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# A note on cut-elimination for classical propositional logic

Gabriele Pulcini[1]

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**Abstract**
In Schwichtenberg (Studies in logic and the foundations of mathematics, vol 90, Elsevier, pp 867–895, 1977), Schwichtenberg fine-tuned Tait's technique (Tait in The syntax and semantics of infinitary languages, Springer, pp 204–236, 1968) so as to provide a simplified version of Gentzen's original cut-elimination procedure for first-order classical logic (Gallier in Logic for computer science: foundations of automatic theorem proving, Courier Dover Publications, London, 2015). In this note we show that, limited to the case of classical propositional logic, the Tait–Schwichtenberg algorithm allows for a further simplification. The procedure offered here is implemented on Kleene's sequent system G4 (Kleene in Mathematical logic, Wiley, New York, 1967; Smullyan in First-order logic, Courier corporation, London, 1995). The specific formulation of the logical rules for G4 allows us to provide bounds on the height of cut-free proofs just in terms of the logical complexity of their end-sequent.

**Keywords** Classical propositional logic · Sequent calculus · Cut elimination

**Mathematics Subject Classification** 03F05 Cut-elimination and normal-form theorems

## 1 Introduction

In [5], Schwichtenberg fine-tuned Tait's technique [7] so as to provide a simplified version of Gentzen's original cut-elimination procedure, which notoriously requires a complex induction on a certain lexicographic order [2]. In particular, Schwichtenberg showed that termination of the cut-elimination procedure can be achieved by resorting to two independent inductions on $\omega$. The Reduction Lemma is proved by induction on the sum of the heights of the two derivations delivering the premises of the cut-

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✉ Gabriele Pulcini
gabriele.pulcini@uniroma2.it

1 Dipartimento di Studi Letterari, Filosofici e di Storia dell'Arte, Università di Roma "Tor Vergata", Rome, Italy

application under consideration [5, *Lemma 2.6*, p. 874] and the final *Hauptsatz* is proved by induction on the cut-rank of the whole proof [5, *Theorem 2.7*, p. 875].

In this note we show that, limited to the case of classical propositional logic, cut-elimination allows for a further simplification. As a matter of fact, the proof of Lemma 4 (our Reduction Lemma) is simply led by cases, whereas Theorem 5 (the *Hauptsatz*) is proved by a double induction on the cut-size of proofs and on the number of maximal cut-applications. The size of a cut-application is just defined as the number of connectives occurring in one of its premises. Accordingly, the cut-size of a proof $\pi$ is defined as the supremum of all the cut-sizes relating to $\pi$.

The algorithm proposed in this note is tailored on the sequent system GS4, the one-sided formulation *à la* Tait of Kleene's G4 [3,6]. The procedure heavily relies on the fact that, for any non-atomic formula $A$, if the sequent $\vdash \Gamma, A$ is provable in GS4, then it is also provable by means of a particular proof in which $A$ occurs as the *principal formula* in the last inference step (Lemma 3). The main advantage of dealing with Kleene's system GS4 lies in the fact that the height of cut-free proofs turns out to be bounded by the number of occurrences of logical connectives in their end-sequent (Theorem 6). Moreover, we prove that any two cut-free proofs ending in the same sequent have always the same height (Theorem 7).

## 2 Preliminary notions and results

Following [7], we limit ourselves to considering only two connectives: conjunction ($\wedge$) and disjunction ($\vee$). In formal languages *à la* Tait, negation comes as primitive on atomic sentences $\mathbf{AT} = \{p, \overline{p}, q, \overline{q}, \ldots\}$ and it extends to compound formulas by means of the following equivalences:

$$\overline{\overline{A}} \Leftrightarrow A \qquad \overline{A \wedge B} \Leftrightarrow \overline{A} \vee \overline{B} \qquad \overline{A \vee B} \Leftrightarrow \overline{A} \wedge \overline{B}$$

The set $\mathcal{F}$ of well-formed formulas is defined accordingly:

$$\mathcal{F} ::= \mathbf{AT} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F}.$$

Logical contexts $\Gamma, \Delta, \ldots$ are taken to be *multisets* of formulas from $\mathcal{F}$. As usual, we write $\Gamma, A$ and $\Gamma, \Delta$ to mean the two multisets $\Gamma \uplus [A]$ and $\Gamma \uplus \Delta$, respectively. We write $\{\Gamma\}$ to indicate the *set* collecting the elements of $\Gamma$.

We call GS4 the one-sided version of Kleene's sequent system G4 whose rules are displayed in Fig. 1 [1,3,4,6]. The height $h(\pi)$ of a proof $\pi$ is given by the number of sequents occurring in one of its longest branches. A subproof $\delta$ of a proof $\pi$ is said to be *direct* in case $\delta$ ends in one of the premises of $\pi$'s last inference. Moreover, we recall that any application of the logical rules displays a *principal formula* in the conclusion: the formula whose principal connective has been introduced by the very inference step under consideration.

