Selection properties of the split interval and the Continuum Hypothesis

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Abstract

We prove that every usco multimap $\Phi : X \multimap Y$ from a metrizable separable space X to a GO-space Y has an F_σ -measurable selection. On the other hand, for the split interval $\mathbb{I}$ and the projection $P : \mathbb{I}^2 \rightarrow \mathbb{I}^2$ of its square onto the unit square $\mathbb{I}^2$, the usco multimap $P^{-1} : \mathbb{I}^2 \multimap \mathbb{I}^2$ has a Borel (F_σ -measurable) selection if and only if the Continuum Hypothesis holds. This CH-example shows that known results on Borel selections of usco maps into fragmentable compact spaces cannot be extended to a wider class of compact spaces.

Keywords Continuum hypothesis · Split interval · Measurable selection · Borel selection · usco multimap · Fragmentable compact · Rosenthal compact · GO-space

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1 Introduction

By a *multimap* $\Phi : X \multimap Y$ between topological spaces X, Y we understand any subset $\Phi \subseteq X \times Y$, which can be thought as a function assigning to every point $x \in X$ the subset $\Phi(x) := \{y \in Y : \langle x, y \rangle \in \Phi\}$ of Y . For a subset $A \subseteq X$ we put $\Phi[A] = \bigcup_{x \in A} \Phi(x)$. Each function $f : X \rightarrow Y$ can be thought as a single-valued multimap $\{\langle x, f(x) \rangle : x \in X\} \subseteq X \times Y$.

For a multimap $\Phi : X \multimap Y$, its *inverse multimap* $\Phi^{-1} : Y \multimap X$ is defined by $\Phi^{-1} := \{\langle y, x \rangle : \langle x, y \rangle \in \Phi\}$.

A multimap $\Phi : X \multimap Y$ is called

- *lower semicontinuous* if for any open set $U \subseteq Y$ the set $\Phi^{-1}[U]$ is open in X ;
- *upper semicontinuous* if for any closed set $F \subseteq Y$ the set $\Phi^{-1}[F]$ is closed in X ;

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- *Borel-measurable* if for any Borel set $B \subseteq Y$ the set $\Phi^{-1}[B]$ is Borel in X ;
- *compact-valued* if for every $x \in X$ the subspace $\Phi(x)$ of Y is compact and non-empty;
- *usco* if Φ is upper semicontinuous and compact-valued.

It is well-known that for any surjective continuous function $f : X \rightarrow Y$ between compact Hausdorff spaces, the inverse multimap $f^{-1} : Y \multimap X$ is usco.

Let $\Phi : X \multimap Y$ be a multimap between topological spaces. A function $f : X \rightarrow Y$ is called a *selection* of Φ if $f(x) \in \Phi(x)$ for every $x \in X$. The Axiom of Choice ensures that every multimap $\Phi : X \multimap Y$ with non-empty values has a selection. The problem is to find selections possessing some additional properties like the continuity or measurability.

One of classical results in this direction is the following theorem of Kuratowski and Ryll-Nardzewski [12] (see also [16, §5.2] or [14, 6.12]).

Theorem 1 *Let X, Y be Polish spaces. Any Borel-measurable multimap $\Phi : X \multimap Y$ with non-empty values has a Borel-measurable selection.*

We recall that a function $f : X \rightarrow Y$ between topological spaces is called *Borel-measurable* (resp. F_σ -measurable) if for every open set $U \subseteq Y$ the preimage $f^{-1}[U]$ is Borel (or type F_σ) in X .

F_σ -Measurable selections of usco multimaps with values in non-metrizable compact spaces were studied by many mathematicians [4–9]. Positive results are known for two classes of compact spaces: fragmentable and linearly ordered.

Let us recall [3, 5.0.1] (see also [15, §6]) that a topological space K is *fragmentable* if K has a metric ρ such that for every $\varepsilon > 0$ each non-empty subset $A \subseteq K$ contains a non-empty relatively open set $U \subseteq A$ of ρ -diameter $< \varepsilon$. By [3, 5.1.12], each fragmentable compact Hausdorff space contains a metrizable dense G_δ -subspace.

The following selection theorem can be deduced from Theorem 1' and Lemma 6 in [9].

Theorem 2 (Hansell, Jayne, Talagrand) *Any usco map $\Phi : X \rightarrow K$ from a perfectly paracompact space X to a fragmentable compact space Y has an F_σ -measurable selection.*

A similar selection theorem holds for usco maps into countably cellular GO-spaces. A Hausdorff topological space X is called a *generalized ordered space* (briefly, a *GO-space*) if X admits a linear order $\leq$ such that the topology of X is generated by a base consisting of open order-convex subsets of X . A subset C of a linearly ordered space X is called *order-convex* if for any points $x \leq y$ in C the order interval $[x, y] := \{z \in X : x \leq z \leq y\}$ is contained in C . We say that the topology of X is generated by the linear order $\leq$ if the topology of X is generated by the subbase $\{(\leftarrow, a), (a, \rightarrow) : a \in X\}$ consisting the the order-convex sets $(\leftarrow, a) := \{x \in X : x < a\}$ and $(a, \rightarrow) := \{x \in X : a < x\}$.

A topological space X is *countably cellular* if every disjoint family of open sets in X is at most countable. It is easy to see that each separable topological space is countably cellular. A topological space is called *F_σ -perfect* if every open set in X is

of type F_σ in X (i.e., can be represented as the countable union of closed sets). For example, every metrizable space is F_σ -perfect.

The following selection theorem will be proved in Sect. 2.

Theorem 3 *Let Y be a GO-space and X be an $(F_\sigma$ -perfect) topological space. If X or Y is countably cellular, then any usco map $\Phi : X \multimap Y$ has a Borel (F_σ -measurable) selection.*

Theorems 2, 3 suggest the following problem.

