




Iterated ultrapowers for the masses

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Received: 19 February 2017 / Accepted: 22 September 2017 / Published online: 5 October 2017
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Abstract We present a novel, perspicuous framework for building iterated ultrapowers. Furthermore, our framework naturally lends itself to the construction of a certain type of order indiscernibles, here dubbed *tight indiscernibles*, which are shown to provide smooth proofs of several results in general model theory.

Keywords Iterated ultrapower · Tight indiscernible sequence · Automorphism

Mathematics Subject Classification Primary 03C20; Secondary 03C50

1 Introduction and preliminaries

One of the central results of model theory is the celebrated Ehrenfeucht–Mostowski theorem on the existence of models with indiscernible sequences. All textbooks in model theory, including those of Chang and Keisler [4], Hodges [16], Marker [27], and Poizat [28] demonstrate this theorem using essentially the same compactness argument

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that was originally presented by Ehrenfeucht and Mostowski in their seminal 1956 paper [6], using Ramsey's partition theorem.

The source of inspiration for this paper is a fundamentally different proof of the Ehrenfeucht–Mostowski theorem that was discovered by Gaifman [14] in the mid-1960s, a proof that relies on the technology of *iterated ultrapowers*, and in contrast with the usual proof, does not invoke Ramsey's theorem. Despite the passage of several decades, Gaifman's proof seems to be relatively unknown among logicians, perhaps due to the forbidding technical features of the existing constructions of iterated ultrapowers in the literature.¹ Here we attempt to remedy this situation by presenting a streamlined account that is sufficiently elementary to be accessible to logicians familiar with the rudiments of model theory.

Our exposition also incorporates novel technical features: we bypass the usual method of building iterated ultrapowers as direct limits, and instead build them in the guise of *dimensional (Skolem) ultrapowers*, in a natural manner reminiscent of the usual construction of (Skolem) ultrapowers. We also isolate the key notion of *tight indiscernibles* to describe a special and useful kind of order indiscernibles that naturally arise from dimensional ultrapowers. In the interest of balancing clarity with succinctness, we have opted for a tutorial style: plenty of motivation is offered, but some of the proofs are left in the form of exercises for the reader, and the solutions are collected in a freely available appendix [12] to this paper.

In the rest of this section we review some preliminaries, introduce the key notion of tight indiscernibles, and state a corresponding existence theorem (Theorem 1.4). Then in Sect. 2 we utilize tight indiscernibles to prove some results in general model theory, including a refinement of the Ehrenfeucht–Mostowski theorem (Theorem 2.7). Having thus demonstrated the utility of tight indiscernibles in Sect. 2, we show how to construct them with the help of dimensional ultrapowers in Sect. 3. Section 4 presents a brief historical background of iterated ultrapowers, with pointers to the rich literature of the subject.

Definitions, notations and conventions 1.1 All the structures considered in this paper are first order structures; we follow the convention of using M , M^* , M_0 , etc. to denote (respectively) the universes of discourse of structures \mathcal{M} , \mathcal{M}^* , \mathcal{M}_0 , etc. We assume the Axiom of Choice in the metatheory, and use ω for the set of non-negative integers.

- (a) Given a structure \mathcal{M} , $\mathcal{L}(\mathcal{M})$ is the *language* of \mathcal{M} , and the *cardinality* of \mathcal{M} refers to the cardinality of M .
- (b) \mathcal{M}_1 is an *expansion* of \mathcal{M} if $M_1 = M$, $\mathcal{L}(\mathcal{M}_1) \supseteq \mathcal{L}(\mathcal{M})$, and \mathcal{M}_1 and \mathcal{M} give the same interpretation to every symbol in $\mathcal{L}(\mathcal{M})$. \mathcal{M}^+ is the expansion $(\mathcal{M}, m)_{m \in M}$ of \mathcal{M} (we follow the practice of confusing the constant symbol in $\mathcal{L}(\mathcal{M}^+)$ interpreted by m with m in the interest of a lighter notation). $\mathcal{M}^\#$ is a *full expansion* of \mathcal{M} if $\mathcal{M}^\# = (\mathcal{M}, R)_{R \in \mathcal{F}}$, where $\mathcal{F} = \{R \subseteq M^n : n \in \omega\}$.

¹ In contrast, iterated ultrapowers are well-known to specialists in the 'large cardinals' area of set theory, albeit in a very specialized form, where (a) the ultrafilters at work have extra combinatorial features, and (b) the iteration is always carried out along a well-ordering (as opposed to a linear ordering).