**Definition 1** The complexity $\mathcal{C}(A)$ of a formula $A$ is given by the number of occurrences of logical connectives in $A$. More formally: $\mathcal{C}(A) = 0$, for any $A \in \mathbf{AT}$, and

*Axiom*

$$\dfrac{}{\vdash \Gamma, p, \overline{p}}\, ax \qquad \text{with } \{\Gamma\} \subset \mathbf{AT}$$

*Logical rules*

$$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\, \wedge \qquad\qquad \dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\, \vee$$

**Fig. 1** The rules of the sequent calculus GS4

$\mathcal{C}(A \wedge B) = \mathcal{C}(A \vee B) = \mathcal{C}(A) + \mathcal{C}(B) + 1$. For any multiset $\Gamma = [A_1, A_2, \ldots, A_n]$, we set $\mathcal{C}(\Gamma) = \mathcal{C}(A_1) + \mathcal{C}(A_2) + \cdots + \mathcal{C}(A_n)$.

***Remark 1*** For any multiset of formulas $\Gamma, C$, we have $\mathcal{C}(\Gamma, C) = \mathcal{C}(\Gamma, \overline{C})$.

Observe that, in the specific formulation adopted here, instances of the $ax$-rule must be *clauses*, i.e., sequents in which only atomic formulas from **AT** are displayed. The next proposition shows that such a linguistic restriction does not affect provability.

**Proposition 1** *GS4 proves the sequent* $\vdash \Gamma, p, \overline{p}$*, for any multiset of formulas* $\Gamma$*, and any* $p \in \mathbf{AT}$.

***Proof*** We proceed by induction on $\mathcal{C}(\Gamma)$. If $\mathcal{C}(\Gamma) = 0$, then $\vdash \Gamma, p, \overline{p}$ is already an instance of the $ax$-rule. As for $\mathcal{C}(\Gamma) > 0$, we distinguish two cases:

- $\Gamma = \Gamma', A \wedge B$. By inductive hypothesis, there are two GS4-proofs $\delta$ and $\rho$ ending in $\vdash \Gamma', A, p, \overline{p}$ and $\vdash \Gamma', B, p, \overline{p}$, respectively. The two proofs $\delta$ and $\rho$ can be then composed by means of an application of the $\wedge$-rule so as to finally get the conclusion $\vdash \Gamma', A \wedge B, p, \overline{p}$.
- $\Gamma = \Gamma', A \vee B$. Similar to the previous case. □

Below, we recall the well-known fact that the structural rule of Weakening is admissible in GS4 (cfr, for instance, [5, *Lemma 2.3.1*, p. 873]):

**Lemma 2** (Weakening admissibility) *If GS4 proves* $\vdash \Gamma$*, then it also proves the sequent* $\vdash \Gamma, A$*, for any formula* $A$.

***Proof*** Let $\pi$ be a GS4-proof ending in $\vdash \Gamma$. Once the formula $A$ is uniformly added to all the sequents occurring in $\pi$, each of $\pi$'s top sequents $\vdash \Gamma, p, \overline{p}$ is turned into the sequent $\vdash \Gamma, A, p, \overline{p}$ which is, by Proposition 1, provable. □

***Notation*** Given a GS4-proof $\pi$ of $\vdash \Gamma$ and a formula $A$, we denote with $\mathcal{W}(\pi, A)$ the GS4-proof of $\vdash \Gamma, A$ obtained from $\pi$ according to the procedure employed in the proof of Lemma 2. If $A \in \Gamma$, then $\mathcal{W}(\pi, A) = \pi$.

The following lemma states a peculiar property of the GS4 system which will prove crucial to attain the results proposed in the next section. Such a property comes as a byproduct of the fact that GS4 logical rules are all reversible in the sense that provability of the conclusion always implies provability of the premise(s) (cfr. [5, *Lemma 2.5*, p. 873]).

**Lemma 3** (Height-preserving permutability) *Assume there is a GS4-proof $\pi$ of $\vdash \Gamma, A$ with $\mathcal{C}(A) > 0$. The sequent $\vdash \Gamma, A$ is also provable by means of a proof $\rho$ such that: $(i)$ the formula $A$ occurs as principal in $\rho$'s last inference, and $(ii)$ $h(\pi) = h(\rho)$.*

***Proof*** If $\mathcal{C}(\Gamma) = 0$, then $\pi$'s last rule must be already the one introducing $A$'s principal connective and so $\rho = \pi$. Otherwise, we proceed by showing that any proof $\pi$ of $\vdash \Gamma, A$ can be turned into a proof $\rho$ of $\vdash \Gamma, A$ having the desired form, simply by permuting downwards along $\pi$ the specific instance of the logical rule introducing $A$'s principal connective. The proof is led by induction on $\mathcal{C}(\Gamma, A)$. We shall be considering the following four possible situations.