Problem 1 Is it true that any usco map $\Phi : M \multimap K$ from a compact metrizable space M to a compact Hausdorff space K has a Borel (F_σ -measurable) selection?

In this paper we prove that this problem has negative answer under the negation of the Continuum Hypothesis (i.e., under $\omega_1 < \mathfrak{c}$). A suitable counterexample will be constructed using the split square $\mathbb{I}^2$, which is the square of the split interval $\mathbb{I}$.

The *split interval* is the linearly ordered space $\mathbb{I} = [0, 1] \times \{0, 1\}$ whose topology is generated by the lexicographic order (defined by $\langle x, i \rangle \leq \langle y, j \rangle$ iff either $x < y$ or else $x = y$ and $i \leq j$). The split interval plays a fundamental role in the theory of separable Rosenthal compacta [17]. Let us recall that a topological space is called *Rosenthal compact* if it is homeomorphic to a compact subspace of the space $B_1(P)$ of functions of the first Baire class on a Polish space P . It is well-known (and easy to see) that the split interval is Rosenthal compact and so is its square. By Theorem 4 of Todorćević [17], each non-metrizable Rosenthal compact space of countable spread contains a topological copy of the split interval. A topological space has *countable spread* if it contains no uncountable discrete subspaces.

By Theorem 3, any usco map $\Phi : X \multimap \mathbb{I}$ from an F_σ -perfect topological space X has an F_σ -measurable selection. In contrast, the split square $\mathbb{I}^2$ has dramatically different selections properties. Let $p : \mathbb{I} \rightarrow \mathbb{I}$, $p : \langle x, i \rangle \mapsto x$, be the natural projection of the split interval onto the unit interval $\mathbb{I} = [0, 1]$, and

$$P : \mathbb{I}^2 \rightarrow \mathbb{I}^2, \quad P : \langle x, y \rangle \mapsto \langle p(x), p(y) \rangle,$$

be the projection of the split square $\mathbb{I}^2$ onto the unit square $\mathbb{I}^2$.

Theorem 4 *The following conditions are equivalent:*

1. *the usco multimap $P^{-1} : \mathbb{I}^2 \rightarrow \mathbb{I}^2$ has a Borel-measurable selection;*
2. *the usco multimap $P^{-1} : \mathbb{I}^2 \rightarrow \mathbb{I}^2$ has an F_σ -measurable selection;*
3. $\omega_1 = \mathfrak{c}$.

The implication (2) $\Rightarrow$ (1) of Theorem 4 is trivial and the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2) are proved in Lemmas 2 and 8, respectively.

Combining Theorem 4 with Theorem 4 in [17], we obtain the following consistent characterization of metrizable compacta.

Corollary 1 *Under $\omega_1 < \mathfrak{c}$ a Rosenthal compact space K is metrizable if and only if K has countable spread and each usco multimap $\Phi : \mathbb{I}^2 \multimap K^2$ has a Borel-measurable selection.*

Proof The “only if” part follows from Theorem 2. To prove the “if” part, assume that a Rosenthal compact K is not metrizable but has countable spread. By Theorem 4 of [17], the space K contains a topological copy of the split interval $\mathbb{I}$. We lose no generality assuming that $\mathbb{I} \subseteq K$. By Theorem 4, under $\omega_1 < \mathfrak{c}$, the usco multimap $P^{-1} : \mathbb{I}^2 \multimap \mathbb{I}^2 \subseteq K^2$ does not have Borel-measurable selections. $\square$

Now we pose some open problems suggested by Theorem 4.

Problem 2 Assume CH. Is it true that each usco map $\Phi : X \rightarrow \mathbb{I}^2$ from a metrizable (separable) space X has a Borel-measurable selection?

Observe that the map $p : \mathbb{I} \rightarrow \mathbb{I}$ is 2-to-1 and its square $P : \mathbb{I}^2 \rightarrow \mathbb{I}^2$ is 4-to-1. A function $f : X \rightarrow Y$ is called n -to-1 for some $n \in \mathbb{N}$ if $|f^{-1}(y)| \leq n$ for any $y \in Y$. By Theorem 3 of Todorćević [17], every Rosenthal compact space of countable spread admits a 2-to-1 map onto a metrizable compact space. Let us observe that the splitted square $\mathbb{I}^2$ contains a discrete subspace of cardinality continuum and hence has uncountable spread.

Problem 3 Let $n \in \{2, 3\}$. Is there an n -to-1 map $f : K \rightarrow M$ from a (Rosenthal) compact space K to a metrizable compact space M such that the inverse multimap $f^{-1} : M \multimap K$ has no Borel selections?

Remark 1 Hris Heunen asked on Mathoverflow [10] whether for any continuous surjective function $\pi : X \rightarrow Y$ between compact Hausdorff spaces and any continuous map $\gamma : \mathbb{I} \rightarrow Y$ there exists a Borel-measurable function $g : \mathbb{I} \rightarrow X$ such that $\gamma = \pi \circ g$. Theorem 4 provides a consistent counterexample to this problem of Heunen. Indeed, consider the projection $P : \mathbb{I}^2 \rightarrow \mathbb{I}^2$ and take any continuous surjective map $\gamma : \mathbb{I} \rightarrow \mathbb{I}^2$ (whose existence was proved by Giuseppe Peano in 1890). By Theorem 2, the inverse multimap $\gamma^{-1} : \mathbb{I}^2 \multimap \mathbb{I}$ has a Borel-measurable selection $s : \mathbb{I}^2 \rightarrow \mathbb{I}$. Assuming that there exists a Borel-measurable function $g : \mathbb{I} \rightarrow \mathbb{I}^2$ with $\gamma = P \circ g$, we conclude that $g \circ s : \mathbb{I}^2 \rightarrow \mathbb{I}^2$ is a Borel-measurable selection of the multimap P^{-1} , which contradicts Theorem 4 under CH.

