Archive for Mathematical Logic (2021) 60:317–327
https://doi.org/10.1007/s00153-020-00747-2

Mathematical Logic

ORIGINAL ARTICLE

Small sets in Mann pairs

Pantelis E. Eleftheriou[1]

Received: 16 July 2019 / Accepted: 10 August 2020 / Published online: 2 September 2020

Abstract
Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, G\rangle$ be an expansion of a real closed field $\mathcal{M}$ by a dense subgroup G of $\langle M^{>0}, \cdot\rangle$ with the Mann property. We prove that the induced structure on G by $\mathcal{M}$ eliminates imaginaries. As a consequence, every small set X definable in $\widetilde{\mathcal{M}}$ can be definably embedded into some G^l, uniformly in parameters. These results are proved in a more general setting, where $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$ is an expansion of an o-minimal structure $\mathcal{M}$ by a dense set $P \subseteq M$, satisfying three tameness conditions.

Keywords Mann pairs · Elimination of imaginaries · Small sets

Mathematics Subject Classification Primary 03C64; Secondary 06F20

1 Introduction

This note is a natural extension of the work in [6]. In that reference, expansions $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$ of an o-minimal structure $\mathcal{M}$ by a dense predicate $P \subseteq M$ were studied, and under three tameness conditions, it was shown that the induced structure P_{ind} on P by $\mathcal{M}$ eliminates imaginaries. The tameness conditions were verified for dense pairs of real closed fields, for expansions of $\mathcal{M}$ by an independent set P, and for expansions of a real closed field $\mathcal{M}$ by a dense subgroup P of $\langle M^{>0}, \cdot\rangle$ with the Mann property (henceforth called *Mann pairs*), assuming P is divisible. As pointed out in [6, Remark 4.10], without the divisibility assumption in the last example, the third tameness condition no longer holds, and in [6, Question 4.11] it was asked whether in that case P_{ind} still eliminates imaginaries. In this note, we prove that it does. Indeed, we replace the third tameness condition by a weaker one, which we verify for arbitrary

Research supported by a Research Grant from the German Research Foundation (DFG) and a Zukunftskolleg Research Fellowship.

✉ Pantelis E. Eleftheriou
panteleimon.eleftheriou@uni-konstanz.de

[1] Department of Mathematics and Statistics, University of Konstanz, Box 216, 78457 Constance, Germany

Mann pairs, and prove that together with the two other tameness conditions it implies elimination of imaginaries for P_{ind}.

Let us fix our setting. Throughout this text, $\mathcal{M} = \langle M, <, +, 0, \ldots \rangle$ denotes an o-minimal expansion of an ordered group with a distinguished positive element 1. We denote by $\mathcal{L}$ its language, and by dcl the usual definable closure operator in $\mathcal{M}$. An '$\mathcal{L}$-definable' set is a set definable in $\mathcal{M}$ with parameters. We write '$\mathcal{L}_A$-definable' to specify that those parameters come from $A \subseteq M$. It is well-known that $\mathcal{M}$ admits definable Skolem functions and eliminates imaginaries ([4, Chapter 6]).

Let $D, P \subseteq M$. The *D-induced structure on P by $\mathcal{M}$*, denoted by $P_{ind(D)}$, is a structure in the language

$$\mathcal{L}_{ind(D)} = \{R_{\phi(x)}(x) : \phi(x) \in \mathcal{L}_D\},$$

whose universe is P and, for every tuple $a \subseteq P$,

$$P_{ind(D)} \models R_\phi(a) \;\Leftrightarrow\; \mathcal{M} \models \phi(a).$$

If $Q \subseteq P^n$, by a *trace on Q* we mean a set of the form $Y \cap Q$, where Y is $\mathcal{L}$-definable. We call $Y \cap P^n$ a *full trace*. We call a set $Q \subseteq P^n$ *fiber-dense* if for every x in the projection $\pi(Q)$ of Q onto the first $n-1$ coordinates, the fiber Q_x is dense in P.