- $A \equiv B \wedge C$ and $\pi$'s last rule is a $\wedge$-rule. Let $D \wedge E$ be the formula occurring as principal in $\pi$'s last inference, and $\pi_1$ and $\pi_2$ the two direct subproofs of $\pi$ ending in $\vdash \Gamma, B \wedge C, D$ and $\vdash \Gamma, B \wedge C, E$, respectively. By inductive hypothesis, there is a proof $\pi'$ shaped as displayed below, such that $h(\pi_1) = max(h(\pi_{\langle 1,1\rangle}), h(\pi_{\langle 1,2\rangle}))+1$ and $h(\pi_2) = max(h(\pi_{\langle 2,1\rangle}), h(\pi_{\langle 2,2\rangle})) + 1$.

$$\dfrac{\dfrac{\dfrac{\pi_{\langle 1,1\rangle}}{\vdash \Gamma, B, D} \quad \dfrac{\pi_{\langle 1,2\rangle}}{\vdash \Gamma, C, D}}{\vdash \Gamma, B \wedge C, D}\wedge \qquad \dfrac{\dfrac{\pi_{\langle 2,1\rangle}}{\vdash \Gamma, B, E} \quad \dfrac{\pi_{\langle 2,2\rangle}}{\vdash \Gamma, C, E}}{\vdash \Gamma, B \wedge C, E}\wedge}{\vdash \Gamma, B \wedge C, D \wedge E}\wedge$$

  The proof $\pi'$ can be then rearranged into the proof $\rho$ reported below, simply by interchanging the two final applications of the logical rules.

$$\dfrac{\dfrac{\dfrac{\pi_{\langle 1,1\rangle}}{\vdash \Gamma, B, D} \quad \dfrac{\pi_{\langle 2,1\rangle}}{\vdash \Gamma, B, E}}{\vdash \Gamma, B, D \wedge E}\wedge \qquad \dfrac{\dfrac{\pi_{\langle 1,2\rangle}}{\vdash \Gamma, C, D} \quad \dfrac{\pi_{\langle 2,2\rangle}}{\vdash \Gamma, C, E}}{\vdash \Gamma, C, D \wedge E}\wedge}{\vdash \Gamma, B \wedge C, D \wedge E}\wedge$$

  We finally observe that:

$$\begin{aligned} h(\pi) &= max(h(\pi_1), h(\pi_2)) + 1 = \\ &= max(max(h(\pi_{\langle 1,1\rangle}), h(\pi_{\langle 1,2\rangle})) + 1, max(h(\pi_{\langle 2,1\rangle}), h(\pi_{\langle 2,2\rangle})) + 1) + 1 \\ &= max(h(\pi_{\langle 1,1\rangle}), h(\pi_{\langle 1,2\rangle}), h(\pi_{\langle 2,1\rangle}), h(\pi_{\langle 2,2\rangle})) + 2 \\ &= max(max(h(\pi_{\langle 1,1\rangle}), h(\pi_{\langle 2,1\rangle})) + 1, max(h(\pi_{\langle 1,2\rangle}), h(\pi_{\langle 2,2\rangle})) + 1) + 1 \\ &= h(\rho) \end{aligned}$$

- $A \equiv B \vee C$ and $\pi$'s last rule is a $\wedge$-rule. Let $D \wedge E$ be the formula occurring as principal in $\pi$'s last inference, and $\pi_1$ and $\pi_2$ the two direct subproofs of $\pi$ ending in $\vdash \Gamma, B \vee C, D$ and $\vdash \Gamma, B \vee C, E$, respectively. By inductive hypothesis, there is a proof $\pi'$ shaped as indicated below, such that $h(\pi_1) = h(\pi_1') + 1$ and $h(\pi_2) = h(\pi_2') + 1$.

$$\dfrac{\dfrac{\begin{array}{c}\pi'_1\\ \vdots\\ \vdash \Gamma, B, C, D\end{array}}{\vdash \Gamma, B \vee C, D}\vee \qquad \dfrac{\begin{array}{c}\pi'_2\\ \vdots\\ \vdash \Gamma, B, C, E\end{array}}{\vdash \Gamma, B \vee C, E}\vee}{\Gamma, B \vee C, D \wedge E}\wedge$$

We interchange the two final applications of the logical rules so as to obtain the proof $\rho$ reported below.

$$\dfrac{\dfrac{\begin{array}{c}\pi'_1\\ \vdots\\ \vdash \Gamma, B, C, D\end{array} \qquad \begin{array}{c}\pi'_2\\ \vdots\\ \vdash \Gamma, B, C, E\end{array}}{\Gamma, B, C, D \wedge E}\wedge}{\Gamma, B \vee C, D \wedge E}\vee$$

Since $h(\pi) = max(h(\pi_1), h(\pi_2)) + 1$, we also have $h(\pi) = max(h(\pi'_1) + 1, h(\pi'_2) + 1) + 1$, thence $h(\pi) = max(h(\pi'_1), h(\pi'_2)) + 2 = h(\rho)$.