- (c) For a formula $\varphi(x_1, \dots, x_k)$ with free variables implicitly ordered as shown, we write $\varphi^{\mathcal{M}}$ for $\{ \langle m_1, \dots, m_k \rangle \in M^k : \mathcal{M} \models \varphi(m_1, \dots, m_k) \}$.
- (d) Suppose $X \subseteq M^k$, where k is a positive integer. X is \mathcal{M} -definable if $X = \varphi^{\mathcal{M}}$ for some $\mathcal{L}(\mathcal{M})$ -formula. X is *parametrically \mathcal{M} -definable* if $X = \varphi^{\mathcal{M}^+}$ for some $\mathcal{L}(\mathcal{M}^+)$ -formula.
- (e) A *parametrically \mathcal{M} -definable* function is a function $f : M^k \rightarrow M$ (where k is a positive integer) such that the graph of f is parametrically \mathcal{M} -definable. If \mathcal{M}^* is an elementary extension of \mathcal{M} (written “ $\mathcal{M}^* > \mathcal{M}$ ” or “ $\mathcal{M} < \mathcal{M}^*$ ”), then any such f extends naturally to a parametrically \mathcal{M}^* -definable function according to the same definition; we may also denote this extension as f .
- (f) \mathcal{M} has *definable Skolem functions* if for every $\mathcal{L}(\mathcal{M})$ -formula $\varphi(x, y_1, \dots, y_k)$, whose free variable(s) include a distinguished free variable x and whose other free variables (if any) are y_1, \dots, y_k , there is an \mathcal{M} -definable function f such that (abusing notation slightly):

$$\mathcal{M} \models \forall y_1 \dots \forall y_k (\exists x \varphi(x, y_1, \dots, y_k) \rightarrow \varphi(f(y_1, \dots, y_k), y_1, \dots, y_k)).$$

- (g) Given a structure \mathcal{M} with definable Skolem functions and a subset S of M , there is a least elementary substructure \mathcal{M}_S of \mathcal{M} that contains S , whose universe is the set of all applications of \mathcal{M} -definable functions to tuples from S . \mathcal{M}_S is called the *submodel of \mathcal{M} generated by S* .
- (h) Suppose $\mathcal{M}^* > \mathcal{M}$ and $(I, <)$ is a linear order with $I \subseteq M^* \setminus M$ (we often omit explicit mention of the order, $<$). We say that $(I, <)$ *forms a set of order indiscernibles over M* if for any $\mathcal{L}(\mathcal{M})$ -formula $\varphi(x_1, \dots, x_n, y_1, \dots, y_k)$, any sequence m_1, \dots, m_n from M , and any two sequences $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$ from I , we have:

$$(\mathcal{M}^*, m)_{m \in M} \models \varphi(m_1, \dots, m_n, i_1, \dots, i_k) \leftrightarrow \varphi(m_1, \dots, m_n, j_1, \dots, j_k).$$

With the above preliminaries in place, we are ready for a definition that plays a central role in this paper.

Definition 1.2 Suppose \mathcal{M} has definable Skolem functions, $\mathcal{M}^* > \mathcal{M}$ and $(I, <)$ is a linear order with $I \subseteq M^* \setminus M$. $(I, <)$ *forms a set of tight indiscernibles generating \mathcal{M}^* over \mathcal{M}* if the following three properties hold. (Note: For brevity we will say that I *forms a set of tight indiscernibles over \mathcal{M}* , when the order and \mathcal{M}^* are clear from the context.)

- (1) $(I, <)$ is a set of order indiscernibles over \mathcal{M} .
- (2) \mathcal{M}^* is generated by $M \cup I$, i.e., every element of \mathcal{M}^* is of the form $f(i_1, \dots, i_k)$ for some i_1, \dots, i_k from I and some parametrically \mathcal{M} -definable function f .
- (3) For all $i_1 < \dots < i_k < j_1 < \dots < j_k$ from I and every parametrically \mathcal{M} -definable function f , if $f(i_1, \dots, i_k) = f(j_1, \dots, j_k)$ then this common value is in M .

