



Combinatorial properties and dependent choice in symmetric extensions based on Lévy collapse

Amitayu Banerjee¹

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Abstract

We work with symmetric extensions based on Lévy collapse and extend a few results of Apter, Cody, and Koepke. We prove a conjecture of Dimitriou from her Ph.D. thesis. We also observe that if V is a model of ZFC, then $DC_{<\kappa}$ can be preserved in the symmetric extension of V in terms of symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, if \mathbb{P} is κ -distributive and \mathcal{F} is κ -complete. Further we observe that if $\delta < \kappa$ and V is a model of $ZF + DC_\delta$, then DC_δ can be preserved in the symmetric extension of V in terms of symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, if \mathbb{P} is $(\delta + 1)$ -strategically closed and \mathcal{F} is κ -complete.

Keywords Dependent choice · Symmetric extensions · Lévy collapse · Large cardinals · Infinitary Chang conjecture

Mathematics Subject Classification 03E25 · 03E55 · 03E35

1 Introduction

Grigorieff [26] proved that symmetric extensions in terms of *symmetric system* $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle^1$ are intermediate models of the form $\text{HOD}(V[a])^{V[G]}$ as a varies over $V[G]$. Apter, Henle, Cody, and Koepke constructed several models of ZF in terms of hereditarily definable sets based on Lévy Collapse (cf. [1, 5, 7, 10, 14]). The purpose of this note is to translate the arguments of a few of those choiceless model constructions to symmetric extensions in terms of symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ and extend a few published results. In particular, we prove the following.

(1) We prove the failure of AC_κ (Every family of κ non-empty sets admits a choice function) in the symmetric extension of [35, Theorem 4.1]. Moreover, we study an

✉ Amitayu Banerjee
banerjee.amitayu@gmail.com

¹ Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, Budapest 1053, Hungary

¹ $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a *symmetric system* if \mathbb{P} is a forcing notion, \mathcal{G} is a group of automorphisms of \mathbb{P} , and \mathcal{F} is a normal filter of subgroups over \mathcal{G} .

argument to preserve the supercompactness of κ in the symmetric model following the methods of [29].

- (2) We reduce the large cardinal assumption of [1, Theorem 2] and [1, Theorem 3].
- (3) We observe an *infinitary Chang conjecture* in the choiceless model constructed in [7, Theorem 11]. Moreover, we prove that \aleph_{ω_1} is an almost Ramsey cardinal in the model.
- (4) Fix an arbitrary $n_0 \in \omega$. We observe that if $\langle S_k : 1 \leq k < \omega \rangle$ is a sequence of stationary sets such that $S_k \subseteq \aleph_{n_0+2(k+1)}$ for every $1 \leq k < \omega$, then $\langle S_k : 1 \leq k < \omega \rangle$ is *mutually stationary* in the choiceless model constructed in [14, Theorem 1]. We also observe an alternating sequence of measurable and non-measurable cardinals in the model. Moreover, we observe that \aleph_ω is an almost Ramsey cardinal in the model.

Secondly, we prove a conjecture of Dimitriou related to the failure of Dependent Choice—or DC—in a symmetric extension based on finite support products of collapsing functions, from [21]. We also study new lemmas related to preserving DC in symmetric extensions inspired by [37, Lemma 1]. In particular, we observe the following.

- (1) Let V be a model of ZFC. If \mathbb{P} is κ -distributive and \mathcal{F} is κ -complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.
- (2) Let $\delta < \kappa$ and V be a model of $ZF + DC_\delta$ where the Axiom of Choice (AC) might fail. If \mathbb{P} is $(\delta + 1)$ -strategically closed and \mathcal{F} is κ -complete, then DC_δ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.

1.1 Preserving dependent choice

Karagila [37, Lemma 1] proved that if \mathbb{P} is κ -closed and \mathcal{F} is κ -complete then $DC_{<\kappa}$ is preserved in the symmetric extension in terms of symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$. We observe that ‘ \mathbb{P} is κ -closed’ can be replaced by ‘ \mathbb{P} is κ -distributive’ in [37, Lemma 1]. This slightly generalize [37, Lemma 1], since there are κ -strategically closed forcing notions which are not κ -closed² and κ -distributivity is weaker than $< \kappa$ -strategic closure.³

Observation 1.1 (Lemma 3.2) *Let V be a model of ZFC. If \mathbb{P} is κ -distributive and \mathcal{F} is κ -complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.*

We also observe that even if we start with a model V , which is a model of $ZF + DC_\delta$ where AC might fail, we can still preserve DC_δ in a symmetric extension of V in certain cases. In particular, we observe the following.

Observation 1.2 (Lemma 3.4) *Let $\delta < \kappa$ and V be a model of $ZF + DC_\delta$. If \mathbb{P} is $(\delta + 1)$ -strategically closed and \mathcal{F} is κ -complete, then DC_δ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.*

² As for an example, the forcing notion $\mathbb{P}(\kappa)$ which adds a *non-reflecting stationary set of cofinality ω ordinals in κ* , is κ -strategically closed but not even ω_2 -closed. (cf. [19, section 6]).

³ As for an example, the forcing notion for *killing a stationary subset of ω_1* , is ω_1 -distributive but not even $< \omega_1$ -strategically closed (cf. [19, section 6]).

1.2 On a question of Apter

Woodin asked in the context of ZFC, that if κ is strongly compact and GCH holds below κ , then must GCH hold everywhere? The problem is still open in the context of ZFC. One variant of this question is if GCH can fail at every limit cardinal less than or equal to a strongly compact cardinal κ where as GCH holds above κ^+ . Apter answered this in the context of ZF. Apter [9, Theorem 3] constructed a model where κ is a regular limit cardinal and a supercompact cardinal, and GCH holds for a limit δ if and only if $\delta > \kappa$. In that model the Countable Choice—or AC_ω —fails. At the end of [9], Apter asked the following question.

Question 1.3 *Is it possible to construct analogs of Theorem 3 in which some weak version of AC holds?*

The author and Karagila constructed a symmetric extension to answer **Question 1.3** in [35, Theorem 4.1].

Theorem 1.4 (cf. [35, Theorem 4.1]) *Let V be a model of ZFC + GCH with a supercompact cardinal κ . Then there is a symmetric extension in which $DC_{<\kappa}$ holds, κ is a regular limit cardinal and supercompact, and GCH holds for a limit cardinal δ if and only if $\delta > \kappa$.*

For the sake of convenience, we call the symmetric extension constructed in [35, Theorem 4.1] as \mathcal{N}_1 . We study an argument to preserve the supercompactness of κ in \mathcal{N}_1 applying the methods of [29] and prove the following in Sect. 4.

Theorem 1.5 *In \mathcal{N}_1 , AC_κ fails.*

1.3 Proving Dimitriou's conjecture

Dimitriou constructed a symmetric extension based on finite support products of collapsing functions. At the end of [21, section 1.4], Dimitriou conjectured that DC would fail in the symmetric extension (cf. [21, Question 1, Chapter 4]). We prove the conjecture. For the sake of convenience, we call this model as Dimitriou's model and prove the following in Sect. 5.

Theorem 1.6 *In Dimitriou's model, AC_ω fails.*

1.4 Reducing the assumption of supercompactness to strong compactness

Apter and Cody [1, Theorem 2] obtained a model of ZF + $\neg AC_\omega$ where \aleph_1 and \aleph_2 are both singular of cofinality ω , and there is a sequence of distinct subsets of \aleph_1 of length equal to any predefined ordinal, assuming a supercompact cardinal κ . In Sect. 6, we observe that applying a recent result of Usuba (cf. [4, Theorem 3.1]) followed by working with a model of ZF + $\neg AC_\omega$ constructed using *strongly compact Prikry forcing*, it is possible to reduce the assumption of a supercompact cardinal κ to a strongly compact cardinal κ .

Observation 1.7 *Suppose that κ is a strongly compact cardinal, GCH holds, and θ is an ordinal. Then there is a model of $ZF + \neg AC_\omega$ in which $cf(\aleph_1) = cf(\aleph_2) = \omega$, and there is a sequence of distinct subsets of \aleph_1 of length θ .*

Similarly, we reduce the large cardinal assumption of [1, Theorem 3] from a supercompact cardinal to a strongly compact cardinal. Apter and Cody [1, Theorem 3] obtained a model of $ZF + \neg AC_\omega$ where \aleph_ω and $\aleph_{\omega+1}$ are both singular with $\omega \leq cf(\aleph_{\omega+1}) < \aleph_\omega$, and there is a sequence of distinct subsets of \aleph_ω of length equal to any predefined ordinal, assuming a supercompact cardinal κ . We prove the following in Sect. 6.

Observation 1.8 *Suppose that κ is a strongly compact cardinal, GCH holds, and θ is an ordinal. Then there is a model of $ZF + \neg AC_\omega$ in which \aleph_ω and $\aleph_{\omega+1}$ are both singular with $\omega \leq cf(\aleph_{\omega+1}) < \aleph_\omega$, and there is a sequence of distinct subsets of \aleph_ω of length θ .*

1.5 Infinitary Chang conjecture from a measurable cardinal

Assuming a measurable cardinal, Apter and Koepke constructed a model \mathcal{N} of ZF based on Lévy collapse in [7, Theorem 11]. In \mathcal{N} , ω_1 is singular, and \aleph_{ω_1} is a Rowbottom cardinal carrying a Rowbottom filter. They mentioned that in \mathcal{N} , AC_ω fails because of the singularity of ω_1 . In Sect. 7, we observe an *infinitary Chang conjecture* in a symmetric extension, which is very similar to \mathcal{N} , except we consider a finite support product construction. We use the observation that it is possible to force a *coherent* sequence of Ramsey cardinals after performing Prikry forcing on a normal measure over a measurable cardinal κ (cf. [7, Theorem 3]). We also use the observation that an infinitary Chang conjecture can be established in a symmetric model, assuming a coherent sequence of Ramsey cardinals. As in the model of [7, Theorem 11], ω_1 is singular and therefore AC_ω fails.

Theorem 1.9 *Let V' be a model of ZFC where there is a measurable cardinal. Then there is a generic extension V of V' , and a symmetric extension $V(G)$ of V such that ω_1 is singular in $V(G)$. Moreover, an infinitary Chang conjecture holds in $V(G)$.*

Similarly, we also observe an infinitary Chang conjecture in the model \mathcal{N} . For the sake of convenience, we call the model \mathcal{N} as Apter and Koepke's model and prove the following in Sect. 7.

Theorem 1.10 *An infinitary Chang conjecture holds in Apter and Koepke's model. Moreover, \aleph_{ω_1} is an almost Ramsey cardinal in the model.*

1.6 Mutual stationarity property from a sequence of measurable cardinals

Foreman and Magidor [23] introduced the idea of *mutual stationarity*. They asked if there is a model of set theory in which every sequence of stationary subsets of the \aleph_n 's of a fixed cofinality is mutually stationary (cf. [23, page 290]). Assuming an ω -sequence of supercompact cardinals, Apter [10, Theorem 1] constructed a model

of ZF + DC in which if $\langle S_n : 1 \leq n < \omega \rangle$ is a sequence of stationary sets such that $S_n \subseteq \aleph_n$ for every $1 \leq n < \omega$, then $\langle S_n : 1 \leq n < \omega \rangle$ is mutually stationary. Apter [14, Theorem 1] further obtained a similar model based on Lévy collapse as constructed in [10], where \aleph_ω carries a Rowbottom filter and $\text{DC}_{\aleph_{n_0}}$ holds for any arbitrary $n_0 \in \omega$, from an ω -sequence of measurable cardinals. For the sake of convenience, we fix an arbitrary $n_0 \in \omega$ in the ground model V , call the model from [14, Theorem 1] as \mathcal{N}_{n_0} , and prove the following in Sect. 8.

Observation 1.11 *The following hold in the model \mathcal{N}_{n_0} .*

- (1) *For each $1 \leq k < \omega$, $\aleph_{n_0+2(k+1)}$ is a measurable cardinal and \aleph_{n_0+2k+1} is not a measurable cardinal. In particular, for each $1 \leq k < \omega$, there are no uniform ultrafilters on \aleph_{n_0+2k+1} .*
- (2) *If $\langle S_k : 1 \leq k < \omega \rangle$ is a sequence of stationary sets such that $S_k \subseteq \aleph_{n_0+2(k+1)}$ for every $1 \leq k < \omega$, then $\langle S_k : 1 \leq k < \omega \rangle$ is mutually stationary.*
- (3) *\aleph_ω is an almost Ramsey cardinal.*

Structure of the paper

- In Sect. 2, we cover the basics.
- In Sect. 3, we prove Observations 1.1 and 1.2.
- In Sect. 4, we prove Theorem 1.5.
- In Sect. 5, we prove Theorem 1.6.
- In Sect. 6, we prove Observations 1.7 and 1.8.
- In Sect. 7, we prove Theorems 1.9 and 1.10.
- In Sect. 8, we prove Observation 1.11.

2 Basics

2.1 Large cardinals

In this section, we recall the definition of inaccessible cardinals in the context of ZFC and other large cardinals in the context of ZF. In ZFC, we say κ is a strongly inaccessible cardinal if it is a regular strong limit cardinal where the definition of “strong limit” is that for all $\alpha < \kappa$, we have $2^\alpha < \kappa$. In the context of ZF, the above definition doesn’t make sense, as 2^α may not be well-ordered. We refer the reader to [16] for details concerning inaccessible cardinals in the context of ZF. We recall some large cardinal definitions in the context of ZF from [34].

Definition 2.1 Let κ be an uncountable cardinal.

- (1) The cardinal κ is *weakly compact* if for all $f : [\kappa]^2 \rightarrow 2$, there is a homogeneous set $X \subseteq \kappa$ for f of order type κ .
- (2) The cardinal κ is *Ramsey* if for all $f : [\kappa]^{<\omega} \rightarrow 2$, there is a homogeneous set $X \subseteq \kappa$ for f of order type κ .
- (3) The cardinal κ is *almost Ramsey* if for all $\alpha < \kappa$ and $f : [\kappa]^{<\omega} \rightarrow 2$, there is a homogeneous set $X \subseteq \kappa$ for f having order type α .

- (4) The cardinal κ is μ -Rowbottom if for all $\alpha < \kappa$ and $f : [\kappa]^{<\omega} \rightarrow \alpha$, there is a homogeneous set $X \subseteq \kappa$ for f of order type κ such that $|f''[X]^{<\omega}| < \mu$. We say that κ is Rowbottom if it is ω_1 -Rowbottom. A filter \mathcal{F} on κ is a *Rowbottom filter* on κ if for any $f : [\kappa]^{<\omega} \rightarrow \lambda$, where $\lambda < \kappa$, there is a set $X \in \mathcal{F}$ such that $|f''[X]^{<\omega}| \leq \omega$.
- (5) The cardinal κ is *measurable* if there is a κ -complete free ultrafilter on κ . A filter \mathcal{F} on a cardinal κ is *normal* if it is closed under diagonal intersections:

$$\text{If } X_\alpha \in \mathcal{F} \text{ for all } \alpha < \kappa, \text{ then } \Delta_{\alpha < \kappa} X_\alpha \in \mathcal{F}.$$

In ZF we have the following lemma.

Lemma 2.2 (cf. [21, Lemma 0.8]) *An ultrafilter \mathcal{U} over κ is normal if and only if for every regressive $f : \kappa \rightarrow \kappa$ there is an $X \in \mathcal{U}$ such that f is constant on X .*

Thus, we say an ultrafilter \mathcal{U} over κ is normal if for every regressive $f : \kappa \rightarrow \kappa$ there is an $X \in \mathcal{U}$ such that f is constant on X .