For the rest of this paper we fix some $P \subseteq M$ and denote $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$. We let $\mathcal{L}(P)$ denote the language of $\widetilde{\mathcal{M}}$; namely, the language $\mathcal{L}$ augmented by a unary predicate symbol P. We denote by $\mathrm{dcl}_{\mathcal{L}(P)}$ the definable closure operator in $\widetilde{\mathcal{M}}$. Unless stated otherwise, by '(A-)definable' we mean (A-)definable in $\widetilde{\mathcal{M}}$, where $A \subseteq M$. We use the letter D to denote an arbitrary, but not fixed, subset of M.

Tameness Conditions (for $\widetilde{\mathcal{M}}$ and D):
(OP) (Open definable sets are $\mathcal{L}$-definable.) For every set A such that $A \setminus P$ is dcl-independent over P, and for every A-definable set $V \subset M^n$, its topological closure $\overline{V} \subseteq M^n$ is $\mathcal{L}_A$-definable.

$(\mathrm{dcl})_D$ Let $B, C \subseteq P$ and

$$A = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap P.$$

Then

$$\mathrm{dcl}(AD) = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD).$$

$(\mathrm{ind})_D$ Let $X \subseteq P^n$ be definable in $P_{ind(D)}$. Then X is a finite union of traces on sets which are $\emptyset$-definable in $P_{ind(D)}$ and fiber-dense. That is, there are $\mathcal{L}$-definable sets $Y_1, \ldots, Y_l \subseteq M^n$, and sets $Q_1, \ldots, Q_l \subseteq P^n$ that are $\emptyset$-definable in $P_{ind(D)}$ and fiber-dense, such that

$$X = \bigcup_i (Y_i \cap Q_i).$$

Conditions (OP) and $(\mathrm{dcl})_D$ are the same with those in [6], and are already known to hold for Mann pairs ([6, Remark 4.11]). Condition $(\mathrm{ind})_D$ is weaker than the corresponding one in [6], in three ways: (a) X is now a *finite union* of traces (instead of a single trace), (b) the traces are on *subsets* of P^n (instead of on the whole P^n), and (c) there is no control in parameters for the Y_i's (although we achieve this in Corollary 3.5 below). These differences result in several non-trivial complications in the proof of our main theorem, which are handled in Sect. 3. For now, let us state the main theorem.

Theorem 1.1 *Assume* (OP)*,* $(\mathrm{dcl})_D$ *and* $(\mathrm{ind})_D$*, and that D is* dcl*-independent over P. Then $P_{ind(D)}$ eliminates imaginaries.*

Condition $(\mathrm{ind})_D$ is modelled after the current literature on Mann pairs, which we now explain. Assume $\mathcal{M} = \langle M, <, +, \cdot, 0, 1\rangle$ is a real closed field, and G a dense subgroup of $\langle M^{>0}, \cdot\rangle$. For every $a_1, \dots, a_r \in M$, a solution $(q_1, \dots, q_r)$ to the equation

$$a_1x_1 + \cdots + a_rx_r = 1$$

is called *non-degenerate* if for every non-empty $I \subseteq \{1, \dots, r\}$, $\sum_{i\in I} a_iq_i \neq 0$. We say that G has the *Mann property*, if for every $a_1, \dots, a_r \in M$, the above equation has only finitely many non-degenerate solutions $(q_1, \dots, q_r)$ in G^r.[1] Let us call such a pair $\langle \mathcal{M}, G\rangle$ a *Mann pair*. Examples of Mann pairs include all multiplicative subgroups of $\langle \mathbb{R}_{>0}, \cdot\rangle$ of finite rank [8], such as $2^{\mathbb{Q}}$ and $2^{\mathbb{Z}}3^{\mathbb{Z}}$. Van den Dries–Günaydin [5, Theorem 7.2] showed that in a Mann pair, where moreover G is divisible (such as $2^{\mathbb{Q}}$), every definable set $X \subseteq G^n$ is a full trace; in particular, $(\mathrm{ind})_D$ from [6] holds. Without the divisibility assumption, however, this is no longer true. Consider for example $G = 2^{\mathbb{Z}}3^{\mathbb{Z}}$ and let X be the subgroup of G consisting of all elements divisible by 2. That is, $X = \{2^{2m}3^{2n} : m, n \in \mathbb{Z}\}$. This set is clearly dense and co-dense in $\mathbb{R}$, and cannot be a trace on any subset of G.