- $A \equiv B \wedge C$ and $\pi$'s last rule is a $\vee$-rule. Let $D \vee E$ be the formula occurring as principal in $\pi$'s last inference and $\pi_1$ the direct subproof of $\pi$ ending in $\vdash \Gamma, B \wedge C, D, E$. By inductive hypothesis, there is a proof $\pi'$ shaped as indicated below and such that $h(\pi_1) = max(h(\pi_{\langle 1,1\rangle}), h(\pi'_{\langle 1,2\rangle})) + 1$.

$$\dfrac{\dfrac{\begin{array}{c}\pi'_{\langle 1,1\rangle}\\ \vdots\\ \vdash \Gamma, B, D, E\end{array} \qquad \begin{array}{c}\pi'_{\langle 1,2\rangle}\\ \vdots\\ \vdash \Gamma, C, D, E\end{array}}{\vdash \Gamma, B \wedge C, D, E}\wedge}{\vdash \Gamma, B \wedge C, D \vee E}\vee$$

The proof $\rho$ can be obtained from $\pi'$ be interchanging the two final applications of the logical rules as indicated below.

$$\dfrac{\dfrac{\begin{array}{c}\pi'_{\langle 1,1\rangle}\\ \vdots\\ \vdash \Gamma, B, D, E\end{array}}{\vdash \Gamma, B, D \vee E}\vee \qquad \dfrac{\begin{array}{c}\pi'_{\langle 1,2\rangle}\\ \vdots\\ \vdash \Gamma, C, D, E\end{array}}{\vdash \Gamma, C, D \vee E}\vee}{\vdash \Gamma, B \wedge C, D \vee E}\wedge$$

Since, $h(\pi) = h(\pi_1) + 1$, we also have $h(\pi) = max(h(\pi'_{\langle 1,1\rangle}), h(\pi'_{\langle 1,2\rangle})) + 2 = max(h(\pi'_{\langle 1,1\rangle}) + 1, h(\pi'_{\langle 1,2\rangle}) + 1) + 1 = h(\rho)$.

- $A \equiv B \vee C$ and $\pi$'s last rule is a $\vee$-rule. Let $D \vee E$ be the formula occurring as principal in $\pi$'s last inference and $\pi_1$ the direct subproof of $\pi$ ending in $\vdash \Gamma, B \vee C, D, E$. By inductive hypothesis, there is a proof $\pi'$ shaped as indicated below and such that $h(\pi_1) = h(\pi'_1) + 1$.

$$\cfrac{\cfrac{\begin{array}{c}\pi_1'\\ \vdots\\ \vdash \Gamma, B, C, D, E\end{array}}{\vdash \Gamma, B \vee C, D, E}\,\vee}{\vdash \Gamma, B \vee C, D \vee E}\,\vee$$

The derivation $\pi'$, in turn, can be easily rewritten into the derivation $\rho$ by interchanging the two final applications of the $\vee$-rule as indicated below.

$$\cfrac{\cfrac{\begin{array}{c}\pi_1'\\ \vdots\\ \vdash \Gamma, B, C, D, E\end{array}}{\vdash \Gamma, B, C, D \vee E}\,\vee}{\vdash \Gamma, B \vee C, D \vee E}\,\vee$$

We finally observe that $h(\pi) = h(\pi_1) + 1 = h(\pi_1') + 2 = h(\rho)$. $\square$

***Notation*** Given a GS4-proof $\pi$ of $\vdash \Gamma, A$ with $\mathcal{C}(A) > 0$, we denote with $\mathcal{P}(\pi, A)$ the proof of $\vdash \Gamma, A$ whose last inference is the one introducing $A$'s principal connective. The proof $\mathcal{P}(\pi, A)$ is intended to be obtained from $\pi$ according to the procedure indicated in the proof of Lemma 3. For $A \equiv B \wedge C$, we indicate with $\mathcal{P}(\pi, A)_{\mathrm{L}}$ and $\mathcal{P}(\pi, A)_{\mathrm{R}}$ the two direct subproofs of $\mathcal{P}(\pi, A)$ ending in $\vdash \Gamma, B$ and $\vdash \Gamma, C$, respectively.

