A version of κ -Miller forcing

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Abstract

We consider a version of κ -Miller forcing on an uncountable cardinal κ . We show that under $2^{<\kappa} = \kappa$ this forcing collapses 2^κ to ω and adds a κ -Cohen real. The same holds under the weaker assumptions that $\text{cf}(\kappa) > \omega$, $2^{2^{<\kappa}} = 2^\kappa$, and forcing with $([\kappa]^\kappa, \subseteq)$ collapses 2^κ to ω .

Keyword Forcing with higher perfect trees

Mathematics Subject Classification Primary 03E05; Secondary 03E04 · 03E15

1 Introduction

Many of the tree forcings on the classical Baire space have various analogues for higher cardinals. Here we are concerned with Miller forcing [4]. In the classical case, a Miller condition is a superperfect subtree of $\omega^{<\omega}$. The subtree is ordered by the end-extension relation on $\omega^{<\omega}$. The forcing order is simply $\subseteq$. A tree is superperfect if each node has an extension that has infinitely many immediate tree successors. Such a node is called a splitting node. We can assume that each node has just one direct successor or infinitely many.

For a κ -version of Miller forcing, superperfectness and splitting are usually interpreted as follows: Above each node $t \in p \subseteq \kappa^{<\kappa}$ there is a node splitting node s . The common interpretation of “ s is a splitting node of p ” is:

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$\{\alpha \in \kappa : s^\wedge \langle \alpha \rangle \in p\}$ contains a club subset of κ .

In order to gain $(< \kappa)$ -closure of the notion of forcing, in addition to the club version of superperfectness one usually requires for conditions that (see, e.g., [2, Section 5.2]) limits of length less than κ of splitting nodes be splitting nodes as well.

In this paper we investigate a version of κ -Miller forcing where the conditions on superperfectness and $(< \kappa)$ -closure of splitting nodes are kept and the definition of “ s is a splitting node of p ” is weakened to

$$|\{\alpha : s^\wedge \langle \alpha \rangle \in p\}| = \kappa.$$

We show: If $\text{cf}(\kappa) > \omega$, $\text{cf}(\kappa) = \kappa$ or $\text{cf}(\kappa) < 2^{\text{cf}(\kappa)} \leq \kappa$, $2^{2^{<\kappa}} = 2^\kappa$, and there is a κ -mad family of size 2^κ , then this variant of Miller forcing is related to the forcing $([\kappa]^\kappa, \subseteq)$ and collapses 2^κ to ω . In particular, if $\omega < \kappa^{<\kappa} = \kappa$, then our four premises are fulfilled. Thus we provide some mathematical justification of the customary choice of higher Miller forcing.

Throughout the paper we let κ be an uncountable cardinal. We do not make the general assumption that $2^{<\kappa} = \kappa$, nor do we assume that κ is regular.

We denote forcing orders in the form $(\mathbb{P}, \leq_{\mathbb{P}})$ and let $q \leq_{\mathbb{P}} p$ mean that q is *stronger* than p .

If $\text{dom}(t)$, i are ordinals, we write $t^\wedge \langle i \rangle$ for the concatenation of t with the singleton function $\{(0, i)\}$, i.e., $t^\wedge \langle i \rangle = t \cup \{(\text{dom}(t), i)\}$. For cardinals κ, λ , we write ${}^{<\lambda}\kappa$ for the set of functions $f: \alpha \rightarrow \kappa$ for some $\alpha < \lambda$. For $s, t \in \kappa^{<\lambda}$ we write $s \leq t$ if $s = t \restriction \text{dom}(s)$, and the corresponding strict order is written as $\triangleleft$. The domain α of f is also called the length of f . The set of subsets of κ of size κ is denoted by $[\kappa]^\kappa$.

Definition 1.1 (1) $\mathbb{Q}_\kappa^1$ is the forcing $([\kappa]^\kappa, \subseteq)$.

(2) $\mathbb{Q}_\kappa^2$ is the following version of κ -Miller forcing: Conditions are trees $T \subseteq {}^{\kappa>\kappa}\kappa$ that are κ *superperfect*: for each $s \in T$ there is $s \leq t$ such that t is a κ -splitting node of T . A node $t \in T$ is called a κ -*splitting node* if

$$\text{set}_p(t) = \{\alpha < \kappa : t^\wedge \langle \alpha \rangle \in T\}$$

has size κ . The set of splitting nodes of T is denoted by $\text{spl}(T)$.

We furthermore require for $p \in \mathbb{Q}_\kappa^2$ that the limit of an $\triangleleft$ -increasing sequence of length less than κ of κ -splitting nodes is a κ -splitting node if it has length less than κ .

For $p, q \in \mathbb{Q}_\kappa^2$ we write $q \leq_{\mathbb{Q}_\kappa^2} p$ if $q \subseteq p$. So subtrees are stronger conditions.

- (3) For $p \in \mathbb{Q}_\kappa^2$ and $\eta \in p$ we let $\text{suc}_p(s) = \{t \in {}^{\kappa>\kappa}\kappa : (\exists \alpha < \kappa)(t = s^\wedge \langle \alpha \rangle \in p)\}$.
- (4) Let $s \in p \in \mathbb{Q}_\kappa^2$. We let $p^{(s)} = \{t \in p : t \leq s \vee s \leq t\}$.
- (5) For $a, b \subseteq \kappa$ we write $a \leq_\kappa^* b$ if $|a \setminus b| < \kappa$.

Each of the two forcing orders $\mathbb{P}$ has a weakest element, denoted by $1_{\mathbb{P}}$. Namely, $\mathbb{Q}_\kappa^1$ has as a weakest element $1_{\mathbb{Q}_\kappa^1} = \kappa$, and $\mathbb{Q}_\kappa^2$ has as a weakest element the full tree ${}^{\kappa>\kappa}\kappa$. We write $\mathbb{P} \Vdash \varphi$ if the weakest condition forces φ .

2 Results about $\mathbb{Q}_\kappa^1$

In this section we consider $\mathbb{Q}_\kappa^1$. The purpose is to provide standardised $\mathbb{Q}_\kappa^1$ -names for collapses. Later these particular $\mathbb{Q}_\kappa^1$ -names shall be translated to $\mathbb{Q}_\kappa^2$ -names.

Definition 2.1 A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called a κ -almost disjoint family if for $A \neq B \in \mathcal{A}$, $|A \cap B| < \kappa$.

Observation 2.2 If $2^{<\kappa} = \kappa$, there is a κ -ad family $\mathcal{A} \subseteq [\kappa]^\kappa$ of size 2^κ .

Proof We let $f: {}^\kappa 2 \rightarrow \kappa$ be an injection. We assign to each branch b of ${}^\kappa 2$ a set $a_b = \{f(s) : s \in b\}$. The resulting family $\{a_b : b \text{ branch of } {}^\kappa 2\}$ is κ -ad. $\square$

Observation 2.3 If $\mathbb{Q}_\kappa^1$ collapses 2^κ to ω , then there is a κ -ad family of size 2^κ .

Proof $\mathbb{Q}_\kappa^1$ cannot have the 2^κ -c.c. Hence there is an antichain of size 2^κ . Since $p \perp_{\mathbb{Q}_\kappa^1} q$ means $|p \cap q| < \kappa$, the antichain is a κ -ad family. $\square$

We will apply the following result for $\chi = 2^\kappa$.

Theorem 2.4 [5, Theorem 0.5] Suppose that there is an antichain in $\mathbb{Q}_\kappa^1$ of size χ . Then the following holds.

- (1) Forcing with $\mathbb{Q}_\kappa^1$ collapses χ to $\aleph_0$ if $\aleph_0 < \text{cf}(\kappa) = \kappa$ or if $\aleph_0 < \text{cf}(\kappa) < 2^{\text{cf}(\kappa)} \leq \kappa$.
- (2) Forcing with $\mathbb{Q}_\kappa^1$ collapses χ to $\aleph_1$ in the case of $\aleph_0 = \text{cf}(\kappa) < \kappa$.