A substitute to [5, Theorem 7.2] was proved by Berenstein-Ealy-Günaydin [1], as follows. Consider, for every $d \in \mathbb{N}$, the set $G^{[d]}$ of all elements of G divisible by d,

$$G^{[d]} = \{x \in G : \exists y \in G,\ x = y^d\}.$$

Under the mild assumption that for every prime p, $G^{[p]}$ has finite index in G, [5, Theorem 7.5] provides a near model completeness result, which is then used in [1] to prove that every definable set $X \subseteq P^n$ is a finite union of traces on $\emptyset$-definable subsets of P^n (Fact 3.10 below). Note this mild assumption is still satisfied by all multiplicative subgroups of $\langle \mathbb{R}_{>0}, \cdot\rangle$ of finite rank (as noted in [9]).

Corollary 1.2 *Assume $\widetilde{\mathcal{M}} = \langle \mathcal{M}, G\rangle$ is a Mann pair, such that for every prime p, $G^{[p]}$ has finite index in G. Let $D \subseteq M$ be* dcl*-independent over P. Then* (OP)*,* $(\mathrm{dcl})_D$ *and* $(\mathrm{ind})_D$ *hold. In particular, $P_{ind(D)}$ eliminates imaginaries.*

[1] The original definition only involved equations with coefficients a_i in the prime field of $\mathcal{M}$, but, by [5, Proposition 5.6], the two definitions are equivalent.

Observe that Corollary 1.2 stands in contrast to the current literature, as it is known that in Mann pairs both existence of definable Skolem functions and elimination of imaginaries (for $\widetilde{\mathcal{M}}$) fail ([2]). Note also that the assumption of D being dcl-independent over P is necessary; namely, without it, $P_{ind(D)}$ need not eliminate imaginaries ([6, Example 5.1]).

Theorem 1.1 has the following important consequence. Recall from [3] that a set $X \subseteq M^n$ is called *P-bound over A* if there is an $\mathcal{L}_A$-definable function $h : M^m \to M^n$ such that $X \subseteq h(P^m)$. The recent work in [7] provides an analysis for all definable sets in terms of '$\mathcal{L}$-definable-like' and P-bound sets. Using Theorem 1.1, we further reduce the study of P-bound sets to that of definable subsets of P^l.

Corollary 1.3 *Assume* (OP)*,* $(\text{dcl})_D$ *and* $(\text{ind})_D$ *hold for every $D \subseteq M$ which is* dcl-*independent over P. Let $X \subseteq M^n$ be an A-definable set. If X is P-bound over A, then there is an $A \cup P$-definable injective map $\tau : X \to P^l$. If A itself is* dcl-*independent over P, then the extra parameters from P can be omitted.*

Note that the assumption of Corollary 1.3 holds for $\widetilde{\mathcal{M}}$ as in Corollary 1.2. Note also that allowing parameters from P is standard practice when studying definability in this context; see for example [7, Lemma 2.5 and Corollary 3.26].

Structure of the paper. In Sect. 2, we fix notation and recall some basic facts. In Sect. 3, we prove our results.

2 Preliminaries

We assume familiarity with the basics of o-minimality and pregeometries, as can be found, for example, in [4] or [10]. Recall that $\mathcal{M} = \langle M, <, +, 0, \ldots \rangle$ is our fixed o-minimal expansion of an ordered group with a distinguished positive element 1 and dcl denotes the usual definable closure operator. We denote the corresponding dimension by dim. If A, B are two sets, we often write AB for $A \cup B$. We denote by $\Gamma(f)$ the graph of a function f. If $T \subseteq M^m \times M^n$ and $x \in M^n$, we write T_x for the fiber

$$\{b \in M^m \ : \ (b, x) \in T\}.$$

The topological closure of a set $Y \subseteq M^n$ is denoted by $\overline{Y}$ and its frontier $\overline{Y} \setminus Y$ by $\text{fr}(Y)$. If $X \subseteq Y$, the relative interior of X in Y is denoted by $\text{int}_Y(X)$. It is not hard to see that:

$$\text{int}_Y(X) = \{x \in X : \text{there is open} B \subseteq M^n \text{containing} x \text{with} B \cap Y \subseteq X\}.$$

We will need the following fact.