- (6) For a set A , we say \mathcal{U} is a *fine measure* on $\mathcal{P}_\kappa(A)$ if \mathcal{U} is a κ -complete ultrafilter and for any $i \in A$, $\{x \in \mathcal{P}_\kappa(A) : i \in x\} \in \mathcal{U}$. We say that \mathcal{U} is a *normal measure* on $\mathcal{P}_\kappa(A)$ if \mathcal{U} is a fine measure and if $f : \mathcal{P}_\kappa(A) \rightarrow A$ is such that $f(X) \in X$ for a set in \mathcal{U} , then f is constant on a set in \mathcal{U} . The cardinal κ is λ -*strongly compact* if there is a fine measure on $\mathcal{P}_\kappa(\lambda)$; it is *strongly compact* if it is λ -strongly compact for all $\kappa \leq \lambda$.
- (7) The cardinal κ is λ -*supercompact* if there is a normal measure on $\mathcal{P}_\kappa(\lambda)$; it is *supercompact* if it is λ -supercompact for all $\kappa \leq \lambda$.

Remark 2.3 We note that the definition of supercompact (similarly strongly compact) is given in the terms of ultrafilters, which is weaker than the definition of supercompact in terms of elementary embedding due to Woodin [45, Definition 220] (e.g. \aleph_1 can be supercompact or strongly compact if we consider the definition of supercompact or strongly compact in terms of ultrafilters, but \aleph_1 can not be the critical point of an elementary embedding (cf. [29])).

Remark 2.4 Ikegami and Trang [30, section 2] defined that an ultrafilter \mathcal{U} on $\mathcal{P}_\kappa X$ is normal if for any set $A \in \mathcal{U}$ and $f : A \rightarrow \mathcal{P}_\kappa X$ with $\emptyset \neq f(\sigma) \subseteq \sigma$ for all $\sigma \in A$, there is an $x_0 \in X$ such that for \mathcal{U} -measure one many σ in A , $x_0 \in f(\sigma)$. They note that their definition of normality is equivalent to the closure under diagonal intersections in ZF, while it may not be equivalent to the definition of normality in our sense without AC.

From now on, all our inaccessible cardinals are strongly inaccessible. We recall that a limit of Ramsey cardinals is an almost Ramsey cardinal in ZF (cf. [6, Proposition 1]).

2.2 Forcing extension and Lévy–Solovay theorem

Let \mathbb{P} be a forcing notion, by which we mean a partially ordered set with a maximum element 1. For $p, q \in \mathbb{P}$, we say that p is stronger than q or p extends q if $p \leq q$. Let

G be a \mathbb{P} -generic filter over V and $V^{\mathbb{P}}$ be a class of all \mathbb{P} -names defined recursively as follows: if τ is a set, $\tau \in V^{\mathbb{P}}$ if and only if $\tau \subseteq V^{\mathbb{P}} \times \mathbb{P}$. The interpretation of a \mathbb{P} -name τ by G is defined recursively as $\tau^G = \{\sigma^G : \exists p \in G((\sigma, p) \in \tau)\}$ and the generic extension is defined as $V[G] = \{\tau^G : \tau \in V^{\mathbb{P}}\}$. We recall that $V[G]$ is the smallest transitive model of ZFC which has the same ordinals as V and contains both V and G (cf. [39]). If \mathbb{P} is our forcing notion and G is a \mathbb{P} -generic filter over V , we will abuse notation somewhat and use both $V^{\mathbb{P}}$ and $V[G]$ to denote the generic extension of V . We state a part of Lévy–Solovay Theorem (cf. [31, Theorem 21.2]) in ZFC.

Theorem 2.5 *Let κ be an infinite cardinal, and let \mathbb{P} be a forcing notion of size less than κ . Let G be a \mathbb{P} -generic filter over V .*

- (1) *If κ is Ramsey in V , then κ is Ramsey in $V[G]$.*
- (2) *If κ is measurable with a κ -complete ultrafilter \mathcal{U} in V , then κ is measurable with a κ -complete ultrafilter $\mathcal{U}_1 = \{X \subseteq \kappa : X \in V[G], \text{ and there is a } Y \in \mathcal{U} \text{ such that } Y \subseteq X\}$ defined in $V[G]$ generated by \mathcal{U} in $V[G]$.*

Proof (1) follows from [31, Theorem 21.2] and (2) follows from [40, Theorem 10]. \square

2.3 Symmetric extension

Symmetric extensions are submodels of the generic extension containing the ground model, where AC can consistently fail. Let \mathbb{P} be a forcing notion, \mathcal{G} be a group of automorphisms of \mathbb{P} and \mathcal{F} be a normal filter of subgroups over \mathcal{G} . We recall the following Symmetry Lemma from [31].

Lemma 2.6 (The Symmetry Lemma; cf. [31, Lemma 14.37]) *Let \mathbb{P} be a forcing notion, φ be a formula of the forcing language with n free variables and let $\sigma_1, \sigma_2, \dots, \sigma_n$ be \mathbb{P} -names. If a is an automorphism of \mathbb{P} , then $p \Vdash \varphi(\sigma_1, \sigma_2, \dots, \sigma_n) \iff a(p) \Vdash \varphi(a(\sigma_1), a(\sigma_2), \dots, a(\sigma_n))$.*

For $\tau \in V^{\mathbb{P}}$, we denote its symmetric group with respect to \mathcal{G} by $\text{sym}^{\mathcal{G}}(\tau) = \{g \in \mathcal{G} : g\tau = \tau\}$ and say τ is symmetric with respect to \mathcal{F} if $\text{sym}^{\mathcal{G}}(\tau) \in \mathcal{F}$. Let $HS^{\mathcal{F}}$ be the class of all hereditary symmetric names. That is, recursively for $\tau \in V^{\mathbb{P}}$,

$\tau \in HS^{\mathcal{F}}$ iff τ is symmetric with respect to \mathcal{F} , and for each $\sigma \in \text{dom}(\tau)$, $\sigma \in HS^{\mathcal{F}}$.

We define the symmetric extension of V with respect to \mathcal{F} as $V(G)^{\mathcal{F}} = \{\tau^G : \tau \in HS^{\mathcal{F}}\}$. For the sake of our convenience we omit the superscript \mathcal{F} sometimes and call $V(G)^{\mathcal{F}}$ as $V(G)$, $HS^{\mathcal{F}}$ as HS , and $\text{sym}^{\mathcal{F}}(\tau)$ as $\text{sym}(\tau)$.

Definition 2.7 (Symmetric system; cf. [27, Definition 2.1]) We say $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system if \mathbb{P} is a forcing notion, \mathcal{G} is a group of automorphisms of \mathbb{P} , and \mathcal{F} is a normal filter of subgroups over \mathcal{G} .

Definition 2.8 (Tenacious system; cf. [36, Definition 4.6]) Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system. A condition $p \in \mathbb{P}$ is \mathcal{F} -tenacious if $\{\pi \in \mathcal{G} : \pi(p) = p\} \in \mathcal{F}$. We say \mathbb{P} is \mathcal{F} -tenacious if there is a dense subset of \mathcal{F} -tenacious conditions. We say $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a tenacious system if \mathbb{P} is \mathcal{F} -tenacious.

Karagila and Hayut proved that every symmetric system is equivalent to a tenacious system (cf. [36, Appendix A]). Thus, it is natural to assume tenacity and work with tenacious system. We recall the following theorem which states that the symmetric extension $V(G)$ is a transitive model of ZF.

Theorem 2.9 (cf. [31, Lemma 15.51]) *If $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system and G is a \mathbb{P} -generic filter over V , then $V(G)$ is a transitive model of ZF and $V \subseteq V(G) \subseteq V[G]$.*

2.4 Terminologies

We recall the terminologies like *Approximation Lemma*, *Approximation property*, and $(\mathcal{G}, \mathcal{I})$ -homogeneous forcing notion, from [21] and [29]. For $E \subseteq \mathbb{P}$, let us define its pointwise stabilizer group to be $\text{fix}_{\mathcal{G}}E = \{g \in \mathcal{G} : \forall p \in E (g(p) = p)\}$, i.e., it is the set of automorphisms which fix E pointwise. We denote $\text{fix}_{\mathcal{G}}E$ by $\text{fix } E$ for the sake of convenience.

Definition 2.10 (*\mathcal{G} -symmetry generator*; [29, Definition 14]) Let \mathbb{P} be a forcing notion and \mathcal{G} be a group of automorphisms of \mathbb{P} . A subset $\mathcal{I} \subseteq \mathcal{P}(\mathbb{P})$ is called a *\mathcal{G} -symmetry generator* if it consists of up-sets, is closed under unions, and if for all $g \in \mathcal{G}$ and $E \in \mathcal{I}$, there is an $E' \in \mathcal{I}$ such that $g(\text{fix}E)g^{-1} \supseteq \text{fix}E'$.

We can see that if \mathcal{I} is a \mathcal{G} -symmetry generator, then the set $\{\text{fix}E : E \in \mathcal{I}\}$ generates a normal filter over \mathcal{G} (cf. [29, Lemma 15]).

Definition 2.11 Let \mathcal{I} be the \mathcal{G} -symmetry generator, we say $E \in \mathcal{I}$ *supports* a name $\sigma \in HS$ if $\text{fix}E \subseteq \text{sym}(\sigma)$.

Definition 2.12 (*Projectable \mathcal{G} -symmetry generator*; [21, Definition 1.25] & [29, Definition 17]) Let \mathbb{P} be a forcing notion, \mathcal{G} be a group of automorphisms of \mathbb{P} , and \mathcal{I} be a \mathcal{G} -symmetry generator. We say \mathcal{I} is *projectable* for the pair $(\mathbb{P}, \mathcal{G})$ if for every $p \in \mathbb{P}$ and every $E \in \mathcal{I}$, there is a $p^* \in E$ that is minimal (with respect to the partial order) and unique such that $p^* \geq p$. We call $p \upharpoonright E = p^*$ the *projection* of p to E .

For the rest of this section, let \mathbb{P} be a forcing notion, \mathcal{G} be a group of automorphisms of \mathbb{P} , and \mathcal{I} be a projectable \mathcal{G} -symmetry generator for the pair $(\mathbb{P}, \mathcal{G})$.

Definition 2.13 (*Approximation property*; [29, Definition 18]) We say that the triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ has the *approximation property* if for any formula φ with n free variables, and names $\sigma_1, \sigma_2, \dots, \sigma_n \in HS$ all with support $E \in \mathcal{I}$, and for any $p \in \mathbb{P}$, $p \Vdash \varphi(\sigma_1, \sigma_2, \dots, \sigma_n)$ implies that $p \upharpoonright E \Vdash \varphi(\sigma_1, \sigma_2, \dots, \sigma_n)$.

Definition 2.14 ($(\mathcal{G}, \mathcal{I})$ -homogeneous forcing notion; [21, Definition 1.26] & [29, Definition 19]) We say that \mathbb{P} is $(\mathcal{G}, \mathcal{I})$ -homogeneous if for every $E \in \mathcal{I}$, every $p \in \mathbb{P}$, and every $q \in \mathbb{P}$ such that $q \leq p \upharpoonright E$, there is an automorphism $a \in \text{fix}E$ such that $a(p) \parallel q$.

Lemma 2.15 ([21, Lemma 1.27] & [29, Lemma 20]) *If \mathbb{P} is $(\mathcal{G}, \mathcal{I})$ -homogeneous, then $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ has the approximation property.*

We note that if $E \in \mathcal{I}$, then E itself is a forcing notion (with the same top element as \mathbb{P}). So, we can say that set of pairs τ is an E -name iff τ is a relation and for every $(\sigma, p) \in \tau$, σ is an E -name and $p \in E$.

Lemma 2.16 (Approximation Lemma; [21, Lemma 1.29] & [29, Lemma 21]) *If the triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ has the approximation property then for all set of ordinals $X \in V(G)$, there exists an $E \in \mathcal{I}$ and an E -name for X . Thus, $X \in V[G \cap E]$.*

2.5 Homogeneity of forcing notions

We recall the definition of *weakly homogeneous* and *cone homogeneous* forcing notion from [20].

Definition 2.17 (*Weakly homogeneous forcing notion*; [20, Definition 2]) We say a set forcing notion \mathbb{P} is *weakly homogeneous* if and only if for any $p, q \in \mathbb{P}$, there is an automorphism $a : \mathbb{P} \rightarrow \mathbb{P}$ such that $a(p)$ and q are compatible.⁴

Definition 2.18 (*Cone homogeneous forcing notion*; [20, Definition 2]) For $p \in \mathbb{P}$, let $Cone(p)$ denote $\{r \in \mathbb{P} : r \leq p\}$, the cone of conditions in \mathbb{P} below p . We say a set forcing notion \mathbb{P} is *cone homogeneous* if and only if for any $p, q \in \mathbb{P}$, there exist $p' \leq p, q' \leq q$, and an isomorphism $\pi : Cone(p') \rightarrow Cone(q')$.

If \mathbb{P} is weakly homogeneous, then it is cone homogeneous too (cf. [20, Fact 1]). Also, the finite support products of weakly (cone) homogeneous forcing notions are weakly (cone) homogeneous. A crucial feature of symmetric extensions using weakly (cone) homogeneous forcings are that they can be approximated by certain intermediate submodel where AC holds.

2.6 Failure of weak choice principles

We use AC_κ to denote the statement “Every family of κ non-empty sets admits a choice function”. We note that if κ^+ is singular, then AC_κ fails. This is due to the following well known fact.

Fact 2.19 *For all successor cardinal λ , AC_κ implies $cf(\lambda) > \kappa$.*

We sketch another way of refuting AC_κ . For sets A and B , we use $AC_A(B)$ to denote the statement “for each set X of non-empty subsets of B , if there is an injection from X to A then there is a choice function for X ”. We recall [21, Lemmas 0.2, 0.3, 0.12].

- Under $AC_A(B)$, if there is a surjection from B to A , then there is an injection from A to B (cf. [21, Lemma 0.2]).
- For every infinite cardinal κ , there is a surjection from $\mathcal{P}(\kappa)$ onto κ^+ in ZF (cf. [21, Lemma 0.3]).

⁴ The Levy collapse $Col(\lambda, < \kappa)$ is weakly homogeneous, given an infinite cardinal κ and a regular cardinal λ .

- If κ is measurable with a normal measure or κ is weakly compact and $\alpha < \kappa$, then there is no injection $f : \kappa \rightarrow \mathcal{P}(\alpha)$ in ZF (cf. [15, Proposition 0.1], [21, Lemma 0.12]).

The following lemma states that if a successor cardinal κ is either measurable with a normal measure or weakly compact then AC_κ fails, which is [15, Corollary 0.3].

Lemma 2.20 *Let $\kappa = \alpha^+$ be a successor cardinal. If κ is measurable with a normal measure or weakly compact then $AC_{\alpha^+}(\mathcal{P}(\alpha))$ fails.*

Proof Let $AC_{\alpha^+}(\mathcal{P}(\alpha))$ holds. We show $\kappa = \alpha^+$ is neither measurable with a normal measure nor weakly compact. In ZF, there is a surjection from $\mathcal{P}(\alpha)$ onto α^+ . Now $AC_{\alpha^+}(\mathcal{P}(\alpha))$ implies there is an injection f' from α^+ to $\mathcal{P}(\alpha)$ which states that $\kappa = \alpha^+$ is neither measurable with a normal measure nor weakly compact. \square

3 Preserving dependent choice in symmetric extensions

Dependent Choice, denoted by DC or DC_ω , is a weaker version of AC which is strictly stronger⁵ than the Countable Choice, denoted by AC_ω . This principle is strong enough to give the basis of analysis as it is equivalent to the Baire Category Theorem which is a fundamental theorem in functional analysis. Further, DC is equivalent to other important theorems like the countable version of the Downward Löwenheim–Skolem theorem and every tree of height ω without a maximum node has an infinite branch etc. On the other hand, AC has several controversial applications like the existence of a non-Lebesgue measurable set of real numbers, Banach–Tarski Paradox and the existence of a well-ordering of real numbers whereas DC does not have such counter-intuitive consequences. Thus it is desirable to preserve DC in symmetric extensions.