Now we start defining tree structures from $\mathbb{Q}_\kappa^1$ -names for collapsing functions. Those trees will later be used to define dense suborders $\mathcal{Q}_T$ of $\mathbb{Q}_\kappa^2$. The idea of $\mathcal{Q}_T$ is that the sets $\text{set}_p(t)$, $t \in \text{spl}(p)$, for $p \in \mathcal{Q}_T$ will be sufficiently strong $\mathbb{Q}_\kappa^1$ conditions.

Lemma 2.5 Suppose that $\mathbb{Q}_\kappa^1$ collapses 2^κ to ω . Then there is a $\mathbb{Q}_\kappa^1$ -name $\tau: \aleph_0 \rightarrow 2^\kappa$ for a surjection, and there is a labelled tree $T = \langle (a_\eta, n_\eta, \varrho_\eta) : \eta \in {}^{\omega>} (2^\kappa) \rangle$ with the following properties

- (a) $a_\emptyset = \kappa$ and for any $\eta \in {}^{\omega>} (2^\kappa)$, $a_\eta \in [\kappa]^\kappa$.
- (b) $\eta_1 \triangleleft \eta_2$ implies $a_{\eta_1} \supseteq a_{\eta_2}$.
- (c) $n_\eta \in [\text{dom}(\eta) + 1, \omega)$.
- (d) If $a \in [\kappa]^\kappa$ then there is some $\eta \in {}^{\omega>} (2^\kappa)$ such that $a \supseteq a_\eta$.
- (e) If $\eta \hat{\langle} \beta \rangle \in T$ then $a_{\eta \hat{\langle} \beta \rangle}$ forces $\tau \restriction n_\eta = \varrho_{\eta \hat{\langle} \beta \rangle}$ for some $\varrho_{\eta \hat{\langle} \beta \rangle} \in {}^{n_\eta} (2^\kappa)$, such that the $\varrho_{\eta \hat{\langle} \beta \rangle}$, $\beta \in 2^\kappa$, are pairwise different. Hence for any $\eta \in {}^{\omega>} (2^\kappa)$, the family $\{a_{\eta \hat{\langle} \alpha \rangle} : \alpha < 2^\kappa\}$ is a κ -ad family in $[a_\eta]^\kappa$.

Proof Let τ be a $\mathbb{Q}_\kappa^1$ -name such that $\mathbb{Q}_\kappa^1 \Vdash \tau: \aleph_0 \rightarrow 2^\kappa$ is onto. For $\alpha < 2^\kappa$ let AP_α be the set of objects $\bar{m}$ satisfying

- (*)₁ (1.1) $\bar{m} = (T, \bar{a}, \bar{n}, \bar{\varrho}) = (T_{\bar{m}}, \bar{a}_{\bar{m}}, \bar{n}_{\bar{m}}, \bar{\varrho}_{\bar{m}})$.
- (1.2) T is a subtree of $({}^{\omega>} (2^\kappa), \triangleleft)$ of cardinality $\leq |\alpha| + \kappa$ and $\langle \rangle \in T$.
- (1.3) $\bar{a} = \langle a_\eta : \eta \in T \rangle$ fulfils $\eta \triangleleft \nu \rightarrow a_\nu \subseteq a_\eta$ and $a_\emptyset = \kappa$ and $a_\eta \in [\kappa]^\kappa$.
- (1.4) $\bar{n} = \langle n_\eta : \eta \in T \rangle$ fulfils $\text{dom}(\varrho_{\eta \hat{\langle} \beta \rangle}) = n_\eta > \text{dom}(\eta)$ for any $\eta \hat{\langle} \beta \rangle \in T$.

(1.5) If $\eta^{\wedge}\langle\beta\rangle \in T$, then $a_{\eta^{\wedge}\langle\beta\rangle}$ forces a value to $\tau \restriction n_{\eta}$, called $\varrho_{\eta^{\wedge}\langle\beta\rangle}$, and for $\beta \neq \gamma$ we have $\varrho_{\eta^{\wedge}\langle\beta\rangle} \neq \varrho_{\eta^{\wedge}\langle\gamma\rangle}$. Hence for any $\eta^{\wedge}\langle\beta\rangle, \eta^{\wedge}\langle\gamma\rangle \in T_{\bar{m}}, \beta \neq \gamma$ implies $a_{\eta^{\wedge}\langle\beta\rangle} \cap a_{\eta^{\wedge}\langle\gamma\rangle} \in [\kappa]^{<\kappa}$.

(1.6) For $\eta \in T_{\bar{m}}$, we let

$$\text{Pos}(a_{\eta}, n_{\eta}) = \{\varrho \in {}^n(2^{\kappa}) : a_{\eta} \Vdash_{\mathbb{Q}_{\kappa}^1} \tau \restriction n_{\eta} \neq \varrho\},$$

and require that the latter has cardinality 2^{κ} .

In the next items we state some properties of AP_{α} that are derived from $(*)_1$.

$(*)_2$ $AP = \bigcup\{AP_{\alpha} : \alpha < 2^{\kappa}\}$ is ordered naturally by $\leq_{AP}$, which means end extension.

$(*)_3$ (a) AP_{α} is not empty and increasing in α .

(b) For infinite α , AP_{α} is closed under unions of increasing sequences of length $< |\alpha|^{+}$.

$(*)_4$ Let $\gamma < 2^{\kappa}$. If $\bar{m} \in AP_{\gamma}$ and $\eta \in T_{\bar{m}}$ and $\eta^{\wedge}\langle\alpha\rangle \notin T_{\bar{m}}$ then there is $\bar{m}' \in AP_{\gamma}$ such that $\bar{m} \leq_{AP} \bar{m}'$ and $T_{\bar{m}'} = T_{\bar{m}} \cup \{\eta^{\wedge}\langle\alpha\rangle\}$.

Proof: For $\eta \in T_{\bar{m}}$,

$$\mathcal{U} = \text{Pos}(a_{\eta}, n_{\eta}) = \{\varrho \in {}^n(2^{\kappa}) : a_{\eta} \Vdash_{\mathbb{Q}_{\kappa}^1} \tau \restriction n_{\eta} \neq \varrho\} \text{ has size } 2^{\kappa},$$

whereas

$$\Lambda_{\eta} = \{\varrho_{\eta^{\wedge}\langle\beta\rangle} \restriction n_{\eta} : \beta \in 2^{\kappa} \wedge \eta^{\wedge}\langle\beta\rangle \in T_{\bar{m}}\}$$

is of size $\leq |T_{\bar{m}}| \leq |\gamma| + \kappa$. Hence we can choose $\varrho_{*} \in \mathcal{U} \setminus \Lambda_{\eta}$ and $b_{*} \in [a_{\eta}]^{\kappa}$ such that $b_{*} \Vdash_{\mathbb{Q}_{\kappa}^1} \varrho_{*} = \tau \restriction n_{\eta}$. We let $\varrho_{\eta^{\wedge}\langle\alpha\rangle} = \varrho_{*}$. Since b_{*} forces a value of $\tau \restriction n_{\eta}$ that is incompatible with the one forced by $a_{\eta^{\wedge}\langle\beta\rangle}$ for any $\eta^{\wedge}\langle\beta\rangle \in T_{\bar{m}}$, the set b_{*} is κ -almost disjoint from $a_{\eta^{\wedge}\langle\beta\rangle}$ for any $\eta^{\wedge}\langle\beta\rangle \in T_{\bar{m}}$. We take $b_{*} = a_{\bar{m}', \eta^{\wedge}\langle\alpha\rangle} \subseteq a_{\bar{m}, \eta}$.