## 3 The cut-elimination algorithm

We call GS4$^+$ the system obtained by adding to the rules of GS4 the cut-rule in its additive one-sided formulation:

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, \overline{A}}{\vdash \Gamma}\,cut$$

When the situation requires it, we will point at specific applications of the cut-rule by adding a subscript $i \in \mathbb{N}$ to the label '*cut*'.

Before going into the details of the cut-elimination algorithm, we need to introduce some key notions to provide a suitable measure for the 'quantity of cut' present in a derivation.

**Definition 2** The size of a cut-application

$$\frac{\vdash \Gamma, C \qquad \vdash \Gamma, \overline{C}}{\vdash \Gamma}\,cut_i$$

is taken to equal the complexity of the multiset of formulas displayed in one of its premises, i.e., $|cut_i| = \mathcal{C}(\Gamma, C) = \mathcal{C}(\Gamma, \overline{C})$ (cfr. Remark 1). Let $\{cut_1, cut_2, \ldots, cut_n\}$ be a complete enumeration of the cut-applications occurring in a GS4$^+$-proof $\pi$. The cut-size of $\pi$ is defined as $|\pi| = max\{|cut_i| + 1 : 1 \leq i \leq n\}$. If $\pi$ is cut-free, then $|\pi| = 0$. A cut-application $cut_i$ is said to be *maximal* in $\pi$ whenever $|cut_i| = |\pi| - 1$.

**Lemma 4** (Reduction Lemma) *Any GS4$^+$-proof $\pi$ of $\vdash \Gamma$ displaying exactly one cut-application can be turned into a GS4$^+$-proof $\pi'$ of the same sequent and such that $|\pi'| < |\pi|$.*

***Proof*** We can limit ourselves to considering a proof $\pi$ whose unique cut-application occurs as $\pi$'s last rule without any loss of generality. Let $\delta$ and $\rho$ be the two direct subproofs of $\pi$ ending in the two premises of the cut-application under consideration:

$$\dfrac{\begin{array}{c}\delta\\ \vdots\\ \vdash \Gamma, C\end{array} \qquad \begin{array}{c}\rho\\ \vdots\\ \vdash \Gamma, \overline{C}\end{array}}{\vdash \Gamma}\ cut$$

Since $\pi$ contains exactly one cut-application, we immediately have that: $(i)$ both $\delta$ and $\rho$ are cut-free, and $(ii)$ $|\pi| = \mathcal{C}(\Gamma, C) + 1 = \mathcal{C}(\Gamma, \overline{C}) + 1$.

If $|\pi| = 1$, then the premises of the cut-application are both introduced as instances of the *ax*-rule; say $C \equiv p$, for some atomic sentence $p \in \mathbf{AT}$. It is easy to see that either $\Gamma = \Gamma', p, \overline{p}$ or $\Gamma = \Gamma', q, \overline{q}$ for some $q \in \mathbf{AT}$. Thence, the proof $\pi$ can be simply rewritten as follows:

$$\dfrac{\dfrac{}{\vdash \Gamma, p}\ ax \qquad \dfrac{}{\vdash \Gamma, \overline{p}}\ ax}{\vdash \Gamma}\ cut \quad \longrightarrow \quad \dfrac{}{\vdash \Gamma}\ ax$$

If $|\pi| > 1$, we need to proceed by cases and subcases as follows.

[Case 1] For $\mathcal{C}(C) > 0$, we consider the two following subcases according to whether $C$'s principal connective is a conjunction or a disjunction. Both of them are treated by means of a two-step reduction. The first step (indicated by $\Longrightarrow$) is an application of Lemma 3 aiming at permuting downwards the logical rules introducing the principal connective of the cut-formulas $C$ and $\overline{C}$. The second step (indicated by $\longrightarrow$) comes as a standard parallel reduction.

[Case 1.1] If $C \equiv A \wedge B$, then we proceed as follows:

$$\dfrac{\begin{array}{c}\delta\\ \vdots\\ \vdash \Gamma, A \wedge B\end{array} \qquad \begin{array}{c}\rho\\ \vdots\\ \vdash \Gamma, \overline{A} \vee \overline{B}\end{array}}{\vdash \Gamma}\ cut \quad \Longrightarrow$$

$$\Longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{P}(\delta, A \wedge B)_{\mathrm{L}}\\ \vdots\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\mathcal{P}(\delta, A \wedge B)_{\mathrm{R}}\\ \vdots\\ \vdash \Gamma, B\end{array}}{\vdash \Gamma, A \wedge B}\ \wedge \qquad \dfrac{\begin{array}{c}\mathcal{P}(\rho, \overline{A} \vee \overline{B})\\ \vdots\\ \vdash \Gamma, \overline{A}, \overline{B}\end{array}}{\vdash \Gamma, \overline{A} \vee \overline{B}}\ \vee}{\vdash \Gamma}\ cut \quad \longrightarrow$$