For an infinite cardinal κ , we denote by DC_κ the principle of Dependent Choice for κ . This principle states that for every non-empty set X , if R is a binary relation such that for each ordinal $\alpha < \kappa$, and each $f : \alpha \rightarrow X$ there is some $y \in X$ such that $f R y$, then there is $f : \kappa \rightarrow X$ such that for each $\alpha < \kappa$, $f \upharpoonright \alpha R f(\alpha)$. We denote the assertion $(\forall \lambda < \kappa)DC_\lambda$ by $DC_{<\kappa}$. We recall that AC is equivalent to $\forall \kappa (DC_\kappa)$ and DC_κ implies AC_κ . We refer the reader to [32, Chapter 8], for details concerning DC_κ and related choice principles.

We recall the definitions of a forcing notion with the κ -chain condition (κ -c.c.) and of forcing notions that are κ -closed and κ -distributive from [19, Definition 5.8]. We also recall the definitions of forcing notions that are $< \kappa$ -strategically closed, κ -strategically closed, and $(\kappa + 1)$ -strategically closed from [19, Definition 5.14, Definition 5.15]. Monro [43, Corollary 5.11] proved that DC_κ is not preserved by generic extensions for any infinite cardinal κ .

Karagila [37, Lemma 1] proved that if \mathbb{P} is κ -closed and \mathcal{F} is κ -complete then $DC_{<\kappa}$ is preserved in the symmetric extension in terms of symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$. The author and Karagila both observed independently that “ \mathbb{P} is κ -closed” can be replaced by “ \mathbb{P} has the κ -c.c.” in [37, Lemma 1]. The author noticed this observation combining

⁵ In Howard–Rubin’s first model (\mathcal{N}_{38} in [28]), AC_ω holds but DC_ω fails.

the role of κ -c.c. forcing notions from [11, Lemma 2.2] and the role of κ -completeness of \mathcal{F} from [37, Lemma 1].

The idea was the following. If \mathbb{P} has the κ -c.c., then any antichain is of size less than κ . So by Zorn’s Lemma in the ground model, there is a maximal antichain of conditions $\mathcal{A} = \{p_\alpha : \alpha < \gamma < \kappa\}$ extending p such that for all $\alpha < \gamma$, $p_\alpha \Vdash \dot{f}(\hat{\alpha}) = \dot{i}_\alpha$ where $\dot{i}_\alpha \in HS$. Then we can follow the proof of [37, Lemma 1] to finish the proof.

In a private conversation with Karagila, the author came to know that they independently observed the same fact. We note that there was a gap in the above observation. Specifically, the author was not aware of the fact that every symmetric system is equivalent to a tenacious system. Karagila fixed this gap. In particular, in [35, Lemma 3.3], Karagila wrote that the natural assumption that $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a tenacious system is also required in the proof.

Lemma 3.1 (Karagila; [35, Lemma 3.3]) *Let V be a model of ZFC. If \mathbb{P} has the κ -c.c. and \mathcal{F} is κ -complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.*

We slightly generalize [37, Lemma 1] and prove **Observation 1.1**.

Lemma 3.2 (Observation 1.1) *Let V be a model of ZFC. If \mathbb{P} is κ -distributive and \mathcal{F} is κ -complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.*

Proof Let G be a \mathbb{P} -generic filter over V . Let $\delta < \kappa$. We show that DC_δ holds in $V(G)$. Let X and R be elements of $V(G)$ as in the assumptions of DC_δ . Since AC is equivalent to $\forall\kappa(DC_\kappa)$ and $V[G]$ is a model of AC, using $\forall\kappa(DC_\kappa)$ in $V[G]$, we can find an $f : \delta \rightarrow X$ in $V[G]$ such that $f \upharpoonright \alpha \Vdash R f(\alpha)$ for all $\alpha < \delta$. We show that this f is in $V(G)$. Let $p_0 \Vdash \dot{f}$ is a function whose domain is δ and range is X which is a subset of $V(G)$. For each $\alpha < \delta$, $D_\alpha = \{p \leq p_0 : (\exists x \in X) p \Vdash \dot{f}(\check{\alpha}) = \dot{x} \text{ where } \dot{x} \in HS\}$ is open dense below p_0 . Consequently by κ -distributivity of \mathbb{P} , $D = \bigcap_{\alpha < \delta} D_\alpha$ is dense below p_0 . So, there is some $p \in D \cap G$. We can see that for each $\alpha < \delta$, there is an x_α such that $p \Vdash \dot{f}(\check{\alpha}) = \dot{x}_\alpha$ where $\dot{x}_\alpha \in HS$. Define $\dot{g} = \{\langle \check{\alpha}, \dot{x}_\alpha \rangle : \alpha < \delta\}$. Now, since each $\dot{x}_\alpha \in HS$, $sym(\dot{x}_\alpha) \in \mathcal{F}$. By κ -completeness of \mathcal{F} , $H = \bigcap_{\alpha < \delta} sym(\dot{x}_\alpha) \in \mathcal{F}$. Next, since H is a subgroup of $sym(\dot{g})$ and \mathcal{F} is a filter, $\dot{g} \in HS$. We can see that $p \Vdash \dot{g} = \dot{f}$. Thus, there is a dense open set of conditions $q \leq p_0$, such that for some $\dot{g} \in HS$, $q \Vdash \dot{g} = \dot{f}$. By genericity, $\dot{f}^G = f \in V(G)$. \square

Remark 3.3 If κ is either a supercompact cardinal or a strongly compact cardinal and $\lambda > \kappa$ is a regular cardinal, there are certain forcing notions like supercompact Prikry forcing [12] and strongly compact Prikry forcing [5] which are known to be non- κ -distributive, but still can preserve DC_κ in the symmetric extension based on such forcings. In particular, Apter communicated to us that, assuming the consistency of a 2^λ -supercompact cardinal κ and a regular cardinal $\lambda > \kappa$, Kofkoulis proved in [38], that in a symmetric extension based on supercompact Prikry forcing, DC_κ was preserved. In particular, DC_κ holds in the symmetric inner model constructed in [12, Theorem 1]. Further applying the methods of Kofkoulis, assuming the consistency of a λ -strongly compact cardinal κ and a measurable cardinal $\lambda > \kappa$, a symmetric

extension based on strongly compact Prikry forcing was constructed in [5] where κ became a singular cardinal of cofinality ω , κ^+ remained a measurable cardinal and DC_κ was preserved. We can also find another exhibition of Kofkoulis’s method with certain modifications in [8].

Next, we prove Observation 1.2.

Lemma 3.4 (Observation 1.2) *Let $\delta < \kappa$ and V be a model of $ZF + DC_\delta$. If \mathbb{P} is $(\delta + 1)$ -strategically closed and \mathcal{F} is κ -complete, then DC_δ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.*

Proof Let $\delta < \kappa$. Let G be a \mathbb{P} -generic filter over V . By [25, Theorem 2.2], DC_δ is preserved in $V[G]$ since \mathbb{P} is $(\delta + 1)$ -strategically closed. We show that DC_δ holds in $V(G)$. Let X and R be elements of $V(G)$ as in the assumptions of DC_δ . Since DC_δ is preserved in $V[G]$, we can find an $f : \delta \rightarrow X$ in $V[G]$ such that $f \restriction \alpha \Vdash R$ for all $\alpha < \delta$. We show that this f is in $V(G)$. Let $p \Vdash \dot{f}$ is a function whose domain is δ and its range is a subset of $V(G)$. Consider a game of length $\delta + 1$, between two players I and II who play at odd stages and even stages respectively such that initially II chooses a trivial condition and I chooses a condition p_0 extending p such that $p_0 \Vdash \dot{f}(\check{0}) = \dot{i}_0$ where \dot{i}_0 is in HS , and at non-limit even stages $2\alpha > 0$, II chooses a condition p_α extending the condition of the previous stage such that $p_\alpha \Vdash \dot{f}(\check{\alpha}) = \dot{i}_\alpha$ where \dot{i}_α is in HS . By $(\delta + 1)$ -strategic closure of \mathbb{P} , II has winning strategy. Thus, we can extend p to $p_0 \geq p_1 \geq \dots \geq p_\alpha \geq \dots \geq p_\delta$ such that $p_\alpha \Vdash \dot{f}(\check{\alpha}) = \dot{i}_\alpha$ where \dot{i}_α is in HS for each $\alpha < \delta$. It is enough to show that $\dot{f} = \{ \langle \check{\beta}, \dot{i}_\beta \rangle : \beta < \delta \}$ is in HS which follows using κ -completeness of \mathcal{F} as done in [37, Lemma 1]. \square

Remark 3.5 We note that we are using the definition of a $(\delta + 1)$ -strategically closed forcing notion from [19, Definition 5.15] which is different from the definition used in [25]. In particular, in our case a forcing notion is δ -strategically closed if in the two person game in which the players construct a descending sequence $\langle p_\alpha : \alpha < \delta \rangle$, where player I plays odd stages and player II plays even and limit stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued; a forcing notion is $(\delta + 1)$ -strategically closed if the corresponding game has length $\delta + 1$. Whereas Gitman and Johnstone defined that a forcing notion is $\leq \delta$ -strategically closed if in the game of ordinal length $\delta + 1$ in which two players alternatively select conditions from it to construct a descending $\delta + 1$ -sequence with the second player playing at even and limit stages, the second player has a strategy that allows her to always continue playing (cf. [25, the paragraph before Theorem 2.2]). Thus in our case if V is a model of $ZF + DC_\delta$ and \mathbb{P} is $(\delta + 1)$ -strategically closed then DC_δ is preserved in $V[G]$ by [25, Theorem 2.2].

Remark 3.6 Let V be a model of $ZF + DC_\kappa$. Suppose \mathbb{P} is well-orderable of order type at most κ and has the κ -c.c. property. We remark that if \mathcal{F} is κ -complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V in terms of the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$. Let G be a \mathbb{P} -generic filter over V . By [25, Theorem 2.1], DC_κ is preserved in $V[G]$. The rest follows from the proof of [35, Lemma 3.3].

Question 3.7 *Let V be a model of $ZF + DC_\kappa$. Suppose that \mathbb{P} is κ -distributive. Can we preserve DC_κ in every forcing extension $V[G]$ by \mathbb{P} ?*

If the answer is in the affirmative, then we can say that if V is a model of $ZF + DC_\kappa$, \mathbb{P} is κ -distributive, and \mathcal{F} is κ -complete, then $DC_{<\kappa}$ is preserved in the symmetric extension in terms of the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ following Lemma 3.2.

3.1 Number of normal measures a successor cardinal can carry and dependent choice

Takeuti [44] and Jech [33] independently proved that if we assume the consistency of “ZFC + there is a measurable cardinal” then the theory “ZF+DC + \aleph_1 is a measurable cardinal” is consistent. Dimitriou [21, section 1.33] modified Jech’s construction and proved that if we assume the consistency of “ZFC + there is a measurable cardinal κ and $\gamma < \kappa$ is a regular cardinal” then the theory “ZF + the cardinality of γ is preserved + γ^+ is a measurable cardinal” is consistent. Apter, Dimitriou, and Koepke [3] constructed symmetric models in which for an arbitrary ordinal ρ , $\aleph_{\rho+1}$ can be the least measurable as well as the least regular uncountable cardinal. Bilinsky and Gitik [17] proved that if we assume the consistency of “ZFC+GCH + there is a measurable cardinal κ ” then we can obtain a symmetric extension where κ is a measurable cardinal without a normal measure. Assuming the consistency of “ZFC+GCH + there is a measurable cardinal”, we can construct models of ZF+DC where successor of regular cardinals like $\aleph_1, \aleph_2, \aleph_{\omega+2}$, as well as \aleph_{ω_1+2} , can carry an arbitrary (non-zero) number of normal measures.

Friedman–Magidor [22, Theorem 1] proved that a measurable cardinal can be forced to carry arbitrary number of normal measures in ZFC.

Lemma 3.8 (Friedman and Magidor; [22, Theorem 1]) *Assume GCH. Suppose that κ is a measurable cardinal and let α be a cardinal at most κ^{++} . Then in a cofinality-preserving forcing extension, κ carries exactly α normal measures.*

We recall the definition of a *symmetric collapse* from [27].

Definition 3.9 (*Symmetric Collapse*; [27, Definition 4.1]) Let $\kappa \leq \lambda$ be two infinite cardinals. The *symmetric collapse* is the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ defined as follows.

- $\mathbb{P} = Col(\kappa, < \lambda)$, so a condition in \mathbb{P} is a partial function p with domain $\{ \langle \alpha, \beta \rangle : \kappa < \alpha < \lambda, \beta < \kappa \}$ such that $p(\alpha, \beta) < \alpha$ for all α and β , $\text{supp}(p) = \{ \alpha < \lambda : \exists \beta, \langle \alpha, \beta \rangle \in \text{dom } p \}$ is bounded below λ and $|p| < \kappa$.
- \mathcal{G} is the group of automorphisms π such that there is a sequence of permutations $\vec{\pi} = \langle \pi_\alpha : \kappa < \alpha < \lambda \rangle$ such that π_α is a permutation of α satisfying $\pi p(\alpha, \beta) = \pi_\alpha(p(\alpha, \beta))$.
- \mathcal{F} is the normal filter of subgroups generated by $\text{fix}(E)$ for bounded $E \subseteq \lambda$, where $\text{fix}(E)$ is the group $\{ \pi : \forall \alpha \in E, \pi p(\alpha, \beta) = p(\alpha, \beta) \}$.

Lemma 3.10 *Let $\kappa \leq \lambda$ be two infinite cardinals such that $cf(\lambda) \geq \kappa$ and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is the symmetric collapse where $\mathbb{P} = Col(\kappa, < \lambda)$. Then, \mathcal{F} is κ -complete.*

Proof Fix $\gamma < \kappa$ and let, for each $\beta < \gamma$, $K_\beta \in \mathcal{F}$. There must be bounded $E_\beta \subseteq \lambda$ for each $\beta < \gamma$ such that $\text{fix } E_\beta \subseteq K_\beta$. Next, $\text{fix}(\bigcup_{\beta < \gamma} E_\beta) \subseteq \bigcap_{\beta < \gamma} \text{fix } E_\beta \subseteq \bigcap_{\beta < \gamma} K_\beta$. Since $cf(\lambda) \geq \kappa$, $\bigcup_{\beta < \gamma} E_\beta$ is a bounded subset of λ . Consequently, $\bigcap_{\beta < \gamma} K_\beta \in \mathcal{F}$. □

We observe that after a symmetric collapse, the successor of a regular cardinal can be a measurable cardinal carrying an arbitrary (non-zero) number of normal measures assuming the consistency of a measurable cardinal. Further we can preserve Dependent Choice in certain cases.

Theorem 3.11 *Let V be a model of $ZFC + GCH$ with a measurable cardinal κ . Let λ be any non-zero cardinal at most κ^{++} and let $\eta \leq \kappa$ be regular. Then, there is a symmetric extension where $\kappa = \eta^+$ is a measurable cardinal carrying λ normal measures. Moreover, AC_κ fails and $DC_{<\eta}$ holds⁶ in the symmetric model.*

Proof Applying Lemma 3.8, we obtain a cofinality-preserving forcing extension V' of V where κ is a measurable cardinal with λ many normal measures. Let $V'(G)$ be the symmetric extension of V' obtained by the symmetric collapse $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ where $\mathbb{P} = Col(\eta, < \kappa)$ and G is a \mathbb{P} -generic filter over V' . In $V'(G)$, $\kappa = \eta^+$. We can also have the following in $V'(G)$.

- By [11, Lemmas 2.4, 2.5], κ remains a measurable cardinal with λ many normal measures.
- Since κ is a successor as well as a measurable cardinal, AC_κ fails using Lemma 2.20.
- We note that \mathcal{F} is η -complete by Lemma 3.10. Since \mathbb{P} is η -closed, $DC_{<\eta}$ holds using [37, Lemma 1].

□

Remark 3.12 The referee pointed out that $DC_{<\kappa}$ is preserved in $V'(G)$. Assuming that λ is regular, the proof of Lemma 3.10 gives that \mathcal{F} is λ -complete. Consequently, since κ is a regular cardinal in V' , \mathcal{F} is κ -complete. Since \mathbb{P} has the κ -c.c., by Lemma 3.1, $DC_{<\kappa}$ is preserved in $V'(G)$.

