively indecomposable ordinals. It follows that

$$ht(\Theta + 1, \mathcal{I}_1) = \sup_{\xi \leq \Theta} (\ell\xi + 1) = \sup_{\xi \leq \Theta \cap \mathbb{H}} (\ell\xi + 1) = \sup_{\xi \leq \Theta \cap \mathbb{H}} (L\xi + 1) = \sup_{\xi \leq \Theta} (L\xi + 1).$$

Since $\alpha \mapsto L\alpha$ is non-decreasing, this last supremum is equal to $L(\Theta)+1$, as claimed. □

In what follows, we write simply $\mathcal{I}_\lambda$ instead of $\mathcal{I}_{0\uparrow\lambda}$. These topologies are important because, as we will see, the completeness theorem can quickly be reduced to the case when the underlying space is an ordinal equipped with a topology of the form $\mathcal{I}_\lambda$.

2.3 d-maps and J-maps

There is an appropriate notion of structure-preserving mappings between scattered spaces. We say that a function between topological spaces is *pointwise discrete* if the preimage of any singleton is a discrete subspace.

Definition 2.11 (d-map) Let X and Y be scattered spaces. A function $f : X \to Y$ is a *d-map* if it is continuous, open, and pointwise discrete.

Clearly, any homeomorphism is a d-map. In particular, ordinal addition and substraction, i.e., functions of the form

$$(-\xi + \cdot)\colon ([\xi, \xi + \Theta], \mathcal{I}_\zeta) \to ([0, \Theta], \mathcal{I}_\zeta),$$

are d-maps. The rank function

$$\rho_\tau\colon (X, \tau) \to ([0, \rho_\tau X], \mathcal{I}_0)$$

is also a d-map. A more interesting example is given by end-logarithms of the form:

$$\ell\colon (\Theta, \mathcal{I}_{1+\zeta}) \to (\Theta, \mathcal{I}_\zeta).$$

A proof of this, and the more general Lemma 2.21 below can be found in Fernández-Duque [16].

Since the composition of d-maps is a d-map, they can be thought of as morphisms in the category of scattered spaces. We will now state various properties of d-maps.

Lemma 2.12 *Let $f : X \to Y$ be a d-map.*

1. *If Y is an ordinal Θ with the initial segment topology, then f is the rank function on X.*
2. *For any $A \subset Y$, $f^{-1}dA = df^{-1}A$.*
3. *$f : (X, \tau_{\uparrow\lambda}) \to (Y, \sigma_{\uparrow\lambda})$ is a d-map for any λ.*
4. *If f is surjective, then for any $\mathcal{L}_1$-formula φ, $X \models \varphi$ implies $Y \models \varphi$.*

Proof Items 1 and 2 appear in Beklemishev-Gabelaia [10]; item 4 appears in Bezhanishvili-Mines-Morandi [12] in the current formulation. Item 3 is proved in [3], but therein a different definition of $\tau_{\uparrow\lambda}$ is used, and we still have not shown that they are equivalent. Nonetheless, the claim can be proved by an easy induction.

Suppose $f : (X, \tau_{\uparrow\lambda}) \to (Y, \sigma_{\uparrow\lambda})$ is a d-map. Note that 2 implies that d-maps are rank-preserving, i.e.,

$$\rho_{\tau_{\uparrow\lambda}}(x) = \rho_{\sigma_{\uparrow\lambda}}(f(x)) \text{ for every } x \in X.$$

It follows that for any $\tau_{\uparrow\lambda}$-open A,

$$f(A \cap (\alpha, \beta)^{\tau_{\uparrow\lambda}}) = f(A) \cap (\alpha, \beta)^{\sigma_{\uparrow\lambda}} \in \sigma_{\uparrow\lambda+1};$$

and for any $\sigma_{\uparrow\lambda}$-open B,

$$f^{-1}(B \cap (\alpha, \beta)^{\sigma_{\uparrow\lambda}}) = f^{-1}(B) \cap (\alpha, \beta)^{\tau_{\uparrow\lambda}} \in \tau_{\uparrow\lambda+1};$$

so that f is $(\tau_{\uparrow\lambda+1}, \sigma_{\uparrow\lambda+1})$-continuous and open. Clearly it is also pointwise discrete. The case for limit λ follows from Fernández-Duque [16, Lemma 5.8]. □

As mentioned in the proof of 2.12.3, Lemma 2.12.2 implies that d-maps are rank-preserving. Also, it follows from 2.12.3 that if the rank of (X, τ) is Θ, then

$$\rho_\tau : (X, \tau_{\uparrow\lambda}) \to (\Theta, \mathcal{I}_\lambda)$$

is a d-map. The main feature of d-maps is as follows:

Lemma 2.13 *GLP_1 is complete with respect to a scattered space (X, τ) if, and only if, for any finite, converse-wellfounded tree T, there exists a τ-open subspace S of X and a surjective d-map $f : (S, \tau) \to T$.*

Proof That completeness follows from the existence of d-maps is independently due to Abashidze [1] and Blass [13]. Note that it immediately follows from Proposition 2.2 and Lemma 2.12.4.

The converse is probably folklore and will not be needed below, but we prove it anyway. Suppose GLP_1 is complete with respect to (X, τ), where τ is scattered. Let $(T, <)$ be a finite, converse wellfounded tree. We define from T a modal formula φ consistent with GLP_1. Let $\{p_t : t \in T\}$ be a set of distinct propositional variables and r be the root of T. Set

$$\begin{aligned}\varphi = p_r \wedge \bigwedge_{s\in T;\, s\neq r} \neg p_s \wedge \left(\bigwedge_{t\in T; t\neq r} \Diamond p_t\right) \wedge \Box\left(\bigvee_{t\in T} p_t\right) \wedge \Box(\neg p_r) \\ \wedge \left(\bigwedge_{s,t\in T;\, s\neq t} \Box(p_s \to \neg p_t)\right) \wedge \left(\bigwedge_{s<t} \Box(p_s \to \Diamond p_t)\right) \\ \wedge \left(\bigwedge_{s\not<t} \Box(p_s \to \neg\Diamond p_t)\right) \wedge \Box \bigwedge_{t\in T} \left(p_t \to \Box \bigvee_{t<s} p_s\right).\end{aligned}$$

Clearly, there is a Kripke model based on T where φ is true in r; namely, any one where each p_t holds only in t. Hence, φ is consistent with GLP_1, whereby it is satisfiable in X. Fix a valuation over X and a point $x_r \in X$ such that $x_r \models \varphi$. Thus, x_r satisfies p_r and $\bigwedge_{s,t \in T;\, s \neq r} \neg p_s$ and x_r is a limit point of points satisfying each of p_t, for $t \neq r$. Moreover, by each of the conjuncts above:

1. there is a punctured neighborhood of x_r where each point satisfies p_t for some $t \in T$;
2. there is a punctured neighborhood of x_r where no point satisfies p_r;
3. for each pair of distinct $s, t \in T$, there is a punctured neighborhood of x_r of points satisfying at most one of p_s and p_t;
4. for each pair of distinct $s, t \in T$ with $s < t$, there is a punctured neighborhood of x_r where all points satisfying p_s are limits of points satisfying p_t;
5. for each pair of distinct $s, t \in T$ with $s \not< t$, there is a punctured neighborhood of x_r where all points satisfying p_s are not limits of points satisfying p_t; and
6. there is a punctured neighborhood of x_r where whenever a point x satisfies p_t, then there is a punctured neighborhood of x where each points satisfies one of p_s, with $t < s$.

Let S be the intersection of all those finitely many open neighborhoods of x_r. Clearly, $x \models p_t$ and $t \neq s$ together imply $x \not\models p_s$. We define $f : S \to T$ by

$$f(x) = t \text{ if, and only if, } x \models p_t.$$

We claim f is a d-map. Let A_t be an open subset of T of the form

$$\{s \in T : t \leq s\},$$

so that

$$f^{-1}(A_t) = \{x \in S : x \models p_s \wedge t \leq s\}.$$

This clearly equals S if $t = r$. Otherwise, for each $x \in S$ with $x \models p_s$ and $t \leq s$, there is an open neighborhood U of x where each point satisfies p_u for some $u > s$. But $t < u$, whence $U \subset f^{-1}(A_t)$. This implies that $f^{-1}(A_t)$ is open, and so f is continuous.

Conversely, suppose $U \subset S$ is open, $x \in U$ is such that $x \models p_t$, and $t < s$. Then x is a limit of points satisfying p_s, so that

$$\{y \in S : y \models p_s\} \cap U \neq \varnothing,$$

whence $s \in f(U)$. Hence, f is open. Finally if $t \in T$, then $f^{-1}(t)$ is discrete, for t is the image of points satisfying p_t and no point in S can satisfy $p_s \wedge \Diamond p_s$ for any s. Therefore, f is a d-map. □

Hence, the need to check whether a given space $\mathfrak{X}$ satisfies a formula is replaced by the definition of a suitable mapping between $\mathfrak{X}$ and some other space which is known to do so. In practice, polymodal analogs of Lemma 2.13 do not even require us to use full d-maps, but rather a weaker form of embeddings, as shown by Beklemishev [7]:

Definition 2.14 (J-frame) A finite polymodal Kripke frame

$$\mathfrak{F} = (W, \{<_n\}_{n<\omega})$$

is called a *J-frame* if each relation is transitive and conversely wellfounded and it satisfies the following two conditions:

(I) For all $x, y \in W$ and all $m < n$: $x <_n y$ implies that for all $z \in W$: $x <_m z$ if, and only if $y <_m z$.
(J) For all $x, y, z \in W$ and all $m < n$: if $x <_m y$ and $y <_n z$, then $x <_m z$.

We call a J-frame a J_n*-frame* if all binary relations past the nth one are empty.

Let $(T, <_0, \ldots, <_N)$ be a frame. Denote by E_n the reflexive, symmetric, and transitive closure of $\bigcup_{n\leq k<\omega} <_k$. The equivalence classes under E_n are called n*-planes*. A natural order is defined on the set of $(n+1)$-planes:

$$\alpha \prec \beta \text{ if, and only if, } x <_n y \text{ for some } x \in \alpha, y \in \beta$$

We say that a J-frame is a *J-tree* if for all n, the $(n+1)$-planes contained in each n-plane form a tree under $<_n$ and if whenever $\alpha < \beta$ for two $(n+1)$-planes α, β, we have $x <_n y$ for all $x \in \alpha$ and $y \in \beta$. This means that each J_n-tree can be thought of as a tree each of whose nodes is itself a J_{n-1}-tree. Below, a node $t \in T$ is a *hereditary* k*-root* if for no $j \geq k$ and no $s \in T$ do we have $s <_j t$. We also write $x \ll_k y$ if $x <_j y$ for some $j \geq k$ and

$$\ll_k(x) = \{x \in T : x \ll_k y\}.$$

Definition 2.15 (J-map) Let $(T, \sigma_0, \ldots, \sigma_n)$ be a J_n-tree and $(X, \tau_0, \ldots, \tau_n)$ be a space with $n+1$ topologies. We say that a function $f : X \to T$ is a J_n*-map* if

(j_1) $f : (X, \tau_n) \to (T, \sigma_n)$ is a d-map;
(j_2) $f : (X, \tau_k) \to (T, \sigma_k)$ is open for each k;
(j_3) $f^{-1}(\ll_k(x)), f^{-1}(\{x\} \cup \ll_k(x)) \in \tau_k$ for each $k < n$ and each hereditary $(k+1)$-root x;
(j_4) $f^{-1}x$ is a τ_k-discrete subspace for each $k < n$ and each hereditary $(k+1)$-root x.

Lemma 2.16 (Beklemishev [7]). *If $f : Y \to Z$ is a J_n-map and $g : X \to Y$ is a d-map, then $f \circ g : X \to Z$ is a J_n-map.*

Lemma 2.17 (Beklemishev-Gabelaia [10]). *For each $\mathcal{L}_{n+1}$-formula φ consistent with GLP, there exists a J_n-tree T such that if $\mathfrak{X}$ is a GLP_{n+1}-space and $f : \mathfrak{X} \to T$ is a surjective J_n-map, then $\mathfrak{X} \not\models \neg\varphi$, i.e., φ is satisfiable in $\mathfrak{X}$.*

We call the tree obtained in Lemma 2.17 the *canonical tree for φ*.