2 Proof of Theorem 3

Theorem 3 follows from Lemmas 2 and 3, proved in this section.

First we prove one lemma, showing that our definition of a GO-space agrees with the original definition of Lutzer [13]. Probably this lemma is known but we could not find the precise reference in the literature.

Lemma 1 *The linear order $\leq$ of any GO-space X is a closed subset of the square $X \times X$.*

Proof Given two elements $x, y \in X$ with $x \not\leq y$, use the Hausdorff property of X and find two disjoint order-convex neighborhoods $O_x, O_y \subseteq X$ of the points x, y , respectively. We claim that the product $O_x \times O_y$ is disjoint with the linear order $\leq$. Assuming that this is not true, find elements $x' \in O_x$ and $y' \in O_y$ such that $x' \leq y'$.

Taking into account that the sets O_x, O_y are disjoint and order-convex, we conclude that $x' < y$ and $x < y'$. It follows from $x \not\leq y$ that $y < x$. Then $x' < y < x < y'$ and hence $[y, x] \subseteq O_x \cap O_y = \emptyset$, which is a desired contradiction. This contradiction shows that the neighborhood $O_x \times O_y$ of the pair $\langle x, y \rangle$ is disjoint with $\leq$ and hence $\leq$ is a closed subset of $X \times X$. $\square$

Lemma 2 Any usco multimap $\Phi : X \multimap Y$ from an $(F_\sigma$ -perfect) topological space X to a countably cellular GO-space space Y has a Borel (F_σ -measurable) selection.

Proof Being a GO-space, Y has a base of the topology consisting of open order-convex subsets with respect to some linear order $\leq$ on Y . By Lemma 1, the linear order $\leq$ is a closed subset of $Y \times Y$. Then for every $a \in Y$ the order-convex set $(\leftarrow, a] = \{y \in Y : y \leq a\}$ is closed in Y , which implies that each non-empty compact subset of Y has the smallest element.

Then for any usco multimap $\Phi : X \multimap Y$ we can define a selection $f : X \rightarrow Y$ of Φ assigning to each point $x \in X$ the smallest element $f(x)$ of the non-empty compact set $\Phi(x) \subseteq Y$. We claim that this selection is F_σ -measurable.

A subset $U \subseteq Y$ is called *upper* if for any $u \in U$ the order-convex set $[u, \rightarrow) = \{y \in Y : u \leq y\}$ is contained in U .

Claim 1 For any upper open set $U \subseteq Y$ the preimage $f^{-1}[U]$ is open in X .

Proof For any $x \in f^{-1}[U]$ we get $\Phi(x) \subseteq [f(x), \rightarrow) \subseteq U$. The upper semicontinuity of Φ yields a neighborhood $O_x \subseteq X$ such that $\Phi[O_x] \subseteq U$. Consequently, $f(O_x) \subseteq \Phi[O_x] \subseteq U$, witnessing that the set $f^{-1}[U]$ is open in X . $\square$

A subset $L \subseteq Y$ is *lower* if for every $a \in L$ the order-convex set $(\leftarrow, a] = \{y \in Y : y \leq a\}$ is contained in L .

Claim 2 For any closed lower set $L \subseteq Y$ the preimage $f^{-1}[L]$ is closed in X .

Proof Observe that the complement $X \setminus L$ is an open upper set in Y . By Claim 1, the preimage $f^{-1}[X \setminus L]$ is open in X and hence its complement $X \setminus f^{-1}[X \setminus L] = f^{-1}[L]$ is closed in X . $\square$

Claim 3 For any lower set $L \subseteq Y$ the preimage $f^{-1}(L)$ is of type F_σ in X .

Proof If L has a largest element λ , then $L = (\leftarrow, \lambda]$ and $f^{-1}[L] = f^{-1}[(\leftarrow, \lambda]]$ is closed by Claim 2. So, we assume that L does not have a largest element. Then the countable cellularity of Y implies that L has a countable cofinal subset $C \subseteq L$ (which means that for every $x \in L$ there exists $y \in C$ with $x \leq y$). By Lemma 2, for every $c \in C$ the preimage $f^{-1}[(\leftarrow, c]]$ is closed in X . Since $L = \bigcup_{c \in C} (\leftarrow, c]$, the preimage $f^{-1}[L] = \bigcup_{c \in C} f^{-1}[(\leftarrow, c]]$ is of type F_σ in X . $\square$

Claim 4 For any open order-convex subset $U \subseteq Y$ the preimage $f^{-1}[U]$ is a Borel subset of X (of type F_σ if the space X is F_σ -perfect).

Proof The order-convexity of U implies that $U = \overleftarrow{U} \cap \overrightarrow{U}$ where $\overleftarrow{U} = \bigcup_{u \in U} (\leftarrow, u]$ and $\overrightarrow{U} = \bigcup_{u \in U} [u, \rightarrow)$. Taking into account that Y has a base of order-convex sets,

one can show that the upper set $\vec{U}$ is open in X . By Claim 1, the preimage $f^{-1}[\vec{U}]$ is open in X (of type F_σ if the space X is F_σ -perfect). By Claim 3, the preimage $f^{-1}[\overleftarrow{U}]$ is of type F_σ in X . Then $f^{-1}(U) = f^{-1}[\overleftarrow{U}] \cap f^{-1}[\vec{U}]$ is Borel (of type F_σ if X is F_σ -perfect). $\square$

Claim 5 For every open set $U \subseteq Y$ the preimage $f^{-1}[U]$ is Borel subset of X (of type F_σ if X is F_σ -perfect).