Remark 3.13 In [11, Theorem 1], starting with a model of “ $ZFC + GCH + o(\kappa) = \delta^{*+}$ ” for $\delta^* \leq \kappa^+$ any finite or infinite cardinal, Apter constructed a model of $ZF + DC_{<\kappa}$ where κ carries exactly δ^* normal measures and $2^\delta = \delta^{++}$ on a set having measure one with respect to every normal measure over κ . We observe that we can obtain the result of [11, Theorem 1] starting from just one measurable cardinal κ if we use Lemma 3.8 instead of passing to an inner model of Mitchell from [42] as done in the proof of [11, Theorem 1]. In particular, we can prove the following.

Corollary 3.14 (of [11, Theorem 1]) *Let V be a model of $ZFC + GCH$ with a measurable cardinal κ and let λ be a cardinal at most κ^{++} . Then there is a model of $ZF + DC_{<\kappa}$ where κ is a measurable cardinal carrying λ many normal measures $\langle U_\alpha^* : \alpha < \lambda \rangle$. Moreover, we have $2^\delta = \delta^{++}$ on a set having measure one with respect to any of the measures U_α^* .*

4 The Proof of Theorem 1.5

In this section we prove Theorem 1.5.

⁶ If we assume $\eta > \omega$.

Proof (of Theorem 1.5) Firstly, we give a description of the symmetric extension constructed in [35, Theorem 4.1] as follows.

(1) **The ground model** (V). At the beginning of the proof of [9, Theorem 3], from the given requirements, Apter constructed a model V where there is an enumeration $\langle \kappa_i : i < \kappa \rangle$ of $C \cup \{\omega\}$ where $C \subseteq \kappa$ is a club of inaccessible and limit cardinals below a supercompact cardinal κ such that $2^{\kappa_i} = \kappa_i^{++}$ holds. We consider V to be our ground model. For reader’s convenience we recall the steps from the proof of [9, Theorem 3] as follows.

- Let V be a model of ZFC + GCH with a supercompact cardinal κ .
- Let \mathbb{Q}_1 be Laver’s partial ordering of [41] which makes κ ’s supercompactness indestructible under κ -directed closed forcing. Since \mathbb{Q}_1 may be defined so that $|\mathbb{Q}_1| = \kappa$, we have $V^{\mathbb{Q}_1 * \text{Add}(\kappa, \kappa^{++})} = V_2$ is a model of “ZFC + κ is supercompact + $2^\kappa = \kappa^{++} + 2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa^+$ ”.
- Let $\mathbb{Q}_3 \in V_2$ be the Radin forcing defined over κ . Taking a suitable measure sequence will enable one to preserve the supercompactness of κ (cf. [24]). Consequently, $V_2^{\mathbb{Q}_3} = \bar{V}$ is a model of “ZFC + κ is supercompact + $2^\kappa = \kappa^{++} + 2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa^+$ + There is a club $C \subseteq \kappa$ composed of inaccessible cardinals and their limits with $2^\delta = 2^{\delta^+} = \delta^{++}$ for every $\delta \in C$ ”.
- For the sake of convenience we consider the ground model to be $\bar{V} = V$. Let $\langle \kappa_i : i < \kappa \rangle \in V$ be the continuous, increasing enumeration of $C \cup \{\omega\}$.

(2) **Defining the symmetric system** $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.

- Let \mathbb{P} be the Easton support product of $\mathbb{P}_\alpha = \text{Col}(\kappa_\alpha^{++}, < \kappa_{\alpha+1})$ where $\alpha < \kappa$.
- Let \mathcal{G} be the Easton support product of the automorphism groups of each \mathbb{P}_α .
- Let \mathcal{F} be the filter generated by the groups of the form $\text{fix}(\alpha)$ for $\alpha < \kappa$, where $\text{fix}(\alpha) = \{\pi \in \prod_{\beta < \kappa} \text{Aut}(\mathbb{P}_\beta) : \pi \upharpoonright \alpha = \text{id}\}$.

(3) **Defining the symmetric extension of V** . Let G be a \mathbb{P} -generic filter over V . We consider the symmetric extension $V(G)^{\mathcal{F}}$ with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ defined above in (2) and denote it by \mathcal{N}_1 for the sake of our convenience.

Since \mathcal{F} is κ -complete, and \mathbb{P} has the κ -c.c., $\text{DC}_{< \kappa}$ is preserved in \mathcal{N}_1 by Lemma 3.1 (cf. [35, section 4.1]). Since each \mathbb{P}_α is weakly homogeneous, the following lemma holds as a corollary of Lemma 2.16. □

Lemma 4.1 *If $A \in \mathcal{N}_1$ is a set of ordinals, then $A \in V[G \upharpoonright \alpha]$ for some $\alpha < \kappa$.*

We apply Lemma 4.1 to prove that κ remains supercompact in \mathcal{N}_1 following the methods of [29]. Inamder [29] proved that if we assume the consistency of “ZFC + there is a supercompact cardinal κ , and $\gamma < \kappa$ is a regular cardinal” then the theory “ZF + the cardinality of γ is preserved + γ^+ is a supercompact cardinal” is consistent. We recall the relevant lemmas and incorporate the arguments from [29] in order to show that κ remains supercompact in \mathcal{N}_1 .

Lemma 4.2 (cf. [29, Lemma 26]) *Let κ be a regular cardinal, let $\gamma \geq \kappa$, and let \mathbb{P} be a forcing notion of size less than κ . Then for every $C \in \mathcal{P}_\kappa(\gamma)^{V[G]}$, there is a $D \in \mathcal{P}_\kappa(\gamma)^V$ such that in $V[G]$, $C \subseteq D$.*

Lemma 4.3 (Lévy–Solovay Lemma; [29, Lemma 27]) *In V , let κ be a regular cardinal, D be a set and \mathcal{U} be a κ -complete ultrafilter on D . Let \mathbb{P} be a forcing notion of size less than κ and G be a \mathbb{P} -generic filter over V . Suppose $V[G] \models f : D \rightarrow V$. Then there is an $S \in \mathcal{U}$ and a $g : S \rightarrow V$ in V such that $V[G] \models f \upharpoonright S = g$.*

Applying Lemmas 4.1 and 4.3 we obtain the following lemma, which is similar to [29, Lemma 33].

Lemma 4.4 *Let D be a set and \mathcal{U} be a κ -complete ultrafilter on D in V . Suppose $\mathcal{N}_1 \models f : D \rightarrow V$. Then there is an $S \in \mathcal{U}$ and a $g : S \rightarrow V$ in V such that $\mathcal{N}_1 \models f \upharpoonright S = g$.*

Proof By Lemma 4.1, for some $\alpha < \kappa$ we get $f \in V[G \upharpoonright \alpha]$. Now we can say $G \upharpoonright \alpha$ is \mathbb{P}' -generic over V where $|\mathbb{P}'| < \kappa$. By Lemma 4.3, we obtain an $S \in \mathcal{U}$ and a $g : S \rightarrow V$ in V such that $V[G \upharpoonright \alpha] \models f \upharpoonright S = g$. So, $\mathcal{N}_1 \models f \upharpoonright S = g$. \square

Similarly [29, Lemma 34], we obtain the following lemma by applying Lemma 4.4.

Lemma 4.5 *In V , let D be a set and \mathcal{U} be a κ -complete ultrafilter on D . Let \mathcal{W} be the filter on D generated by \mathcal{U} in \mathcal{N}_1 . Then \mathcal{W} is a κ -complete ultrafilter.*

We follow the proof of [29, Theorem 35] and refer the reader to [29] for further details.

Lemma 4.6 *In \mathcal{N}_1 , κ is supercompact.*

Proof Let $\gamma \geq \kappa$ be arbitrary. Since κ is supercompact in V , there is a normal measure \mathcal{U} on $\mathcal{P}_\kappa(\gamma)$ in V . Let \mathcal{V} be the κ -complete measure it generates on $\mathcal{P}_\kappa(\gamma)^V$ in \mathcal{N}_1 . Let \mathcal{W} be the filter generated by \mathcal{V} on $\mathcal{P}_\kappa(\gamma)$ in \mathcal{N}_1 . Since \mathcal{W} is generated by a κ -complete ultrafilter on $\mathcal{P}_\kappa(\gamma)^V \subseteq \mathcal{P}_\kappa(\gamma)$, \mathcal{W} is a κ -complete ultrafilter by Lemma 4.5.

Fineness: Let $X \in \mathcal{P}_\kappa(\gamma)^{\mathcal{N}_1}$. By Lemma 4.1, for some $\alpha < \kappa$ we have $X \in V[G \upharpoonright \alpha]$. Since κ is not collapsed while going from V to $V[G \upharpoonright \alpha]$, $X \in \mathcal{P}_\kappa(\gamma)^{V[G \upharpoonright \alpha]}$. By Lemma 4.2 (and following the arguments in the proof of [29, Theorem 35]), $\hat{X} \in \mathcal{V}'$, where \mathcal{V}' is the fine measure that \mathcal{U} generates on $\mathcal{P}_\kappa(\gamma)^{V[G \upharpoonright \alpha]}$. Now $\mathcal{U} \subseteq \mathcal{V}' \subseteq \mathcal{W}$ since $\mathcal{P}_\kappa(\gamma)^{V[G \upharpoonright \alpha]} \subseteq \mathcal{P}_\kappa(\gamma)^{\mathcal{N}_1}$. Consequently \mathcal{W} is fine.

Choice function: Let $\mathcal{N}_1 \models f : \mathcal{P}_\kappa(\gamma) \rightarrow \gamma$ and $\mathcal{N}_1 \models \forall X \in \mathcal{P}_\kappa(\gamma)(f(X) \in X)$. By Lemma 4.1, for some $\alpha < \kappa$ we get $h = f \upharpoonright \mathcal{P}_\kappa(\gamma)^V \in V[G \upharpoonright \alpha]$. By Lemma 4.3, we get $Y \in \mathcal{U}$ and $(g : Y \rightarrow \gamma)^V$ such that $V[G \upharpoonright \alpha] \models h \upharpoonright Y = g$. Now by normality of \mathcal{U} in V we get a set x in \mathcal{U} such that g is constant on x , and so h is constant on a set in \mathcal{U} . Hence, we obtain a set y in \mathcal{W} such that f is constant on y . \square

Lemma 4.7 *In \mathcal{N}_1 , AC_κ fails.*

Proof Since the cardinality of κ_α^{++} is preserved in \mathcal{N}_1 for $\alpha < \kappa$, we can define in \mathcal{N}_1 the set $X_\alpha = \{x \subseteq \kappa_\alpha^{++} : x \text{ codes a well ordering of } (\kappa_\alpha^{+++})^V \text{ of order type } \kappa_\alpha^{++}\}$. We claim that $\langle X_\alpha : \alpha < \kappa \rangle \in \mathcal{N}_1$. The sets X_α have fully symmetric names \dot{X}_α (any permutation of a name for an element of \dot{X}_α returns a name for an element of \dot{X}_α). Let $\dot{X} = \{\dot{X}_\alpha : \alpha < \kappa\}$. Consequently, $\text{sym}(\dot{X}) \in \mathcal{F}$, i.e., \dot{X} is symmetric with respect

to \mathcal{F} . Since all the names appearing in \dot{X} are from HS , $\dot{X} \in HS$. Consequently, $\langle X_\alpha : \alpha < \kappa \rangle \in \mathcal{N}_1$.

Although each $X_\alpha \neq \emptyset$, we claim that $(\prod_{\alpha < \kappa} X_\alpha)^{\mathcal{N}_1} = \emptyset$. Otherwise let $y \in (\prod_{\alpha < \kappa} X_\alpha)^{\mathcal{N}_1}$. Since y is a sequence of sets of ordinals, so can be coded as a set of ordinals. Then there is a $\gamma < \kappa$ such that $y \in V[G \upharpoonright \gamma]$ by Lemma 4.1 and $V[G \upharpoonright \gamma]$ is \mathbb{P} -generic over V such that $|\mathbb{P}| < \kappa$. So there is a final segment of the sequence $\langle (\kappa_\alpha^{+++}) : \alpha < \kappa \rangle$ which remains a sequence of cardinals in $V[G \upharpoonright \gamma]$ which is a contradiction. □

Remark 4.8 Since GCH implies AC, GCH is weakened to a form which states that *there is no injection from δ^{++} into $\mathcal{P}(\delta)$* in [9, Theorem 3]. We follow this weakened version of GCH in our case. We follow the explanation given in [35, section 4.1] by Karagila, to see that in \mathcal{N}_1 , GCH holds for a limit cardinal δ if and only if $\delta > \kappa$.

The referee suggested us to remark the following. In the context of ZF, there are two reasonable definitions for the statement “GCH at μ ”.

- (1) There is no injection $\mu^{++} \rightarrow^{inj} \mathcal{P}(\mu)$.
- (2) There is no surjection $\mathcal{P}(\mu) \rightarrow^{sur} \mu^{++}$.

In ZF, it is possible that there is no $\mu^+ \rightarrow^{inj} \mathcal{P}(\mu)$, but there is always a surjection $\mathcal{P}(\mu) \rightarrow^{sur} \mu^+$. In our case the above two definitions behave the same, so the referee suggested us to remark that both definitions (1) and (2) work, by the same proof.

5 Proving Dimitriou’s conjecture

Fix an arbitrary $n_0 \in \omega$. Apter [14, Theorem 1] obtained a model of $ZF + \neg DC_{\aleph_{n_0+1}}$ where \aleph_ω carries a Rowbottom filter and $DC_{\aleph_{n_0}}$ holds, from an ω -sequence of measurable cardinals. In Sect. 8, we observe that there is an alternating sequence of measurable and non-measurable cardinals in that model. Apter constructed the model based on Easton support products of Lévy collapse. Consequently, $DC_{\aleph_{n_0}}$ was preserved (cf. [14, Lemma 1.4]). Dimitriou [21, section 1.4] constructed a similar model with an alternating sequence of measurable and non-measurable cardinals, excluding the singular limits. She constructed the model based on finite support products of collapsing functions, unlike the model from [14]. In [21], Dimitriou claimed that by using such a finite support product construction, a lot of arguments could be made easier. In particular, she used finite support products of *injective tree-Prikry forcings*, in several constructions from [21, Chapter 2]. There are many models of ZF constructed using the finite support products of Lévy collapse. Hayut and Karagila [27, Theorem 5.6] considered a symmetric extension constructed using the finite support products of Lévy collapse. In Sect. 6, we encounter two models of ZF constructed using the finite support products of Lévy collapse due to Apter and Cody from [1] (cf. [5, Theorem 2] as well). On the other hand, there is a downside to this method. Specifically, Dimitriou conjectured that DC_ω would fail in the model. In this section, we prove that AC_ω fails in the model and thus prove the conjecture of Dimitriou. In other words, we prove Theorem 1.6. We refer the reader to the terminologies from Sect. 2.4.

Proof (of Theorem 1.6) Firstly, we give a description of the symmetric extension constructed in [21, section 1.4] as follows.

(1) **The ground model** (V). Let V be a model of ZFC, ρ be an ordinal, and $\mathcal{K} = \langle \kappa_\epsilon : 0 < \epsilon < \rho \rangle$ be a sequence of measurable cardinals. Let κ_0 be a regular cardinal below all the measurable cardinals in \mathcal{K} .

(2) **Defining the triple** $(\mathbb{P}, \mathcal{G}, \mathcal{I})$.

- Let $\kappa'_1 = \kappa_0$. For each $1 < \epsilon < \rho$ we define the following cardinals,
 - $\kappa'_\epsilon = \kappa_{\epsilon-1}^+$ if ϵ is a successor ordinal,
 - $\kappa'_\epsilon = (\bigcup_{\zeta < \epsilon} \kappa'_\zeta)^+$ if ϵ is a limit ordinal and $\bigcup_{\zeta < \epsilon} \kappa'_\zeta$ is singular,
 - $\kappa'_\epsilon = (\bigcup_{\zeta < \epsilon} \kappa'_\zeta)^{++}$ if ϵ is a limit ordinal and $\bigcup_{\zeta < \epsilon} \kappa'_\zeta = \kappa_\epsilon$ is regular,
 - $\kappa'_\epsilon = \bigcup_{\zeta < \epsilon} \kappa'_\zeta$ if ϵ is a limit ordinal and $\bigcup_{\zeta < \epsilon} \kappa'_\zeta < \kappa_\epsilon$ is regular.