Remark 1.3 Property (3) of tight indiscernibles is not implied by the other two properties: Start with any structure \mathcal{M} with definable Skolem functions that also

has a definable bijection π from M^2 to M . Use the Ehrenfeucht–Mostowski theorem to build a proper chain $\mathcal{M} \prec \mathcal{M}_1 \prec \mathcal{M}_2$ such that for some linear order $(I_0, <)$, $I_0 \subseteq M_2 \setminus M_1$, and $(I_0, <)$ is a set of order indiscernibles in \mathcal{M}_2 over \mathcal{M}_1 . Then fix some $a \in M_1 \setminus M$ and let $I = \{\pi(a, i) : i \in I_0\}$, with the obvious order inherited from I . Finally, let \mathcal{M}^* be the elementary submodel of \mathcal{M}_2 generated by $M \cup I$. In this example properties (1) and (2) of Definition 1.2 are satisfied, but not property (3), since $a \notin M$ and yet $f(i) = a$ for every $i \in I$, where f is the definable function given by $f(x) = y$ iff y is the unique element y such that $\exists z \pi(y, z) = x$.

Theorem 1.4 (Existence of tight indiscernibles) *Every infinite structure \mathcal{M} has an expansion \mathcal{M}_1 with definable Skolem functions such that for any ordered set $(I, <)$, with I disjoint from M_1 , there is $\mathcal{M}^* \succ \mathcal{M}_1$ such that $(I, <)$ forms a set of tight indiscernibles generating \mathcal{M}^* over \mathcal{M}_1 . Moreover, $|\mathcal{L}(\mathcal{M}_1)| = \max\{|\mathcal{L}(\mathcal{M})|, \aleph_0\}$.*

The proof of Theorem 1.4 will be presented in Sect. 3, using the technology of dimensional ultrapowers for all infinite structures \mathcal{M} . However, if both $\mathcal{L}(\mathcal{M})$ and M are countable, then a straightforward proof can be given by fine-tuning the Ehrenfeucht–Mostowski method of building order indiscernibles:

Proof of Theorem 1.4 when $|\mathcal{L}(\mathcal{M})| \leq |M| = \aleph_0$. Let $\mathcal{M}_1 = (\mathcal{M}, \triangleleft)$, where \triangleleft is an ordering of M of order type ω . Note that \mathcal{M}_1 has definable Skolem functions. Let $\langle \varphi_i : i \in \omega \rangle$ be an enumeration $\mathcal{L}(\mathcal{M}_1^+)$ -formulae with at least one free variable, and suppose the set of free variables of φ_i is $\{x_1, \dots, x_{n_i}\}$. Use Ramsey’s partition theorem to construct a decreasing sequence $\langle X_i : i \in \omega \rangle$ of infinite subsets of M such that for all $i \in \omega$ either φ_i or $\neg\varphi_i$ holds for all increasing $a_1 \triangleleft \dots \triangleleft a_{n_i}$ from X_i . Fix an ordered set $(I, <)$ disjoint from M . Let T be the corresponding Ehrenfeucht–Mostowski blueprint, i.e., the result of augmenting the elementary diagram of \mathcal{M}_1 with all sentences $\varphi(i_1, \dots, i_n)$, where $i_1 < \dots < i_n$ are elements of I , φ is allowed to have parameters from M , and for some $k \in \omega$ and all $a_1 \triangleleft \dots \triangleleft a_n$ from X_k , $\varphi(a_1, \dots, a_n)$ holds in \mathcal{M} . T is easily seen to be consistent by compactness considerations. We obtain the desired $\mathcal{M}^* \succ \mathcal{M}_1$ in which I forms a set of tight indiscernibles by starting with any model of T and taking the submodel generated by M and I . Now suppose that in \mathcal{M}^* we have strictly increasing sequences from I , $c_1 < \dots < c_k$ and $d_1 < \dots < d_k$, with $c_k < d_1$ such that:

$$[*] \quad f(c_1, \dots, c_k) = f(d_1, \dots, d_k).$$

We shall show that this (common) value is in M . For some X_i , we have $f(a_1, \dots, a_k) = f(b_1, \dots, b_k)$ for all $a_1 < \dots < a_k < b_1 < \dots < b_k$ from X_i (since if instead the equality were false on X_i , $[*]$ would fail). Let $m \in M$ be the common value of all such $f(a_1, \dots, a_k)$. Then for some X_j , $m = f(a_1, \dots, a_k)$ for all $a_1 \triangleleft \dots \triangleleft a_k$ from X_j , since otherwise this equality is false for all $a_1 \triangleleft \dots \triangleleft a_k$ from some X_j , which is impossible by considering what holds in X_s for $s = \max\{i, j\}$. So $f(c_1, \dots, c_k) = m \in M$. □