Proof By the definition of the topology of Y , each point $x \in U$ has an open order-convex neighborhood $O_x \subseteq U$. By the Kuratowski-Zorn Lemma, each open order-convex subset of U is contained in a maximal open order-convex subset of U . Let $\mathcal{C} \subseteq \mathcal{B}$ be the family of maximal open order-convex subsets of U . Observe that $U = \bigcup \mathcal{C}$ and any distinct sets $C, D \in \mathcal{C}$ are disjoint: otherwise the union $C \cup D$ would be an open order-convex subset of U and by the maximality of C and D , $C = C \cup D = D$. Since the space Y is countably cellular, the family $\mathcal{C}$ is at most countable. By Claim 4, for every $C \in \mathcal{C}$ the preimage $f^{-1}(C)$ is Borel (of type F_σ -set if X is F_σ -perfect) and so is the countable union $f^{-1}[U] = \bigcup_{C \in \mathcal{C}} f^{-1}(C)$.

Claim 5 completes the proof of Lemma 2. $\square$

Lemma 3 Every usco multimap $\Phi : X \multimap Y$ from a countably cellular (F_σ -perfect) topological space X into a GO-space Y has a Borel (F_σ -measurable) selection.

Proof The Kuratowski-Zorn Lemma implies that the usco map Φ contains a minimal usco map $\Psi : X \multimap Y$. We claim that the image $\Psi[X] \subseteq Y$ is a countably cellular subspace of Y . Assuming the opposite, we can find an uncountable disjoint family $(U_\alpha)_{\alpha \in \omega_1}$ of non-empty open subsets in $\Psi[X]$. For every $\alpha \in \omega_1$, find $x_\alpha \in X$ such that $\Phi(x_\alpha) \cap U_\alpha \neq \emptyset$. By Lemma 3.1.2 [3], the minimality of the usco map Ψ implies that $\Psi[V_\alpha] \subseteq U_\alpha$ for some non-empty open set $V_\alpha \subseteq X$. Taking into account that the family $(U_\alpha)_{\alpha \in \omega_1}$ is disjoint, we conclude that the family $(V_\alpha)_{\alpha \in \omega_1}$ is disjoint, witnessing that the space X is not countably cellular. But this contradicts our assumption. This contradiction shows that the GO-subspace $\Psi[X]$ of Y is countably cellular. By Lemma 2, the usco map $\Psi : X \multimap \Psi[X]$ has a Borel (F_σ -measurable) selection, which is also a selection of the usco map Φ . $\square$

Finally, let us prove one selection property of the split interval, which will be used in the proof of Lemma 8.

Lemma 4 Any selection $s : \mathbb{I} \rightarrow \mathbb{I}$ of the multimap $p^{-1} : \mathbb{I} \rightarrow \mathbb{I}$ is F_σ -measurable.

Proof Given any open subset $U \subseteq \mathbb{I}$, we need to show that $s^{-1}[U]$ is of type F_σ in $\mathbb{I}$. For every $x \in s^{-1}[U]$, find an open order-convex set $I_x \subseteq U$ containing $s(x)$. It is well-known (see e.g. [2, 3.10.C(a)]) that the split interval $\mathbb{I}$ is hereditarily Lindelöf. Consequently, there exists a countable set $C \subseteq s^{-1}[U]$ such that $\bigcup_{x \in s^{-1}[U]} I_x = \bigcup_{x \in C} I_x$ and hence $s^{-1}[U] = \bigcup_{x \in C} s^{-1}[I_x]$. For every $x \in C$ the order-convexity of the interval $I_x \subseteq \mathbb{I}$ implies that its preimage $s^{-1}[I_x]$ is a convex subset of $\mathbb{I}$, containing x . Since convex subsets of $\mathbb{I}$ are of type F_σ , the countable union $s^{-1}[U] = \bigcup_{x \in C} s^{-1}[I_x]$ is an F_σ -set in $\mathbb{I}$. $\square$

3 Selection properties of the split square $\mathbb{I}^2$ under the negation of CH

In this section we study the selection properties of the split square $\mathbb{I}^2$ under the negation of the Continuum Hypothesis.

By $\langle x, y \rangle$ we denote the ordered pair of points x, y . In this way we distinguish ordered pairs from the order intervals $(x, y) := \{z : x < z < y\}$ in linearly ordered spaces.

The split interval $\mathbb{I} = \mathbb{I} \times \{0, 1\}$ carries the lexicographic order defined by $\langle x, i \rangle \leq \langle y, j \rangle$ iff either $x < y$ or $(x = y \text{ and } i \leq j)$. It is well-known that the topology generated by the lexicographic order on $\mathbb{I}$ is compact and Hausdorff, see [2, 3.10.C(b)]. By $p : \mathbb{I} \rightarrow \mathbb{I}, p : \langle x, i \rangle \mapsto x$, we denote the coordinate projection and by $P : \mathbb{I}^2 \rightarrow \mathbb{I}^2, P : \langle x, y \rangle \mapsto \langle p(x), p(y) \rangle$ the square of the map p .

The following lemma proves the implication (1) $\Rightarrow$ (3) of Theorem 4.

Lemma 5 *If $\omega_1 < \mathfrak{c}$, then the multimap $P^{-1} : \mathbb{I}^2 \multimap \mathbb{I}^2$ has no Borel selections.*

Proof To derive a contradiction, assume that the multimap P^{-1} has a Borel-measurable selection $s : \mathbb{I}^2 \rightarrow \mathbb{I}^2$.