2.4 Arithmetic, II

We will need the definition of hyperlogarithms and hyperexponentials, due to Fernández-Duque and Joosten [17]:

Definition 2.18

1. The *hyperlogarithms* $\{\ell^\xi\}_{\xi \in Ord}$ are the unique family of pointwise maximal initial[2] functions that satisfy:
   (a) $\ell^1 = \ell$, and
   (b) $\ell^{\alpha+\beta} = \ell^\beta \ell^\alpha$.
2. Let the function e be defined by $\xi \mapsto -1 + \omega^\xi$. The *hyperexponentials* $\{e^\zeta\}_{\zeta \in Ord}$ are the unique pointwise minimal family of normal functions that satisfy
   (a) $e^1 = e$, and
   (b) $e^{\alpha+\beta} = e^\alpha e^\beta$ for all α and β.

One can verify by induction that the sequence $\{\ell^\xi \gamma : \xi \in Ord\}$ is non-increasing for any ordinal γ. If we set e^0 to be the identity function and $e^\xi 0 = 0$ for all ξ, then one can also describe hyperexponentials recursively by condition 2.18.2(b), together with the following normality clause:

$$\text{for any } \xi \text{ and any limit } \lambda : e^\xi \lambda = \lim_{\eta \to \lambda} e^\xi \eta; \tag{2.3}$$

and the following fixed-point clause:

$$\text{for any } \xi \text{ and any limit } \lambda : e^\lambda(\xi + 1) = \lim_{\eta \to \lambda} e^\eta (e^\lambda \xi + 1). \tag{2.4}$$

The hyperexponential family refines the Veblen hierarchy. We mention some more properties of hyperlogarithms and—exponentials.

Lemma 2.19 (see Fernández-Duque and Joosten [17] *and Fernández-Duque* [16]).

1. *If ξ and δ are nonzero, then $\ell^\xi(\gamma + \delta) = \ell^\xi \delta$; if $\gamma < \delta$ as well, then $\ell^\xi(-\gamma + \delta) = \ell^\xi \delta$. Moreover, if $1 < \xi$, then $\ell^\xi(\gamma\delta) = \ell^\xi \delta$;*
2. *If $\xi < \zeta$, then $\ell^\xi e^\zeta = e^{-\xi+\zeta}$ and $\ell^\zeta e^\xi = \ell^{-\xi+\zeta}$. Furthermore, if $\alpha < e^\xi \beta$, then $\ell^\xi \alpha < \beta$.*

Proof *(Sketch of 2.19)* That $\ell^\xi(\gamma + \delta) = \ell^\xi \delta$ is proved by induction on ξ using 2.18.2(b). From this follows that if $\gamma < \delta$, then

$$\ell^\xi(\delta) = \ell^\xi(\gamma + (-\gamma + \delta)) = \ell^\xi(-\gamma + \delta).$$

Finally, it is proved by induction that $\ell(\gamma\delta) = L\gamma + \ell\delta$, so that if $1 < \xi$, then

$$\ell^\xi(\gamma\delta) = \ell^{-1+\xi} \ell(\gamma\delta) = \ell^{-1+\xi}(L\gamma + \ell\delta) = \ell^{-1+\xi} \ell\delta = \ell^\xi \delta,$$

[2] That is, sending initial segments of Ord onto initial segments of Ord.

as desired. □

We now give an alternative characterization of topologies $\tau_{\uparrow\lambda}$ and their rank functions:

Lemma 2.20 *Let (X, τ) be a scattered space of rank Θ.*

1. *The topologies $\tau_{\uparrow\lambda}$ are computed as follows:*
   - *$\tau_{\uparrow 0}$ is equal to τ*
   - *$\tau_{\uparrow\lambda}$ generated by τ and all sets of the form*

   $$(\alpha, \beta]^\tau_\xi := \{x \in X : \alpha < \ell^\xi \rho_\tau x \leq \beta\},$$

   for some $-1 \leq \alpha < \beta \leq \Theta$ and some $\xi < \lambda$.
2. *If (X, τ) is a scattered space, then $\rho_{\tau_{\uparrow\lambda}} = \ell^\lambda \circ \rho_\tau$. In particular, the rank function of $\mathcal{I}_\lambda$ is ℓ^λ.*

Sets of the form $[\alpha, \beta]^\tau_\xi$, $[\alpha, \beta)^\tau_\xi$, and $(\alpha, \beta)^\tau_\xi$ are defined in the obvious way. In particular, note that $(\alpha, \beta)^\tau_0 = (\alpha, \beta)^\tau$.

Proof The second claim follows from Lemmas 2.12.1 and 2.12.3. We use this to prove the first claim by induction. Suppose $\tau_{\uparrow\lambda}$ is generated by τ and all sets of the form

$$(\alpha, \beta]^\tau_\xi := \{x \in X : \alpha < \ell^\xi \rho_\tau x \leq \beta\},$$

for $\xi < \lambda$. By definition, $\tau_{\uparrow(\lambda+1)} = (\tau_{\uparrow\lambda})_{\uparrow 1}$ is generated by $\tau_{\uparrow\lambda}$ and all sets of the form

$$(\alpha, \beta)^{\tau_{\uparrow\lambda}} := \{x \in X : \alpha < \rho_{\tau_{\uparrow\lambda}} x < \beta\},$$

but $\rho_{\tau_{\uparrow\lambda}} = \ell^\lambda \circ \rho_\tau$ by induction hypothesis. So $(\tau_{\uparrow\lambda})_{\uparrow 1}$ is generated by all sets of the form

$$(\alpha, \beta]^\tau_\xi := \{x \in X : \alpha < \ell^\xi \rho_\tau x \leq \beta\},$$

for $\xi < \lambda + 1$. The limit case is immediate. □

The following lemma provides the key relationship between arithmetic and topology for ordinals:

Lemma 2.21 (Fernández-Duque [16]). *Hyperlogarithms*

$$\ell^\xi : (\Theta, \mathcal{I}_{\xi+\zeta}) \to (\Theta, \mathcal{I}_\zeta)$$

are d-maps.

We will make use of the following two lemmata from [3]:

Lemma 2.22 *Let (X, τ) be a scattered space and λ be an ordinal. Any x in $(X, \tau_{\uparrow\lambda})$ has a λ-neighborhood U such that whenever $x \neq y \in U$, $\ell^{\lambda}\rho_0 y < \ell^{\lambda}\rho_0 x$.*

Lemma 2.23 *Let $1 < \lambda$ be an additively indecomposable ordinal and $x \in X$ be such that $\rho_\tau x = e^{\lambda}\Theta > 0$. Then for any $\tau_{\uparrow\lambda}$-neighborhood V of x, there exist*

- *a set $U \in \tau$, and*
- *ordinals $\eta < e^{\lambda}\Theta$ and $\zeta < \lambda$,*

such that V contains the set $U \cap (\eta, e^{\lambda}\Theta]^{\tau}_{\zeta}$.

For ranks not of the form $e^{\lambda}\Theta$, we have a more general result, also from [3]:

Lemma 2.24 *Let (X, τ) be a scattered space. Suppose $0 < \lambda$, $0 < \ell^{\lambda}\xi$, and $\rho_\tau x = \xi$. Then for any $\tau_{\uparrow\lambda}$-neighborhood V of x, there exist*

- *a set $U \in \tau$, and*
- *a finite partial function $r : \lambda \to Ord$ such that letting*

$$B^X_r(x) = \bigcap_{\zeta \in dom(r)} (r(\zeta), \ell^{\zeta}\xi]^{\tau}_{\zeta},$$

we have $U \cap B^X_r(x) \subset V$.

We conclude this section with a final observation on logarithms:

Lemma 2.25 *Suppose that λ is additively indecomposable, ζ is of the form $e^{\lambda}\zeta_0$, and $\ell^{\lambda}\xi < \zeta_0$. Let*

$$\eta = \min\{\beta : \ell^{\beta}\xi < \zeta\}.$$

Then η is a successor ordinal or zero.

Proof This is proved by induction on ξ. Suppose towards a contradiction that ξ is least such that η is a limit; clearly $\zeta < \xi$. If η is additively decomposable, say

$$\eta = \eta_0 + \omega^{\rho} > \omega^{\rho},$$

then

$$\ell^{\eta}\xi = \ell^{\omega^{\rho}}\ell^{\eta_0}\xi.$$

Now, we must have $\ell^{\eta_0}\xi < \xi$, for otherwise

$$\xi = \ell^{\eta_0}\xi \leq \ell^{\omega^{\rho}}\xi = \ell^{\omega^{\rho}}\ell^{\eta_0}\xi \leq \xi,$$

contradicting the fact $\ell^{\omega^{\rho}}\ell^{\eta_0}\xi < \zeta$; thus, $\ell^{\eta_0}\xi < \xi$. But then, $\zeta \leq \ell^{\eta_0}\xi$ and

$$\ell^{\lambda}\ell^{\eta_0}\xi = \ell^{\eta_0+\lambda}\xi \leq \ell^{\lambda}\xi < \zeta_0,$$

so by the induction hypothesis applied to $\ell^{\eta_0}\xi$, the least η' such that

$$\ell^{\eta'}\ell^{\eta_0}\xi < \zeta$$

is a successor ordinal. However, this ordinal is ω^ρ, which is a contradiction.

Thus η is additively indecomposable. Fernández-Duque and Joosten [17] computed that, letting

$$\eta^* = \arg\min\{\ell^\nu\xi : 0 \leq \nu < \eta\},$$

i.e., letting η^* be the least ordinal ν which minimizes $\ell^\nu\xi$ in $[0, \eta)$, we have

1. if $0 < \eta^*$, then

$$\ell^\eta\xi = \ell^\eta\ell^{\eta^*}\xi;$$

2. if $0 = \eta^*$, then

$$\ell^\eta\xi = \sup_{\beta\in[0,\xi)} \big(\ell^\eta\beta + 1\big).$$

If $0 < \eta^*$, then $\ell^{\eta^*}\xi < \xi$, then one reaches a contradiction as above, using the induction hypothesis on $\ell^{\eta^*}\xi$; thus $0 = \eta^*$. Now, note that $\eta \leq \lambda$, since $\ell^\lambda\xi < \zeta_0 \leq \zeta$. In fact, we must have $\eta < \lambda$, by the displayed equation above. Since λ is additively indecomposable, we have

$$\ell^\eta e^\lambda\zeta_0 = e^\lambda\zeta_0 = \zeta < \xi,$$

and thus

$$\begin{aligned}\ell^\eta\xi &= \sup_{\beta\in[0,\xi)} \big(\ell^\eta\beta + 1\big)\\ &\geq \ell^\eta\zeta + 1\\ &= \ell^\eta e^\lambda\zeta_0 + 1\\ &= e^\lambda\zeta_0 + 1,\\ &= \zeta + 1,\end{aligned}$$

which is again a contradiction. This proves the lemma. □

3 $\vec{\vartheta}$-polytopologies

In this section, we state our completeness theorem and prove it modulo the *product lemma*, which will be proved in the next section. Let us begin with some motivation by recalling the constructions from [10] and [16]. Let $\mathfrak{X} = (X, \tau)$ be a scattered space;

by [3], GL is complete with respect to each topology $\tau_{\uparrow\lambda}$, with $0 < \lambda$, provided $\mathfrak{X}$ is tall enough. Thus, one would attempt to prove completeness of GLP with respect to the polytopology

$$\{\tau_{\uparrow\lambda} : 0 < \lambda < \Lambda\}.$$

However, this is not a GLP-space and thus does not validate the axioms of GLP. The idea is then to replace each topology $\tau_{\uparrow\lambda}$ by a rank-preserving extension and prove completeness for that space. It is not known whether this is possible for arbitrary Λ. What we will do here is, given a formula ϕ consistent with GLP, say, with occurrences of modalities $\lambda_0, \ldots, \lambda_n$, and a (tall enough) scattered space $\mathfrak{X} = (X, \tau)$, we produce a sequence of topologies $\tau_0, \ldots, \tau_n$ such that

1. $(X, \tau_0, \ldots, \tau_n)$ satisfies ϕ, and
2. each τ_i is a rank-preserving extension of $\tau_{\uparrow 1+\lambda_i}$.