Let $\mathbb{P} = \prod_{0 < i < \rho} \mathbb{P}_i$ be the Easton support product of $\mathbb{P}_i = Fn(\kappa'_i, \kappa_i, \kappa'_i)$ ordered componentwise where for each $0 < i < \rho$, $Fn(\kappa'_i, \kappa_i, \kappa'_i) = \{p : \kappa'_i \rightarrow \kappa_i : |p| < \kappa'_i \text{ and } p \text{ is an injection}\}$ ordered by reverse inclusion. We denote by $p : \kappa'_i \rightarrow \kappa_i$ a partial function from κ'_i to κ_i .

- Let $\mathcal{G} = \prod_{0 < i < \rho} \mathcal{G}_i$ where for each $0 < i < \rho$, \mathcal{G}_i is the full permutation group of κ_i that can be extended to \mathbb{P}_i by permuting the range of its conditions, i.e., for all $a \in \mathcal{G}_i$ and $p \in \mathbb{P}_i$, $a(p) = \{(\psi, a(\beta)) : (\psi, \beta) \in p\}$.
- For every finite sequence of ordinals $e = \{\alpha_i : 1 \leq i \leq m\}$ such that for every $1 \leq i \leq m$ there is a distinct $0 < \epsilon_i < \rho$ such that $\alpha_i \in (\kappa'_{\epsilon_i}, \kappa_{\epsilon_i})$, we define $E_e = \{(\emptyset, \dots, p_{\epsilon_1} \cap (\kappa'_{\epsilon_1} \times \alpha_1), \emptyset, \dots, p_{\epsilon_2} \cap (\kappa'_{\epsilon_2} \times \alpha_2), \emptyset, \dots, p_{\epsilon_i} \cap (\kappa'_{\epsilon_i} \times \alpha_i), \emptyset, \dots, p_{\epsilon_m} \cap (\kappa'_{\epsilon_m} \times \alpha_m), \emptyset, \dots)\}$; $\vec{p} \in \mathbb{P}$ and $\mathcal{I} = \{E_e : e \in \prod_{0 < i < \rho}^{fin} (\kappa'_i, \kappa_i)\}$ where $\prod_{0 < i < \rho}^{fin} (\kappa'_i, \kappa_i)$ is the finite support product.

(3) **Defining the symmetric extension of V** . Clearly, \mathcal{I} is a projectable symmetry generator with projections $\vec{p} \upharpoonright E_e = \{(\emptyset, \dots, p_{\epsilon_1} \cap (\kappa'_{\epsilon_1} \times \alpha_1), \emptyset, \dots, p_{\epsilon_2} \cap (\kappa'_{\epsilon_2} \times \alpha_2), \emptyset, \dots, p_{\epsilon_m} \cap (\kappa'_{\epsilon_m} \times \alpha_m), \emptyset, \dots)\}$. Let \mathcal{I} generate a normal filter $\mathcal{F}_{\mathcal{I}}$ over \mathcal{G} . Let G be a \mathbb{P} -generic filter over V . We consider the symmetric model $V(G)^{\mathcal{F}_{\mathcal{I}}}$ as our desired symmetric extension.

It is possible to see that \mathbb{P} is $(\mathcal{G}, \mathcal{I})$ -homogeneous and so $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ has the approximation property. Consequently, by Lemma 2.16 for all set of ordinals $X \in V(G)^{\mathcal{F}_{\mathcal{I}}}$, there exists an $E_e \in \mathcal{I}$ such that $X \in V[G \cap E_e]$. Following [21, Lemma 1.35], for every $0 < \epsilon < \rho$, $(\kappa'_\epsilon)^+ = \kappa_\epsilon$ in $V(G)^{\mathcal{F}_{\mathcal{I}}}$. We prove that AC_ω fails in $V(G)^{\mathcal{F}_{\mathcal{I}}}$. For the sake of convenience we call $V(G)^{\mathcal{F}_{\mathcal{I}}}$ as $V(G)$, $HS^{\mathcal{F}_{\mathcal{I}}}$ as HS , and $\mathcal{F}_{\mathcal{I}}$ as \mathcal{F} . \square

Lemma 5.1 *In $V(G)$, AC_ω fails.*

Proof Since the cardinality of κ'_n is preserved in $V(G)$ for $n < \omega$, we can define in $V(G)$ the set $X_n = \{x \subseteq \kappa'_n : x \text{ codes a well ordering of } ((\kappa'_n)^+)^V \text{ of order type } \kappa'_n\}$. We claim that $\langle X_n : n < \omega \rangle \in V(G)$. The sets X_n have fully symmetric names \check{X}_n (any permutation of a name for an element of \check{X}_n returns a name for an element of \check{X}_n). Let $\check{X} = \{\check{X}_n : n < \omega\}$. Consequently, $sym(\check{X}) \in \mathcal{F}$, i.e., \check{X} is symmetric with respect to \mathcal{F} . Since all the names appearing in \check{X} are from HS , $\check{X} \in HS$. Consequently, $\langle X_n : n < \omega \rangle \in V(G)$.

Although $X_n \neq \emptyset$, we claim that $(\prod_{n < \omega} X_n)^{V(G)} = \emptyset$. Otherwise let $y \in (\prod_{n < \omega} X_n)^{V(G)}$. Since y is a sequence of sets of ordinals, so can be coded as a set

of ordinals. Thus, there is an $e = \{\alpha_1, \dots, \alpha_m\}$ such that $y \in V[G \cap E_e]$ by Lemma 2.16. There are distinct ϵ_i such that $\alpha_i \in (\kappa'_{\epsilon_i}, \kappa_{\epsilon_i})$ and let l be $\max\{\epsilon_i : \alpha_i \in e\}$ such that l is an integer. Next let $M = \{i : \epsilon_i \leq l\}$ and $M' = \{i : \epsilon_i > l\}$. Then $V[G \cap E_e]$ is $\prod_{i \in M} Fn(\kappa'_{\epsilon_i}, \alpha_i, \kappa'_{\epsilon_i}) \times \prod_{i \in M'} Fn(\kappa'_{\epsilon_i}, \alpha_i, \kappa'_{\epsilon_i})$ -generic over V . By closure properties of $\prod_{i \in M'} Fn(\kappa'_{\epsilon_i}, \alpha_i, \kappa'_{\epsilon_i})$, all elements of the sequence $\langle (\kappa'_n)^+ : n < \omega \rangle$ remain cardinals after forcing with $\prod_{i \in M'} Fn(\kappa'_{\epsilon_i}, \alpha_i, \kappa'_{\epsilon_i})$. Next, since M is finite we can find $j < \omega$ such that for all $r \geq j$, $|\prod_{i \in M} Fn(\kappa'_{\epsilon_i}, \alpha_i, \kappa'_{\epsilon_i})| < \kappa_r$. Thus, a final segment of the sequence $\langle (\kappa'_n)^+ : n < \omega \rangle$ remains a sequence of cardinals in $V[G \cap E_e]$ which is a contradiction. \square

Remark 5.2 Hayut and Karagila [27, Theorem 5.6] proved the following.

- Assuming the existence of countably many measurable cardinals, it is consistent that there is a uniform ultrafilter on \aleph_ω but for all $0 < n < \omega$, there are no uniform ultrafilters on \aleph_n .

They considered a symmetric extension M based on finite support product of the symmetric collapses $Col(\kappa_n, < \kappa_{n+1})$. Following the proof of Lemma 5.1, we can say that AC_ω fails in the symmetric extension M . We consider another similar symmetric extension. Let V_1 be a model of ZFC where $\langle \kappa_n : 1 \leq n < \omega \rangle$ is a countable sequence of supercompact cardinals. Let \mathbb{Q} be the forcing notion (see [10, 13]) which makes the supercompactness of each κ_n indestructible under κ_n -directed closed forcing notions. Let H be a \mathbb{Q} -generic filter over V_1 and $V = V_1[H]$ be our ground model. Let $\kappa_0 = \omega$ in V . Consider the symmetric extension \mathcal{N} obtained by taking the finite support product of the symmetric collapses $Col(\kappa_n, < \kappa_{n+1})$. In the resulting model \mathcal{N} the following hold:

- (1) Since the forcing notions involved are weakly homogeneous, if A is a set of ordinals in \mathcal{N} , then A was added by an intermediate submodel where AC holds.
- (2) For $n > 0$, each κ_n becomes \aleph_n in \mathcal{N} .

Following [27, Theorem 4.3], we can observe that for each $1 \leq n < \omega$, there are no uniform ultrafilters on \aleph_n in \mathcal{N} . Consequently for each $1 \leq n < \omega$, \aleph_n can not be a measurable cardinal in \mathcal{N} . Since we are considering symmetric extension based on finite support products, AC_ω fails following the proof of Lemma 5.1. We can see that each \aleph_n remains a Ramsey cardinal for $1 \leq n < \omega$ in \mathcal{N} . Fix $1 \leq n < \omega$. Let $f : [\kappa_n]^{<\omega} \rightarrow 2$ is in \mathcal{N} . Since f can be coded by a set of ordinals, f was added by an intermediate submodel (say V') where AC holds. Without loss of generality, we can say that $V' = V[G_1][G_2]$ where G_1 is \mathbb{Q}_1 -generic over V such that \mathbb{Q}_1 is κ_n -directed closed and G_2 is \mathbb{Q}_2 -generic over $V[G_1]$ such that $|\mathbb{Q}_2| < \kappa_n$. Since \mathbb{Q}_1 is κ_n -directed closed, κ_n remains supercompact in $V[G_1]$ as the supercompactness of κ_n was indestructible under κ_n -directed closed forcing notions in V . Consequently, κ_n remains a Ramsey cardinal in $V[G_1]$. Since $|\mathbb{Q}_2| < \kappa_n$, κ_n remains Ramsey in $V[G_1][G_2]$ by Theorem 2.5. There is then a set $X \in [\kappa_n]^{\kappa_n}$ homogeneous for f in V' , and since $V' \subseteq \mathcal{N}$, $X \in [\kappa_n]^{\kappa_n}$ is homogeneous for f in \mathcal{N} . Consequently, for each $1 \leq n < \omega$, κ_n is Ramsey in \mathcal{N} .

6 Reducing the assumption of supercompactness to strong compactness

6.1 Strongly compact Prikry forcing

Suppose $\lambda > \kappa$ and κ be a λ -strongly compact cardinal in the ground model V . Let \mathcal{U} be a κ -complete fine ultrafilter over $\mathcal{P}_\kappa(\lambda)$.

Definition 6.1 (cf. [24, Definition 1.51]) A set T is called a \mathcal{U} -tree with trunk t if and only if the following hold.

- (1) T consists of finite sequences $\langle P_1, \dots, P_n \rangle$ of elements of $\mathcal{P}_\kappa(\lambda)$ so that $P_1 \subseteq P_2 \subseteq \dots \subseteq P_n$.
- (2) $\langle T, \trianglelefteq \rangle$ is a tree, where \trianglelefteq is the order of the end extension of finite sequences.
- (3) t is a trunk of T , i.e., $t \in T$ and for every $\eta \in T$, $\eta \trianglelefteq t$ or $t \trianglelefteq \eta$.
- (4) For every $t \trianglelefteq \eta$, $Suc_T(\eta) = \{Q \in \mathcal{P}_\kappa(\lambda) : \eta \frown \langle Q \rangle \in T\} \in \mathcal{U}$.

The set $\mathbb{P}_\mathcal{U}$ consists of all pairs $\langle t, T \rangle$ such that T is a \mathcal{U} -tree with trunk t . If $\langle t, T \rangle, \langle s, S \rangle \in \mathbb{P}_\mathcal{U}$, we say that $\langle t, T \rangle$ is stronger than $\langle s, S \rangle$, and denote this by $\langle t, T \rangle \geq \langle s, S \rangle$, if and only if $T \subseteq S$. We call $\mathbb{P}_\mathcal{U}$ with the ordering defined above as the *strongly compact Prikry forcing* with respect to \mathcal{U} .⁷

Let G be a $\mathbb{P}_\mathcal{U}$ -generic filter over V . We summarize the necessary properties of $\mathbb{P}_\mathcal{U}$ from [24].

- By a Prikry like lemma and a similar proof as in the ordinary Prikry forcing,⁸ $\mathbb{P}_\mathcal{U}$ does not add new bounded subsets to κ (cf. [5, Lemma 1.1], [24, Theorem 1.52]).
- Every cardinal in $(\kappa, \lambda]$ is collapsed to have size κ in $V[G]$ (cf. [24, Lemma 1.50] and the arguments before [24, Theorem 1.52]).
- Every $\delta \in [\kappa, \mu]$ of cofinality $\geq \kappa$ (in V) changes its cofinality to ω in $V[G]$ (cf. [24, Lemma 1.50, Theorem 1.52]), where $\mu = \lambda$ if $cf(\lambda) \geq \kappa$ and $\mu = \lambda^+$ if $cf(\lambda) < \kappa$.
- Any two conditions with the same trunk, i.e. of the form $\langle t, T \rangle$ and $\langle t, S \rangle$ are compatible. Also there are $\lambda^{<\kappa}$ many possibilities for trunks for members of $\mathbb{P}_\mathcal{U}$.

⁷ Alternatively, we also recall the definition of a strongly compact Prikry forcing $\mathbb{P}_\mathcal{U}$ from [5]. Let \mathcal{U} be a fine measure on $\mathcal{P}_\kappa(\lambda)$ and $\mathcal{F} = \{f : f \text{ is a function from } [\mathcal{P}_\kappa(\lambda)]^{<\omega} \text{ to } \mathcal{U}\}$. In particular, $\mathbb{P}_\mathcal{U}$ is the set of all finite sequences of the form $\langle p_1, \dots, p_n, f \rangle$ satisfying the following properties.

- $\langle p_1, \dots, p_n \rangle \in [\mathcal{P}_\kappa(\lambda)]^{<\omega}$.
- for $0 \leq i < j \leq n$, $p_i \cap \kappa \neq p_j \cap \kappa$.
- $f \in \mathcal{F}$.

The ordering on $\mathbb{P}_\mathcal{U}$ is given by $\langle q_1, \dots, q_m, g \rangle \leq \langle p_1, \dots, p_n, f \rangle$ if and only if we have the following.

- $n \leq m$.
- $\langle p_1, \dots, p_n \rangle$ is the initial segment of $\langle q_1, \dots, q_m \rangle$.
- For $i = n + 1, \dots, m$, $q_i \in f(\langle p_1, \dots, p_n, q_{n+1}, \dots, q_{i-1} \rangle)$.
- For $\vec{s} \in [\mathcal{P}_\kappa(\lambda)]^{<\omega}$, $g(\vec{s}) \subseteq f(\vec{s})$.

For any regular $\delta \in [\kappa, \lambda]$, we denote $r \upharpoonright \delta = \{\langle p_0 \cap \delta, \dots, p_n \cap \delta \rangle : \exists f \in \mathcal{F}[\langle p_0, \dots, p_n, f \rangle \in G]\}$. In $V[r \upharpoonright \kappa] \subseteq V[G]$, κ is a singular cardinal having cofinality ω . Since any two conditions having the same stems are compatible, i.e. any two conditions of the form $\langle p_1, \dots, p_n, f \rangle$ and $\langle p_1, \dots, p_n, g \rangle$ are compatible., $\mathbb{P}_\mathcal{U}$ is $(\lambda^{<\kappa})^+$ -c.c.

⁸ i.e., the arguments of [24, Lemma 1.9].

Consequently, $\mathbb{P}_{\mathcal{U}}$ satisfies the $(\lambda^{<\kappa})^+$ -c.c. (cf. [24, Lemma 1.48] and the arguments before [24, Theorem 1.52]).

6.2 The Proofs of Observations 1.7 and 1.8

In this subsection, we prove Observations 1.7 and 1.8.