Fact 2.1 ([4, Ch. 4 (1.3)]) *Let $X \subseteq Y \subseteq M^n$ be two $\mathcal{L}$-definable sets. Then*

$$\dim(X \setminus \text{int}_Y(X)) < \dim Y.$$

2.1 Elimination of imaginaries

We recall that a structure $\mathcal{N}$ *eliminates imaginaries* if for every $\emptyset$-definable equivalence relation E on N^n, there is a $\emptyset$-definable map $f : N^n \to N^l$ such that for every $x, y \in N^n$,

$$E(x, y) \;\Leftrightarrow\; f(x) = f(y).$$

In the ordered setting, we have the following criterion (extracted from [10, Section 3]; for a proof see [6, Fact 2.2]).

Fact 2.2 *Let $\mathcal{N}$ be a sufficiently saturated structure with two distinct constants in its language. Suppose the following property holds.*

(*) Let $B, C \subseteq N$ and $A = \mathrm{dcl}_{\mathcal{N}}(B) \cap \mathrm{dcl}_{\mathcal{N}}(C)$. If $X \subseteq N^n$ is B-definable and C-definable, then X is A-definable.

Then $\mathcal{N}$ eliminates imaginaries.

2.2 The induced structure

Recall from the introduction that

$$P_{ind(D)} = \langle P, \{R \cap P^l : R \subseteq M^l \mathcal{L}_D - \text{definable}, l \in \mathbb{N}\}\rangle.$$

Remark 2.3 For $A \subseteq P$, we have:

(1) if $Q \subseteq P^n$ is A-definable in $P_{ind(D)}$, and $Y \subseteq M^n$ is $\mathcal{L}_{AD}$-definable, then $Q \cap Y$ is A-definable in $P_{ind(D)}$. Indeed, $Q \cap Y = Q \cap (Y \cap P^n)$.
(2) in general, if $Q \subseteq P^n$ is A-definable in $P_{ind(D)}$, then it is AD-definable. The converse will be true for Mann pairs, by Corollary 3.11 below.

3 Proofs of the results

In this section we prove elimination of imaginaries for $P_{ind(D)}$ under our assumptions (Theorem 1.1) and deduce Corollaries 1.2 and 1.3 from it. Our goal is to establish (*) from Fact 2.2 for $\mathcal{N} = P_{ind(D)}$ (Lemma 3.8 below). As in [6], the strategy is to reduce the proof of (*) to [10, Proposition 2.3], which is an assertion of (*) for $\mathcal{M}$. This reduction takes place in the proof of Lemma 3.8 below, and requires the key Lemma 3.4. The analogous key lemma in [6] (namely, [6, Lemma 3.1]) cannot help us here, because its assumptions are not met in the proof of Lemma 3.8. Furthermore, the proof of Lemma 3.4 requires an entirely new technique.

We begin with some preliminary observations.

Fact 3.1 *Assume* (OP). *Then for every $A \subseteq P$,* $\mathrm{dcl}_{\mathcal{L}(P)}(A) = \mathrm{dcl}(A)$.

Proof Take $x \in \mathrm{dcl}_{\mathcal{L}(P)}(A)$. That is, the set $\{x\}$ is A-definable in $\widetilde{\mathcal{M}}$. By (OP), we have that $\overline{\{x\}}$ is $\mathcal{L}_A$-definable. But $\overline{\{x\}} = \{x\}$. □

Lemma 3.2 *Assume* (OP). *Let* $X \subseteq M^n$ *be an* $\mathcal{L}$*-definable set which is also* C*-definable, for some* $C \subseteq M$ *with* $C \backslash P$ dcl*-independent over* P. *Then* X *is* $\mathcal{L}_C$*-definable.*

Proof We work by induction on $k = \dim X$. If $X = \emptyset$, the statement is obvious. Assume $k \geq 0$. By (OP), $\overline{X}$ is $\mathcal{L}_C$-definable. By o-minimality, $\dim \operatorname{fr}(X) < k$. Since $\operatorname{fr}(X) = \overline{X} \backslash X$ is both $\mathcal{L}$-definable and C-definable, by inductive hypothesis, it is $\mathcal{L}_C$-definable. So $X = \overline{X} \backslash \operatorname{fr}(X)$ is $\mathcal{L}_C$-definable. □