Definition 3.1 (ϑ-maximal topology) Let ϑ be a nonzero ordinal and (X, τ) be a scattered topological space. We say that τ_* is a ϑ-*extension* of τ if

1. $\tau \subset \tau_*$,
2. $\rho_{\tau_*} = \rho_\tau$, and
3. the identity function $id : (X, \tau) \to (X, \tau_*)$ is continuous at all points x such that

$$\ell^{\vartheta} \rho_\tau(x) = 0.$$

We say that τ_* is an ϑ-*maximal topology* if there are no proper ϑ-extensions of τ_*.

In particular, when $\vartheta = 1$, ϑ-maximality coincides with the notion of ℓ-maximality from [10]. If $\vec{\vartheta} = \{\vartheta_i : 0 < i < n\}$ is a finite increasing sequence of non-zero ordinals, we write

$$\partial\vec{\vartheta} := \{\partial\vartheta_{i+1} : 0 < i < n\},$$

where $\partial\vartheta_{i+1} = -\vartheta_i + \vartheta_{i+1}$ for $0 < i$. For such a sequence $\vec{\vartheta}$, we also write $\partial\vartheta_1 = \vartheta_1$.

Definition 3.2 Let us call a polytopological space $\mathfrak{X} = (X, \tau_0, \ldots, \tau_n)$ a $\vec{\vartheta}$-*polytopology* over (X, τ) if $\vec{\vartheta} = \{\vartheta_i : 0 < i \leq n\}$ is an increasing sequence of non-zero ordinals and

1. τ_0 is a ϑ_1-maximal extension of τ,
2. τ_{i+1} is a $\partial\vartheta_{i+2}$-maximal extension of $(\tau_i)_{\uparrow\partial\vartheta_{i+1}}$, for $i + 1 < n$, and
3. $\tau_n = (\tau_{n-1})_{\uparrow\partial\vartheta_n}$.

We remark the following consequence of the definition.

Lemma 3.3 *Let* $\mathfrak{X} = (X, \tau_0, \ldots, \tau_n)$ *be a* $\vec{\vartheta}$*-polytopology over* (X, τ). *Then, for each* i *with* $1 \leq i \leq n$,

$$\rho_{\tau_i} = \rho_{\tau_{\uparrow\vartheta_i}} = \ell^{\vartheta_i} \circ \rho_\tau.$$

Let us refer to the polytopologies considered in [10] and [16] as *BG-polytopologies*. We will not need that notion below, so we do not define them. $\vec{\vartheta}$-polytopologies are weak versions of BG-polytopologies. For example, suppose $\mathfrak{X}$ is a $\{\omega_1\}$-polytopology over the interval topology on an ordinal. Then τ_1 is a rank-preserving extension of $\mathcal{I}_{\omega_1}$ obtained just by adding sets that would be already included in the corresponding rank-preserving extension of $\mathcal{I}_{\omega_1}$ in any corresponding BG-space of length $\geq \omega_1$ over $\mathcal{I}_1$. We now state the completeness theorem we shall prove:

Theorem 3.4 *Let $\vec{\vartheta}$ be an increasing sequence of nonzero ordinals. Denote by $GLP \upharpoonright \vec{\vartheta}$ the fragment of GLP whose only modalities appear in $\vec{\vartheta}$.*

1. *(Soundness) All theorems of $GLP \upharpoonright \vec{\vartheta}$ hold in every $\vec{\vartheta}$-space.*
2. *(Completeness) Let $\Lambda \geq \vartheta_n$ be a limit ordinal and (X, τ) be any scattered space of height $\geq e^{\Lambda}1$. Suppose φ only contains modalities in $\vec{\vartheta}$ and is consistent with $GLP \upharpoonright \vec{\vartheta}$. Then, there is an open subset U of X such that φ is satisfied on a $\vec{\vartheta}$-polytopology over $(U, \tau_{\uparrow 1})$.*

The result also holds also for successor ordinals by replacing $e^{\Lambda}1$ with $e^{1+\Lambda}\omega$. In fact, this general version is what we will prove; the smaller bound in the statement of the theorem follows from the fact that, for limit Λ,

$$\lim_{\lambda\to\Lambda} e^{\lambda}\omega = \lim_{\lambda\to\Lambda} e^{\lambda+1}1 = \lim_{\lambda\to\Lambda} e^{\lambda}1 \leq \lim_{\lambda\to\Lambda} e^{\Lambda}1 = e^{\Lambda}1$$

These bounds are sharp (this follows from Lemma 2.13).

Notice that in Theorem 3.4.3.4, we satisfy the consistent formula on a $\vec{\vartheta}$-polytopology over a subspace $(X, \tau_{\uparrow 1})$. We cannot in general replace this with $(X, \tau_{\uparrow 0})$—consider an ordinal with the initial-segment topology. It is still useful to consider polytopologies of this sort. We will call $\vec{\vartheta}$-polytopologies over the initial-segment topology *improper*.

In the remainder of this article, we prove Theorem 3.4. Soundness follows from Lemma 3.6 below, which in turn follows from Lemma 3.5. The proofs are the same as in the case $\vartheta = 1$ from [10]. They can also be found in [2].

Lemma 3.5 *(X, τ) is a ϑ-maximal space if, and only if, for all $x \in X$ whose rank ρ is such that $\ell^{\vartheta}\rho > 0$ and all $V \in \tau$ with $V \subset [0, \rho)^{\tau}_0$, one of the following holds:*

1. *$V \cup \{x\} \in \tau$, or*
2. *$\rho_\tau(U \cap V) < \rho$ for some τ-neighborhood U of x.*

Lemma 3.6 *Suppose (X, τ) is ϑ-maximal and $\lambda \geq \vartheta$. Then $\{dA : A \subset X\} \subset \tau_{\uparrow\lambda}$.*

It follows from Lemma 3.6 that all $\vec{\vartheta}$-polytopologies are GLP_n-spaces, which implies soundness.

Lemma 3.7 *(pullback) Suppose $\mathfrak{Y} = (Y, \mathcal{S}_0, \ldots, \mathcal{S}_n)$ is a (possibly improper) $\vec{\vartheta}$-polytopology over (Y, σ) and $f : (X, \tau) \to (Y, \sigma)$ is a d-map. Then, there exists a $\vec{\vartheta}$-polytopology $\mathfrak{X} = (X, \mathcal{T}_0, \ldots, \mathcal{T}_n)$ over (X, τ) such that*

$$f : (X, \mathcal{T}_i) \to (Y, \mathcal{S}_i) \tag{3.1}$$

is a d-map for each $i \le n$.

Proof This is essentially the same proof as for the case $\vartheta = 1$ (see [10, Lemma 8.5]). The key point is the following claim:

Claim 1 *Suppose $\mathcal{T}$ and $\mathcal{S}$ are topologies such that $f\colon (X, \mathcal{T}) \to (Y, \mathcal{S})$ is a d-map and $\mathcal{S}'$ is a ϑ-maximal extension of $\mathcal{S}$. Then, the topology generated by $\mathcal{T}$ and the family*

$$f^{-1}\mathcal{S}' := \{f^{-1}S\colon S \in \mathcal{S}'\}$$

is a ϑ-extension of $\mathcal{T}$. Moreover, $f\colon (X, \mathcal{T}') \to (Y, \mathcal{S}')$ is a d-map for any ϑ-extension $\mathcal{T}'$ of this topology.

To see this suffices, suppose the claim holds. Then, letting $\mathcal{T}_0$ be any ϑ_1-maximal extension of the topology given by $\mathcal{T}$ and $f^{-1}\mathcal{S}_0$, we obtain that (3.1) holds for $i = 0$. Inductively, suppose (3.1) holds for some i. By Lemma 2.12.3,

$$f\colon (X, \mathcal{T}_{i\uparrow\partial\vartheta_{i+1}}) \to (Y, \mathcal{S}_{i\uparrow\partial\vartheta_{i+1}})$$

is a d-map. If $i + 1 = n$, then we are done; otherwise, by definition, $\mathcal{S}_{i+1}$ is a $\partial\vartheta_{i+2}$-maximal extension of $\mathcal{S}_{i\uparrow\partial\vartheta_{i+1}}$, whereby the claim yields that (3.1) holds for $i + 1$ if we set $\mathcal{T}_{i+1}$ to be some $\partial\vartheta_{i+2}$-maximal extension of the topology given by $\mathcal{T}_{i\uparrow\partial\vartheta_{i+1}}$ and $f^{-1}\mathcal{S}_{i+1}$. Hence, it suffices to prove the claim.

Proof of the claim Let $\mathcal{R}$ be the topology given by $\mathcal{T}$ and $f^{-1}\mathcal{S}'$. Using the fact that

$$f\colon (X, \mathcal{T}) \to (Y, \mathcal{S}) \text{ is a d-map,} \tag{3.2}$$

it is not hard to see that $f\colon (X, \mathcal{R}) \to (Y, \mathcal{S}')$ is also a d-map. By definition, $\mathcal{T} \subset \mathcal{R}$, whence $\mathcal{R}$ is a rank-preserving extension of $\mathcal{T}$. Let $x \in X$ be such that $\ell^\vartheta \rho_\mathcal{T} x = 0$. We need to show that $id\colon (X, \mathcal{T}) \to (X, \mathcal{R})$ is continuous at x. This follows from the fact that $\mathcal{S}'$ is a ϑ-extension of $\mathcal{S}$: for any $\mathcal{R}$-neighborhood of x of the form $f^{-1}V$, we have that V is a $\mathcal{S}'$-neighborhood of $f(x)$ and $\ell^\vartheta \rho_{\mathcal{S}'} f(x) = 0$, so that $f(x) \in U$ for some $U \in \mathcal{S}$ with $U \subset V$. Therefore, $x \in f^{-1}U \subset f^{-1}V$ and $f^{-1}U \in \mathcal{T}$, by (3.2).

Now let $\mathcal{T}'$ be any ϑ-extension of $\mathcal{R}$. Clearly, $f\colon (X, \mathcal{T}') \to (Y, \mathcal{S}')$ is continuous and pointwise discrete. Suppose towards a contradiction that $x \in X$ and $W \in \mathcal{T}'$ witness a failure of f being open. Let

$$\rho := \rho_\mathcal{T} x = \rho_{\mathcal{T}'} x = \rho_{\mathcal{S}'} f(x).$$

Note that we must have $0 < \ell^\vartheta \rho$. Without loss of generality, we may assume ρ is the least possible rank of a counterexample and W contains no other point of rank $\ge \rho$, so that $W = W_0 \cup \{x\}$, for some $W_0 \in \mathcal{T}$ with $W_0 \subset [0, \rho)_0^\mathcal{T}$. We will arrive at a contradiction using Lemma 3.5: since f is rank-preserving, we have that $\ell^\vartheta \rho > 0$, $f(W_0) \in \mathcal{S}'$, and $f(W_0) \subset [0, \rho)_0^{\mathcal{S}'}$. Hence, by Lemma 3.5, one of the following holds:

1. $f(W_0) \cup \{f(x)\} \in \mathcal{S}'$, or
2. $\rho_{\mathcal{S}'}(U \cap f(W_0)) < \rho$ for some $\mathcal{S}'$-neighborhood U of $f(x)$.

It must be the second one that holds, for $f(W_0) \cup \{f(x)\} = f(W)$ is not $\mathcal{S}'$-open by hypothesis. Observe that $f^{-1}U \cap W$ is a $\mathcal{T}'$-neighborhood of x and thus contains points with rank of every ordinal up to, and including, ρ. Because W contains only one point of rank ρ and f is rank-preserving,

$$(f^{-1}U \cap W) \setminus f^{-1}(x) = f^{-1}U \cap W_0.$$

It follows that the set

$$f^{-1}U \cap W_0$$

contains points with rank of every ordinal up to, but not including, ρ. However, this is impossible by 3 above, because ρ is a limit ordinal. This finishes the proof of the claim and the lemma. □

Lemma 3.7 is still true in the degenerate case $\sigma = \mathcal{I}_0$. In this case, notice that $\mathcal{I}_0$ is already ϑ-maximal for every ϑ, for there is only one point of each rank. The proof of Theorem 3.8 below is postponed to the next section. It is stated in a way that directly gives all information we will need when applying it below.