For a real number $x \in \mathbb{I}$ by x_0 and x_1 we denote the points $\langle x, 0 \rangle$ and $\langle x, 1 \rangle$ of the split interval $\mathbb{I}$. Then $\mathbb{I} = \mathbb{I}_0 \cup \mathbb{I}_1$ where $\mathbb{I}_i = \{x_i : x \in \mathbb{I}\}$ for $i \in \{0, 1\}$.

For any numbers $i, j \in \{0, 1\}$ consider the set

$$Z_{ij} = \{z \in \mathbb{I}^2 : s(z) \in \mathbb{I}_i \times \mathbb{I}_j\}$$

and observe that $\mathbb{I}^2 = \bigcup_{i,j=0}^1 Z_{ij}$.

For a point $a \in \mathbb{I}$, let $[0_0, a)$ and $(a, 1_1]$ be the order intervals in $\mathbb{I}$ with respect to the lexicographic order. Observe that for any $x \in \mathbb{I}$ we have

$$\begin{aligned} p([0_0, x_0]) &= [0, x), & p([0_0, x_1]) &= [0, x], \\ p((x_0, 1_1]) &= [x, 1], & p((x_1, 1_1]) &= (x, 1]. \end{aligned}$$

For every $a \in (0, 2) \subseteq \mathbb{R}$ consider the lines

$$L_a = \{\langle x, y \rangle \in \mathbb{R}^2 : x + y = a\} \quad \text{and} \quad \Gamma^a = \{\langle x, y \rangle \in \mathbb{R}^2 : y - x = a\}$$

on the plane.

Claim 6 *For every $a \in \mathbb{R}$ the intersection $L_a \cap Z_{00}$ is at most countable.*

Proof If for some $a \in \mathbb{R}$ the intersection $L_a \cap Z_{00}$ is uncountable, then we can choose a non-Borel subset $B \subseteq L_a \cap Z_{00}$ of cardinality $|B| = \omega_1$. For every point $\langle x, y \rangle \in B \subseteq Z_{00}$, the definition of the set Z_{00} ensures that $s(\langle x, y \rangle) = \langle x_0, y_0 \rangle$ and hence the set $U_{\langle x, y \rangle} = [0_0, x_1] \times [0_0, y_1] = [0_0, x_0] \times [0_0, y_0]$ is an open neighborhood of $s(\langle x, y \rangle)$ in $\mathbb{I}^2$. Observe that $\langle x, y \rangle \in s^{-1}(U_{\langle x, y \rangle}) \subseteq p(U_{\langle x, y \rangle}) = [0, x] \times [0, y]$ and hence $L_a \cap s^{-1}(U_{\langle x, y \rangle}) = \{\langle x, y \rangle\}$. Then for the open set $U = \bigcup_{\langle x, y \rangle \in B} U_{\langle x, y \rangle}$ the preimage $s^{-1}[U]$ is not Borel in $\mathbb{I}^2$ because the intersection $s^{-1}[U] \cap L_a = B$ is not Borel. But this contradicts the Borel measurability of s . $\square$

By analogy we can prove the following claims.

Claim 7 For every $a \in \mathbb{R}$ the intersection $L_a \cap Z_{11}$ is at most countable.

Claim 8 For every $b \in \mathbb{R}$ the intersection $\Gamma^b \cap (Z_{01} \cup Z_{10})$ is at most countable.

Now fix any subset $\Omega \subseteq [\frac{1}{2}, \frac{3}{2}]$ of cardinality $|\Omega| = \omega_1$. By Claims 6, 7, for every $a \in \Omega$ the intersection $L_a \cap (Z_{00} \cup Z_{11})$ is at most countable. Consequently the union

$$U = \bigcup_{a \in \Omega} L_a \cap (Z_{00} \cup Z_{11})$$

has cardinality $|U| \leq \omega_1$. Since $|U| \leq \omega_1 < \mathfrak{c}$, there exists a real number $b \in [\frac{1}{2}, \frac{3}{2}]$ such that the line Γ^b does not intersect the set U . Since $\{b\} \cup \Omega \subseteq [\frac{1}{2}, \frac{3}{2}]$ for every $a \in \Omega$ the intersection $\Gamma^b \cap L_a \cap \mathbb{I}^2$ is not empty. Then the set $X = \bigcup_{a \in \Omega} L_a \cap \Gamma^b \subset \mathbb{I}^2$ is uncountable and $X \subseteq \Gamma^b \setminus U \subseteq \Gamma^b \cap (Z_{01} \cup Z_{10})$. But this contradicts Claim 8.

4 Selection properties of the split square $\mathbb{I}^2$ under the Continuum Hypothesis

In this section we shall prove that under the continuum hypothesis the usco multimap $P^{-1} : \mathbb{I}^2 \multimap \mathbb{I}^2$ has an F_σ -measurable selection.

First we introduce some terminology related to monotone functions.

A subset $f \subseteq \mathbb{I}^2$ is called

- a *function* if for any $\langle x, y \rangle, \langle x', y' \rangle \in f$ the equality $x = x'$ implies $y = y'$;
- *strictly increasing* if for any $\langle x, y \rangle, \langle x', y' \rangle \in f$ the strict inequality $x < x'$ implies $y < y'$;
- *strictly decreasing* if for any $\langle x, y \rangle, \langle x', y' \rangle \in f$ the inequality $x < x'$ implies $y > y'$;
- *strictly monotone* if f is strictly increasing or strictly decreasing.

Lemma 6 Each strictly increasing function $f \subset \mathbb{I}^2$ is a subset of a Borel strictly increasing function $\tilde{f} \subset \mathbb{I}^2$.