Since $\text{cf}(2^{\kappa}) > \aleph_0$ and since

$$|\{\text{range}(\varrho) : \varrho \in {}^{\omega>}(2^{\kappa}) \wedge b_{*} \Vdash_{\mathbb{Q}_{\kappa}^1} \tau \restriction n \neq \varrho\}| = 2^{\kappa},$$

there is an n such that

$$\text{Pos}(b_{*}, n) = \{\varrho \in {}^n(2^{\kappa}) : b_{*} \Vdash_{\mathbb{Q}_{\kappa}^1} \tau \restriction n \neq \varrho\}$$

has cardinality 2^{κ} . We take the minimal one and let it be $n_{\eta^{\wedge}\langle\alpha\rangle}$.

$(*)_5$ If $\bar{m} \in AP_{\alpha}$ and $a \in [\kappa]^{\kappa}$ then there is some $\bar{m}' \geq \bar{m}$, such that there is $\eta \in T_{\bar{m}'}$ with $a_{\bar{m}', \eta} \subseteq a$.

Let

$$\mathcal{U}_a = \{\varrho \in {}^{\omega>}(2^{\kappa}) : a \Vdash_{\mathbb{Q}_{\kappa}^1} \varrho \not\leq \tau\},$$

i.e.

$$\mathcal{U}_a = \{\varrho \in {}^{\omega>}(2^\kappa) : (\exists b \geq_{\mathbb{Q}_\kappa^1} a)(b \Vdash_{\mathbb{Q}_\kappa^1} \varrho \triangleleft \tau)\}.$$

This set has cardinality 2^κ because $\mathbb{Q}_\kappa^1 \Vdash \tau : \omega \rightarrow 2^\kappa$ is onto. We take n minimal such that

$$\mathcal{U}_{a,n} = \{\varrho \in {}^n(2^\kappa) : (\exists b \geq_{\mathbb{Q}_\kappa^1} a)(b \Vdash_{\mathbb{Q}_\kappa^1} \varrho \triangleleft \tau)\}$$

has size 2^κ . We let

$$\text{set}_n^+(\bar{m}) = \{\varrho_\eta : \eta \in T_{\bar{m}}, \text{dom}(\varrho_\eta) \geq n\}.$$

Clearly $|\text{set}_n^+(\bar{m})| \leq |T_{\bar{m}}| \leq |\gamma| + \kappa$. Thus we can take $\varrho_a \in \mathcal{U}_{a,n}$ that is incompatible with every element of $\text{set}_n^+(\bar{m})$. We take some $b_a \in [a]^\kappa$ such that $b_a \Vdash_{\mathbb{Q}_\kappa^1} \varrho_a \trianglelefteq \tau$. The set

$$\Lambda_a = \{\eta \in T_{\bar{m}} : b_a \subseteq_\kappa^* a_\eta\}$$

is $\triangleleft$ -linearly ordered by $(*)_1$ clauses 1.3 and 1.5 and $\langle \rangle \in \Lambda_a$. Since b_a does not pin down τ , Λ_a has a $\triangleleft$ -maximal member η_a . Now we take $\alpha_* = \min\{\beta : \eta_a \hat{\langle} \beta \rangle \notin T_{\bar{m}}\}$. For any $\eta_a \hat{\langle} \beta \rangle \in T_{\bar{m}}$ we have $\varrho_{\eta_a \hat{\langle} \beta \rangle}$ and ϱ_a are incompatible, and hence $a_{\eta_a \hat{\langle} \beta \rangle} \cap b_a \in [\kappa]^{<\kappa}$. Now we choose $b_a^1 \in [b_a]^\kappa$ and ϱ_a^* such that $b_a^1 \Vdash_{\mathbb{Q}_\kappa^1} \varrho_a^* \triangleleft \tau$ and $\text{dom}(\varrho_a^*) \geq n_{\bar{m}, \eta_a} > \text{dom}(\eta_a)$.

We let

$$\begin{aligned} T_{\bar{m}'} &= T_{\bar{m}} \cup \{\eta_a \hat{\langle} \alpha_* \rangle\}, \\ a_{\eta_a \hat{\langle} \alpha_* \rangle} &= b_a^1, \end{aligned}$$

We let $n_{\eta_a \hat{\langle} \alpha_* \rangle}$ be the minimal n such that $|\text{Pos}(b_a^1, n)| \geq 2^\kappa$. So $(*)_5$ holds.

Now we are ready to construct $\mathcal{T}$ as in the statement of the lemma. We do this by recursion on $\alpha \leq 2^\kappa$. First we enumerate $[\kappa]^\kappa$ as $\langle c_\alpha : \alpha < 2^\kappa \rangle$, and we enumerate ${}^{\omega>}(2^\kappa)$ as $\langle \eta_\alpha : \alpha < 2^\kappa \rangle$ such that $\eta_\alpha \triangleleft \eta_\beta$ implies $\alpha < \beta$. We choose an increasing sequence $\bar{m}_\alpha$ by induction on $\alpha < 2^\kappa$. We start with the tree $\{\langle \rangle\}$, $a_{\langle \rangle} = \kappa$, $\varrho_{\langle \rangle} = \emptyset$, $n_{\langle \rangle}$ be minimal such that $|\text{Pos}(\kappa, n)| = 2^\kappa$. In the odd successor steps we take $\bar{m}_{2\alpha+1} \geq_{AP} \bar{m}_\alpha$ so that $a_\eta \subseteq c_\alpha$ for some $\eta \in T_{2\alpha+1}$. This is done according to $(*)_5$. In the even successor steps we take $\bar{m}_{2\alpha+2} \geq_{AP} \bar{m}_{2\alpha+1}$ such that $\eta_\alpha \in T_{2\alpha+2}$. Since all initial segments of η_α appeared among the η_β , $\beta < \alpha$, $\bar{m}_{2\alpha+2}$ is found according to $(*)_4$. In the limit steps we take unions. Then $\mathcal{T}$ that is given by the last three components of $\bar{m}_{2^\kappa}$ has properties (a) to (e). $\square$

Since $\tau = \tau[G]$ is not in $\mathbf{V}$, for any $\mathcal{T}$ as in Lemma 2.5, for any $f \in {}^\omega(2^\kappa) \cap \mathbf{V}$, the branch $\langle (a_f \restriction m, n_f \restriction m, \varrho_f \restriction m) : m \in \omega \rangle$ of $\mathcal{T}$ has a no $\subseteq_\kappa^*$ -lower bound for its first coordinate.

3 Transfer to $\mathbb{Q}_\kappa^2$

In this section we use the tree $\mathcal{T}$ from Lemma 2.5 for finding $\mathbb{Q}_\kappa^2$ -names. First we establish a dense subforcing $\mathcal{Q}_\mathcal{T}$ of $\mathbb{Q}_\kappa^2$. Then we construct $\mathcal{Q}_\mathcal{T}$ -names that are based on a $\mathbb{Q}_\kappa^1$ -name of a collapse and on the equation $2^{2^{<\kappa}} = 2^\kappa$.

Definition 3.1 Let μ, λ be cardinals. For $v, v' \in {}^\lambda \mu$ we write $v \perp v'$ if $v \not\leq v'$ and $v' \not\leq v$.

Typical pairs (λ, μ) are $(\omega, 2^\kappa)$ and (κ, κ) .

An important tool for the analysis of $\mathbb{Q}_\kappa^2$ is the following particular kind of fusion sequence $\langle p_\alpha : \alpha < \kappa^{<\kappa} \rangle$ in $\mathbb{Q}_\kappa^2$. Since we do not suppose $\kappa^{<\kappa} = \kappa$, a fusion sequence can be longer than κ . An important property is that for each $v \in {}^\kappa \mu$ there is at most one $\alpha < \kappa^{<\kappa}$ such that $\text{set}_{p_\alpha}(v) \supsetneq \text{set}_{p_{\alpha+1}}(v)$.