Proof (of Observation 1.7) We perform the construction in two stages. In the first stage, we consider a model similar to the choiceless model constructed in [5, section 2].

- (1) **The ground model** (V): We start with a model V_0 of ZFC where κ is a strongly compact cardinal, θ is an ordinal, and GCH holds. By [4, Theorem 3.1] we can obtain a forcing extension V_1 in which the strong compactness of κ is indestructible under $Add(\kappa, \theta)$ for all θ . Then forcing with $Add(\kappa, \theta)$, we may assume without loss of generality that κ is strongly compact and $2^\kappa = \theta$ in a forcing extension V of V_1 . Let λ be a cardinal in V such that $\kappa < \lambda$ and $(cf(\lambda))^V < \kappa$.
- (2) **Defining an inner model of a forcing extension of V :**

- Let \mathcal{U} be a fine measure on $\mathcal{P}_\kappa(\lambda)$ and $\mathbb{P} = \mathbb{P}_{\mathcal{U}}$ be the strongly compact Prikrý forcing. Let G be a $\mathbb{P}_{\mathcal{U}}$ -generic filter over V .
- We consider a model similar to the choiceless model constructed in [5, section 2]. In particular, we consider our model \mathcal{N} to be the least model of ZF extending V and containing $r \restriction \delta$ for each regular $\delta \in [\kappa, \lambda)$ where $r \restriction \delta = \{\langle p_0 \cap \delta, \dots, p_n \cap \delta \rangle : \exists f \in \mathcal{F}[\langle p_0, \dots, p_n, f \rangle \in G]\}$ but not the λ -sequence of $r \restriction \delta$'s.

□

We follow the homogeneity of strongly compact Prikrý forcing mentioned in [5, Lemma 2.1] to observe the following lemma.

Lemma 6.2 *If $A \in \mathcal{N}$ is a set of ordinals, then $A \in V[r \restriction \delta]$ for some regular $\delta \in [\kappa, \lambda)$.*

Lemma 6.3 *In \mathcal{N} , κ is a strong limit cardinal.*

Proof Since $V \subseteq \mathcal{N} \subseteq V[G]$ and \mathbb{P} does not add bounded subsets to κ , V and \mathcal{N} have the same bounded subsets of κ .⁹ Consequently, in \mathcal{N} , κ is a limit of inaccessible cardinals and thus a strong limit cardinal as well. □

As explained in the introduction, our definitions of “strong limit cardinal” and “inaccessible cardinal” generally do not make sense in choiceless models. In spite of that, we can see that the assertion in Lemma 6.3 makes sense (see the paragraph after [1, Theorem 1]). Since \mathcal{N} and V have the same bounded subsets of κ , the usual definitions of “ κ is a strong limit cardinal” and “ $\delta < \kappa$ is an inaccessible cardinal” make sense in \mathcal{N} .

⁹ We can observe another argument from [5, Lemma 2.2].

Lemma 6.4 *If $\gamma \geq \lambda$ is a cardinal in V , then γ remains a cardinal in \mathcal{N} .*

Proof For the sake of contradiction, let γ is not a cardinal in \mathcal{N} . There is then a bijection $f : \alpha \rightarrow \gamma$ for some $\alpha < \gamma$ in \mathcal{N} . Since f can be coded by a set of ordinals, by Lemma 6.2 we have $f \in V[r \upharpoonright \delta]$ for some regular $\delta \in [\kappa, \lambda)$. Since GCH is assumed in V_0 we have $(\delta^{<\kappa})^{V_0} = \delta$, and since $Add(\kappa, \theta)$ preserves cardinals and adds no sequences of ordinals of length less than κ , we conclude that $(\delta^{<\kappa})^V = (\delta^{<\kappa})^{V_0} = \delta$. Now $\mathbb{P}_{\mathcal{U} \upharpoonright \delta}$ is $(\delta^{<\kappa})^+$ -c.c. in V and hence δ^+ -c.c. in V . Consequently, γ is a cardinal in $V[r \upharpoonright \delta]$ which is a contradiction. \square

Lemma 6.5 *In \mathcal{N} , $cf(\kappa) = \omega$. Moreover, $(\kappa^+)^{\mathcal{N}} = \lambda$ and $cf(\lambda)^{\mathcal{N}} = cf(\lambda)^V$.*

Proof For each regular $\delta \in [\kappa, \lambda)$, we have $V[r \upharpoonright \delta] \subseteq \mathcal{N}$. Consequently, $cf(\kappa)^{\mathcal{N}} = \omega$ since $cf(\kappa)^{V[r \upharpoonright \kappa]} = \omega$. Following [5, Lemma 2.4], every ordinal in (κ, λ) which is a cardinal in V collapses to have size κ in \mathcal{N} , and so $(\kappa^+)^{\mathcal{N}} = \lambda$. Since V and \mathcal{N} have same bounded subsets of κ , we see that $cf(\lambda)^{\mathcal{N}} = cf(\lambda)^V < \kappa$. \square

We can see that since, $V \subseteq \mathcal{N}$ and $(2^\kappa = \theta)^V$, there is a θ -sequence of distinct subsets of κ in \mathcal{N} . Since $cf(\kappa^+)^{\mathcal{N}} < \kappa$ we can also see that AC_κ fails in \mathcal{N} .

In the second stage, we consider an inner model of a forcing extension of \mathcal{N} based on a product of Lévy collapses as done in the proof of [1, Theorem 2].

- (1) **The ground model (\mathcal{N}).** Consider the ground model to be \mathcal{N} . Let $\langle \kappa_n : n < \omega \rangle$ be a sequence of inaccessible cardinals less than κ which is cofinal in κ .
- (2) **Defining an inner model of a forcing extension of \mathcal{N} .**
 - Let $\mathbb{P} = Col(\omega, < \kappa)$, and let G be \mathbb{P} -generic over \mathcal{N} . Let $\mathbb{P}_n = Col(\omega, < \kappa_n)$. Following the proof of [1, Theorem 2], $G_n = G \cap \mathbb{P}_n$ is \mathbb{P}_n -generic over \mathcal{N} .
 - Let \mathcal{M} be the least model of ZF extending \mathcal{N} containing each G_n , but not G as constructed in [1, Theorem 2].

Following the proof of [1, Theorem 2], we have the following in \mathcal{M} .

- (1) Since \mathcal{M} contains G_n for each n , cardinals in $[\omega, \kappa)$ are collapsed to have size ω and so $\aleph_1^{\mathcal{M}} \geq \kappa$.
- (2) If $x \in \mathcal{M}$ is a set of ordinals, then $x \in \mathcal{N}[G_n]$ for some $n < \omega$.
- (3) Since $Col(\omega, < \kappa_n)$ is canonically well-orderable in \mathcal{N} with order type κ_n , cardinals and cofinalities greater than or equal to κ are preserved to $\mathcal{N}[G_n]$.
- (4) Since κ is not collapsed, $\kappa = \aleph_1^{\mathcal{M}}$, $cf(\aleph_1)^{\mathcal{M}} = cf(\aleph_2)^{\mathcal{M}} = \omega$. Consequently, AC_ω fails in \mathcal{M} .
- (5) There is a sequence of distinct subsets of \aleph_1 of length θ .

\square

Proof (of Observation 1.8) We recall the model \mathcal{N} from the previous proof. We consider an inner model of the forcing extension of \mathcal{N} as done in the proof of [1, Theorem 3].

- (1) **The ground model (\mathcal{N}).** Consider the ground model to be \mathcal{N} as in the previous proof. Let $\langle \kappa_n : n < \omega \rangle$ be a sequence of inaccessible cardinals less than κ which is cofinal in κ .

(2) **Defining an inner model of a forcing extension of \mathcal{N} .**

- Let $\mathbb{P}_0 = Col(\omega, < \kappa_0)$, $\mathbb{P}_i = Col(\kappa_{i-1}, < \kappa_i)$ for $1 \leq i < \omega$. Let $\mathbb{P} = \prod_{i < \omega}^{fin} \mathbb{P}_i$ be a finite support product of \mathbb{P}_i . For each $n < \omega$, we can factor \mathbb{P} as $\mathbb{P} \cong \mathbb{P}_n^* \times \mathbb{P}^n$ where $\mathbb{P}_n^* = \prod_{0 \leq i \leq n}^{fin} \mathbb{P}_i$ and $\mathbb{P}^n = \prod_{n+1 \leq i < \omega}^{fin} \mathbb{P}_i$. Let $G \cong G_n^* \times G^n$ be a \mathbb{P} -generic filter over \mathcal{N} . Following [1, Theorem 3], each G_n^* is \mathbb{P}_n^* -generic over \mathcal{N} .
- Let \mathcal{M} be the least model of ZF extending \mathcal{N} containing each G_n^* , but not $\langle G_n^* : n < \omega \rangle$ as constructed in [1, Theorem 3].

Following the proof of [1, Theorem 3], we have the following in \mathcal{M} .

- (1) Since $G_n^* \in \mathcal{M}$ for each $n < \omega$, we have $\aleph_\omega \geq \kappa$ and hence $\aleph_{\omega+1} \geq (\kappa^+)^{\mathcal{N}}$ in \mathcal{M} .
- (2) If x is a set of ordinals in \mathcal{M} , then $x \in \mathcal{N}[G_n^*]$ for some $n < \omega$ (cf. [1, Lemma 6]).
- (3) Since \mathcal{N} and V contain the same bounded subsets of κ , and $V \subseteq \mathcal{N}$, \mathbb{P}_n^* can be well-ordered in both V and \mathcal{N} with order type less than κ . Therefore, cardinals and cofinalities greater than or equal to κ are preserved.
- (4) $\kappa = \aleph_\omega$ and $(\kappa^+)^{\mathcal{N}} = \aleph_{\omega+1}$ are both singular with $\omega \leq cf(\aleph_{\omega+1}) < \aleph_\omega$.
- (5) AC_ω fails in \mathcal{M} .
- (6) There is a sequence of distinct subsets of \aleph_ω of length θ .

□

7 Infinitary Chang conjecture from a measurable cardinal

7.1 Infinitary Chang conjecture

We recall the required definitions and relevant lemmas from [21, Chapter 3]. For the sake of our convenience we denote a structure \mathcal{A} on domain A as $\mathcal{A} = \langle A, \dots \rangle$.

Definition 7.1 (*Set of good indiscernibles*; [21, Definition 3.2]) For a structure $\mathcal{A} = \langle A, \dots \rangle$ with $A \subseteq Ord$, a set $I \subseteq A$ is a *set of indiscernibles* if for all $n < \omega$, all n -ary formula ϕ in the language for \mathcal{A} , and every $\alpha_1, \dots, \alpha_n, \alpha'_1, \dots, \alpha'_n$ in I , if $\alpha_1 < \dots < \alpha_n$ and $\alpha'_1 < \dots < \alpha'_n$ then

$$\mathcal{A} \models \phi(\alpha_1, \dots, \alpha_n) \text{ if and only if } \mathcal{A} \models \phi(\alpha'_1, \dots, \alpha'_n).$$

The set I is a *set of good indiscernibles* if and only if it is a set of indiscernibles and we allow parameters that lie below $\min\{\alpha_1, \dots, \alpha_n, \alpha'_1, \dots, \alpha'_n\}$, i.e., if for every $x_1, \dots, x_m \in A$ such that $x_1, \dots, x_m \leq \min\{\alpha_1, \dots, \alpha_n, \alpha'_1, \dots, \alpha'_n\}$ and every $(n + m)$ -ary formula ϕ ,

$$\mathcal{A} \models \phi(x_1, \dots, x_m, \alpha_1, \dots, \alpha_n) \text{ if and only if } \mathcal{A} \models \phi(x_1, \dots, x_m, \alpha'_1, \dots, \alpha'_n).$$

Definition 7.2 (α -Erdős cardinal; [21, Definition 0.14]) The partition relation $\alpha \rightarrow (\beta)_\delta^\gamma$ for ordinals $\alpha, \beta, \gamma, \delta$ means for all $f : [\alpha]^\gamma \rightarrow \delta$ there is an $X \in [\alpha]^\beta$ such that X is homogeneous for f . For an infinite ordinal α , the α -Erdős cardinal $\kappa(\alpha)$ is the least κ such that $\kappa \rightarrow (\alpha)_2^{<\omega}$.

For cardinals $\kappa > \lambda$ and ordinal $\theta < \kappa$ we mean $\kappa \rightarrow^\theta (\lambda)_2^{<\omega}$ if for every first order structure $\mathcal{A} = \langle \kappa, \dots \rangle$ with a countable language, there is a set $I \in [\kappa \setminus \theta]^\lambda$ of good indiscernibles for \mathcal{A} (cf. [21, Definition 3.7]).

Definition 7.3 (Infinitary Chang conjecture; [21, Definition 3.10]) Let $\langle \kappa_n : n < \omega \rangle$ and $\langle \lambda_n : n < \omega \rangle$ be two increasing sequences of cardinals such that $\kappa_n > \lambda_n$ for every $n < \omega$. The *infinitary Chang conjecture* for these sequences, written as $(\kappa_n)_{n \in \omega} \rightarrow (\lambda_n)_{n \in \omega}$, is the statement “for every first order structure $\mathcal{A} = \langle \bigcup_{n < \omega} \kappa_n, \dots \rangle$ there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ with domain B of cardinality $\bigcup_{n < \omega} \lambda_n$ such that for every $n \in \omega$, $|B \cap \kappa_n| = \lambda_n$ ”.

Definition 7.4 (cf. [21, Definition 3.14]) Let $\langle \kappa_i : i < \omega \rangle$ and $\langle \lambda_i : 0 < i < \omega \rangle$ be two increasing sequences of cardinals. Let $\kappa = \bigcup_{i < \omega} \kappa_i$. We say $\langle \kappa_i : i < \omega \rangle$ is a *coherent sequence of cardinals* with the property $\kappa_{i+1} \rightarrow^{\kappa_i} (\lambda_{i+1})_2^{<\omega}$ if and only if for every structure $\mathcal{A} = \langle \kappa, \dots \rangle$ with a countable language, there is a $\langle \lambda_i : 0 < i < \omega \rangle$ -coherent sequence of good indiscernibles for \mathcal{A} with respect to $\langle \kappa_i : i < \omega \rangle$, i.e., a sequence $\langle A_i : 0 < i < \omega \rangle$ with the following properties.

- (1) for every $0 < i < \omega$, $A_i \in [\kappa_i \setminus \kappa_{i-1}]^{\lambda_i}$,
- (2) if $x, y \in [\kappa]^{<\omega}$ are such that $x = \{x_1, \dots, x_n\}$, $y = \{y_1, \dots, y_n\}$, $x, y \subseteq \bigcup_{0 < i < \omega} A_i$, and for every $0 < i < \omega$, $|x \cap A_i| = |y \cap A_i|$ then for every $(n + l)$ -ary formula ϕ in the language of \mathcal{A} and every $z_1, \dots, z_l < \min\{x_1, \dots, x_n, y_1, \dots, y_n\}$, $\mathcal{A} \models \phi(z_1, \dots, z_l, x_1, \dots, x_n) \iff \mathcal{A} \models \phi(z_1, \dots, z_l, y_1, \dots, y_n)$.

Lemma 7.5 (ZF; Dimitriou; [21, Corollary 3.15]) Let $\langle \kappa_i : i < \omega \rangle$ and $\langle \lambda_i : 0 < i < \omega \rangle$ be two increasing sequences of cardinals. Let $\kappa = \bigcup_{i < \omega} \kappa_i$. If $\langle \kappa_i : i < \omega \rangle$ is a coherent sequence of cardinals with the property $\kappa_{i+1} \rightarrow^{\kappa_i} (\lambda_{i+1})_2^{<\omega}$ then the infinitary Chang Conjecture $(\kappa_n)_{n \in \omega} \rightarrow (\lambda_n)_{n \in \omega}$ holds.