Proof It follows that the strictly increasing function f is a strictly increasing bijective function between the sets $\text{pr}_1[f] = \{x \in \mathbb{I} : \exists y \in \mathbb{I} \langle x, y \rangle \in f\}$ and $\text{pr}_2[f] = \{y \in \mathbb{I} : \exists x \in \mathbb{I} \langle x, y \rangle \in f\}$. It is well-known that monotone functions of one real variable have at most countably many discontinuity points. Consequently, the sets of discontinuity points of the strictly monotone functions f and f^{-1} are at most countable. This allows us to find a countable set $D_f \subseteq f$ such that the set $f \setminus D_f$ coincides with the graph of some increasing homeomorphism between subsets of $\mathbb{I}$. Replacing D_f by a larger countable set, we can assume that $D_f = f \cap (\text{pr}_1[D_f] \times \text{pr}_2[D_f])$, where $\text{pr}_1, \text{pr}_2 : \mathbb{I}^2 \rightarrow \mathbb{I}$ are coordinate projections. By the Lavrentiev Theorem [11, 3.9], the homeomorphism $f \setminus D_f$ extends to a (strictly increasing) homeomorphism $h \subseteq \mathbb{I}^2$ between G_δ -subsets of $\mathbb{I}^2$ such that $f \setminus D_f$ is dense in h . It is easy to check that the Borel subset $\tilde{f} = (h \setminus (\text{pr}_1[D_f] \times \text{pr}_2[D_f])) \cup D_f$ is a strictly increasing function extending f . $\square$

By analogy we can prove

Lemma 7 *Each strictly decreasing function $f \subset \mathbb{I}^2$ is a subset of a Borel strictly decreasing function $\tilde{f} \subset \mathbb{I}^2$.*

Now we are ready to prove the main result of this section.

Lemma 8 *Under $\omega_1 = \mathfrak{c}$ the multifunction $P^{-1} : \mathbb{I}^2 \multimap \check{\mathbb{I}}^2$ has an F_σ -measurable selection.*

Proof Let $\mathcal{M}$ be the set of infinite strictly monotone Borel functions $f \subset \mathbb{I}^2$. Since $\omega_1 = \mathfrak{c}$, the set $\mathcal{M}$ can be written as $\mathcal{M} = \{f_\alpha\}_{\alpha < \omega_1}$. It is clear $\bigcup_{\alpha < \omega_1} f_\alpha = \mathbb{I}^2$. So, for any point $z \in \mathbb{I}^2$ we can find the smallest ordinal $\alpha_z < \omega_1$ such that $z \in f_{\alpha_z}$. Consider the sets

$$\begin{aligned} L &:= \{z \in \mathbb{I}^2 : f_{\alpha_z} \text{ is strictly increasing}\} \text{ and} \\ \Gamma &:= \{z \in \mathbb{I}^2 : f_{\alpha_z} \text{ is strictly decreasing}\} = \mathbb{I}^2 \setminus L. \end{aligned}$$

Define a selection $s : \mathbb{I}^2 \rightarrow \check{\mathbb{I}}^2$ of the multimap $P^{-1} : \mathbb{I}^2 \multimap \check{\mathbb{I}}^2$ letting

$$s(\langle x, y \rangle) = \begin{cases} \langle x_1, y_1 \rangle & \text{if } \langle x, y \rangle \in L, \\ \langle x_1, y_0 \rangle & \text{if } \langle x, y \rangle \in \Gamma, \end{cases}$$

for $\langle x, y \rangle \in \mathbb{I}^2$.

We claim that the function $s : \mathbb{I}^2 \rightarrow \check{\mathbb{I}}^2$ is F_σ -measurable. Given any open set $U \subseteq \check{\mathbb{I}}^2$, we should prove that its preimage $s^{-1}[U]$ of type F_σ in $\mathbb{I}^2$. Consider the open subset $V := U \cap (0_1, 1_0)^2 \subseteq \check{\mathbb{I}}^2$ of U . Using Lemma 4, it can be shown that the set $s^{-1}[U \setminus V] \subseteq \mathbb{I}^2 \setminus (0, 1)^2$ is of type F_σ in $\mathbb{I}^2$. Therefore, it remains to show that the preimage $s^{-1}[V]$ is of type F_σ in $\mathbb{I}^2$.

Let $\mathbb{Q} := \{\frac{n}{m} : n, m \in \mathbb{N}, n < m\}$ be the set of rational numbers in the interval $(0, 1)$.

Consider the subsets $L_V := L \cap s^{-1}(V)$ and $\Gamma_V := \Gamma \cap s^{-1}(V)$. For every $\langle x, y \rangle \in L_V$ we have $s(\langle x, y \rangle) = \langle x_1, y_1 \rangle \in V$ and by the definition of the topology of the split interval, we can find rational numbers $a(x, y), b(x, y) \in \mathbb{Q}$ such that $x < a(x, y)$, $y < b(x, y)$ and $s(\langle x, y \rangle) = \langle x_1, y_1 \rangle \in [x_1, a(x, y)_0] \times [y_1, b(x, y)_0] \subseteq V$. Then

$$[x, a(x, y)] \times [y, b(x, y)] = s^{-1}([x_1, a(x, y)_0] \times [y_1, b(x, y)_0]) \subseteq s^{-1}[V].$$

On the other hand, for every $\langle x, y \rangle \in \Gamma_V$ there are rational numbers $a(x, y), b(x, y) \in \mathbb{Q}$ such that $x < a(x, y)$, $b(x, y) < y$ and $s(\langle x, y \rangle) = \langle x_1, y_0 \rangle = [x_1, a(x, y)_0] \times (b(x, y)_1, y_0] \subseteq V$. In this case

$$[x, a(x, y)] \times (b(x, y), y] = s^{-1}([x_1, a(x, y)_0] \times (b(x, y)_1, y_0]) \subseteq s^{-1}[V].$$

It follows that

$$s^{-1}[V] = \left(\bigcup_{\langle x, y \rangle \in L_V} [x, a(x, y)] \times [y, b(x, y)] \right) \cup \left(\bigcup_{\langle x, y \rangle \in \Gamma_V} [x, a(x, y)] \times (b(x, y), y] \right)$$

$$\cup \left(\bigcup_{\langle x, y \rangle \in \Gamma_V} [x, a(x, y)) \times (b(x, y), y] \right).$$

This equality and the following claim imply that the set $s^{-1}[V]$ is of type F_σ in $\mathbb{I}^2$.