Lemma 3.2 Let $\langle v_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be an injective enumeration of $\kappa^{<\kappa}$ such that

$$v_\alpha \triangleleft v_\beta \rightarrow \alpha < \beta. \quad (3.1)$$

Let $\langle p_\alpha, v_\alpha, c_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be a sequence such that for any $\alpha \leq \lambda$ the following holds:

- (a) $p_0 \in \mathbb{Q}_\kappa^2$.
- (b1) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $v_\beta \in \text{spl}(p_\beta)$, then

$$c_\beta \in [\text{suc}_{p_\beta}(v_\beta)]^\kappa \text{ and} \\ p_\alpha = p_\beta(v_\beta, c_\beta) := \bigcup \{p_\beta^{(v_\beta \hat{\ } i)} : i \in c_\beta\} \cup \bigcup \{p_\beta^{(\eta)} : \eta \not\leq v_\beta \wedge v_\beta \not\leq \eta\}$$

- (b2) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $v_\beta \notin \text{spl}(p_\beta)$ then $p_\alpha = p_\beta$.
- (c) $p_\alpha = \bigcap \{p_\beta : \beta < \alpha\}$ for limit $\alpha \leq \kappa^{<\kappa}$.

Then for any $\lambda \leq \kappa^{<\kappa}$, $p_\lambda \in \mathbb{Q}_\kappa^2$ and $\forall \beta < \lambda$, $p_\lambda \leq_{\mathbb{Q}_\kappa^2} p_\beta$.

Proof We go by induction on λ . The case $\lambda = 0$ and the successor steps are obvious. So we assume that $\lambda \leq \kappa^{<\kappa}$ is a limit ordinal and $p_\alpha \in \mathbb{Q}_\kappa^2$ for $\alpha < \lambda$. Since $\emptyset \in p_\lambda$, p_λ is not empty, and p_λ clearly is a tree. Let $t \in p_\lambda$. We show that there is $t' \supseteq t$ that is a splitting node in p_λ .

We fix the smallest α such that $v_\alpha \supseteq_{p_0} t$ is a splitting node in p_0 . Then in p_0 there are no splitting nodes in $\{s : t \triangleleft s \triangleleft v_\alpha\}$. Hence $v_\alpha \in \text{spl}(p_\beta)$ for any $\beta \in [0, \lambda]$.

Now we show that the limit of splitting nodes in p_λ is a splitting node. Let $\gamma < \lambda$ and let $\langle v^i : i < \gamma \rangle$ be an $\triangleleft$ -increasing sequence of splitting nodes of p_λ with union $v \in \kappa^{<\kappa}$. Then v is a splitting node of each p_α , $\alpha < \lambda$, and also in p_λ since $\langle \text{set}_{p_\alpha}(v) : \alpha < \lambda \rangle$ has at most two entries and their intersection has size κ . $\square$

We use yet another, richer type of fusion sequence.

Definition 3.3 Let $p \in \mathbb{Q}_\kappa^2$ and let $v \in \text{spl}(p)$.

- (1) We say η is the *shortest splitting node above v in p* and write $\eta = \text{sucspl}_p(v)$ if η is the shortest splitting node in p such that $\eta \supseteq v$. Equality is allowed and occurs if v is a splitting node.
- (2) We say $F \subseteq p$ is the *front of next splitting nodes above v in p* , if

$$F = \{\eta' \in \text{spl}(p) : \exists(\eta \in \text{suc}_p(v))(\eta' = \text{sucspl}_p(\eta))\}.$$

Lemma 3.4 Let $\langle v_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be an injective enumeration of $\kappa^{<\kappa}$ such that

$$v_\alpha \triangleleft v_\beta \rightarrow \alpha < \beta. \quad (3.2)$$

Let $\langle p_\alpha, v_\alpha, c_\alpha, F_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be a sequence such that for any $\alpha \leq \lambda$ the following holds:

- (a) $p_0 \in \mathbb{Q}_\kappa^2$.
- (b1) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $v_\beta \in \text{sp}(p_\beta)$, then $c_\beta \in [\text{suc}_{p_\beta}(v_\beta)]^\kappa$, F_β contains for each $i \in c_\beta$ exactly one $\eta \in \text{spl}(p_\beta^{(v_\beta \hat{\wedge} \langle i \rangle)})$, and

$$p_\alpha = p_\beta(v_\beta, c_\beta, F_\beta) := \bigcup \{p_\beta^{(\eta)} : i \in c_\beta, \eta \in F_\beta\} \\ \cup \bigcup \{p_\beta^{(\eta)} : \eta \not\triangleleft v_\beta \wedge v_\beta \not\triangleleft \eta\}.$$

Note that this implies that F_β is the front of next splitting nodes of p_α above v_β .

- (b2) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $v_\beta \notin \text{spl}(p_\beta)$ then $p_\alpha = p_\beta$.
- (c) $p_\alpha = \bigcap \{p_\beta : \beta < \alpha\}$ for limit $\alpha \leq \kappa^{<\kappa}$.

Then for any $\lambda \leq \kappa^{<\kappa}$, $p_\lambda \in \mathbb{Q}_\kappa^2$ and $\forall \beta < \lambda$, $p_\lambda \leq_{\mathbb{Q}_\kappa^2} p_\beta$.

Proof We go by induction on λ . The case $\lambda = 0$ and the successor steps are obvious. So we assume that $\lambda \leq \kappa^{<\kappa}$ is a limit ordinal and $p_\alpha \in \mathbb{Q}_\kappa^2$ for $\alpha < \lambda$. Since $\emptyset \in p_\lambda$, p_λ is not empty, and p_λ clearly is a tree. Let $t \in p_\lambda$. We show that there is $t' \supseteq t$ that is a splitting node in p_λ .

We fix the smallest α such that $v_\alpha \supseteq_{p_0} t$ is a splitting node in p_0 . Then in p_0 there are no splitting nodes in $\{s : t \trianglelefteq s \triangleleft v_\alpha\}$. Hence $v_\alpha \in \text{spl}(p_\beta)$ for any $\beta \in [0, \lambda]$.

Now we show that the limit of splitting nodes in p_λ is a splitting node. Let $\gamma < \lambda$ and let $\langle v^i : i < \gamma \rangle$ be an $\triangleleft$ -increasing sequence of splitting nodes of p_λ with union $v \in \kappa^{<\kappa}$. Then v is a splitting node of each p_α , $\alpha < \lambda$, and also in p_λ since $\langle \text{set}_{p_\alpha}(v) : \alpha < \lambda \rangle$ has at most two entries and their intersection has size κ . $\square$

In the special case $F_\beta = \{v_\beta \hat{\wedge} \langle j \rangle : j \in c_\beta\}$, the construction of Lemma 3.4 coincides with the simpler construction from Lemma 3.2.

Definition 3.5 We assume $\mathbb{Q}_\kappa^1$ collapses 2^κ to ω . Let τ and $\mathcal{T} = \langle (a_\eta, n_\eta, \mathcal{Q}) : \eta \in {}^{\omega>} (2^\kappa) \rangle$ be as in Lemma 2.5. Now let $\mathcal{Q}_\mathcal{T}$ be the set of κ -Miller trees p such that for every $v \in \text{spl}(p)$ there is $\eta_{p,v} = \eta_v \in {}^{\omega>} (2^\kappa)$ such that

$$\text{set}_p(v) = \{\varepsilon \in \kappa : v \hat{\wedge} \langle \varepsilon \rangle \in p\} = a_{\eta_v}. \quad (3.3)$$

By the properties of $\mathcal{T}$, the node $\eta_{p,v}$ is unique.