Lemma 7.6 (Dimitriou; [21, Proposition 3.50]) Let us assume that $V \models ZFC + “\kappa = \kappa(\lambda)$ exists”, \mathbb{P} is a forcing notion such that $|\mathbb{P}| < \kappa$, and \mathbb{Q} is a forcing notion that doesn’t add subsets to κ . If G is $\mathbb{P} \times \mathbb{Q}$ generic then for every $\theta < \kappa$, $V[G] \models \kappa \rightarrow^\theta (\lambda)_2^{<\omega}$.

Lemma 7.7 (Dimitriou; [21, Lemma 3.52]) Let $\langle \kappa_i : i < \omega \rangle$ and $\langle \lambda_i : 0 < i < \omega \rangle$ be two increasing sequences of cardinals such that $\langle \kappa_i : 0 < i < \omega \rangle$ is a coherent sequence of Erdős cardinals with respect to $\langle \lambda_i : 0 < i < \omega \rangle$. If \mathbb{P}_1 is a forcing notion of cardinality $< \kappa_1$ and G is \mathbb{P}_1 -generic, then in $V[G]$, $\langle \kappa_i : i < \omega \rangle$ is a coherent sequence of cardinals with the property $\kappa_{i+1} \rightarrow^{\kappa_i} (\lambda_{i+1})_2^{<\omega}$.

7.2 The Proofs of Theorems 1.9 and 1.10

In this subsection, we prove Theorems 1.9 and 1.10.

Proof (of Theorem 1.9)

(1) **The ground model** (V) . Let κ be a measurable cardinal in a model V' of ZFC. By Prikry forcing it is possible to make κ singular with cofinality ω where an end segment $\langle \kappa_i : 1 \leq i < \omega \rangle$ of the Prikry sequence $\langle \delta_i : 1 \leq i < \omega \rangle$ is a *coherent* sequence of Ramsey cardinals by [7, Theorem 3]. Now Ramsey cardinals κ_i are exactly the κ_i -Erdős cardinals. Thus we obtain a generic extension (say V) where $\langle \kappa_i : 1 \leq i < \omega \rangle$ is a coherent sequence of cardinals with supremum κ such that for all $1 \leq i < \omega$, $\kappa_i = \kappa(\kappa_i)$. We define the following cardinals.

- (a) $\kappa'_0 = \omega$ and $\kappa_0 = \aleph_\omega$.
- (b) $\kappa'_1 = \aleph_{\omega+1}$.
- (c) $\kappa'_i = \kappa_{i-1}^{+\omega_{i-1}+1}$ for each $1 < i < \omega$.

(2) **Defining the triple** $(\mathbb{P}, \mathcal{G}, \mathcal{I})$. We consider a triple similar to the one constructed in Sect. 5.

- Let $\mathbb{P} = \prod_{i < \omega} \mathbb{P}_i$ be the Easton support product¹⁰ of $\mathbb{P}_i = Fn(\kappa'_i, \kappa_i, \kappa'_i)$ ordered componentwise where for each $i < \omega$, $Fn(\kappa'_i, \kappa_i, \kappa'_i) = \{p : \kappa'_i \dashv \rightarrow \kappa_i : |p| < \kappa'_i \text{ and } p \text{ is an injection}\}$ ordered by reverse inclusion.
- $\mathcal{G} = \prod_{i < \omega} \mathcal{G}_i$ where for each $i < \omega$, \mathcal{G}_i is the full permutation group of κ_i that can be extended to \mathbb{P}_i by permuting the range of its conditions, i.e., for all $a \in \mathcal{G}_i$ and $p \in \mathbb{P}_i$, $a(p) = \{(\psi, a(\beta)) : (\psi, \beta) \in p\}$.
- For $m \in \omega$ and $e = \{\alpha_1, \dots, \alpha_m\}$ a sequence of ordinals such that for each $1 \leq i \leq m$, there is a distinct $\epsilon_i < \omega$ such that $\alpha_i \in (\kappa'_{\epsilon_i}, \kappa_{\epsilon_i})$, we define $E_e = \{(\emptyset, \dots, p_{\epsilon_1} \cap (\kappa'_{\epsilon_1} \times \alpha_1), \emptyset, \dots, p_{\epsilon_2} \cap (\kappa'_{\epsilon_2} \times \alpha_2), \emptyset, \dots, p_{\epsilon_m} \cap (\kappa'_{\epsilon_m} \times \alpha_m), \emptyset, \dots\}$; $\vec{p} \in \mathbb{P}$ and $\mathcal{I} = \{E_e : e \in \prod_{i < \omega}^{fin} (\kappa'_i, \kappa_i)\}$.

(3) **Defining symmetric extension of V** . Let \mathcal{I} generate a normal filter $\mathcal{F}_{\mathcal{I}}$ over \mathcal{G} . Let G be a \mathbb{P} -generic filter. We consider the symmetric model $V(G)^{\mathcal{F}_{\mathcal{I}}}$. We denote $V(G)^{\mathcal{F}_{\mathcal{I}}}$ by $V(G)$ for the sake of convenience.

□

Since the forcing notions involved are weakly homogeneous, the following holds.

Lemma 7.8 *If $A \in V(G)$ is a set of ordinals, then $A \in V[G \cap E_e]$ for some $E_e \in \mathcal{I}$.*

Following the arguments in [21, Lemma 1.35], we can see that in $V(G)$, $(\kappa'_i)^+ = \kappa_i$ for every $i < \omega$. Similar to the arguments from the proof of [7, Theorem 11], it is possible to see that in $V(G)$, $\kappa = \aleph_{(\aleph_\omega)^V}$ and $(\aleph_\omega)^V = \aleph_1$. Consequently $\kappa = \aleph_{\omega_1}$ and $cf(\kappa) = \omega$ in $V(G)$. Further ω_1 is singular in $V(G)$. Following Fact 2.19, AC_ω fails in $V(G)$. We prove that an infinitary Chang conjecture holds in $V(G)$.

Lemma 7.9 *In $V(G)$, an infinitary Chang conjecture holds.*

Proof Let $\mathcal{A} = \langle \kappa, \dots \rangle$ be a structure in a countable language in $V(G)$. Let $\{\phi_n : n < \omega\}$ be an enumeration of the formulas of the language of \mathcal{A} such that each ϕ_n has $k(n) \leq n$ many free variables. Define $f : [\kappa]^{<\omega} \rightarrow 2$ by,

$$f(\epsilon_1, \dots, \epsilon_n) = 1 \text{ if and only if } \mathcal{A} \models \phi_n(\epsilon_1, \dots, \epsilon_{k(n)}) \text{ and } f(\epsilon_1, \dots, \epsilon_n) = 0 \text{ otherwise.}$$

¹⁰ In this case, it is equivalent to full support.

By Lemma 7.8, there is an $E_e \in \mathcal{I}$ such that $f \in V[G \cap E_e]$. Fix an arbitrary $1 \leq i < \omega$. We can write $V[G \cap E_e] = V[G_1][G_2]$ where G_1 is \mathbb{Q}_1 -generic over V such that $|\mathbb{Q}_1| < \kappa_i$, and G_2 is \mathbb{Q}_2 -generic over $V[G_1]$ such that G_2 adds no subsets of κ_i . Consequently, by Lemma 7.6, $\kappa_i \rightarrow^{\kappa_{i-1}} (\kappa_i)_2^{<\omega}$ in $V[G \cap E_e]$. So, for all $1 \leq i < \omega$, $\kappa_i \rightarrow^{\kappa_{i-1}} (\kappa_i)_2^{<\omega}$ in $V[G \cap E_e]$.

Let $e = \{\alpha_1, \dots, \alpha_m\}$ where for each $i \in \{1, \dots, m\}$, there is a distinct ϵ_i such that $\alpha_i \in (\kappa'_{\epsilon_i}, \kappa_{\epsilon_i})$. Consider j to be $\max\{\epsilon_i : \alpha_i \in e\}$. If $G \cap E_e$ is \mathbb{P} -generic over V then since $|\mathbb{P}| < \kappa_j$, by Lemma 7.7, $\langle \kappa_i : j - 1 \leq i < \omega \rangle$ is a coherent sequence of cardinals with the property $\kappa_i \rightarrow^{\kappa_{i-1}} (\kappa_i)_2^{<\omega}$ for all $j \leq i < \omega$. By Definition 7.4, there is a $\langle \kappa_i : j \leq i < \omega \rangle$ -coherent sequence $\langle A_n : j \leq n < \omega \rangle$ of good indiscernibles for \mathcal{A} with respect to $\langle \kappa_i : j - 1 \leq i < \omega \rangle$. We obtain a $\langle \kappa_i : j - 1 \leq i < \omega \rangle$ -coherent sequence $\langle A_n : j - 1 \leq n < \omega \rangle$ of good indiscernibles for \mathcal{A} with respect to $\langle \kappa_i : j - 2 \leq i < \omega \rangle$ as follows.

- Since $\kappa_{j-1} \rightarrow^{\kappa_{j-2}} (\kappa_{j-1})_2^{<\omega}$, we obtain a set $A_{j-1} \in [\kappa_{j-1} \setminus \kappa_{j-2}]^{\kappa_{j-1}}$ of indiscernibles for \mathcal{A} with respect to parameters below κ_{j-2} . Consequently, we obtain a $\langle \kappa_i : j - 1 \leq i < \omega \rangle$ -coherent sequence $\langle A_n : j - 1 \leq n < \omega \rangle$ of good indiscernibles for \mathcal{A} with respect to $\langle \kappa_i : j - 2 \leq i < \omega \rangle$.

If we continue in this manner step by step for the remaining cardinals $\kappa_1, \dots, \kappa_{j-2}$, then since $\kappa_i \rightarrow^{\kappa_{i-1}} (\kappa_i)_2^{<\omega}$ for each $1 \leq i \leq j - 2$, we can obtain a $\langle \kappa_i : 0 < i < \omega \rangle$ -coherent sequence $A = \langle A_n : 0 < n < \omega \rangle$ of good indiscernibles for \mathcal{A} with respect to $\langle \kappa_i : i < \omega \rangle$ and $A \in V[G \cap E_e] \subseteq V(G)$. Therefore for all $1 \leq i < \omega$, $\kappa_i \rightarrow^{\kappa_{i-1}} (\kappa_i)_2^{<\omega}$ and $\langle \kappa_i : 1 \leq i < \omega \rangle$ is a coherent sequence of cardinals in $V(G)$ by Definition 7.4. Using Lemma 7.5, we can obtain an infinitary Chang conjecture in $V(G)$ as Lemma 7.5 can be proved in ZF. \square

Proof (of Theorem 1.10) Let \mathcal{N} be the choiceless model constructed in [7, Theorem 11]. We first translate the arguments in terms of a symmetric extension based on a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.

- Consider V, \mathbb{P} , and \mathcal{G} as mentioned in the previous construction (used for proving Theorem 1.9).
- Let $\mathcal{I} = \{E_e : e \in \prod_{i < \omega} (\kappa'_i, \kappa_i)\}$ where for every $e = \{\alpha_i : i < \omega\} \in \prod_{i \in \omega} (\kappa'_i, \kappa_i)$, $E_e = \{\langle p_i \cap (\kappa'_i \times \alpha_i) : i < \omega \rangle : \vec{p} \in \mathbb{P}\}$. Let \mathcal{I} generate a normal filter $\mathcal{F}_{\mathcal{I}}$ over \mathcal{G} . We define \mathcal{F} to be $\mathcal{F}_{\mathcal{I}}$.

Let G be a \mathbb{P} -generic filter. We consider the symmetric model $V(G)^{\mathcal{F}}$. We denote $V(G)^{\mathcal{F}}$ by $V(G)$ for the sake of convenience. The model $V(G)$ is analogous to the choiceless model \mathcal{N} constructed in [7, Theorem 11]. Since the forcing notions involved are weakly homogeneous, the following holds. \square

Lemma 7.10 *If $A \in V(G)$ is a set of ordinals, then $A \in V[G \cap E_e]$ for some $E_e \in \mathcal{I}$.*

Similar to Lemma 7.9, we observe an infinitary Chang conjecture in $V(G)$.

Lemma 7.11 *In $V(G)$, an infinitary Chang conjecture holds.*

Proof Let $\mathcal{A} = \langle \kappa, \dots \rangle$ be a structure in a countable language in $V(G)$. Let $\{\phi_n : n < \omega\}$ be an enumeration of the formulas of the language of \mathcal{A} such that each ϕ_n has $k(n) \leq n$ many free variables. Define $f : [\kappa]^{<\omega} \rightarrow 2$ by,

$$f(\epsilon_1, \dots, \epsilon_n) = 1 \text{ if and only if } \mathcal{A} \models \phi_n(\epsilon_1, \dots, \epsilon_{k(n)}) \text{ and}$$

$$f(\epsilon_1, \dots, \epsilon_n) = 0 \text{ otherwise.}$$

By Lemma 7.10, there is an $E_e \in \mathcal{I}$ such that $f \in V[G \cap E_e]$. Fix an arbitrary $1 \leq i < \omega$. We can write $V[G \cap E_e] = V[G_1][G_2]$ where G_1 is \mathbb{Q}_1 -generic over V such that $|\mathbb{Q}_1| < \kappa_i$, and G_2 is \mathbb{Q}_2 -generic over $V[G_1]$ such that G_2 adds no subsets of κ_i . Consequently, by Lemma 7.6, $\kappa_i \rightarrow^{\kappa_i-1} (\kappa_i)_2^{<\omega}$ in $V[G \cap E_e]$. So, for all $1 \leq i < \omega$, $\kappa_i \rightarrow^{\kappa_i-1} (\kappa_i)_2^{<\omega}$ in $V[G \cap E_e]$. Thus, we obtain a set $A_i \in [\kappa_i \setminus \kappa_{i-1}]^{\kappa_i}$ of good indiscernibles for \mathcal{A} for each $1 \leq i < \omega$, in $V[G \cap E_e]$. Consequently, we obtain a $\langle \kappa_i : 0 < i < \omega \rangle$ -coherent sequence $A = \langle A_i : 0 < i < \omega \rangle$ of good indiscernibles for \mathcal{A} with respect to $\langle \kappa_i : i < \omega \rangle$ and $A \in V[G \cap E_e] \subseteq V(G)$. The rest is the same as in the proof of Lemma 7.9. □

Applying [7, Theorem 4] and [6, Proposition 1], we prove that \aleph_{ω_1} is an almost Ramsey cardinal in $V(G)$.

Lemma 7.12 *In $V(G)$, \aleph_{ω_1} is an almost Ramsey cardinal.*

Proof Following the terminologies from the proof of [7, Theorem 11] we have $\kappa = \aleph_{\omega_1}$ in $V(G)$. We show that κ is an almost Ramsey cardinal in $V(G)$. Let $f : [\kappa]^{<\omega} \rightarrow 2$ be in $V(G)$. Since f can be coded by a set of ordinals, $f \in V[G \cap E_e]$ for some $E_e \in \mathcal{I}$ by Lemma 7.10. Now, in V , κ is the supremum of a coherent sequence of Ramsey cardinals $\langle \kappa_i : i < \omega \rangle$. By [7, Theorem 4] we can see that $\langle \kappa_i : i < \omega \rangle$ stays a coherent sequence of Ramsey cardinals in $V[G \cap E_e]$. Also κ is the supremum of $\langle \kappa_i : i < \omega \rangle$ in $V[G \cap E_e]$. Thus κ is an almost Ramsey cardinal in $V[G \cap E_e]$ by [6, Proposition 1]. Thus for all $\beta < \kappa$, there is a set $X_\beta \in V[G \cap E_e] \subseteq V(G)$ which is homogeneous for f and has order type at least β . Hence, κ is almost Ramsey in $V(G)$ since f was arbitrary. □

8 Mutual stationarity property from a sequence of measurable cardinals

8.1 Mutual stationarity

We recall the idea of *mutual stationarity* introduced by Foreman and Magidor in [23] and a theorem due to Cummings, Foreman, and Magidor from [18].

Definition 8.1 (cf. [23, Definition 6] & [10, Definition 1.1]) Let \mathcal{K} be a set of regular cardinals with supremum λ . Suppose that $S_\kappa \subseteq \kappa$ for each $\kappa \in \mathcal{K}$. Then $\langle S_\kappa : \kappa \in \mathcal{K} \rangle$ is *mutually stationary* if and only if for all algebras \mathcal{A} on λ , there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ such that for all $\kappa \in \mathcal{B} \cap \mathcal{K}$, $\text{sup}(\mathcal{B} \cap \kappa) \in S_\kappa$.