$$\longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{W}(\mathcal{P}(\delta, A \wedge B)_{\mathrm{L}}, \overline{B})\\ \vdots\\ \vdash \Gamma, A, \overline{B}\end{array} \qquad \begin{array}{c}\mathcal{P}(\rho, \overline{A} \vee \overline{B})\\ \vdots\\ \vdash \Gamma, \overline{A}, \overline{B}\end{array}}{\vdash \Gamma, \overline{B}}\ cut_1 \qquad \begin{array}{c}\mathcal{P}(\delta, A \wedge B)_{\mathrm{R}}\\ \vdots\\ \vdash \Gamma, B\end{array}}{\vdash \Gamma}\ cut_2$$

By definition, $|cut| = \mathcal{C}(\Gamma, A \wedge B)$, $|cut_1| = \mathcal{C}(\Gamma, A, \overline{B})$, and $|cut_2| = \mathcal{C}(\Gamma, \overline{B})$. Since $\mathcal{C}(B) = \mathcal{C}(\overline{B})$, we can conclude that $|cut_2| \leq |cut_1| < |cut|$.
[CASE 1.2] $C \equiv A \vee B$. Symmetric with respect to the previous one.

[CASE 2] If $\mathcal{C}(C) = 0$, since $\mathcal{C}(\Gamma) > 0$, there will be a formula $D \in \Gamma$ such that $\mathcal{C}(D) > 0$. We need now to distinguish two subcases according to whether $D$'s principal connective is a conjunction or a disjunction. As for the previous case, we provide a list of two-step reductions. The first reduction ($\Longrightarrow$) is still an application of Lemma 3 which allows us to permute downward the logical rule introducing the principal connective of $D$. By performing the second step ($\longrightarrow$) we permute upwards the cut-application under consideration.

[CASE 2.1] $D \equiv A \vee B$

$$\dfrac{\begin{array}{c}\delta\\ \vdots\\ \vdash \Gamma, A \vee B, p\end{array} \qquad \begin{array}{c}\rho\\ \vdots\\ \vdash \Gamma, A \vee B, \overline{p}\end{array}}{\vdash \Gamma, A \vee B}\ cut \quad \Longrightarrow$$

$$\Longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{P}(\delta, A \vee B)\\ \vdots\\ \vdash \Gamma, A, B, p\end{array}}{\vdash \Gamma, A \vee B, p}\ \vee \qquad \dfrac{\begin{array}{c}\mathcal{P}(\rho, A \vee B)\\ \vdots\\ \vdash \Gamma, A, B, \overline{p}\end{array}}{\vdash \Gamma, A \vee B, \overline{p}}\ \vee}{\vdash \Gamma, A \vee B}\ cut \quad \longrightarrow$$

$$\longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{P}(\delta, A \vee B)\\ \vdots\\ \vdash \Gamma, A, B, p\end{array} \qquad \begin{array}{c}\mathcal{P}(\rho, A \vee B)\\ \vdots\\ \vdash \Gamma, A, B, \overline{p}\end{array}}{\vdash \Gamma, A, B}\ cut_1}{\vdash \Gamma, A \vee B}\ \vee$$

Since $|cut| = \mathcal{C}(\Gamma, A \wedge B, p)$ and $|cut_1| = \mathcal{C}(\Gamma, A, B, p)$, we have that $|cut_1| < |cut|$.
[CASE 2.2] $D \equiv A \wedge B$

$$\dfrac{\begin{array}{c}\delta\\ \vdots\\ \vdash \Gamma, A \wedge B, p\end{array} \qquad \begin{array}{c}\rho\\ \vdots\\ \vdash \Gamma, A \wedge B, \overline{p}\end{array}}{\vdash \Gamma, A \wedge B}\ cut \quad \Longrightarrow$$

$$\Longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{P}(\delta, A \wedge B)_{\mathrm{L}}\\ \vdots\\ \vdash \Gamma, A, p\end{array} \qquad \begin{array}{c}\mathcal{P}(\delta, A \wedge B)_{\mathrm{R}}\\ \vdots\\ \vdash \Gamma, B, p\end{array}}{\vdash \Gamma, A \wedge B, p}\ \wedge \qquad \dfrac{\begin{array}{c}\mathcal{P}(\rho, A \wedge B)_{\mathrm{L}}\\ \vdots\\ \vdash \Gamma, A, \overline{p}\end{array} \qquad \begin{array}{c}\mathcal{P}(\rho, A \wedge B)_{\mathrm{R}}\\ \vdots\\ \vdash \Gamma, B, \overline{p}\end{array}}{\vdash \Gamma, A \wedge B, \overline{p}}\ \wedge}{\vdash \Gamma, A \wedge B}\ cut \quad \longrightarrow$$