Theorem 3.8 *(Product Lemma) Assume ς is a nonzero additively indecomposable ordinal, $([0,\kappa], \mathcal{T}_0, \ldots, \mathcal{T}_n)$ is a $\vec{\vartheta}$-polytopology over $\mathcal{I}_\varsigma$, and $([0,\lambda], \mathcal{S}_0, \ldots, \mathcal{S}_n)$ is a $\vec{\vartheta}$-polytopology over $\mathcal{I}_0$. Suppose moreover that*

$$\kappa < e^{\varsigma+\vartheta_n}\omega$$

and that

$$\lambda < e^{\vartheta_n}\omega.$$

Let $\xi = \ell^\varsigma[0,\kappa]$, let $\Theta = e^\varsigma(\xi+\lambda) < e^{\varsigma+\vartheta_n}\omega$, and define $X_\uparrow = [0,\Theta] \cap [\xi,\infty)_\varsigma$ and $X_\downarrow = [0,\Theta] \cap [0,\xi)_\varsigma$. Then, there exist:

1. *A $\vec{\vartheta}$-polytopology $([0,\Theta], \mathcal{R}_0, \ldots, \mathcal{R}_n)$ over $\mathcal{I}_\varsigma$.*
2. *Functions $\pi_0\colon [0,\Theta] \to [0,\kappa]$ and $\pi_1\colon [0,\Theta] \to [0,\lambda]$ such that:*
   - *$\pi_0\colon (X_\downarrow, \mathcal{R}_i) \to ([0,\kappa], \mathcal{T}_i)$ is a surjective d-map for each i;*
   - *$\pi_1\colon (X_\uparrow, \mathcal{R}_i) \to ([0,\lambda], \mathcal{S}_i)$ is a surjective d-map for each i;*
   - *$X_\uparrow \subset d_{\mathcal{R}_0}\pi_0^{-1}(\alpha)$ for any $\alpha \leq \kappa$;*
   - *$\pi_1 = -\xi + \ell^\varsigma$;*
   - *the polytopology $(\mathcal{R}_0, \ldots, \mathcal{R}_n)$, when restricted to $X_\uparrow$, is the one obtained from Lemma 3.7 by pulling back via π_1;*
   - *$\pi_1^{-1}(\{\lambda\}) = \{\Theta\}$.*

Theorem 3.8 is the main new ingredient of our proof. With it, we can adapt the usual proofs to obtain completeness. First, we need an embedding lemma:

Lemma 3.9 *Let $(T, <_0, <_1, \ldots, <_n)$ be a finite J_n-tree with root r and $\vec{\vartheta}$ be an increasing n-sequence of nonzero ordinals. Then, for any $\varsigma > 0$, there exist*

- *a $\vec{\vartheta}$-polytopology $([0,\Theta], \mathcal{T}_0, \ldots, \mathcal{T}_n)$ over $([0,\Theta], \mathcal{I}_\varsigma)$ such that $\Theta < e^{\varsigma+\vartheta_n}\omega$; and*
- *a surjective J_n-map $f : ([0,\Theta], \mathcal{T}_0, \ldots, \mathcal{T}_n) \to T$ such that $f^{-1}(r) = \{\Theta\}$.*

Proof The proof is by induction on n. The base case follows from [3, Theorem 6.11], so we assume that the result holds for all $m < n$ and proceed by a subsidiary double induction on

1. ς, which we decompose as $\varsigma_0 + \omega^\rho$; and
2. $hgt_0(T)$, the height of $<_0$,

in that order. Let $hgt_i(T)$ be the height of $<_i$. We need to consider various cases:

Case I: $\omega^\rho < \varsigma$. By the induction hypothesis (for ς) applied to ω^ρ, there are:

- a $\vec{\vartheta}$-polytopology $([0,\Theta_0], \mathcal{S}_0, \ldots, \mathcal{S}_n)$ over $([0,\Theta_0], \mathcal{I}_{\omega^\rho})$ such that $\Theta_0 < e^{\omega^\rho+\vartheta_n}\omega$; and
- a surjective J$_n$-map $g : ([0,\Theta_0], \mathcal{S}_0, \ldots, \mathcal{S}_n) \to T$ such that $g^{-1}(r) = \{\Theta_0\}$.

By Lemma 2.21,

$$\ell^{\varsigma_0} : ([0, e^{\varsigma_0}\Theta_0], \mathcal{I}_\varsigma) \to ([0,\Theta_0], \mathcal{I}_{\omega^\rho})$$

is a d-map, whence by Lemma 3.7, there is a $\vec{\vartheta}$-polytopology

$$\mathfrak{X} = ([0, e^{\varsigma_0}\Theta_0], \mathcal{T}_0, \ldots, \mathcal{T}_n)$$

over $([0, e^{\varsigma_0}\Theta_0], \mathcal{I}_\varsigma)$ such that

$$\ell^{\varsigma_0} : ([0, e^{\varsigma_0}\Theta_0], \mathcal{T}_i) \to ([0,\Theta_0], \mathcal{S}_i)$$

is a d-map for each $i \leq n$. It follows that $f := g \circ \ell^{\varsigma_0}$ is a surjective J$_n$-map from $\mathfrak{X}$ onto T. It remains to check the bound on $\Theta := e^{\varsigma_0}\Theta_0$. Since $\Theta_0 < e^{\omega^\rho+\vartheta_n}\omega$, we have

$$e^{\varsigma_0}\Theta_0 < e^{\varsigma_0}e^{\omega^\rho+\vartheta_n}\omega = e^{\varsigma_0+\omega^\rho+\vartheta_n}\omega = e^{\varsigma+\vartheta_n}\omega,$$

as needed.

Case II: ς is additively indecomposable and $0 = hgt_0(T)$, so that $<_0 = \varnothing$. Let

$$\vec{\vartheta}^* = -\vartheta_1 + \vec{\vartheta} \upharpoonright [2,n] = (-\vartheta_1 + \vartheta_2, -\vartheta_1 + \vartheta_3, \ldots, -\vartheta_1 + \vartheta_n).$$

By induction hypothesis (applied to n), there are:

- a $\vec{\vartheta}^*$-polytopology $([0,\Theta_0], \mathcal{S}_1, \ldots, \mathcal{S}_n)$ over $([0,\Theta_0], \mathcal{I}_{\vartheta_1})$ such that

$$\Theta_0 < e^{\vartheta_1+(-\vartheta_1+\vartheta_n)}\omega = e^{\vartheta_n}\omega;$$

- a surjective J_{n-1}-map $g : ([0, \Theta_0], \mathcal{S}_1, \ldots, \mathcal{S}_n) \to (T, <_1, \ldots, <_n)$ such that $g^{-1}(r) = \{\Theta_0\}$.

Note that each $\mathcal{S}_i$ is a rank-preserving extension of $\mathcal{I}_{\vartheta_i}$. Thus, a simple computation shows that $([0, \Theta_0], \mathcal{I}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n)$ is a(n improper) $\vec{\vartheta}$-polytopology. Clearly,

$$\ell^\varsigma : ([0, e^\varsigma \Theta_0], \mathcal{I}_\varsigma) \to ([0, \Theta_0], \mathcal{I}_0)$$

is a d-map. By Lemma 3.7, there exists a $\vec{\vartheta}$-polytopology $\mathfrak{X}$ of the form $([0, e^\varsigma \Theta_0], \mathcal{T}_0, \ldots, \mathcal{T}_n)$ over $([0, e^\varsigma \Theta_0], \mathcal{I}_\varsigma)$ such that

$$\ell^\varsigma : ([0, e^\varsigma \Theta_0], \mathcal{T}_i) \to ([0, \Theta_0], \mathcal{S}_i)$$

is a d-map for each $i \leq n$. Let $\Theta := e^\varsigma \Theta_0 < e^{\varsigma + \vartheta_n}\omega$. Let $f := g \circ \ell^\varsigma$. We have that

$$f : ([0, \Theta], \mathcal{T}_1, \ldots \mathcal{T}_n) \to (T, <_1, \ldots, <_n) \text{ is a } J_{n-1}\text{-map.} \tag{3.3}$$

We claim (3.3) holds for the full space $\mathfrak{X}$ and the structure $(T, <_0, \ldots, <_n)$, i.e., f is already a J_n-map: condition (j_1) is given by definition; (j_2) is satisfied trivially since the topology induced by $<_0$ is discrete; (j_3) and (j_4) hold for $k \neq 0$ because of (3.3) and for $k = 0$ because r is the sole hereditary 1-root of T.

Case III: ς is additively indecomposable and $0 < hgt_0(T)$. Let $r_1, \ldots, r_m$ be all $<_0$-successors of r that are hereditary 1-roots and (following earlier notation) let $\ll_0(r_i)$ denote the generated subtrees. Also let $\ll_1(r)$ denote the subtree consisting of all nodes that are $<_0$-incomparable with r (i.e., the $<_0$-roots). By induction hypothesis (applied to $hgt_0(T)$), there exist $\vec{\vartheta}$-polytopologies

$$\mathfrak{X}_i = ([0, \kappa_i], \mathcal{T}_0^i, \ldots, \mathcal{T}_n^i)$$

over $([0, \kappa_i], \mathcal{I}_\varsigma)$ and surjective J_n-maps

$$f_i : \mathfrak{X}_i \to (\ll_0(r_i), <_0, \ldots, <_n)$$

for $0 < i$. Let

$$\kappa = \kappa_1 + 1 + \kappa_2 + 1 + \ldots + 1 + \kappa_m$$

and

$$\mathfrak{X} = ([0, \kappa], \mathcal{T}_0, \ldots \mathcal{T}_n)$$

be the topological sum. We also denote by

$$f^* : \mathfrak{X} \to \left(\bigcup_{0 < i \leq n} \ll_0(r_i), <_0, \ldots, <_n \right)$$

the sum of the functions f_i. We may define an improper $\vec{\vartheta}$-polytopology

$$\mathfrak{Y}' = ([0,\lambda], \mathcal{I}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n)$$

over $\mathcal{I}_0$ as in Case II in such a way that there is a J_{n-1}-map

$$g\colon ([0,\lambda], \mathcal{S}_1, \ldots, \mathcal{S}_n) \to (\lll_1(r), <_1, \ldots, <_n)$$

such that, letting $f_0 = g \circ \ell^{\varsigma}$ and

$$\mathfrak{Y} = [0, e^{\varsigma}\lambda]$$

then

$$f_0\colon \mathfrak{Y} \to (\lll_1(r), <_0, \ldots, <_n)$$

is a J_n-map.