Lemma 3.6 Assume that $\mathbb{Q}_\kappa^1$ collapses 2^κ to ω , let $\mathcal{T}$ be chosen as in Lemma 2.5, and let $Q_{\mathcal{T}}$ be defined from $\mathcal{T}$ as above. Then $Q_{\mathcal{T}}$ is dense in $\mathbb{Q}_\kappa^2$.

Proof Let $p_0 = T \in \mathbb{Q}_\kappa^2$. Let $\langle v_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be an injective enumeration of $\kappa^{<\kappa}$ with property (3.1). We now define fusion sequence $\langle p_\alpha, v_\alpha, c_\alpha : \alpha \leq \kappa^\kappa \rangle$ according to the pattern in Lemma 3.2 in order to find $p_{\kappa^{<\kappa}} \geq T$ such that $p_{\kappa^{<\kappa}} \in Q_{\mathcal{T}}$.

Suppose that p_α and v_α are given. If v_α is not in p_α or is not a splitting node in p_α , then we let $p_{\alpha+1} = p_\alpha$. If $v_\alpha \in \text{spl}(p_\alpha)$, then according to Lemma 2.5 clause (d) there is $\eta \in {}^{\omega>} (2^\kappa)$ such that $\text{suc}_{p_\alpha}(v_\alpha) \supseteq a_\eta$. We choose such an η of minimal length and call it $\eta(\alpha)$.

Then we strengthen p_α to

$$p_{\alpha+1} = \bigcup \{ p_\alpha^{(v')} : v' = v_\alpha \hat{\ } i \wedge i \in a_{\eta(\alpha)} \} \cup \bigcup \{ p_\alpha^{(\eta)} : \eta \not\leq v_\alpha \wedge v_\alpha \not\leq \eta \}. \quad (3.4)$$

Now we have that

$$\eta_{p_{\alpha+1}, v_\alpha} = \eta(\alpha), c_\alpha = a_{\eta(\alpha)}.$$

For limit ordinals $\lambda \leq \kappa^{<\kappa}$, we let $p_\lambda = \bigcap \{ p_\beta : \beta < \lambda \}$. Since the sequence $\langle p_\alpha, v_\alpha, c_\alpha : \alpha \leq \kappa^{<\kappa} \rangle$ matches the pattern in Lemma 3.2, we have $p_{\kappa^{<\kappa}} \in \mathbb{Q}_\kappa^2$. By construction, for any $\alpha < \kappa^{<\kappa}$ for any $\delta \in [\alpha + 1, \kappa^{<\kappa})$, $v_\alpha \in \text{spl}(p_\delta)$ implies

$$\text{set}_{p_{\alpha+1}}(v_\alpha) = \text{set}_{p_\delta}(v_\alpha) = a_{\eta(\alpha)}.$$

Hence the condition $p = p_{\kappa^{<\kappa}}$ fulfils Equation (3.3) in its splitting node v_α with witness $\eta_{p, v_\alpha} = \eta(\alpha)$. Since all nodes are enumerated, we have $p_{\kappa^{<\kappa}} \in Q_{\mathcal{T}}$. $\square$

We use only the inclusion $\text{set}_p(v) \subseteq a_{\eta_v}$ from Definition 3.5.

Definition 3.7 We assume that $\mathbb{Q}_\kappa^1$ collapses 2^κ to ω and the $\mathcal{T}$ is as in Lemma 2.5. For $T \in Q_{\mathcal{T}}$ and a splitting node v of T we set $\varrho_{T,v} := \varrho_{\eta_{T,v}} \in {}^{\omega>} (2^\kappa)$. Recall $\eta_{T,v}$ is defined in Def. 3.5, and ϱ is a component of $\mathcal{T}$.

For $p \in Q_{\mathcal{T}}$, the relation $v \trianglelefteq v' \in p$ does neither imply $\eta_v \trianglelefteq \eta_{v'}$ nor $\varrho_v \trianglelefteq \varrho_{v'}$. However, $\eta_v \triangleleft \eta_{v'}$ implies $a_{\eta_v} \supset a_{\eta_{v'}}$ and $\varrho_v \triangleleft \varrho_{v'}$.

Observation 3.8 We assume that $\mathbb{Q}_\kappa^1$ collapses 2^κ to ω . Let $p_1, p_2 \in Q_{\mathcal{T}}$. If $p_1 \leq_{\mathbb{Q}_\kappa^2} p_2$ then for $v \in \text{spl}(p_2)$ we have $v \in \text{spl}(p_1)$ and $\varrho_{p_1,v} \trianglelefteq \varrho_{p_2,v}$.

We introduce dense sets:

Definition 3.9 We assume that $\mathbb{Q}_\kappa^1$ collapses 2^κ to ω . Let $n \in \omega$.

$$D_n = \{ p \in Q_{\mathcal{T}} : (\forall v \in \text{spl}(p)) (\text{dom}(\varrho_{p,v}) > n) \}.$$

D_n is open dense in $Q_{\mathcal{T}}$ and the intersection of the D_n is empty.

Recall, by Lemma 3.6 we can work with the dense subforcing $Q_{\mathcal{T}}$ of $\mathbb{Q}_{\kappa}^2$. The following technical lemma is the next step of a transformation of a $\mathbb{Q}_{\kappa}^1$ -name of a surjection from ω onto 2^{κ} into a $Q_{\mathcal{T}}$ -name of such a surjection. The coordinate $\bar{\gamma}_{\alpha}$ and the clauses (d), (e), (f) are used for a counting argument in the induction steps. Later, only the coordinates p_{α} , n_{α} , and clauses (a), (b), (c) and Remark 3.11 will be used.

Lemma 3.10 *We assume that $\mathbb{Q}_{\kappa}^1$ collapses 2^{κ} to ω , $\text{cf}(\kappa) > \omega$ and $2^{(\kappa^{<\kappa})} = 2^{\kappa}$. Let $\langle T_{\alpha} : \alpha < 2^{\kappa} \rangle$ enumerate $\mathbb{Q}_{\kappa}^2$ such that each Miller tree appears 2^{κ} times. There is $\langle (p_{\alpha}, n_{\alpha}, \bar{\gamma}_{\alpha}) : \alpha < 2^{\kappa} \rangle$ such that*

- (a) $n_{\alpha} < \omega$,
- (b) $p_{\alpha} \in D_{n_{\alpha}} \subseteq Q_{\mathcal{T}}$ and $p_{\alpha} \geq T_{\alpha}$.
- (c) If $\beta < \alpha$ and $n_{\beta} \geq n_{\alpha}$ then $p_{\beta} \perp p_{\alpha}$.
- (d) $\bar{\gamma}_{\alpha} = \langle \gamma_{\alpha, v} : v \in \text{spl}(p_{\alpha}) \rangle$.
- (e) $(\forall v \in \text{spl}(p_{\alpha}))(a_{\eta_{p_{\alpha}, v}} \Vdash_{\mathbb{Q}_{\kappa}^1} \gamma_{\alpha, v} \in \text{range}(\varrho_{p_{\alpha}, v}))$.
- (f) $\gamma_{\alpha, v} \in 2^{\kappa} \setminus W_{<\alpha, v}$ with

$$W_{<\alpha, v} = \bigcup \{ \text{range}(\varrho_{p_{\beta}, v}) : \beta < \alpha, v \in \text{spl}(p_{\beta}) \}.$$

Proof Assume that $\langle (p_{\beta}, n_{\beta}, \bar{\gamma}_{\beta}) : \beta < \alpha \rangle$ has been defined and we are to define $(p_{\alpha}, n_{\alpha}, \bar{\gamma}_{\alpha})$. Note that the p_{β} need not be increasing in strength.