Lemma 3.3 *Let* $C \subseteq M$ *and*

$$X = \bigcup_{i=1}^{m} (Z_i \cap R_i),$$

where $Z_1, \ldots, Z_m \subseteq M^n$ *are* $\mathcal{L}_C$*-definable sets, and* $R_1, \ldots, R_m \subseteq P^n$ *are* $\emptyset$*-definable in* $P_{ind(D)}$ *and fiber-dense. Then*

$$X = \bigcup_{i=1}^{l} (Y_i \cap Q_i),$$

for some $\mathcal{L}_C$*-definable* ***disjoint*** *sets* $Y_1, \ldots, Y_l \subseteq M^n$*, and sets* $Q_1, \ldots, Q_l \subseteq P^n$ *which are* $\emptyset$*-definable in* $P_{ind(D)}$ *and fiber-dense.*

Proof For $\sigma \subseteq \{1, \ldots, m\}$, let

$$Q_\sigma = \bigcup_{i \in \sigma} R_i$$

and

$$Y_\sigma = \left(\bigcap_{i \in \sigma} Z_i \right) \backslash \left(\bigcup_{j \notin \sigma} Z_j \right).$$

Clearly, Q_σ is fiber-dense. It is also easy to check that for any two distinct $\sigma, \tau \subseteq \{1, \ldots, m\}$, we have $Y_\sigma \cap Y_\tau = \emptyset$, and that

$$X = \bigcup_{\sigma \subseteq \{1, \ldots, m\}} (Y_\sigma \cap Q_\sigma),$$

as required. □

Now, the key technical lemma.

Lemma 3.4 *Assume* (OP) *and* $(\mathrm{ind})_D$*, and that* D *is* dcl*-independent over* P. *Let* $B, C \subseteq P$ *and* $X \subseteq P^n$ *be* B*-definable and* C*-definable in* $P_{ind(D)}$. *Then there* *are $W_1, \dots, W_l \subseteq M^n$, that are both $\mathcal{L}_{BD}$-definable and $\mathcal{L}_{CD}$-definable, and sets $S_1, \dots, S_k \subseteq P^n$, that are $\emptyset$-definable in $P_{ind(D)}$ and fiber-dense, such that*

$$X = \bigcup_{i=1}^{l} W_i \cap S_i.$$

Proof First note that X is both BD-definable and CD-definable in $\langle \mathcal{M}, P \rangle$. Since $B, C \subseteq P$, by (OP) it follows that $\overline{X}$ is $\mathcal{L}_{BD}$-definable and $\mathcal{L}_{CD}$-definable.

We perform induction on the dimension of $\overline{X}$. For $X = \emptyset$, the statement is obvious. Suppose now that $\dim \overline{X} = k \geq 0$. By $(\text{ind})_D$ and Lemma 3.3, there are $\mathcal{L}$-definable disjoint sets $Z_1, \dots, Z_m \subseteq M^n$, and sets $R_1, \dots, R_l \subseteq P^n$, each $\emptyset$-definable in $P_{ind(D)}$ and fiber-dense, such that

$$X = \bigcup_{i=1}^{l} (Z_i \cap R_i).$$

For every i, define

$$T_i = \{x \in \overline{X} : \text{ there is relatively open} V \subseteq \overline{X} \text{around} x, \text{with} V \cap R_i \subseteq X\}.$$

Let $T = \bigcup_i T_i$. It is immediate from the definition, that each T_i, and hence T, is relatively open in $\overline{X}$. Therefore, by (OP), it is $\mathcal{L}$-definable. On the other hand, each T_i is BD-definable and CD-definable, because X is, and R_i is D-definable. Hence, by Lemma 3.2, each T_i, and hence T, is $\mathcal{L}_{BD}$-definable and $\mathcal{L}_{CD}$-definable.

Claim. $\dim \overline{X \setminus \bigcup_i (T_i \cap R_i)} < k$.