Similarly, suppose ξ is any ordinal and let

$$f_\xi(\zeta) = g\Big(-\xi + \ell^{\varsigma}(\zeta)\Big).$$

Since

$$-\xi + \ell^{\varsigma} : \Big(\big[0, e^{\varsigma}(\xi+\lambda)\big] \cap \big[\xi, \infty\big)_{\varsigma}, \mathcal{I}_{\varsigma}\Big) \to \big([0,\lambda], \mathcal{I}_0\big)$$

is a d-map (it is the rank function), by applying Lemma 3.7 we obtain a polytopology $\hat{\mathcal{S}}_0, \ldots, \hat{\mathcal{S}}_n$ on $\big[0, e^{\varsigma}(\xi+\lambda)\big] \cap \big[\xi, \infty\big)_{\varsigma}$ such that

$$f_\xi : \Big(\big[0, e^{\varsigma}(\xi+\lambda)\big] \cap \big[\xi, \infty\big)_{\varsigma}, \hat{\mathcal{S}}_i\Big) \to \big([0,\lambda], \mathcal{S}_i\big)$$

is a d-map for each i. Arguing as in Case II, we see that

$$f_\xi : \Big(\big[0, e^{\varsigma}(\xi+\lambda)\big] \cap \big[\xi, \infty\big)_{\varsigma}, \hat{\mathcal{S}}_0, \ldots, \hat{\mathcal{S}}_n\Big) \to \big(\lll_1(r), <_0, \ldots, <_n\ \big)$$

is also a J_n-map. We now fix

$$\xi = \text{ the unique ordinal equal to } \ell^{\varsigma}[0,\kappa].$$

Let $\Theta = e^{\varsigma}(\xi+\lambda)$ and write $X_{\uparrow} = [0,\Theta] \cap [\xi,\infty)_{\varsigma}$ and $X_{\downarrow} = [0,\Theta] \cap [0,\xi)_{\varsigma}$. By the Product Lemma, there are:

1. A $\vec{\vartheta}$-polytopology $([0,\Theta], \mathcal{R}_0, \ldots, \mathcal{R}_n)$ over $\mathcal{I}_{\varsigma}$.
2. Functions $\pi_0\colon [0,\Theta] \to [0,\kappa]$ and $\pi_1\colon [0,\Theta] \to [0,\lambda]$ such that:

   - $\pi_0\colon (X_{\downarrow}, \mathcal{R}_i) \to ([0,\kappa], \mathcal{T}_i)$ is a surjective d-map for each i;

- $\pi_1 : (X_\uparrow, \mathcal{R}_i) \to ([0, \lambda], \mathcal{S}_i)$ is a surjective d-map for each i;
- $X_\uparrow \subset d_{\mathcal{R}_0}\pi_0^{-1}(\alpha)$ for any $\alpha \leq \kappa$, and in particular when

$$\alpha = \kappa_1 + 1 + \ldots + 1 + \kappa_i,$$

  for $i < m$;
- $\pi_1 = -\xi + \ell^\varsigma$;
- the polytopology $(\mathcal{R}_0, \ldots, \mathcal{R}_n)$, when restricted to $X_\uparrow$, is the one obtained from Lemma 3.7 by pulling back via π_1;
- $\pi_1^{-1}(\{\lambda\}) = \{\Theta\}$.

Thus, when restricted to $X_\uparrow$, the topologies $\mathcal{R}_i$ and $\hat{\mathcal{S}}_i$ coincide (directly by definition), so

$$f_\xi : (X_\uparrow, \mathcal{R}_0, \ldots, \mathcal{R}_n) \to (\ll_1(r), <_0, \ldots, <_n)$$

is a J_n-map. We define a function

$$f : ([0, \Theta], \mathcal{R}_0, \ldots, \mathcal{R}_n) \to (T, <_0, \ldots, <_n)$$

given by:

$$f(x) = \begin{cases} f^*(\pi_0(x)) & \text{if } x \in X_\downarrow \\ f_\xi & \text{if } x \in X_\uparrow \end{cases}$$

Since $X_\uparrow$ and $X_\downarrow$ are $\mathcal{I}_{\varsigma+1}$-clopen and $1 \leq \partial\vartheta_1$, it follows that $X_\uparrow$ and $X_\downarrow$ are $\mathcal{R}_i$-clopen for all $0 < i$. The facts that f_ξ and f^* are J_n-maps and that the projection π_0 is a d-map yield condition (j_1), as well as conditions (j_2)–(j_4) for $0 < i$. We verify the remaining ones:

(j_2) Let U be $\mathcal{R}_0$-open. If $U \subset X_\downarrow$, then $f(U)$ is $<_0$-open, as $\pi_0 \circ f^*$ is a J_n-map. If $U \cap X_\uparrow \neq \varnothing$, then we claim $f(U)$ is $<_0$-open in T. Indeed, since $X_\uparrow \subset d_{\mathcal{R}_0}\pi_0^{-1}(\kappa_i)$ for any $0 < i \leq m$, then there are ordinals $\xi_0, \ldots, \xi_m \in U$ such that $\pi_0(\xi_i) = \kappa_i$ for each i. But then $\pi_0(U \cap X_\downarrow)$ contains a neighborhood U_i of each κ_i and by choice of f^*, $f^*(U_i) = \ll_0(r_i)$.

(j_3) Any hereditary 1-root x is either r or in some T_i. In the former case, $f^{-1}(\ll_0(r))$ and $f^{-1}(\{r\} \cup \ll_0(r)) \in \tau_0$ equal $[1, \Theta)$ and $[1, \Theta]$, respectively. In the latter case, the result follows from the continuity of π_0 and the fact that f^* is a J_n-map.

(j_4) Again, $f^{-1}(\{r\}) = \{\Theta\}$ and for any hereditary 1-root $x \neq r$, $f^{-1}(x) = \pi_0^{-1} f^{*-1}(x)$ is discrete because $f^{*-1}(x)$ is discrete and π_0 is pointwise discrete.

Therefore, f is a indeed a surjective J_n-map.
Since we have considered all cases, the lemma follows. □

We can now finish the proof of Theorem 3.4. Let $\Lambda \geq \vartheta_n$ and (X, τ) be any scattered space of height $\geq e^{1+\Lambda}\omega$. Suppose φ only contains modalities in $\vec{\vartheta}$ and is consistent with $GLP \upharpoonright \vec{\vartheta}$. We need to show that φ is satisfied on a $\vec{\vartheta}$-polytopology over $(X, \tau_{\uparrow 1})$.

We use Lemmata 2.17 and 3.9 to find

1. the canonical tree T for φ with root r,
2. a $\vec{\vartheta}$-polytopology $([0, \Theta], \mathcal{T}_0, \ldots, \mathcal{T}_n)$ over $([0, \Theta], \mathcal{I}_1)$ such that $\Theta < e^{1+\vartheta_n}\omega$, and
3. a surjective J_n-map $f : ([0, \Theta], \mathcal{T}_0, \ldots, \mathcal{T}_n) \to T$ such that $f^{-1}(r) = \{\Theta\}$.

By Lemma 2.17,

$$([0, \Theta], \mathcal{T}_0, \ldots, \mathcal{T}_n) \not\models \neg\varphi,$$

i.e., φ is satisfiable in $([0, \Theta], \mathcal{T}_0, \ldots, \mathcal{T}_n)$. Now, by assumption, (X, τ) is a scattered space such that

$$e^{1+\vartheta_n}\omega \leq e^{1+\Lambda}\omega \leq hgt(X, \tau).$$

Let U be the neighborhood of x consisting of all points of rank $\leq\Theta$, so (U, τ) is a scattered space of height $\Theta + 1$. By Lemma 3.7, there is a $\vec{\vartheta}$-polytopolgy

$$\mathfrak{U} = (U, \mathcal{S}_0, \ldots, \mathcal{S}_n)$$

over $(U, \tau_{\uparrow 1})$ such that

$$\rho_\tau : (U, \mathcal{S}_i) \to ([0, \Theta], \mathcal{T}_i)$$

is a d-map for each $i \leq n$. By Lemmata 2.17 and 2.16, φ is then satisfiable in $\mathfrak{U}$. This completes the proof of Theorem 3.4.

4 Proof of the product lemma

For convenience, we restate the lemma:

Theorem 4.1 *(Product Lemma) Assume ς is a nonzero additively indecomposable ordinal, $([0, \kappa], \mathcal{T}_0, \ldots, \mathcal{T}_n)$ is a $\vec{\vartheta}$-polytopology over $\mathcal{I}_\varsigma$, and $([0, \lambda], \mathcal{S}_0, \ldots, \mathcal{S}_n)$ is a $\vec{\vartheta}$-polytopology over $\mathcal{I}_0$. Suppose moreover that*

$$\kappa < e^{\varsigma+\vartheta_n}\omega$$

and that

$$\lambda < e^{\vartheta_n}\omega.$$

Let $\xi = \ell^\varsigma[0, \kappa]$, let $\Theta = e^\varsigma(\xi + \lambda) < e^{\varsigma+\vartheta_n}\omega$, and define $X_\uparrow = [0, \Theta] \cap [\xi, \infty)_\varsigma$ and $X_\downarrow = [0, \Theta] \cap [0, \xi)_\varsigma$. Then, there exist:

1. *A* $\vec{\vartheta}$*-polytopology* $([0,\Theta],\mathcal{R}_0,\ldots,\mathcal{R}_n)$ *over* $\mathcal{I}_\varsigma$.
2. *Functions* $\pi_0\colon[0,\Theta]\to[0,\kappa]$ *and* $\pi_1\colon[0,\Theta]\to[0,\lambda]$ *such that:*
   - $\pi_0\colon(X_\downarrow,\mathcal{R}_i)\to([0,\kappa],\mathcal{T}_i)$ *is a surjective d-map for each i;*
   - $\pi_1\colon(X_\uparrow,\mathcal{R}_i)\to([0,\lambda],\mathcal{S}_i)$ *is a surjective d-map for each i;*
   - $X_\uparrow\subset d_{\mathcal{R}_0}\pi_0^{-1}(\alpha)$ *for any* $\alpha\le\kappa$*;*
   - $\pi_1=-\xi+\ell^\varsigma$*;*
   - *the polytopology* $(\mathcal{R}_0,\ldots,\mathcal{R}_n)$*, when restricted to* $X_\uparrow$*, is the one obtained from Lemma 3.7 by pulling back via* π_1*;*
   - $\pi_1^{-1}(\{\lambda\})=\{\Theta\}$.

Let ς, κ, and λ be as in the statement. Fix $\vec{\vartheta}$-polytopologies $([0,\kappa],\mathcal{T}_0,\ldots,\mathcal{T}_n)$ and $([0,\lambda],\mathcal{S}_0,\ldots,\mathcal{S}_n)$ over $\mathcal{I}_\varsigma$ and $\mathcal{I}_0$, respectively and let

$$\xi=\ell^\varsigma[0,\kappa].$$

Since logarithms map initial segments of Ord to initial segments of Ord (by definition), ξ is an ordinal. In fact, it is a successor ordinal. To see this, suppose towards a contradiction that it were a limit ordinal. Thus, for each $\zeta<\xi$, we have $\zeta\in\ell^\varsigma[0,\kappa]$, so $e^\varsigma\zeta<\kappa$. By continuity, $e^\varsigma\xi\le\kappa$, so $\xi\in\ell^\varsigma[0,\kappa]$—a contradiction.

Thus ξ is a successor ordinal. Write ζ for its predecessor. We have

$$e^\varsigma\zeta\le\kappa<e^\varsigma\xi$$

for otherwise $e^\varsigma\xi$ belongs to the interval $[0,\kappa]$ and so $\xi\in\ell^\varsigma[0,\kappa]$, contrary to its definition. The case $\varsigma>1$ will require most of our efforts and will be considered first, over the next few sections. The case $\varsigma=1$ will be covered in Sect. 4.4.

4.1 The partition

Definition 4.2 As in the statement of the theorem, we define:

$$\begin{aligned}\Theta=&\text{ least }\alpha\text{ such that }-\xi+\ell^\varsigma\alpha=\lambda\\ &=e^\varsigma(\xi+\lambda)\\ &=e^\varsigma(\ell^\varsigma[0,\kappa]+\lambda).\end{aligned}$$

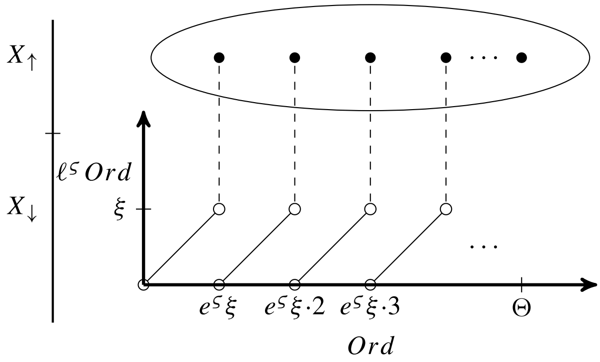
Observe that if $\lambda<e^{\vartheta_n}\omega$ and $\kappa<e^{\varsigma+\vartheta_n}\omega$, then

$$e^\varsigma(\xi+\lambda)<e^{\varsigma+\vartheta_n}\omega,$$

as desired. We also set:

1. $X_\downarrow=[0,\Theta]\cap[0,\xi)_\varsigma$;0
2. $X_\uparrow=[0,\Theta]\cap[\xi,\infty)_\varsigma$.