Claim 9 *There are countable subsets $L' \subseteq L_V$ and $\Gamma' \subseteq \Gamma_V$ such that*

$$\bigcup_{\langle x, y \rangle \in L_V} [x, a(x, y)) \times [y, b(x, y)) = \bigcup_{\langle x, y \rangle \in L'} [x, a(x, y)) \times [y, b(x, y))$$

and

$$\bigcup_{\langle x, y \rangle \in \Gamma_V} [x, a(x, y)) \times (b(x, y), y] = \bigcup_{\langle x, y \rangle \in \Gamma'} [x, a(x, y)) \times (b(x, y), y].$$

We shall show how to find the countable set $L' \subseteq L_V$. The countable set $\Gamma' \subseteq \Gamma_V$ can be found by analogy.

For rational numbers $r, q \in \mathbb{Q}$, consider the set

$$L_{r,q} = \{\langle x, y \rangle \in L_V : a(x, y) = r, b(x, y) = q\}$$

and observe that $L_V = \bigcup_{r,q \in \mathbb{Q}} L_{r,q}$.

Claim 10 *For any rational numbers $r, q \in \mathbb{Q}$ there exists a countable subset $L'_{r,q} \subseteq L_{r,q}$ such that*

$$\bigcup_{\langle x, y \rangle \in L'_{r,q}} [x, r) \times [y, q) = \bigcup_{\langle x, y \rangle \in L_{r,q}} [x, r) \times [y, q).$$

Proof For every rational numbers $r' \leq r$ and $q' \leq q$, consider the numbers

$$\underline{y}(r') := \inf\{y : \langle x, y \rangle \in L_{r,q}, x < r'\} \text{ and } \underline{x}(q') := \inf\{x : \langle x, y \rangle \in L_{r,q}, y < q'\}.$$

Choose countable subsets $L_{r,q}^{r',0}, L_{r,q}^{0,q'} \subseteq L_{r,q}$ such that

$$\underline{y}(r') = \inf\{y : \langle x, y \rangle \in L_{r,q}^{r',0}, x < r'\} \text{ and } \underline{x}(q') = \inf\{x : \langle x, y \rangle \in L_{r,q}^{0,q'}, y < q'\}$$

and moreover,

$$\underline{y}(r') = \min\{y : \langle x, y \rangle \in L_{r,q}^{r',0}, x < r'\} \text{ if } \underline{y}(r') = \min\{y : \langle x, y \rangle \in L_{r,q}, x < r'\}.$$

and

$$\underline{x}(q') = \min\{x : \langle x, y \rangle \in L_{r,q}^{0,q'}, y < q'\} \text{ if } \underline{x}(q') = \min\{x : \langle x, y \rangle \in L_{r,q}, y < q'\}.$$

Consider the countable subset

$$L''_{r,q} := \bigcup \{L_{r,q}^{r',0} \cup L_{r,1}^{0,q'} : r', q' \in \mathbb{Q}, r' < r, q' < q\}$$

of $L_{r,q}$.

Claim 11 $\bigcup_{\langle x,y \rangle \in L_{r,q}} ([x, r) \times [y, q)) \setminus \{\langle x, y \rangle\} \subseteq \bigcup_{\langle x,y \rangle \in L''_{r,q}} [x, r) \times [y, q).$

Proof Fix any pairs $\langle x, y \rangle \in L_{r,q}$ and $\langle x', y' \rangle \in ([x, r) \times [y, q)) \setminus \{\langle x, y \rangle\}$. Three cases are possible:

1. $x < x' < r$ and $y < y' < q$;
2. $x = x'$ and $y < y' < q$;
3. $x < x' < r$ and $y = y'$.

In the first case there exist rational numbers r', q' such that $x < r' < x' < r$ and $y < q' < y' < q$. The definition of $\underline{x}(q')$ ensures that $\underline{x}(q') \leq x < x'$. By the choice of the family $L_{r,q}^{0,q'}$, there exists $\langle x'', y'' \rangle \in L_{r,q}^{0,q'} \subset L''_{r,q}$ such that $x'' < x' < r$ and $y'' < q' < y' < q$. Then $\langle x', y' \rangle \in [x'', r) \times [y'', q)$.

Next, assume that $x = x'$ and $y < y' < q$. In this case we can choose a rational number q' such that $y < q' < y'$. It follows that $\underline{x}(q') \leq x = x'$. If $\underline{x}(q') < x'$, then by the definition of the family $L_{r,q}^{0,q'}$, there exists $\langle x'', y'' \rangle \in L_{r,q}^{0,q'} \subseteq L_{r,q}$ such that $x'' < x' < r$ and $y'' < q' < y' < q$. Then $\langle x', y' \rangle \in [x'', r) \times [y'', q)$.

So, we assume that $\underline{x}(q') = x' = x$ and hence $\underline{x}(q') = x = \min\{x'' : \langle x'', y'' \rangle \in L_{p,q} : y'' < q'\}$. In this case $x' = \underline{x}(q') = x''$ for some $\langle x'', y'' \rangle \in L_{r,q}^{0,q'} \subset L''_{r,q}$ with $y'' < q' < y' < q$. Then $\langle x', y' \rangle \in [x'', r) \times [y'', q)$.