Proof Observe first that $X \subseteq \bigcup_i Z_i$, and hence it suffices to show that for each i,

$$\dim \big((Z_i \cap X) \setminus (T_i \cap R_i) \big) < k.$$

We may write

$$(Z_i \cap X) \setminus (T_i \cap R_i) = \big((Z_i \cap X) \setminus \operatorname{int}_{\overline{X}}(Z_i \cap X) \big) \cup \big(\operatorname{int}_{\overline{X}}(Z_i \cap X) \setminus (T_i \cap R_i) \big),$$

By Fact 2.1, it suffices to show that $\operatorname{int}_{\overline{X}}(Z_i \cap X) \subseteq (T_i \cap R_i)$. Clearly, $\operatorname{int}_{\overline{X}}(Z_i \cap X) \subseteq \operatorname{int}_{\overline{X}}(Z_i) \cap X$, and hence it suffices to show:

$$\operatorname{int}_{\overline{X}}(Z_i) \cap X \subseteq T_i \cap R_i.$$

Let $x \in \operatorname{int}_{\overline{X}}(Z_i) \cap X$. Since $x \in \operatorname{int}_{\overline{X}}(Z_i)$, there is a relatively open $V \subseteq \overline{X}$ containing x, with $V \subseteq Z_i$, and hence $V \cap R_i \subseteq Z_i \cap R_i \subseteq X$. Therefore $x \in T_i$. Since $x \in X \cap Z_i$ and the Z_j's are disjoint, we must also have $x \in R_i$. Hence $x \in T_i \cap R_i$, as needed. □

By Remark 2.3(1), the set $X\setminus\bigcup_i(T_i\cap R_i)$ is both B-definable and C-definable in $P_{ind(D)}$. Hence, by inductive hypothesis and the claim, the conclusion holds for this set. Now, for each i, by definition of T_i, we have $T_i\cap R_i\subseteq X$. Hence

$$X=\left(X\setminus\bigcup_i(T_i\cap R_i)\right)\cup\bigcup_i(T_i\cap R_i),$$

and we are done. □

Corollary 3.5 *Assume* (OP) *and* (ind)$_D$*, and that D is* dcl*-independent over P. Let $A\subseteq P$ and $X\subseteq P^n$ be A-definable in $P_{ind(D)}$. Then there are $\mathcal{L}_{AD}$-definable sets $W_1,\dots,W_l\subseteq M^n$, and sets $S_1,\dots,S_l\subseteq P^k$ that are $\emptyset$-definable in $P_{ind(D)}$ and fiber-dense, such that*

$$X=\bigcup_i(W_i\cap S_i).$$

Proof By Lemma 3.4 for $B=C=A$. □

Our next goal is to prove the promised Lemma 3.8. Denote by cl_D the definable closure operator in $P_{ind(D)}$. We first prove that, under (OP) and (ind)$_D$, cl_D defines a pregeometry (Corollary 3.7).

Lemma 3.6 *Assume* (OP) *and* (ind)$_D$*, and that D is* dcl*-independent over P. Let $f:P^n\to P$ be an A-definable map in $P_{ind(D)}$. Then there are $\mathcal{L}_{AD}$-definable maps $F_1,\dots,F_l:M^n\to M^k$, such that for every $x\in P^n$, there is $i=1,\dots,l$, with $f(x)=F_i(x)$.*

Proof By Corollary 3.5, there are finitely many $\mathcal{L}_{AD}$-definable sets $W_1,\dots,W_l\subseteq M^{n+1}$ and sets $S_1,\dots,S_l\subseteq P^{n+1}$ which are $\emptyset$-definable in $P_{ind(D)}$ and fiber-dense, such that $\Gamma(f)=\bigcup_i W_i\cap S_i$. Fix i, and let f_i be the map whose graph equals $W_i\cap S_i$. It suffices to prove that the graph of f_i is contained in finitely many graphs of $\mathcal{L}_{AD}$-definable maps. To simplify the notation, we set $f=f_i$, $S=S_i$ and $W=W_i$. So $\Gamma(f)=W\cap S$, where $f:X\subseteq P^n\to P$ is A-definable, $W\subseteq M^{n+1}$ is $\mathcal{L}_{AD}$-definable, and $S\subseteq P^{n+1}$ is $\emptyset$-definable in $P_{ind(D)}$ and fiber-dense. By o-minimality, for every x in the projection $\pi(W)$ of W onto the first n coordinates, each fiber W_x is a finite union of open intervals and points. Since for every $x\in X$, $(W\cap S)_x=\{f(x)\}$, and the fiber S_x is dense in P, it follows that W_x is finite, with only one of its elements belonging to P. Let