$$\longrightarrow \dfrac{\dfrac{\begin{array}{c}\mathcal{P}(\delta, A \wedge B)_{\mathrm{L}} \\ \vdots \\ \vdash \Gamma, A, p\end{array} \quad \begin{array}{c}\mathcal{P}(\rho, A \wedge B)_{\mathrm{L}} \\ \vdots \\ \vdash \Gamma, A, \overline{p}\end{array}}{\vdash \Gamma, A}\, cut_1 \quad \dfrac{\begin{array}{c}\mathcal{P}(\delta, A \wedge B)_{\mathrm{R}} \\ \vdots \\ \vdash \Gamma, B, p\end{array} \quad \begin{array}{c}\mathcal{P}(\rho, A \wedge B)_{\mathrm{R}} \\ \vdots \\ \vdash \Gamma, B, \overline{p}\end{array}}{\vdash \Gamma, B}\, cut_2}{\vdash \Gamma, A \wedge B}\, \wedge$$

In this case we have $|cut| = \mathcal{C}(\Gamma, A \wedge B, p)$, $|cut_1| = \mathcal{C}(\Gamma, A, p)$, and $|cut_2| = \mathcal{C}(\Gamma, B, p)$. Therefore, $|cut_1| < |cut|$ and $|cut_2| < |cut|$. □

We are now ready to apply the Reduction Lemma to finally prove the following theorem:

**Theorem 5** (Hauptsatz) *Any GS4$^+$-proof $\pi$ of $\vdash \Gamma$ can be turned into a GS4-proof $\pi'$ ending in the same sequent.*

***Proof*** The proof is led by a double induction: the principal one is on $|\pi|$, whereas the side induction is on the number of maximal cut-applications. If $|\pi| = 1$, then we just keep reducing the topmost cut-applications as indicated in the proof of Lemma 4 till a completely cut-free derivation is achieved.

If $|\pi| > 1$, we consider an arbitrarily selected topmost maximal cut-application $cut_i$. Let $\delta$ be the subproof of $\pi$ whose last inference is the cut-application under consideration. In particular, let $\delta_1$ and $\delta_2$ denote the two direct subproofs of $\delta$ ending in the two premises of $cut_i$:

$$\dfrac{\begin{array}{c}\delta_1 \\ \vdots \\ \vdash \Delta, C\end{array} \quad \begin{array}{c}\delta_2 \\ \vdots \\ \vdash \Delta, \overline{C}\end{array}}{\vdash \Delta}\, cut_i$$

Since $cut_i$ occurs as a topmost maximal cut-application, we have $|\delta_1|, |\delta_2| < |\pi|$. By inductive hypothesis, there are two GS4-proofs $\delta_1'$ and $\delta_2'$ ending in $\vdash \Delta, C$ and $\vdash \Delta, \overline{C}$, respectively. Consider now the proof $\delta'$ obtained from $\delta$ by replacing $\delta_1$ with $\delta_1'$ and $\delta_2$ with $\delta_2'$:

$$\dfrac{\begin{array}{c}\delta_1' \\ \vdots \\ \vdash \Delta, C\end{array} \quad \begin{array}{c}\delta_2' \\ \vdots \\ \vdash \Delta, \overline{C}\end{array}}{\vdash \Delta}\, cut_i$$

By Lemma 4, there is a GS4$^+$-proof $\delta''$ ending in $\vdash \Delta$ and such that $|\delta''| < |\delta|$.

Let $\pi_1$ be the proof obtained from $\pi$ by replacing the subproof $\delta$ with $\delta''$. The proofs $\pi_1$ and $\pi$ end in the same sequent, but $\pi_1$ contains one maximal cut-application less than $\pi$. So, it suffices to keep focussing on topmost maximal cut-applications and reiterate the procedure till a proof $\pi_k$ of $\vdash \Gamma$ such that $|\pi_k| < |\pi|$ is finally achieved. At this point, our inductive hypothesis guarantees the existence of a cut-free proof $\pi'$ ending in $\vdash \Gamma$. □

***Remark 2*** (*First-order logic*) The following rules for quantifiers prove reversible in the sense already specified [8].

$$\frac{\vdash \Gamma, \exists x A, A[^x/_t]}{\vdash \Gamma, \exists x A}\,\exists \qquad \frac{\vdash \Gamma, A[^x/_y]}{\vdash \Gamma, \forall x A}\,\forall$$

Unfortunately, this fact doesn't mean that the technical machinery deployed in this section can be straightforwardly extended so as to prove cut-elimination for the whole first-order system. The reason is simple: for any instance of the $\exists$-rule in which $A(t)$ is non-atomic, $\mathcal{C}(\Gamma, \exists x A, A[^x/_t]) > \mathcal{C}(\Gamma, \exists x A)$.