See the following picture:



The purpose of this section is to partition $X_\downarrow$ into smaller clopen cells. The idea is that the projection π_0 will be defined differently on different cells. We begin by defining a partition of the interval $[0, e^\varsigma \xi)$ into smaller intervals $[\alpha_\iota, \beta_\iota]$ and then we lift it to all of $X_\downarrow$. What α_ι and β_ι are, as well as the set over which ι ranges, will depend on various parameters.

Condition (2.4) states that

$$e^\varsigma(\zeta + 1) = \lim_{\alpha \to \varsigma} e^\alpha(e^\varsigma \zeta + 1).$$

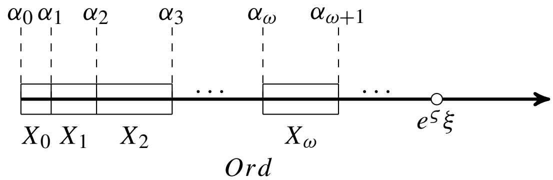
One of the cases we consider is that $e^\varsigma \zeta \leq \kappa < e(e^\varsigma \zeta + 1)$. Otherwise, $e(e^\varsigma \zeta + 1) \leq \kappa < e^\varsigma \xi$, and so

$$e^\alpha(e^\varsigma \zeta + 1) \leq \kappa$$

for some greatest ordinal $0 < \alpha < \varsigma$, which we will denote by ν. Thus,

$$e^\nu(e^\varsigma \zeta + 1) \leq \kappa < e^{\nu+1}(e^\varsigma \zeta + 1).$$

We are not assuming that ς is multiplicatively indecomposable, but it is a limit of ordinals of the form $\nu \cdot \iota$. This is because ς is additively indecomposable, so if $\nu \cdot \iota < \varsigma$, then $\nu \cdot (\iota + 1) < \varsigma$ as well.



4.1.1 When $e(e^\varsigma \zeta + 1) \leq \kappa$

Definition 4.3 For each ι such that $\nu \cdot \iota < \varsigma$, we define:

- $\alpha_0 = 0$;

- $\alpha_{\iota+1} = \beta_\iota + 1$;
- $\alpha_\iota = e^{\nu\cdot\iota}(e^\varsigma\zeta + 1)$, at limit stages; and
- $\beta_\iota = e^{\nu\cdot\iota}\kappa$.

Let $X_\iota = [\alpha_\iota, \beta_\iota]$. Then X_ι is clearly $\mathcal{I}_\varsigma$-clopen.

Lemma 4.4 *Suppose ς is a limit ordinal and $e(e^\varsigma\zeta + 1) \leq \kappa$. Then, the sets $\{X_\gamma : \gamma < \varsigma\}$ in Definition 4.3 form a partition of $e^\varsigma\xi$.*

Proof Recall that ς is additively indecomposable and thus a limit of ordinals of the form $\nu\cdot\iota$. If $\varsigma = \nu\cdot\omega$, then X_γ is only defined for finite γ. We have

$$\begin{aligned}\lim_{n<\omega}\beta_n &= \lim_{n<\omega} e^{\nu\cdot n}\kappa\\ &\geq \lim_{n<\omega} e^{\nu\cdot n} e(e^\varsigma\zeta + 1)\\ &= \lim_{n<\omega} e^{\nu\cdot n}(e^\varsigma\zeta + 1)\\ &= \lim_{\gamma<\varsigma} e^\gamma(e^\varsigma\zeta + 1)\\ &= e^\varsigma\xi.\end{aligned}$$

Since $\beta_n < e^\varsigma\xi$ for all $n < \omega$, we have $\lim_{n<\omega}\beta_n = e^\varsigma\xi$.

Suppose now that $\nu\cdot\omega < \varsigma$. Then, ς is a limit of ordinals of the form $\nu\cdot\iota$, where ι is a limit ordinal. In that case, we have

$$\sup\{\alpha_\iota : \nu\cdot\iota < \varsigma, \iota \in Lim\} = e^\varsigma\xi,$$

so it suffices to show that

$$\lim_{\iota\to\gamma}\alpha_\iota = e^{\nu\cdot\gamma}(e^\varsigma\zeta + 1) \text{ for each limit} \gamma \text{ such that } \nu\cdot\gamma < \varsigma.$$

Let us assume that ς is multiplicatively indecomposable for notational simplicity and show that

$$\lim_{\iota\to\gamma}\alpha_\iota = e^{\nu\cdot\gamma}(e^\varsigma\zeta + 1) \text{ for each limit } \gamma < \varsigma; \tag{4.1}$$

the general case is similar. Let thus $\gamma < \varsigma$ be a limit ordinal and decompose it as $\gamma = \gamma^* + \omega^\rho$, where ρ is nonzero. Recall that the functions e^ι are normal. First, we have:

$$\lim_{\iota\to\gamma}\alpha_\iota = \lim_{\iota\to\gamma}\beta_\iota = \lim_{\iota\to\gamma} e^{\nu\cdot\iota}\kappa.$$

Using the decomposition of γ:

$$\begin{aligned}\lim_{\iota\to\gamma}\alpha_\iota &= \lim_{\iota\to\gamma} e^{\nu\cdot\iota}\kappa\\ &= \lim_{\iota\to\omega^\rho} e^{\nu\cdot(\gamma^*+\iota)}\kappa\\ &= \lim_{\iota\to\omega^\rho} e^{\nu\cdot\gamma^*+\nu\cdot\iota}\kappa\\ &= e^{\nu\cdot\gamma^*}\left(\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}\kappa\right).\end{aligned} \tag{4.2}$$

By choice of ν, we have

$$e^\nu(e^\varsigma\zeta+1)\le\kappa<e^{\nu+1}(e^\varsigma\zeta+1),$$

so normality implies that

$$\begin{aligned}\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}\Big(e^\nu(e^\varsigma\zeta+1)\Big) &\le \lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}\kappa\\ &\le \lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}e^{\nu+1}(e^\varsigma\zeta+1)\\ &= \lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(e^\varsigma\zeta+1)\\ &= \lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}\Big(e^\nu(e^\varsigma\zeta+1)\Big).\end{aligned}$$

This implies that

$$\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}\kappa = \lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(e^\varsigma\zeta+1). \tag{4.3}$$

Now, recall that if ν^* is any limit, then, for every ordinal γ, we have

$$e^{\nu^*}(\gamma+1)=\lim_{\iota\to\nu^*} e^\iota(e^{\nu^*}\gamma+1).$$

Using this (for $\nu^*=\nu\cdot\omega^\rho$ and $\gamma=e^\varsigma\zeta$) and the additive indecomposability of ς, we obtain:

$$\begin{aligned}\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(e^\varsigma\zeta+1) &= \lim_{\iota\to\nu\cdot\omega^\rho} e^\iota(e^\varsigma\zeta+1)\\ &= \lim_{\iota\to\nu\cdot\omega^\rho} e^\iota(e^{\nu\cdot\omega^\rho+\varsigma}\zeta+1)\\ &= \lim_{\iota\to\nu\cdot\omega^\rho} e^\iota(e^{\nu\cdot\omega^\rho}e^\varsigma\zeta+1)\\ &= e^{\nu\cdot\omega^\rho}(e^\varsigma\zeta+1).\end{aligned}$$

Putting this together with Eqs. (4.2) and (4.3),

$$\begin{aligned} e^{\nu\cdot\gamma}(e^{\varsigma}\zeta+1) = e^{\nu\cdot\gamma^*}e^{\nu\cdot\omega^\rho}(e^{\varsigma}\zeta+1) &= e^{\nu\cdot\gamma^*}\left(\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(e^{\varsigma}\zeta+1)\right)\\ &= e^{\nu\cdot\gamma^*}\left(\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}\kappa\right)\\ &= \lim_{\iota\to\gamma}\alpha_\iota, \end{aligned}$$

which proves Eq. (4.1). This finishes the proof of the lemma. □

4.1.2 When $e^{\varsigma}\zeta \leq \kappa < e(e^{\varsigma}\zeta + 1)$

In this case, the partition is simpler than before:

Definition 4.5 For each $\iota < \varsigma$, we define:

- $\alpha_0 = 0$;
- $\alpha_{\iota+1} = \beta_\iota + 1$;
- $\alpha_\iota = e^{\iota}(e^{\varsigma}\zeta + 1)$, at limit stages; and
- $\beta_\iota = e^{\iota+1}(e\kappa + 1)$.

Let $X_\iota = [\alpha_\iota, \beta_\iota]$. As before, X_ι is $\mathcal{I}_\varsigma$-clopen.

Remark 4.6 If $e^{\varsigma}\zeta < \kappa$, then β_ι could also be defined to be $e^{\iota+1}\kappa$, and the argument below will go through. We have chosen to define $\beta_\iota = e^{\iota+1}(e\kappa + 1)$ in order to merge the case $e^{\varsigma}\zeta = \kappa$ with the case $e^{\varsigma}\zeta < \kappa < e(e^{\varsigma}\zeta + 1)$.

Lemma 4.7 *Suppose ς is a limit ordinal and $e^{\varsigma}\zeta \leq \kappa < e(e^{\varsigma}\zeta + 1)$. Then, the sets $\{X_\gamma : \gamma < \varsigma\}$ in Definition 4.5 form a partition of $e^{\varsigma}\xi$.*

Proof If $\varsigma = \omega$, then X_γ is only defined for finite γ. We have

$$\begin{aligned} \lim_{n<\omega}\beta_n &= \lim_{n<\omega} e^n(e\kappa+1)\\ &\geq \lim_{n<\omega} e^n(e^{\varsigma}\zeta+1)\\ &= e^{\varsigma}\xi. \end{aligned}$$

Since $\beta_n < e^{\varsigma}\xi$ for all $n < \omega$, we have $\lim_{n<\omega}\beta_n = e^{\varsigma}\xi$.

Suppose now that $\omega < \varsigma$. Then, ς is a limit of limit ordinals. In that case, we have

$$\sup_{\iota\in\varsigma\cap Lim} \alpha_\iota = e^{\varsigma}\xi,$$

so it suffices to show that

$$\lim_{\iota\to\gamma}\alpha_\iota = e^{\gamma}(e^{\varsigma}\zeta+1) \text{ for each limit } \gamma < \varsigma. \tag{4.4}$$

However, this is immediate from [3, Lemma 2.8]. □

4.2 Projections

Definition 4.8 (Projections) We define the functions π_0 and π_1:

($\downarrow$) $\pi_0 \colon [0, e^{\varsigma}\xi) \to [0, \kappa]$ is defined by:

$$\pi_0(\alpha) = \begin{cases} \ell^{\iota+2}\alpha, & \text{if } e^{\varsigma}\zeta \le \kappa < e(e^{\varsigma}\zeta + 1); \\ \ell^{\nu\cdot\iota}\alpha, & \text{if } e(e^{\varsigma}\zeta + 1) \le \kappa; \end{cases}$$

where ι is the unique ordinal such that $\alpha \in X_\iota$.

($\downarrow$) The function π_0 is extended to all of $X_\downarrow$: given $\alpha \in X_\downarrow$, let η be least such that $\ell^{\eta}\alpha < e^{\varsigma}\xi$. Then, $\pi_0(\alpha) = \pi_0(\ell^{\eta}\alpha)$.

($\uparrow$) $\pi_1 \colon X_\uparrow \to [0, \lambda]$ is defined by

$$\pi_1\alpha = -\xi + \ell^{\varsigma}\alpha.$$

Being the rank function,

$$\pi_1 : (X_\uparrow, \mathcal{I}_\varsigma) \to ([0, \lambda], \mathcal{I}_0)$$

is a surjective d-map. It is not immediately clear whether π_0 is a d-map; this we verify below.

Observe first that, since $X_\downarrow = [0, \Theta]_0 \cap [0, \xi)_\varsigma$ by definition, the ordinal η in the second clause above must be strictly smaller than ς.