Theorem 8.2 (Cummings, Foreman, and Magidor; cf. [18, Theorem 5.2]) *Let $\langle \kappa_i : i < \delta \rangle$ be an increasing sequence of measurable cardinals, where $\delta < \kappa_0$ is a regular cardinal. Let $S_i \subseteq \kappa_i$ be stationary for each $i < \delta$. Then $\langle S_i : i < \delta \rangle$ is mutually stationary.*

8.2 The Proof of Observation 1.11

We recall the choiceless model constructed in [14, Theorem 1] and the terminologies from [14]. In particular we fix an arbitrary $n_0 \in \omega$ and assume an increasing sequence of measurable cardinals $\langle \chi_k : k < \omega \rangle$ in the ground model V of ZFC. Then we consider the choiceless model constructed in [14, Theorem 1]. For the sake of convenience we call the model \mathcal{N}_{n_0} and recall the relevant lemmas from [14].

Lemma 8.3 (cf. [14, Lemma 1.1]) *If $X \in \mathcal{N}_{n_0}$ is a set of ordinals, then $X \in V[G \upharpoonright f]$ for some $f \in K$.*

Lemma 8.4 (cf. [14, Lemma 1.2]) *Let $\lambda = \bigcup_{k < \omega} \chi_k$. Then $\lambda = \aleph_\omega$ in \mathcal{N}_{n_0}*

Proof (of Observation 1.11) We note that in \mathcal{N}_{n_0} , $\chi_k = \aleph_{n_0+2(k+1)}$ for each $k < \omega$. Let $\lambda = \bigcup_{k < \omega} \chi_k$ in V .

- (1) Following the arguments in [21, Lemma 1.36], $\aleph_{n_0+2(k+1)}$ is a measurable cardinal in \mathcal{N}_{n_0} , for each $1 \leq k < \omega$. Following [27, Theorem 4.3], for each $1 \leq k < \omega$, there are no uniform ultrafilters on \aleph_{n_0+2k+1} in \mathcal{N}_{n_0} . Consequently for each $1 \leq k < \omega$, \aleph_{n_0+2k+1} can not be a measurable cardinal in \mathcal{N}_{n_0} .
- (2) We prove that in the model \mathcal{N}_{n_0} , if $\langle S_k : 1 \leq k < \omega \rangle$ is a sequence of stationary sets such that $S_k \subseteq \chi_k$ for every $1 \leq k < \omega$, then $\langle S_k : 1 \leq k < \omega \rangle$ is mutually stationary. By Lemma 8.4, $\lambda = \aleph_\omega$ in \mathcal{N}_{n_0} .

Suppose $\mathcal{N}_{n_0} \models$ “ \mathcal{A} is an algebra on λ and $\langle S_k : 1 \leq k < \omega \rangle$ is a sequence of stationary sets such that $S_k \subseteq \chi_k$ for every $1 \leq k < \omega$ ”. Since both \mathcal{A} and $\langle S_k : 1 \leq k < \omega \rangle$ can be coded by set of ordinals, by Lemma 8.3, there exists some $f \in K$ for which both $\langle S_k : 1 \leq k < \omega \rangle \in V[G \upharpoonright f]$ and $\mathcal{A} \in V[G \upharpoonright f]$. Following [14, Lemma 1.3], χ_k remains measurable in $V[G \upharpoonright f]$ for every $1 \leq k < \omega$. We can observe that S_k is a stationary subset of χ_k in $V[G \upharpoonright f]$ for every $1 \leq k < \omega$. Fix any $1 \leq k < \omega$. Let C be any club subset of χ_k in $V[G \upharpoonright f]$. Since the notion of club subset of χ_k is upward absolute and $V[G \upharpoonright f] \subseteq \mathcal{N}_{n_0}$, C is also a club subset of χ_k in \mathcal{N}_{n_0} . Since in \mathcal{N}_{n_0} , S_k is a stationary subset of χ_k we have $S_k \cap C \neq \emptyset$. Thus, S_k is a stationary subset of χ_k in $V[G \upharpoonright f]$ for every $1 \leq k < \omega$. By Theorem 8.2, $\langle S_k : 1 \leq k < \omega \rangle$ is mutually stationary in $V[G \upharpoonright f]$.

We note that \mathcal{A} is an algebra on λ in $V[G \upharpoonright f]$. Thus there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ in $V[G \upharpoonright f]$ such that for all $k < \omega$, $sup(\mathcal{B} \cap \chi_k) \in S_k$ by Definition 8.1. So there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ in \mathcal{N}_{n_0} such that for all $k < \omega$, $sup(\mathcal{B} \cap \chi_k) \in S_k$. Hence in \mathcal{N}_{n_0} , $\langle S_k : 1 \leq k < \omega \rangle$ is mutually stationary.

- (3) We recall that $\lambda = \aleph_\omega$ in \mathcal{N}_{n_0} . We can see that λ is an almost Ramsey cardinal in \mathcal{N}_{n_0} by a well-known argument from [2, Lemma 2.5]. For reader’s convenience,

we provide a sketch of the proof. Let $f : [\lambda]^{<\omega} \rightarrow 2$ be in \mathcal{N}_{n_0} . Since f can be coded by a set of ordinals, $f \in V[G \upharpoonright f]$ for some $f \in K$ by Lemma 8.3. Following [14, Lemma 1.3], χ_k remains measurable in $V[G \upharpoonright f]$ for every $1 \leq k < \omega$. Consequently, χ_k is Ramsey in $V[G \upharpoonright f]$ for every $1 \leq k < \omega$. Now, in $V[G \upharpoonright f]$, λ is the supremum of Ramsey cardinals $\langle \chi_i : 1 \leq i < \omega \rangle$. Thus λ is an almost Ramsey cardinal in $V[G \upharpoonright f]$ by [6, Proposition 1]. Thus for all $\beta < \lambda$, there is a set $X_\beta \in V[G \upharpoonright f] \subseteq \mathcal{N}_{n_0}$ which is homogeneous for f and has order type at least β . Hence, λ is almost Ramsey in \mathcal{N}_{n_0} since f was arbitrary.

□

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Declarations

Conflict of interests We declare no conflicts of interest.

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References

1. Apter, A., Cody, B.: Consecutive singular cardinals and the continuum function. *Notre Dame J. Form. Logic* **54**(2), 125–136 (2013). <https://doi.org/10.1215/00294527-1960434>
2. Apter, A., Dimitriou, I., Koepke, P.: All uncountable cardinals in the Gitik model are almost Ramsey and carry Rowbottom filters. *MLQ Math. Log. Q.* **62**(3), 225–231 (2016). <https://doi.org/10.1002/malq.201400050>
3. Apter, A., Dimitriou, I., Koepke, P.: The first measurable cardinal can be the first uncountable cardinal at any successor height. *MLQ Math. Log. Q.* **60**(6), 471–486 (2014). <https://doi.org/10.1002/malq.201110007>
4. Apter, A., Dimopoulos, S., Usuba, T.: Strongly compact cardinals and the continuum function. *Ann. Pure Appl. Logic* **172**(9), 103013 (2021). <https://doi.org/10.1016/j.apal.2021.103013>
5. Apter, A., Henle, J.: Relative consistency results via strong compactness. *Fundam. Math.* **139**(2), 133–149 (1991). <https://doi.org/10.4064/fm-139-2-133-149>
6. Apter, A., Koepke, P.: Making all cardinals almost Ramsey. *Arch. Math. Logic* **47**(7–8), 769–783 (2008). <https://doi.org/10.1007/s00153-008-0107-1>

7. Apter, A., Koepke, P.: The consistency strength of \aleph_ω and \aleph_{ω_1} being rowbottom cardinals without the axiom of choice. *Arch. Math. Logic* **45**(6), 721–737 (2006). <https://doi.org/10.1007/s00153-006-0005-3>
8. Apter, A., Magidor, M.: Instances of dependent choice and the measurability of $\aleph_{\omega+1}$. *Ann. Pure Appl. Logic* **74**(3), 203–219 (1995). [https://doi.org/10.1016/0168-0072\(94\)00039-6](https://doi.org/10.1016/0168-0072(94)00039-6)
9. Apter, A.: On some questions concerning strong compactness. *Arch. Math. Logic* **51**(7–8), 819–829 (2012). <https://doi.org/10.1007/s00153-012-0300-0>
10. Apter, A.: On a problem of Foreman and Magidor. *Arch. Math. Logic* **44**(4), 493–498 (2005). <https://doi.org/10.1007/s00153-004-0259-6>
11. Apter, A.: Some remarks on normal measures and measurable cardinals. *Math. Log. Q.* **47**(1), 35–44 (2001). [https://doi.org/10.1002/1521-3870\(200101\)47:1<35::AID-MALQ35>3.0.CO;2-C](https://doi.org/10.1002/1521-3870(200101)47:1<35::AID-MALQ35>3.0.CO;2-C)
12. Apter, A.: Successors of singular cardinals and measurability. *Adv. Math.* **55**(3), 228–241 (1985). [https://doi.org/10.1016/0001-8708\(85\)90092-1](https://doi.org/10.1016/0001-8708(85)90092-1)
13. Apter, A.: Some results on consecutive large cardinals. *Ann. Pure Appl. Logic* **25**(1), 1–17 (1983). [https://doi.org/10.1016/0168-0072\(83\)90051-9](https://doi.org/10.1016/0168-0072(83)90051-9)
14. Apter, A.: On a problem of silver. *Fundam. Math.* **116**(1), 33–38 (1983). <https://doi.org/10.4064/fm-116-1-33-38>
15. Bull, E.L., Jr.: Successive large cardinals. *Ann. Math. Logic* **15**(2), 161–191 (1978). [https://doi.org/10.1016/0003-4843\(78\)90019-0](https://doi.org/10.1016/0003-4843(78)90019-0)
16. Blass, A., Dimitriou, I., Löwe, B.: Inaccessible cardinals without the axiom of choice. *Fundam. Math.* **194**(2), 179–189 (2007). <https://doi.org/10.4064/fm194-2-3>
17. Bilinsky, E., Gitik, M.: A model with a measurable which does not carry a normal measure. *Arch. Math. Logic* **51**(7–8), 863–876 (2012). <https://doi.org/10.1007/s00153-012-0302-y>
18. Cummings, J., Foreman, M., Magidor, M.: Canonical structure in the universe of set theory: part two. *Ann. Pure Appl. Logic* **142**(1–3), 55–75 (2006). <https://doi.org/10.1016/j.apal.2005.11.007>
19. Cummings, J.: Iterated forcing and elementary embeddings. In: *Handbook of Set Theory*, vol. 1, 2, 3. Springer, Dordrecht, pp. 775–883 (2010). https://doi.org/10.1007/978-1-4020-5764-9_13
20. Dobrinen, N., Friedman, S.D.: Homogeneous iteration and measure one covering relative to HOD. *Arch. Math. Logic* **47**(7–8), 711–718 (2008). <https://doi.org/10.1007/s00153-008-0103-5>
21. Dimitriou, I.: Symmetric models, singular cardinal patterns, and indiscernibles. Ph.D. thesis, Universitäts- und Landesbibliothek Bonn (2011)
22. Friedman, S.D., Magidor, M.: The number of normal measures. *J. Symb. Log.* **74**(3), 1069–1080 (2009). <https://doi.org/10.2178/jsl/1245158100>
23. Foreman, M., Magidor, M.: Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $\mathcal{P}_\kappa(\lambda)$. *Acta Math.* **186**(2), 271–300 (2001). <https://doi.org/10.1007/BF02401842>
24. Gitik, M.: Prikry-type forcings. In: *Handbook of Set Theory*, vol. 1, 2, 3. Springer, Dordrecht, pp. 1351–1447 (2010). https://doi.org/10.1007/978-1-4020-5764-9_17
25. Gitman, V., Johnstone, T.A.: On ground model definability. In: *Infinity, Computability, and Metamathematics: Festschrift in Honour of the 60th Birthdays of Peter Koepke and Philip Welch*, Series: Tributes. College Publications, London, GB, pp. 205–227 (2014). [arXiv: 1311.6789](https://arxiv.org/abs/1311.6789) [math.LO]
26. Grigorieff, S.: Intermediate submodels and generic extensions in set theory. *Ann. Math. (2)* **101**(3), 447–490 (1975). <https://doi.org/10.2307/1970935>
27. Hayut, Y., Karagila, A.: Spectra of uniformity. *Comment. Math. Univ. Carolin.* **60**(2), 285–298 (2019). <https://doi.org/10.14712/1213-7243.2019.008>
28. Howard, P., Rubin, J.E.: Consequences of the axiom of choice. *Math. Surv. Monogr.* (1998). <https://doi.org/10.1090/surv/059>
29. Inamdar, T.: Successor large cardinals in symmetric extensions, preprint. ILLC Publications X-2013-02 (2013). <https://eprints.illc.uva.nl/id/eprint/689>
30. Ikegami, D., Trang, N.: On supercompactness of ω_1 . [arXiv:1904.01815](https://arxiv.org/abs/1904.01815) [math.LO] (2019)
31. Jech, T.: *Set Theory*. Springer Monographs in Mathematics, The third millennium edition, revised and expanded Springer, Berlin (2003). <https://doi.org/10.1007/3-540-44761-X>
32. Jech, T.: *The Axiom of Choice. Studies in Logic and the Foundations of Mathematics*, vol. 75. North-Holland Publishing Co., Amsterdam (1973)
33. Jech, T.: ω_1 can be measurable. *Isr. J. Math.* **6**(4), 363–367 (1968). <https://doi.org/10.1007/BF02771215>
34. Kanamori, A.: *The Higher Infinite, Large Cardinals in Set Theory from Their Beginnings*. Springer Monographs in Mathematics. Springer, Berlin (2003). <https://doi.org/10.1007/978-3-540-88867-3>

35. Karagila, A.: Preserving dependent choice. *Bull. Pol. Acad. Sci. Math.* **67**(1), 19–29 (2019). <https://doi.org/10.4064/ba8169-12-2018>
36. Karagila, A.: Iterating symmetric extensions. *J. Symb. Log.* **84**(1), 123–159 (2019). <https://doi.org/10.1017/jsl.2018.73>
37. Karagila, A.: Embedding orders into the cardinals with DC_κ . *Fundam. Math.* **226**, 143–156 (2014). <https://doi.org/10.4064/fm226-2-4>
38. Kofkoulis, G.: Homogeneous sequences of cardinals for OD partition relations. Doctoral dissertation, California Institute of Technology (1990)
39. Kunen, K.: *Set Theory: An Introduction to Independence Proofs*. Studies in Logic and the Foundations of Mathematics, vol. 102. North-Holland Publishing Co., Amsterdam (1980)
40. Lévy, A., Solovay, R.M.: Measurable cardinals and the continuum hypothesis. *Isr. J. Math.* **5**(4), 234–248 (1967). <https://doi.org/10.1007/BF02771612>
41. Laver, R.: Making the supercompactness of κ indestructible under κ -directed closed forcing. *Isr. J. Math.* **29**, 385–388 (1978). <https://doi.org/10.1007/BF02761175>
42. Mitchell, W.J.: Sets constructible from sequences of ultrafilters. *J. Symb. Log.* **39**(1), 57–66 (1974). <https://doi.org/10.2307/2272343>
43. Monro, G.P.: On generic extensions without the axiom of choice. *J. Symb. Log.* **48**(1), 39–52 (1983). <https://doi.org/10.2307/2273318>
44. Takeuti, G.: A relativization of axioms of strong infinity to ω_1 . *Ann. Jpn. Assoc. Philos. Sci.* **3**(5), 191–204 (1970). <https://doi.org/10.4288/jafpos1956.3.191>
45. Woodin, W.H.: Suitable extender models I. *J. Math. Log.* **10**(1–2), 101–339 (2010). <https://doi.org/10.1142/S021906131000095X>

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