- $(\oplus)_1$ The choice of the a_{η} in Lemma 2.5 and the choice $Q_{\mathcal{T}}$ and of $\eta_{p_{\beta}, v}$ for $v \in \text{spl}(p_{\beta})$, $\beta < \alpha$, imply that the set $W_{<\alpha, v}$ is well defined and of cardinality $\leq |\alpha| + \aleph_0 < 2^{\kappa}$. Hence we can choose $\gamma_{\alpha, v} \in 2^{\kappa} \setminus W_{<\alpha, v}$.
- $(\oplus)_2$ With the fusion Lemma 3.2 we choose $q_{\alpha} \geq T_{\alpha}$, $q_{\alpha} \in Q_{\mathcal{T}}$, such that

$$(\forall v \in \text{spl}(q_{\alpha}))(a_{\eta_{q_{\alpha}, v}} \Vdash_{\mathbb{Q}_{\kappa}^1} \gamma_{\alpha, v} \in \text{range}(\varrho_{q_{\alpha}, v})).$$

- $(\oplus)_3$ Let $q \in \mathbb{Q}_{\kappa}^2$. For $n \in \omega$ and $v \in \text{spl}(q)$ we let

$$\mathcal{U}_{\alpha, v, n}(q) = \{ \beta < \alpha : n_{\beta} = n, v \in \text{spl}(p_{\beta}) \wedge |\text{set}_q(v) \cap \text{set}_{p_{\beta}}(v)| = \kappa \}.$$

$$\mathcal{U}_{\alpha, v}(q) = \bigcup \{ \mathcal{U}_{\alpha, v, n}(q) : n \in \omega \}.$$

- $(\oplus)_4$ (a) If $n \in \omega$ and $v \in \text{spl}(q_{\alpha})$ then

$$\beta \in \mathcal{U}_{\alpha, v}(q_{\alpha}) \rightarrow \varrho_{p_{\beta}, v} \leq \varrho_{q_{\alpha}, v}.$$

This is seen as follows. We let $a = \text{set}_{p_{\beta}}(v) \cap \text{set}_{q_{\alpha}}(v)$. Since $\beta \in \mathcal{U}_{\alpha, v}(q_{\alpha})$, $a \in [\kappa]^{\kappa}$. Clearly $a \Vdash_{\mathbb{Q}_{\kappa}^1} \tau \triangleright \varrho_{p_{\beta}, v}, \varrho_{q_{\alpha}, v}$. So either $\varrho_{p_{\beta}, v} \triangleleft \varrho_{q_{\alpha}, v}$ or $\varrho_{p_{\beta}, v} \supseteq \varrho_{q_{\alpha}, v}$. However, since $\gamma_{\alpha, v} \in \text{range}(\varrho_{q_{\alpha}, v}) \setminus W_{<\alpha, v}$, only $\varrho_{q_{\alpha}, v} \triangleright \varrho_{p_{\beta}, v}$ is possible.

(b) So for $v \in \text{spl}(q_{\alpha})$, the set $\{ \varrho_{p_{\beta}, v} : \beta \in \mathcal{U}_{\alpha, v}(q_{\alpha}) \}$ has at most $\text{dom}(\varrho_{q_{\alpha}, v})$ elements.

(c) The assignment $\beta \mapsto \varrho_{p_{\beta}, v}$ is defined between $\mathcal{U}_{\alpha, v}(q_{\alpha})$ and $\{ \varrho_{p_{\beta}, v} : \beta \in$

$\mathcal{U}_{\alpha,v}(q_\alpha)\}$. According to properties (e) and (f) in the induction hypothesis, the assignment is injective, and hence

$$|\mathcal{U}_{\alpha,v}(q_\alpha)| \leq \text{dom}(\mathcal{Q}_{q_\alpha,v}).$$

(d) We state for further use that $\mathcal{U}_{\alpha,v}(q_\alpha)$ is finite and for any $q \leq q_\alpha$, $\mathcal{U}_{\alpha,v}(q) \subseteq \mathcal{U}_{\alpha,v}(q_\alpha)$.

($\oplus$)₅ We look at the cone above q_α and show:

$$\begin{aligned} & (\forall q \geq q_\alpha)(\forall v \in \text{spl}(q))(\exists r_{\alpha,v} \leq_{\mathbb{Q}_\kappa^2} q) \\ & (\exists c \in [\text{set}_q(v)]^\kappa)(\exists F \subseteq \{\eta \in \text{spl}(q) : \eta \triangleright v\}) \\ & (r_{\alpha,v} = q(v, c, F) \wedge (\forall \beta \in \mathcal{U}_{\alpha,v}(q_\alpha))(r_{\alpha,v}^{(v)} \perp p_\beta^{(v)} \vee p_\beta^{(v)} \geq r_{\alpha,v}^{(v)})). \end{aligned} \quad (3.5)$$

How do we find $r_{\alpha,v} = r_{\alpha,v}(q)$? Given $q \leq_{\mathbb{Q}_\kappa^2} q_\alpha$, $v \in \text{spl}(q)$ we enumerate $\mathcal{U}_{\alpha,v}(q_\alpha)$ as $\beta_0, \dots, \beta_{k-1}$. We let $r_0 = q$ and by induction on $i \leq k$ we define r_i , increasing in strength, with $v \in \text{spl}(r_i)$ and $c_i = \text{set}_{r_i}(v)$. Thus the c_i are $\subseteq$ -decreasing sets of size κ . Given r_i , we distinguish cases:

First case: $\beta_i \notin \mathcal{U}_{\alpha,v}(r_i)$. Then there is $c_{i+1} \in [\text{set}_{r_i}(v)]^\kappa$, $c_{i+1} \cap \text{set}_{p_{\beta_i}}(v) = \emptyset$.

We let $r_{i+1} = r_i(v, c_{i+1})$ and thus have $r_{i+1}^{(v)} \perp p_{\beta_i}$.

Second case: $\beta_i \in \mathcal{U}_{\alpha,v}(r_i)$. We let

$$c_i = \{j \in \text{set}_{r_i}(v) : r_i^{(v \wedge \langle j \rangle)} \leq p_{\beta_i}^{(v \wedge \langle j \rangle)}\} \cup \{j \in \text{set}_{r_i}(v) : r_i^{(v \wedge \langle j \rangle)} \not\leq p_{\beta_i}^{(v \wedge \langle j \rangle)}\}.$$

If $c_{i,1} = \{j \in \text{set}_{r_i}(v) : r_i^{(v \wedge \langle j \rangle)} \leq p_{\beta_i}^{(v \wedge \langle j \rangle)}\}$ has size κ , then we let $c_{i+1} = c_{1,i}$ and $r_{i+1} = r_i(v, c_{i+1})$ and thus get $r_{i+1}^{(v)} \perp p_{\beta_i}$.

If $|c_{i,1}| < \kappa$, then $c_{i,2} = \{j \in \text{set}_{r_i}(v) : r_i^{(v \wedge \langle j \rangle)} \not\leq p_{\beta_i}^{(v \wedge \langle j \rangle)}\}$ has size κ , and we let $c_{i+1} = c_{i,2}$. For $j \in c_{i+1}$, $r_i^{(v \wedge \langle j \rangle)} \not\leq p_{\beta_i}^{(v \wedge \langle j \rangle)}$. Thus we can find a node in the $r_i^{(v \wedge \langle j \rangle)} \setminus p_{\beta_i}^{(v \wedge \langle j \rangle)}$ and above this node we find a splitting node of r_i . We take this latter splitting node into r_{i+1} as the direct successor splitting node to $v \wedge \langle j \rangle$. Doing so for every $j \in c_{i+1}$ we get $F_{v,i}$, a front strictly above v in $r_{i+1} = r_i(v, c_{i+1}, F_{v,i})$. Again we get $r_{i+1}^{(v)} \perp p_{\beta_i}$.