By analogy, in the third case ($x < x' < r$ and $y = y'$) we can find a pair $\langle x'', y'' \rangle \in L''_{r,q}$ such that $\langle x', y' \rangle \in [x'', r) \times [y'', q)$.

Claim 11 implies that the set

$$D_{r,q} = \left(\bigcup_{\langle x,y \rangle \in L_{r,q}} [x, r) \times [y, q) \right) \setminus \left(\bigcup_{\langle x,y \rangle \in L''_{r,q}} [x, r) \times [y, q) \right)$$

is contained in $L_{r,q}$. □

Claim 12 *The set $D_{r,q}$ is a strictly decreasing function.*

Proof First we show that $D_{r,q}$ is a function. Assuming that $D_{r,q}$ is not a function, we can find two pairs $\langle x, y \rangle, \langle x, y' \rangle \in D_{r,q}$ with $y < y'$. Applying Claim 11, we conclude that

$$\langle x, y' \rangle \in ([x, r) \times [y, q)) \setminus \{\langle x, y \rangle\} \subseteq \bigcup_{\langle x'', y'' \rangle \in L''_{r,q}} [x'', r) \times [y'', q)$$

and hence $\langle x, y' \rangle \notin D_{r,q}$, which contradicts the choice of the pair $\langle x, y' \rangle$. This contradiction shows that $D_{r,q}$ is a function.

Assuming that $D_{r,q}$ is not strictly decreasing, we can find pairs $\langle x, y \rangle, \langle x', y' \rangle \in D_{r,q}$ such that $x < x'$ and $y \leq y'$. Applying Claim 11, we conclude that

$$\langle x', y' \rangle \in ([x, r) \times [y, q)) \setminus \{\langle x, y \rangle\} \subseteq \bigcup_{\langle x'', y'' \rangle \in L''_{r,q}} [x'', r) \times [y'', q)$$

and hence $\langle x', y' \rangle \notin D_{r,q}$, which contradicts the choice of the pair $\langle x, y \rangle$. This contradiction shows that $D_{r,q}$ is strictly decreasing. $\square$

Claim 13 *The set $D_{r,q}$ is at most countable.*

Proof To derive a contradiction, assume that $D_{r,q}$ is uncountable. By Lemma 7, the strictly decreasing function $D_{r,q}$ is contained in some Borel strictly decreasing function, which is equal to f_α for some ordinal $\alpha < \omega_1$. Since the intersection of a strictly increasing function and a strictly decreasing function contains at most one point, the set

$$D'_{r,q} = \bigcup \{D_{r,q} \cap f_\beta : \beta \leq \alpha, f_\beta \text{ is strictly increasing}\}$$

is at most countable. We claim that $D_{r,q} = D'_{r,q}$. To derive a contradiction, assume that $D_{r,q} \setminus D'_{r,q}$ contains some pair $z = \langle x, y \rangle$. It follows from $z \in D_{r,q} \subseteq f_\alpha$ that $\alpha_z \leq \alpha$. Since $z \notin D'_{r,q}$, the strictly monotone function $f_{\alpha_z} \ni z$ is not strictly increasing and hence f_{α_z} is strictly decreasing. Then the definition of the set L guarantees that $z \notin L$, which contradicts the inclusion $z \in D_{r,q} \subseteq L_{r,q} \subseteq L$. $\square$

Now consider the countable subset $L'_{r,q} := L''_{r,q} \cup D_{r,q}$ of $L_{r,q}$ and observe that

$$\bigcup_{\langle x,y \rangle \in L_{r,q}} ([x, r) \times [y, q)) \subseteq \bigcup_{\langle x,y \rangle \in L''_{r,q}} [x, r) \times [y, q).$$

This completes the proof of Claim 10.

Claim 14 *There exists a countable subset $L' \subseteq L_V$ such that*

$$\bigcup_{\langle x,y \rangle \in L'} ([x, a(x, y)) \times [y, b(x, y))) = \bigcup_{\langle x,y \rangle \in L_V} ([x, a(x, y)) \times [y, b(x, y))).$$

Proof By Claim 10, for any rational numbers $r, q \in \mathbb{Q}$ there exists a countable subset $L'_{r,q} \subseteq L_{r,q}$ such that

$$\begin{aligned} \bigcup_{\langle x,y \rangle \in L'_{r,q}} ([x, a(x, y)) \times [y, b(a, y))) &= \bigcup_{\langle x,y \rangle \in L'_{r,q}} ([x, r) \times [y, q)) = \\ &= \bigcup_{\langle x,y \rangle \in L_{r,q}} ([x, r) \times [y, q)) = \bigcup_{\langle x,y \rangle \in L_{r,q}} ([x, a(x, y)) \times [y, b(x, y))). \end{aligned}$$

Since $L_V = \bigcup_{r,q \in \mathbb{Q}} L_{r,q}$, the countable set $L' := \bigcup_{r,q \in \mathbb{Q}} L'_{r,q}$ has the required property. $\square$

By analogy with Claim 14 we can prove

Claim 15 *There exists a countable subset $\Gamma' \subseteq \Gamma_V$ such that*

$$\bigcup_{\langle x, y \rangle \in \Gamma'} ([x, a(x, y)) \times (b(x, y), y]) = \bigcup_{\langle x, y \rangle \in \Gamma_V} ([x, a(x, y)) \times (b(x, y), y]).$$

Claims 14 and 15 complete the proof of Claim 9 and the proof of Lemma 8.

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