Let us look at the definition of π_0 a bit more closely. Consider a typical element α^* of $X_\downarrow$ and generate the sequence

$$\{\ell^{\nu}\alpha^* : \nu \in Ord\}.$$

Let η be least such that $\alpha := \ell^{\eta}\alpha^* < e^{\varsigma}\xi$. Then α belongs to some cell

$$X_\iota = [\alpha_\iota, \beta_\iota].$$

Let us suppose first that $e^{\varsigma}\zeta \le \kappa < e(e^{\varsigma}\zeta + 1)$. Then,

$$\beta_\iota = e^{\iota+1}(e\kappa + 1),$$

so $\pi_0\alpha = \ell^{\iota+2}\alpha \in [0, \kappa]$ and

$$\pi_0[\alpha_\iota, \beta_\iota] = \ell^{\iota+2}[\alpha_\iota, \beta_\iota] = [0, \kappa],$$

with the maximum being attained at the value $e^{\iota+2}\kappa \in [\alpha_\iota, \beta_\iota]$.

Let us suppose now that $e(e^{\varsigma}\zeta + 1) \le \kappa$. Then

$$\beta_\iota = e^{\nu\cdot\iota}\kappa,$$

so $\pi_0\alpha = \ell^{\nu\cdot\iota}\alpha \in [0,\kappa]$ and again

$$\pi_0[\alpha_\iota, \beta_\iota] = \ell^{\nu\cdot\iota}[\alpha_\iota, \beta_\iota] = [0,\kappa],$$

with the maximum being attained at the value β_ι.

Lemma 4.9 $\pi_0 \colon (X_\downarrow, \mathcal{I}_\varsigma) \to ([0,\kappa], \mathcal{I}_\varsigma)$ *is a surjective d-map.*

Proof Surjectivity we verified in the discussion preceding this lemma; we verify that it is a d-map. Consider cells of the form

$$\begin{aligned} X_\iota^\eta &= \big\{x \in X_\downarrow : \ell^{\eta+1}x < e^\varsigma\xi \text{ and } e^\varsigma\xi \leq \ell^\eta x \text{ and } \ell^{\eta+1}x \in X_\iota\big\} \\ &= \big\{x \in X_\downarrow : \ell^{\eta+1}x \leq e^\varsigma\xi \text{ and } e^\varsigma\xi < \ell^\eta x \text{ and } \ell^{\eta+1}x \in X_\iota\big\} \\ &= X_\downarrow \cap [0, e^\varsigma\xi]_{\eta+1} \cap (e^\varsigma\xi, \infty)_\eta \cap [\alpha_\iota, \beta_\iota]_{\eta+1}. \end{aligned}$$

(The second equality follows from the definition of $X_\downarrow$.) Observe that $[\alpha_\iota, \beta_\iota]_{\eta+1}$ is an $\mathcal{I}_\varsigma$-clopen interval if $\eta < \varsigma$, even when α_ι is a limit ordinal; this is because α_ι is always an isolated point in $\mathcal{I}_\varsigma$ by its definition. The definition of π_0 is the same within each X_ι^η and in each of those sets, π_0 is defined as a logarithm and is thus a d-map (recall that ς is additively indecomposable). Additionally, by Lemma 2.25, if $x \in X_\downarrow$, then the least η such that $\ell^\eta x < e^\varsigma\xi$ is a successor ordinal or zero. If we additionally define $X_\iota^{-1} := X_\iota$, then the collection of all X_ι^η forms a clopen partition of $X_\downarrow$. Since π_0 is a d-map on each cell, it is a d-map on all of $X_\downarrow$. □

Lemma 4.10 $\pi_0^{-1}\alpha$ *is* $\mathcal{I}_\varsigma$*-dense in* $X_\uparrow$ *for any* $\alpha \leq \kappa$.

Proof Let $\beta \in X_\uparrow$, so that β has $\mathcal{I}_\varsigma$-rank $\rho := \ell^\varsigma\beta \geq \xi$. It is enough to consider the case $\rho = \xi$, as any $\mathcal{I}_\varsigma$-neighborhood of any point of higher rank contains a point of rank ξ. Let U be an $\mathcal{I}_\varsigma$-neighborhood of β and fix $\alpha \leq \kappa$ – we show U contains some point χ such that $\pi_0\chi = \alpha$. We distinguish three cases:

Case I: $\beta = e^\varsigma\xi$ and $e(e^\varsigma\zeta + 1) \leq \kappa$. We can apply Lemma 2.23 (over the interval topology) to obtain a $\mathcal{I}_\varsigma$-neighborhood base of $e^\varsigma\xi$ consisting of sets of the form

$$(\eta, e^\varsigma\xi]_{\nu\cdot\gamma}$$

for $\eta < e^\varsigma\xi$ and $\nu \cdot \gamma < \varsigma$.

Hence, we may assume U is a neighborhood of $e^\varsigma\xi$ of the form $(\eta, e^\varsigma\xi]_{\nu\cdot\gamma}$. Let μ be an ordinal large enough so that $\eta < \alpha_\mu$. This ordinal certainly exists, as it follows from Lemma 4.4 that:

$$\lim_{\iota\to\varsigma} e^{\nu\cdot\iota}\kappa = \lim_{\iota\to\varsigma} \alpha_\iota = e^\varsigma\xi.$$

Claim 2 *Let* $\chi := \alpha_{\gamma+\mu} + e^{\nu\cdot\gamma}(\alpha_\mu + e^{\nu\cdot\mu}\alpha)$. *Then* $\chi \in U \cap X_{\gamma+\mu}$.

Proof We first verify that $\chi \in [\alpha_{\gamma+\mu}, \beta_{\gamma+\mu}]$. Clearly $\alpha_{\gamma+\mu} \leq \chi$. Moreover, $e^{\nu\cdot\mu}\kappa$ is additively indecomposable, so

$$\alpha_\mu + e^{\nu\cdot\mu}\alpha \leq e^{\nu\cdot\mu}\kappa.$$

$e^{\nu\cdot\gamma}e^{\nu\cdot\mu}\kappa$ is likewise additively indecomposable, and thus

$$\begin{aligned}\chi &= \alpha_{\gamma+\mu} + e^{\nu\cdot\gamma}(\alpha_\mu + e^{\nu\cdot\mu}\alpha)\\ &\leq e^{\nu\cdot\gamma}e^{\nu\cdot\mu}\kappa\\ &= e^{\nu\cdot(\gamma+\mu)}\kappa\\ &= \beta_{\gamma+\mu},\end{aligned}$$

so $\chi \in [\alpha_{\gamma+\mu}, \beta_{\gamma+\mu}]$. We now verify that $\chi \in U$. To see this, it suffices to note that, by choice of μ,

$$\begin{aligned}\eta < \alpha_\mu &< \alpha_\mu + e^{\nu\cdot\mu}\alpha\\ &= \ell^{\nu\cdot\gamma}e^{\nu\cdot\gamma}(\alpha_\mu + e^{\nu\cdot\mu}\alpha)\\ &= \ell^{\nu\cdot\gamma}\Big(\alpha_{\gamma+\mu} + e^{\nu\cdot\gamma}(\alpha_\mu + e^{\nu\cdot\mu}\alpha)\Big)\\ &= \ell^{\nu\cdot\gamma}\chi.\end{aligned}$$

This proves the claim. □

A direct computation shows

$$\begin{aligned}\pi_0\chi = \ell^{\nu\cdot(\gamma+\mu)}\chi &= \ell^{\nu\cdot\mu}\ell^{\nu\cdot\gamma}\Big(\alpha_{\gamma+\mu} + e^{\nu\cdot\gamma}(\alpha_{\gamma+\mu} + e^{\nu\cdot\mu}\alpha)\Big)\\ &= \ell^{\nu\cdot\mu}\ell^{\nu\cdot\gamma}e^{\nu\cdot\gamma}(\alpha_{\gamma+\mu} + e^{\nu\cdot\mu}\alpha)\\ &= \ell^{\nu\cdot\mu}(\alpha_{\gamma+\mu} + e^{\nu\cdot\mu}\alpha)\\ &= \ell^{\nu\cdot\mu}e^{\nu\cdot\mu}\alpha\\ &= \alpha.\end{aligned}$$

Since U was arbitrary, this finishes the proof in this case.

Case II: $\beta = e^{\varsigma}\xi$ and $e^{\varsigma}\zeta \leq \kappa < e(e^{\varsigma}\zeta + 1)$. As before, we may assume U is of the form

$$(\eta, e^{\varsigma}\xi]_\gamma$$

for $\eta < e^{\varsigma}\xi$ and $\gamma < \varsigma$. Let μ be an ordinal large enough so that $\eta < \alpha_\mu$.

Claim 3 *Let* $\chi := \alpha_{\gamma+\mu} + e^{\gamma}(\alpha_\mu + e^{\mu+2}\alpha)$. *Then,* $\chi \in U \cap X_{\gamma+\mu}$.

Proof Clearly $\alpha_{\gamma+\mu} \leq \chi$. Since $\alpha \leq \kappa$ and $e^{\mu+1}(e\kappa+1)$ is additively indecomposable, we have

$$\alpha_\mu + e^{\mu+2}\alpha \leq e^{\mu+1}(e\kappa + 1),$$

so that

$$\alpha_{\gamma+\mu} + e^\gamma(\alpha_\mu + e^{\mu+2}\alpha) \leq e^{\gamma+\mu+1}(e\kappa + 1) = \beta_{\gamma+\mu},$$

and thus $\chi \in X_{\gamma+\mu}$.

As before, we have

$$\begin{aligned}\eta < \alpha_\mu &< \alpha_\mu + e^{\mu+2}\alpha \\ &= \ell^\gamma e^\gamma(\alpha_\mu + e^{\mu+2}\alpha) \\ &= \ell^\gamma\Big(\alpha_{\gamma+\mu} + e^\gamma(\alpha_\mu + e^{\mu+2}\alpha)\Big) \\ &= \ell^\gamma\chi.\end{aligned}$$

This proves the claim. □

A direct computation shows

$$\begin{aligned}\pi_0\chi = \ell^{\gamma+\mu+2}\chi &= \ell^{\mu+2}\ell^\gamma\Big(\alpha_{\gamma+\mu} + e^\gamma(\alpha_\mu + e^{\mu+2}\alpha)\Big) \\ &= \ell^{\mu+2}\ell^\gamma e^\gamma(\alpha_\mu + e^{\mu+2}\alpha) \\ &= \ell^{\mu+2}(\alpha_\mu + e^{\mu+2}\alpha) \\ &= \ell^{\mu+2}e^{\mu+2}\alpha \\ &= \alpha.\end{aligned}$$

Since U was arbitrary, this finishes the proof in this case.

Case III: $e^\varsigma\xi < \beta$. As observed by Fernández-Duque and Joosten [17], there is a least $\eta^* < \varsigma$ such that

$$\ell^{\eta^*}\beta = e^\varsigma\xi.$$

Since ς is additively indecomposable, η^* must be a successor ordinal, say $\eta+1$. Thus,

$$\ell^{\eta+1}\beta = e^\varsigma\xi < \ell^\eta\beta,$$

and so $\ell^\eta\beta$ must be of the form $\beta' + e^\varsigma\xi$. The Hyperexponential Normal Form theorem [3, Proposition 2.13] states that every ordinal γ can be uniquely written in the form $e^{\gamma_0}\gamma_1$, where γ_1 is either additively decomposable or 1. If $\gamma_1 = 1$, then let us call the expression $e^{\gamma_0}1$ the *normal form expansion* of γ. Inductively, the normal form expansion of

$$\gamma = e^{\gamma_0}(\gamma_1 + \gamma_2)$$

is defined to be

$$\mathtt{nf}(\gamma) = e^{\gamma_0}(\gamma_1 + \mathtt{nf}(\gamma_2)).$$

Observe that $\eta < \varsigma$, for otherwise we would have

$$\xi < e^{\varsigma}\xi \leq \ell^{\varsigma}\beta,$$

contradicting the choice of β. Thus, if one writes out the normal form expansion of β, one obtains an expression of the form

$$e^{\beta_0}\Big(\beta_1 + e^{\beta_2}\big(\ldots(\beta' + \mathtt{nf}(e^{\varsigma}\xi))\ldots\big)\Big), \tag{4.5}$$

where $e^{\varsigma}\xi < \beta' + e^{\varsigma}\xi$ and all the exponents to the left of $\beta' + \mathtt{nf}(e^{\varsigma}\xi)$ add up to η. Consider the sequence $\{\beta(\iota) : \iota < e^{\varsigma}\xi\}$, where $\beta(\iota)$ is the ordinal one obtains if one substitutes ι for the rightmost occurrence of $\mathtt{nf}(e^{\varsigma}\xi)$ in the normal form expansion (4.5) of β. If U is a $\mathcal{I}_{\varsigma}$-neighborhood of β, then, by Lemma 2.24, U contains a set of the form

$$B_r(\beta) = \bigcap_{i \in \mathrm{dom}(r)} (r(i), \ell^i\beta]_i,$$

where $r : \varsigma \to \beta + 1$ is a finite partial function. It follows from this and from the continuity of exponentials that every such set U, if nonempty, contains cofinally many ordinals of the form $\beta(\iota)$. Since $e^{\varsigma}\xi < \beta' + e^{\varsigma}\xi$ and $e^{\varsigma}\xi$ is additively indecomposable, we must have $e^{\varsigma}\xi < \beta'$, so it follows that for each $\beta(\iota)$,

$$\pi_0\beta(\iota) = \pi_0\ell(\beta' + \iota) = \pi_0\ell\iota.$$

An argument as in Case I or Case II (according as $\kappa < e(e^{\varsigma}\zeta + 1)$ or $e(e^{\varsigma}\zeta + 1) \leq \kappa$) shows that there is some $\iota < e^{\varsigma}\xi$ such that $\beta(\iota) \in U$ and $\pi_0\beta(\iota) = \alpha$. □