In the end we let $r_{\alpha,v} = r_k$. There is a front F that contains for each $j \in c_k$ the shortest splitting node of r_k above $v \wedge \langle j \rangle$. Thus we have $r_k = r_{\alpha,v} = q(v, c_k, F)$ and $r_{\alpha,v}$ fulfils (3.5).

($\oplus$)₆ Now we use ($\oplus$)₅ iteratively along all $v \in \kappa^{<\kappa}$ to find a fusion sequence $\langle r_{\alpha,v}, v, c_v, F_v : v < \kappa^{<\kappa} \rangle$ with starting point $q_\alpha = r_{0,v_0}$. In this sequence, $r_{\alpha,v}$ is chosen as $r_{\alpha,v}(q)$ in $\oplus_5$ for $q = \bigcap_{\beta < \alpha} r_\beta$, if $v \in \text{spl}(q)$. If $v \notin \text{spl}(q)$, then $r_{\alpha,v} = q$. Then we apply the fusion Lemma 3.4 and get an lower bound r_α of $r_{\alpha,v}$, $v \in \kappa^{>\kappa}$. Note $r_\alpha^{(v)} \perp p_\beta^{(v)}$ iff $r_\alpha^{(v)} \perp p_\beta^{(v)}$ and $r_\alpha^{(v)} \leq p_\beta^{(v)}$ iff $r_\alpha^{(v)} \leq p_\beta^{(v)}$. Hence $r_\alpha \geq q_\alpha$ and

$$(\forall v \in \text{spl}(r_\alpha))(\forall \beta \in \mathcal{U}_{\alpha,v}(q_\alpha))(r_\alpha^{(v)} \perp p_\beta \vee p_\beta \geq r_\alpha^{(v)}).$$

($\oplus$)₇ Finally we choose n_α and p_α . There are k and ν such that $n < \omega$ and $\nu \in \text{spl}(r_\alpha)$ such that $p_\alpha = r_\alpha^{(\nu)}$ fulfils

$$(\forall \beta < \alpha)(n_\beta \geq k \rightarrow p_\alpha \perp p_\beta).$$

Proof of existence. By induction on $k \in \omega$ we try to find $\langle \nu_k, \beta_k : k \in \omega \rangle$ such that

(a) $\nu_k \in \text{spl}(r_\alpha)$,

(b) $\nu_k \triangleleft \nu_m$ for $k < m$,

(c) $\beta_k < \alpha$ and $n_{\beta_k} \geq k$ and $r_\alpha^{(\nu_k)} \leq p_{\beta_k}$.

If we succeed, then $\nu_* = \bigcup \{\nu_k : k \in \omega\} = \nu^* \in \text{spl}(r_\alpha)$ by Definition 1.1 (2).

Here we use that $\text{cf}(\kappa) > \omega$. Hence

$$r_\alpha^{(\nu^*)} \in \mathcal{Q}_T \cap \bigcap \{D_k : k < \omega\} \text{ and}$$

$$a_{\eta_{r_\alpha}, \nu^*} \text{ determines in } \Vdash_{\mathbb{Q}_\kappa^1} \text{ for any } k < \omega \text{ the value of } \tau \restriction k.$$

This is a contradiction.

So there is a smallest k such that ν_k cannot be defined. We let $n_\alpha = k$. We let p_α be a strengthening of $r_\alpha^{(\nu_{k-1})}$ such that $p_\alpha \in D_{n_\alpha}$. For finding such a strengthening we again invoke the fusion Lemma 3.2.

We show that $p_\alpha \perp p_\beta$ for $\beta < \alpha$ with $n_\beta \geq k$. Otherwise, having arrived at $r_\alpha^{(\nu_{k-1})}$ we find some β_k, α such that $n_{\beta_k} \geq k$ and $r_\alpha^{(\nu_{k-1})}$ is compatible with p_{β_k} . Then we can prolong ν_{k-1} to a splitting node $\nu_k \in \text{spl}(p_{\beta_k}) \cap \text{spl}(r_\alpha)$. By the choice of r_α the latter implies that $r_\alpha^{(\nu_k)} \leq p_{\beta_k}$. However, now we would have found ν_k, β_k as required in contradiction to the choice of k . $\square$

Remark 3.11 Conditions (a) to (c) of Lemma 3.10 yield: For any $k < \omega$,

$$\{p_\alpha : n_\alpha \geq k\} \text{ is dense in } \mathbb{Q}_\kappa^2.$$

Proof Let k and p be given. There is α_0 such that $T_{\alpha_0} \in D_0$ and $T_{\alpha_0} \leq_{\mathbb{Q}_\kappa^2} p$. Then $p_{\alpha_0} \leq T_{\alpha_0}$ and $n_{\alpha_0} \geq 0$. Then there is $\alpha_1 > \alpha_0$ such that $T_{\alpha_1} \leq_{\mathbb{Q}_\kappa^2} p_{\alpha_0}$. Then $p_{\alpha_1} \leq T_{\alpha_1}$ and hence by condition (c), $n_{\alpha_1} > n_{\alpha_0} \geq 0$. We can repeat the argument $k - 1$ times. $\square$

Now we drop the component $\bar{\gamma}_\alpha$ from a sequence $\langle p_\alpha, n_\alpha, \bar{\gamma}_\alpha : \alpha < 2^\kappa \rangle$ given by Lemma 3.10. Then we get a sequence with properties (a), (b), and a weakening (c) with the property stated in the remark. This sequence, combined with $2^{(2^{<\kappa})} = 2^\kappa$, allows to define a $\mathbb{Q}_\kappa^2$ -name of a collapse.

Lemma 3.12 We assume that $\mathbb{Q}_\kappa^1$ collapses 2^κ to ω , $\text{cf}(\kappa) > \omega$ and $2^{(2^{<\kappa})} = 2^\kappa$. Let $\langle T_\alpha : \alpha < 2^\kappa \rangle$ enumerate all Miller trees that such each tree appears 2^κ times. Assume that $\langle (p_\alpha, n_\alpha) : \alpha < 2^\kappa \rangle$ are such that

(a) $n_\alpha < \omega$,

- (b) $p_\alpha \in D_{n_\alpha} \subseteq Q_{\mathcal{T}}$ and $p_\alpha \geq T_\alpha$,
- (c) if $\beta < \alpha$ and $n_\beta = n_\alpha$ then $p_\beta \perp p_\alpha$,
- (d) for any $k \in \omega$, $\{p_\alpha : n_\alpha \geq k\}$ is dense in $\mathbb{Q}_\kappa^2$.

Then there is a $\mathbb{Q}_\kappa^2$ -name τ' for a surjection of ω onto 2^κ .

Proof Let G be a $\mathbb{Q}_\kappa^2$ -generic filter over $\mathbf{V}$. We define $\tau(n)$, a $\mathbb{Q}_\kappa^2$ -name by $\tau(n)[G] = \alpha$ if $p_\alpha \in G$ and $n_\alpha = n$. The name τ is a name of a function by (c). By (d), the domain of τ is forced to be infinite. For any $p \in \mathbb{Q}_\kappa^2$ we let $U_p = \{\alpha : T_\alpha = p\}$. U_p is of size 2^κ , in particular for $\alpha \in 2^\kappa$ we have $|U_{p_\alpha}| = 2^\kappa$ and U_{p_α} contains the antichain $\{p_{\delta(\alpha, i)} : i < 2^\kappa\}$. Hence there is $f : 2^\kappa \rightarrow 2^\kappa$ in $\mathbf{V}[G]$ such that for any $\gamma, \alpha \in 2^\kappa$ there is $\beta = \delta(\alpha, \gamma) \in U_{p_\alpha}$ for some function $\delta : 2^\kappa \times 2^\kappa \rightarrow 2^\kappa$ with $f(\beta) = \gamma$ forced by $p_{\delta(\alpha, \gamma)}$. We let

$$\tau' = f(\tau) = \{ \langle (n_\alpha, \gamma), p_{\delta(\alpha, \gamma)} \rangle : \alpha, \gamma \in 2^\kappa \}.$$