$$T=\{x\in\pi(W):W_x\text{ is finite}\}.$$

So, $X\subseteq T$. Moreover, T is $\mathcal{L}_{AD}$-definable. Now, by o-minimality $W\cap(T\times M)$ is a finite union of $\mathcal{L}_{AD}$-definable maps that contains $\Gamma(f)$, as needed. □

Corollary 3.7 *Assume* (OP) *and* (ind)$_D$*, and that D is* dcl*-independent over P. Then for every $A\subseteq P$, $\mathrm{cl}_D(A)=\mathrm{dcl}(AD)\cap P$. In particular, cl_D defines a pregeometry.*

Proof The inclusion $\supseteq$ is immediate from the definitions, whereas the inclusion $\subseteq$ is immediate from Lemma 3.6. Since $\mathrm{dcl}(-D)$ defines a pregeometry in $\mathcal{M}$, it follows easily that so does $cl_D(-)$ in $P_{ind(D)}$. □

Lemma 3.8 *Assume* (OP), $(\mathrm{dcl})_D$ *and* $(\mathrm{ind})_D$, *and that D is* dcl-*independent over P. Let $B, C \subseteq P$ and $A = cl_D(B) \cap cl_D(C)$. If $X \subseteq P^n$ is B-definable and C-definable in $P_{ind(D)}$, then X is A-definable in $P_{ind(D)}$.*

Proof Let $X \subseteq P^n$ be B-definable and C-definable in $P_{ind(D)}$. By Lemma 3.4, there are $W_1, \dots, W_l \subseteq M^n$, each both $\mathcal{L}_{BD}$-definable and $\mathcal{L}_{CD}$-definable, and $S_1, \dots, S_k \subseteq P^n$, each $\emptyset$-definable in $P_{ind(D)}$, such that

$$X = \bigcup_{i=1}^{l} W_i \cap S_i.$$

By [10, Proposition 2.3], each W_i is $\mathcal{L}$-definable over $\mathrm{dcl}(BD) \cap \mathrm{dcl}(CD)$. By $(\mathrm{dcl})_D$, W_i is $\mathcal{L}$-definable over $\mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap PD$. Hence X is definable over $\mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap P$ in $P_{ind(D)}$. But

$$\mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap P = cl_D(B) \cap cl_D(C) = A,$$

and hence X is A-definable in $P_{ind(D)}$. □

We can now conclude our results.

Proof of Theorem 1.1 By Fact 2.2 and Lemma 3.8.

For the proof of Corollary 1.3, we additionally need the following lemma.

Lemma 3.9 *Assume* (OP) *and* $(\mathrm{ind})_D$, *and that D is* dcl-*independent over P. Let $\mathcal{M}'$ be the expansion of $\mathcal{M}$ with constants for all elements in P, and $\widetilde{\mathcal{M}}' = \langle \mathcal{M}', P \rangle$. Then* $(\mathrm{ind})_D$ *holds for $\widetilde{\mathcal{M}}'$ and D.*

Proof Denote by $P'_{ind(D)}$ the D-induced structure on P by $\mathcal{M}'$. Let $X \subseteq P^n$ be A-definable in $P'_{ind(D)}$. It follows that X is AP-definable in $P_{ind(D)}$. By Corollary 3.5, there are $\mathcal{L}_{APD}$-definable sets $Y_1, \dots, Y_l \subseteq M^n$, and $Q_1, \dots, Q_l \subseteq P^k$, which are $\emptyset$-definable in $P_{ind(D)}$, such that

$$X = \bigcup_i (Y_i \cap Q_i).$$

Such Y_i's are $\mathcal{L}_{AD}$-definable in $\mathcal{M}$, and the Q_i's are of course $\emptyset$-definable in $P'_{ind(D)}$, as required. □

Proof of Corollary 1.3 The proof when A is dcl-independent over P is identical to that of [6, Theorem B]. The proof of the general case is identical to that of [6, Corollary 1.4], after replacing in [6, Lemma 3.4] the clause about $(\mathrm{ind})_D$ with Lemma 3.9 above.