4.3 The polytopology

It remains to define a $\vec{\vartheta}$-polytopology $(\mathcal{R}_0, \ldots, \mathcal{R}_n)$ on $([0, \Theta], \mathcal{I}_{\varsigma})$ such that the projection mappings

$$\pi_0 : (X_{\downarrow}, \mathcal{R}_i) \to ([0, \kappa], \mathcal{T}_i)$$

and

$$\pi_1 : (X_\uparrow, \mathcal{R}_i) \to ([0, \lambda], \mathcal{S}_i)$$

remain d-maps. Recall that n denotes the length of $\vec{\vartheta}$.

To begin, we observe that since

$$\pi_0 : (X_\downarrow, \mathcal{I}_\varsigma) \to ([0, \kappa], \mathcal{I}_\varsigma)$$

is a d-map by Lemma 4.9, we may apply Lemma 3.7 to obtain a $\vec{\vartheta}$-polytopology $(X_\downarrow, \hat{\mathcal{T}}_0, \ldots, \hat{\mathcal{T}}_n)$ over $\mathcal{I}_\varsigma$ such that

$$\pi_0 : (X_\downarrow, \hat{\mathcal{T}}_i) \to ([0, \kappa], \mathcal{T}_i)$$

is a d-map for each $0 \leq i \leq n$.

This $\vec{\vartheta}$-polytopology $(\hat{\mathcal{T}}_0, \ldots, \hat{\mathcal{T}}_n)$ is not, however, a topology on $[0, \Theta]$, so we need to extend it. For each i, let $\hat{\mathcal{R}}_i$ be the smallest topology on $[0, \Theta]$ extending $\mathcal{I}_{\varsigma+\vartheta_i}$ and containing all sets in $\hat{\mathcal{T}}_i$. Since $X_\uparrow$ is $\mathcal{I}_{\varsigma+1}$-clopen, $\hat{\mathcal{R}}_i$ is simply equal to $\mathcal{I}_{\varsigma+\vartheta_i}$ when restricted to $X_\uparrow$, for $0 < i$. We are closer to our goal, but not done yet, since the space

$$([0, \Theta], \hat{\mathcal{R}}_0, \ldots, \hat{\mathcal{R}}_n)$$

might not be a $\vec{\vartheta}$-polytopology, as $\hat{\mathcal{R}}_0$ might not be ϑ_1-maximal around points in $X_\uparrow$.

The projection π_1, which was defined as the function

$$\alpha \mapsto -\xi + \ell^\varsigma \alpha$$

is the rank function of $(X_\uparrow, \mathcal{I}_\varsigma)$ (viewed as a subspace of $([0, \Theta], \mathcal{I}_\varsigma)$), so

$$\pi_1 : (X_\uparrow, \mathcal{I}_\varsigma) \to ([0, \lambda], \mathcal{I}_0)$$

is a d-map. Having only one point of each rank, the space $([0, \lambda], \mathcal{I}_0)$ has no proper rank-preserving extensions, and in particular is ϑ_1-maximal. By the claim within the proof of Lemma 3.7, if $(X_\uparrow, \mathcal{R})$ is any ϑ_1-extension of $(X_\uparrow, \mathcal{I}_\varsigma)$, then

$$\pi_1 : (X_\uparrow, \mathcal{R}) \to ([0, \lambda], \mathcal{I}_0)$$

remains a d-map. Let $([0, \Theta], \mathcal{R}_0)$ be a ϑ_1-maximal extension of $([0, \Theta], \hat{\mathcal{R}}_0)$. Then, $\mathcal{R}_0$ only adds neighborhoods around points of rank some ρ such that $0 < \ell^{\vartheta_1} \rho$ and, moreover, only neighborhoods around points in $X_\uparrow$, since $(X_\downarrow, \hat{\mathcal{T}}_0)$ was already ϑ_1-maximal. Given a point $x \in X_\uparrow$, and recalling that ξ, the minimum $\mathcal{I}_\varsigma$-rank of points in $X_\uparrow$, is a successor ordinal, we see that

$$\begin{aligned} 0 < \ell^{\vartheta_1} \rho_{([0,\Theta],\mathcal{R}_0)} x &\text{ if, and only if, } 0 < \ell^{\vartheta_1} \ell^\varsigma x \\ &\text{ if, and only if, } 0 < \ell^{\vartheta_1}(-\xi + \ell^\varsigma x) \end{aligned}$$

$$\text{if, and only if, } 0 < \ell^{\vartheta_1} \rho_{(X_\uparrow, \mathcal{I}_\varsigma)} x.$$

Thus, the space $(X_\uparrow, \mathcal{R}_0)$ is a ϑ_1-extension of $(X_\uparrow, \mathcal{I}_\varsigma)$. It follows that

$$\pi_1 : (X_\uparrow, \mathcal{R}_0) \to ([0, \lambda], \mathcal{I}_0)$$

remains a d-map. We may now apply Lemma 3.7 to obtain a $\vec{\vartheta}$-polytopology

$$(X_\uparrow, \mathcal{R}_0, \hat{\mathcal{S}}_1, \ldots, \hat{\mathcal{S}}_n)$$

over $(X_\uparrow, \mathcal{R}_0)$ such that

$$\pi_1 : (X_\uparrow, \hat{\mathcal{S}}_i) \to ([0, \lambda], \mathcal{S}_i)$$

is a d-map for each $1 \le i \le n$. For each $1 \le i \le n$, we let $\mathcal{R}_i$ be the disjoint union

$$(X_\downarrow, \hat{\mathcal{T}}_i) \sqcup (X_\uparrow, \hat{\mathcal{S}}_i).$$

The sets $X_\uparrow$ and $X_\downarrow$ are $\mathcal{R}_{0\uparrow\vartheta_1}$-clopen and so it follows that

$$([0, \Theta], \mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_n)$$

is a $\vec{\vartheta}$-polytopology. Moreover, we have seen that

$$\pi_0 : (X_\downarrow, \hat{\mathcal{T}}_i) \to ([0, \kappa], \mathcal{T}_i)$$

is a d-map for each $0 \le i \le n$ and that

$$\pi_1 : (X_\uparrow, \mathcal{R}_i) \to ([0, \lambda], \mathcal{S}_i)$$

is also a d-map for each $0 \le i \le n$. The other conditions in the statement of the Product Lemma we have shown already, so its proof is complete.

4.4 When $\varsigma = 1$

The plan to prove the lemma in the case $\varsigma = 1$ is similar to the case $1 < \varsigma$, but simpler. We follow the same proof outline from the preceding sections. The essence of the construction below is the same as that of the d-product from Beklemishev-Gabelaia [10], but more direct (and, in return, less general). Let $\ell[0, \kappa] = \xi = \zeta + 1$, $\Theta = \omega^{\xi+\lambda}$, $X_\downarrow = [0, \Theta] \cap [0, \xi)_1$, and $X_\uparrow = [0, \Theta] \cap [\xi, \infty)_1$. Then, $X_\uparrow$ consists of all ordinals in $[0, \Theta]$ which are of the form $\omega^\xi \cdot \eta$, for some $\eta \le \omega^\lambda$, and $X_\downarrow$ consists of all ordinals in $[0, \Theta]$ which are of the form $\omega^\xi \cdot \eta + \gamma$, for some $\eta < \omega^\lambda$ and some $\gamma \in [1, \omega^\xi)$.

Lemma 4.11 $(1 + \kappa) \cdot \omega = \omega^\xi$.

Proof Since $\ell[0, \kappa] = \xi = \zeta + 1$, we must have

$$\omega^\zeta \leq \kappa < \omega^\xi.$$

The ordinal ω^ξ is additively indecomposable and thus $(1+\kappa)\cdot n < \omega^\xi$ for all $n < \omega$, so

$$\omega^\xi = \lim_{n<\omega} \omega^\zeta \cdot n \leq \lim_{n<\omega}(1+\kappa)\cdot n \leq \omega^\xi,$$

which proves the lemma. □

Thus, a typical element of $X_\downarrow$ has the form $\omega^\xi \cdot \eta + (1+\kappa)\cdot n + 1 + \gamma$, where $\eta < \omega^\lambda$, $n < \omega$, and $0 \leq \gamma \leq \kappa$. For such η, n, and γ, we define

$$\pi_0(\omega^\xi \cdot \eta + (1+\kappa)\cdot n + 1 + \gamma) = \gamma;$$

for $\eta \leq \omega^\lambda$, we let

$$\pi_1(\omega^\xi \cdot \eta) = -\xi + \ell\eta.$$

Being the rank function, $\pi_1 : (X_\uparrow, \mathcal{I}_1) \to ([0, \lambda], \mathcal{I}_0)$ is a surjective d-map. Similarly, $\pi_0 : (X_\downarrow, \mathcal{I}_1) \to ([0, \kappa], \mathcal{I}_1)$ is a surjective d-map, since $X_\downarrow$ is a disjoint union of copies of the space $[0, \kappa]$ indexed by ordinals $\eta < \omega^\lambda$ and numbers $n < \omega$ and π_0 is a homeomorphism when restricted to any one of these copies.

Lemma 4.12 *$\pi_0^{-1}\alpha$ is $\mathcal{I}_1$-dense in $X_\uparrow$ for any $\alpha \leq \kappa$.*

Proof Fix $\beta = \omega^\xi \cdot \eta \in X_\uparrow$. Suppose first that η is a successor ordinal, so β is of the form $\omega^\xi \cdot \bar\eta + \omega^\xi$. Let U be a $\mathcal{I}_1$-neighborhood of β; we may assume that

$$U = (\omega^\xi \cdot \bar\eta + \gamma, \beta]_0,$$

where $\gamma < \omega^\xi$. By Lemma 4.11, we may find n large enough so that

$$\gamma < (1+\kappa)\cdot n.$$

Therefore, letting

$$\hat\alpha = \omega^\xi \cdot \bar\eta + (1+\kappa)\cdot n + 1 + \alpha,$$

we have $\hat\alpha \in U$ and $\pi_0\hat\alpha = \alpha$, as desired.

If η is a limit ordinal, then, β is a limit of ordinals of the form $\omega^\xi \cdot \bar\eta$, for any $\mathcal{I}_1$-neighborhood U of β, the previous argument applied to any sufficiently large enough $\bar\eta$ shows that U contains some $\hat\alpha$ such that $\pi_0\hat\alpha = \alpha$. This proves the lemma. □

The argument in Sect. 4.3 does not depend on whether $\varsigma = 1$, so one may proceed similarly to construct the polytopology $(\mathcal{R}_0, \ldots \mathcal{R}_n)$ and argue that the projection mappings

$$\pi_0 : (X_{\downarrow}, \mathcal{R}_i) \to ([0, \kappa], \mathcal{T}_i)$$

and

$$\pi_1 : (X_{\uparrow}, \mathcal{R}_i) \to ([0, \lambda], \mathcal{S}_i)$$

remain d-maps. This completes the proof of the theorem.

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