Next we show

$$\mathbb{Q}_\kappa^2 \Vdash \text{range}(\tau') = 2^\kappa.$$

Suppose $p \in Q_{\mathcal{T}}$ and $\gamma < 2^\kappa$ are given. By construction the sequence $\{p_\beta : \beta < 2^\kappa\}$ is dense. Let $p \leq p_\alpha$. Then there is $\beta = \delta(\alpha, \gamma) \in U_{p_\alpha}$ $p_{\delta(\alpha, \gamma)} \leq p_\gamma$, with $f(\alpha) = \gamma$. However, $\delta(\alpha, \gamma) = \beta \in U_{p_\alpha}$ means $T_\beta = p_\alpha \geq p_\beta$ by construction. By the definition of τ , $p_\beta \Vdash \tau(n_\alpha) = \alpha$, so $p_\beta \Vdash (f(\tau))(n_\alpha) = \gamma$. $\square$

So we can sum up:

Theorem 3.13 *We assume that $\mathbb{Q}_\kappa^1$ collapses 2^κ to ω and $\text{cf}(\kappa) > \omega$ and $2^{(\kappa^{<\kappa})} = 2^\kappa$. Then the forcing with $\mathbb{Q}_\kappa^2$ collapses 2^κ to $\aleph_0$.*

4 κ -Cohen reals and the Levy collapse

Many κ -tree forcings add a κ -Cohen real, sometimes even if their ω -version does not add a Cohen real. Also our forcing $\mathbb{Q}_\kappa^2$ is of this kind. Classical Miller forcing preserves P -points and hence does not add a Cohen real. In this section we show that under the above conditions, $\mathbb{Q}_2^\kappa$ add a κ -Cohen real and is equivalent to the Levy collapse of 2^κ to $\aleph_0$.

Lemma 4.1 *If $\mathbb{Q}_\kappa^2$ collapses 2^κ to $\aleph_0$, $\text{cf}(\kappa) > \aleph_0$, and $2^{2^{<\kappa}} = 2^\kappa$, then $\mathbb{Q}_\kappa^2$ adds a κ -Cohen real.*

Proof Let G be $\mathbb{Q}_\kappa^2$ -generic over $\mathbf{V}$. Let $f : \omega \rightarrow 2^{<\kappa}$ be a function in $\mathbf{V}[G]$, such that $(\forall \eta \in 2^{<\kappa})(\exists^\infty k f(k) = \eta)$. Such a function exists since $2^{<\kappa} \leq 2^\kappa$.

Since $2^{2^{<\kappa}} = 2^\kappa$, we can enumerate all antichains in $\mathbb{C}(\kappa)$ in $\alpha_* \leq 2^\kappa$ many steps. In $\mathbf{V}[G]$, α_* is countable. We list it as $\langle \alpha_n : n < \omega \rangle$. Now we choose $\eta_n \in \mathbb{C}(\kappa)^\mathbf{V}$ by induction on n in $\mathbf{V}[G]$: $\eta_0 = \emptyset$. Given η_n we choose k_n such that $f(k_n) = \eta_n$ and then we choose $\eta_{n+1} \supseteq \eta_n$, such that $\eta_{n+1} \in I_{\alpha_n}$. Then $\{\eta : (\exists n < \omega)(\eta \sqsubseteq f(k_n))\}$ is a $\mathbb{C}(\kappa)$ -generic filter over $\mathbf{V}$ and it exists in $\mathbf{V}[G]$, since it is definable from $\{f(k_n) : n < \omega\}$. $\square$

Two forcings $\mathbb{P}_1, \mathbb{P}_2$ are said to be equivalent if their regular open algebras $\text{RO}(\mathbb{P}_i)$ coincide (for a definition of the regular open algebra of a poset, see, e.g., [3, Corollary 14.12]). Some forcings are characterised up to equivalence just by their size and their collapsing behaviour.

Lemma 4.2 [3, Lemma 26.7]. *Let $(Q, <)$ be a notion of forcing such that $|Q| = \lambda > \aleph_0$ and such that Q collapses λ onto $\aleph_0$, i.e.,*

$$1_Q \Vdash_Q |\check{\lambda}| = \aleph_0.$$

Then $\text{RO}(Q) = \text{Levy}(\aleph_0, \lambda)$.

Lemma 4.3 *If $\mathbb{Q}_\kappa^1$ collapses 2^κ to $\aleph_0$, then $\mathbb{Q}_\kappa^1$ is equivalent to $\text{Levy}(\aleph_0, 2^\kappa)$.*

Proof $\mathbb{Q}_\kappa^1$ has size 2^κ . Hence Lemma 4.2 yields $\text{RO}(\mathbb{Q}_\kappa^1) = \text{Levy}(\aleph_0, 2^\kappa)$. $\square$

Definition 4.4 A Boolean algebra is (θ, λ) -nowhere distributive if there are antichains $\bar{p}^\varepsilon = \langle p_\alpha^\varepsilon : \alpha < \alpha_\varepsilon \rangle$ of $\mathbb{P}$ for $\varepsilon < \theta$ such that for every $p \in \mathbb{P}$ for some $\varepsilon < \theta$

$$|\{\alpha < \alpha_\varepsilon : p \not\leq p_\alpha^\varepsilon\}| \geq \lambda.$$

Definition 4.5 Let B be a Boolean algebra. We write $B^+ = B \setminus \{0\}$. A subset $D \subseteq B^+$ is called *dense* if $(\forall b \in B^+)(\exists d \in D)(d \leq b)$. The *density* of a Boolean algebra B is the least size of a dense subset of B . A Boolean algebra B has uniform density if for every $a \in B^+$, $B \restriction a$ has the same density. The *density* of a forcing order $(\mathbb{P}, \leq_\mathbb{P})$ is the density of the regular open algebra $\text{RO}(\mathbb{P})$.

Lemma 4.6 [1, Theorem 1.15] *Let $\theta < \lambda$ be regular cardinals.*

(1) *Suppose that $\mathbb{P}$ has the following properties (a) to (c).*

- (a) $\mathbb{P}$ is a (θ, λ) -nowhere distributive forcing notion,
- (b) $\mathbb{P}$ has density λ ,
- (c) *in case $\theta > \aleph_0$, $\mathbb{P}$ has a θ -complete dense subset S . The latter means: $(\forall B \in [S]^{<\theta})(\exists s \in S)(\forall b \in B)(b \leq_\mathbb{P} s)$.*

Then $\mathbb{P}$ is equivalent to $\text{Levy}(\theta, \lambda)$.

(2) *Under (a) and (b) $\mathbb{P}$ collapses λ to θ (and may or may not collapse λ to $\aleph_0$).*

Proposition 4.7 *If there is a κ -mad family of size 2^κ the forcing $\mathbb{Q}_\kappa^1$ is $(\aleph_0, 2^\kappa)$ -nowhere distributive.*

Proof Lemma 2.5 gives $\mathcal{T}$ such that $\bar{p}^n = \{a_\eta : \eta \in {}^n(2^\kappa)\}$, $n \in \omega$, witnesses $(\aleph_0, 2^\kappa)$ -nowhere distributivity. $\square$

By Lemma 4.2 and Theorem 3.13 we get:

Proposition 4.8 *If $\mathbb{Q}_\kappa^1$ collapses 2^κ to $\aleph_0$, $\text{cf}(\kappa) > \aleph_0$ and $2^{(\kappa^{<\kappa})} = 2^\kappa$ then $\mathbb{Q}_\kappa^2$ is equivalent to $\text{Levy}(\aleph_0, 2^\kappa)$.*

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