We finally turn to our targeted example of Mann pairs. The proof of Corollary 1.2 will be complete after we recall the fact below, which is extracted from [1]. First, observe that if $\widetilde{\mathcal{M}} = \langle \mathcal{M}, G\rangle$ is a Mann pair, then for every $d \in \mathbb{N}$, $G^{[d]}$ is $\emptyset$-definable in $P_{ind(\emptyset)}$. Indeed, $G^{[d]}$ is the projection onto the first coordinate of the set $\{(x^d, x) : x \in M\} \cap G^2$.

Fact 3.10 *Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, G\rangle$ be a Mann pair, such that for every prime p, $G^{[p]}$ has finite index in G. Let $X \subseteq G^n$ a definable set. Then X is a finite union of traces on sets which are $\emptyset$-definable in $G_{ind(\emptyset)}$ and fiber-dense. That is,* $(\mathrm{ind})_D$ *holds.*

Proof By [1, Corollary 57], X is as a finite union of traces on sets of the form $g(G^{[d]})^n$, $d \in \mathbb{N}$. As pointed out in the proof of [1, Theorem 1], each such g can be chosen to be $\emptyset$-definable (in $\widetilde{\mathcal{M}}$). By Fact 3.1, $g \in \mathrm{dcl}(\emptyset)$. By the above observation, $g(G^{[d]})^n$ is $\emptyset$-definable in $G_{ind(\emptyset)}$. It is also fiber-dense. □

***Proof of Corollary* 1.2** By Fact 3.10, $(\mathrm{ind})_D$ hold. By [6], as explained in Remark 4.11 therein, (OP) and $(\mathrm{dcl})_D$ holds. By Theorem 1.1, we are done.

A byproduct of our work is the following corollary.

Corollary 3.11 *Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, G\rangle$ be a Mann pair, such that for every prime p, $G^{[p]}$ has finite index in G. Let $D \subseteq M$ be* dcl*-independent over P. Let $X \subseteq P^n$ be AD-definable, with $A \subseteq P$. Then X is A-definable in $P_{ind(D)}$. In particular, the conclusion of* Corollary 3.5 *holds.*

Proof By Corollaries 1.2 and 3.5 . □

Acknowledgements I thank the referee for several comments that helped to improve the presentation of the paper.

Funding Open Access funding provided by Projekt DEAL.

References

1. Berenstein, A., Ealy, C., Günaydin, A.: Thorn independence in the field of real numbers with a small multiplicative group. Ann. Pure Appl. Logic **150**, 1–18 (2007)
2. Dolich, A., Miller, C., Steinhorn, C.: Structures having o-minimal open core. Trans. AMS **362**, 1371–1411 (2010)
3. van den Dries, L.: Dense pairs of o-minimal structures. Fundam. Math. **157**, 61–78 (1988)
4. van den Dries, L.: Tame Topology and o-minimal Structures. Cambridge University Press, Cambridge (1998)

5. van den Dries, L., Günaydın, A.: The fields of real and complex numbers with a small multiplicative group. Proc. Lond. Math. Soc. **93**, 43–81 (2006)
6. Eleftheriou, P.: Small sets in dense pairs, Israel J. Math, Israel Journal of Mathematics, Online first, pp. 1–27 (2019)
7. Eleftheriou, P., Günaydin, A., Hieronymi, P.: Structure theorems in tame expansions of o-minimal structures by dense sets. Israel Journal of Mathematics (to appear), arXiv:1510.03210v4
8. Everste, J.-H., Schlickewei, H.P., Schmidt, W.M.: Linear equations in variables which lie in a multiplicative subgroup. Ann. Math. **155**, 807–836 (2002)
9. Günaydin, A.: Model theory of fields with multiplicative subgroups, PhD thesis, University of Illinois at Urbana-Champaign (2008)
10. Pillay, A.: Some remarks on definable equivalence relations in o-minimal structures. J. Symb. Logic **51**(3), 709–714 (1986)