## 4 Bounds

One of the main advantages of dealing with Kleene's system GS4 lies in the fact that the height of cut-free proofs turns out to be bounded by the complexity of their end-sequent. In particular:

**Theorem 6** *For any GS4-proof $\pi$ ending in $\vdash \Gamma$, $h(\pi) \leq \mathcal{C}(\Gamma) + 1$.*

***Proof*** We proceed by induction on $\mathcal{C}(\Gamma)$. If $\mathcal{C}(\Gamma) = 0$, then $\pi$ is just an instance of the $ax$-rule and so $h(\pi) = 1$. In case $\mathcal{C}(\Gamma) > 0$, we need to distinguish the following two cases.

- The last inference in $\pi$ is an application of the $\wedge$-rule. With $\pi_1$ and $\pi_2$ we refer to the two direct subproofs of $\pi$ ending in $\vdash \Gamma, A$ and $\vdash \Gamma, B$, respectively. By inductive hypothesis, $h(\pi_1) \leq \mathcal{C}(\Gamma, A)+1$ and $h(\pi_2) \leq \mathcal{C}(\Gamma, B)+1$. Since $h(\pi) = max(h(\pi_1), h(\pi_2)) + 1$, we can finally conclude that $h(\pi) \leq \mathcal{C}(\Gamma, A \wedge B) + 1$.
- The last inference in $\pi$ is an application of the $\vee$-rule. Let $\pi_1$ be the direct subproof of $\pi$ ending in $\vdash \Gamma, A, B$. By inductive hypothesis, $h(\pi_1) \leq \mathcal{C}(\Gamma, A, B) + 1$. It is also the case that $\mathcal{C}(\Gamma, A \vee B) = \mathcal{C}(\Gamma, A, B) + 1$. We then conclude that $h(\pi) = h(\pi_1) + 1 \leq \mathcal{C}(\Gamma, A, B) + 2 = \mathcal{C}(\Gamma, A \vee B) + 1$. □

A further fact can be also established:

**Theorem 7** *If $\pi$ and $\rho$ are two GS4-proofs ending in the same sequent $\vdash \Gamma$, then $h(\pi) = h(\rho)$.*

***Proof*** We proceed by induction on $\mathcal{C}(\Gamma)$. If $\mathcal{C}(\Gamma) = 0$, then $\vdash \Gamma$ is just an instance of the $ax$-rule and so $\pi = \rho$. If $\mathcal{C}(\Gamma) > 0$, then there is a multiset $\Gamma'$ and a formula $A$ such that $\Gamma = \Gamma', A$ with $\mathcal{C}(A) > 0$. We distinguish the following two cases:

- $A \equiv B \wedge C$. Consider the two proofs $\pi'$ (the one on the right) and $\rho'$ (the one on the left) displayed below.

$$\dfrac{\begin{array}{c}\mathcal{P}(\pi, B \wedge C)_{\mathrm{L}} \\ \vdots \\ \vdash \Gamma', B\end{array} \qquad \begin{array}{c}\mathcal{P}(\pi, B \wedge C)_{\mathrm{R}} \\ \vdots \\ \vdash \Gamma', C\end{array}}{\vdash \Gamma', B \wedge C}\,\wedge \qquad\qquad \dfrac{\begin{array}{c}\mathcal{P}(\rho, B \wedge C)_{\mathrm{L}} \\ \vdots \\ \vdash \Gamma', B\end{array} \qquad \begin{array}{c}\mathcal{P}(\rho, B \wedge C)_{\mathrm{R}} \\ \vdots \\ \vdash \Gamma', C\end{array}}{\vdash \Gamma', B \wedge C}\,\wedge$$

By inductive hypothesis, $h(\mathcal{P}(\pi, B \wedge C)_{\mathrm{L}}) = h(\mathcal{P}(\rho, B \wedge C)_{\mathrm{L}})$ and $h(\mathcal{P}(\pi, B \wedge C)_{\mathrm{R}}) = h(\mathcal{P}(\rho, B \wedge C)_{\mathrm{R}})$, thence $h(\pi') = h(\rho')$. Moreover, by Lemma 3, $h(\pi) = h(\pi')$ and $h(\rho) = h(\rho')$. The combination of these facts allows us to conclude that $h(\pi) = h(\rho)$.

- $A \equiv B \vee C$. Similar to the previous case. □

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