2 Tight indiscernibles at work

In this section we show that the existence of tight indiscernibles provides straightforward proofs of several results of general model theory. The following lemma earns us a key strengthening of property (3) of tight indiscernibles.²

Tightness Lemma 2.1 *Suppose that $(I, <)$ is an infinite linear order and I forms a set of tight indiscernibles generating \mathcal{M}^* over \mathcal{M} . Also suppose that f and g are parametrically \mathcal{M} -definable functions and there are disjoint subsets $S_0 = \{i_1, \dots, i_p\}$ and $S_1 = \{j_1, \dots, j_q\}$ of I such that $f(i_1, \dots, i_p) = g(j_1, \dots, j_q)$. Then $f(i_1, \dots, i_p) \in M$.*

Proof We may assume (by tweaking f and g) that $i_1 < \dots < i_p$ and $j_1 < \dots < j_q$. Let the (disjoint) union $S_0 \cup S_1$ be ordered as $k_1 < \dots < k_r$, where $r = p + q$. Since $(I, <)$ is infinite, it contains a strictly increasing or strictly decreasing ω -sequence (by Ramsey’s theorem, or as one of us was told years ago: the order is either well-founded or it’s not). Without loss of generality assume there is a decreasing such sequence, as the other case is entirely analogous. By indiscernibility, we may assume that the elements k_1, \dots, k_r are contained in this sequence; hence we may choose $k'_1 < \dots < k'_r$ such that $k'_r < k_1$.

Our plan is to shift k_1 way to the left, then shift k_2 way to the left (but still to the right of the new k_1), and so on, so that the new k_r is shifted to the left of the original k_1 . So for each natural number $j \leq r$, define the sequence $\langle n_{\{j,i\}} : 1 \leq i < r \rangle$ as follows: $n_{\{j,i\}} = k'_i$ for $i \leq j$, and $n_{\{j,i\}} = k_i$ for $i > j$. In particular, we have

$$\begin{aligned} \langle n_{\{0,i\}} : 1 \leq i \leq r \rangle &= \langle k_i : 1 \leq i \leq r \rangle \text{ and} \\ \langle n_{\{r,i\}} : 1 \leq i \leq r \rangle &= \langle k'_i : 1 \leq i \leq r \rangle. \end{aligned}$$

Let I_0, J_0 be the indices i for which $k_i \in S_0$ or $k_i \in S_1$, respectively. Let $m = f(i_1, \dots, i_p)$. Then indiscernibility and a straightforward induction on j (and a slight but clear abuse of notation) establish the following. (*Hint*: disjointness of S_0 and S_1 guarantee that when moving from j to $j + 1$ for the induction step, either the application of f or the application of g remains unchanged.)

$$m = f(n_{\{j,s\}} : s \in I_0) = g(n_{\{j,s\}} : s \in J_0).$$

In particular, instantiating with $j = 0$ and $j = r$ we see that

$$m = f(n_{\{0,s\}} : s \in I_0) = f(n_{\{r,s\}} : s \in I_0).$$

But $n_{\{r,s_1\}} < n_{\{0,s_2\}}$ for all s_1 and s_2 ; so property (3) of tight indiscernibles immediately implies that $m \in M$. □

² An abstract formalization of this lemma has been verified with the ACL2 theorem prover, cf. [20].

Remark 2.2 After reading our paper, Jim Schmerl observed that Lemma 2.1 holds for all ordered sets I whose size exceeds $p + q$. We leave the proof to the reader. *Hint:* Assuming without loss of generality that $p \leq q$, first observe that by shifting the given arguments of f and g , we see that the value of f is preserved when the only change is to shift just one of the arguments without changing their ordering. Now successively shift all arguments of f to the left with maximum argument u , and then again to the right so that all arguments are beyond u . The definition of tightness now applies.

We are now ready to use tight indiscernibles to establish the following theorems.

Theorem 2.3 (Applications of tight indiscernibles) *Let \mathcal{M} be any infinite structure.*

- (A) *There is a proper elementary extension \mathcal{M}^* of \mathcal{M} such that for some automorphism α of \mathcal{M}^* , α and all its finite iterates α^n fix each element of M and move every element in $M^* \setminus M$. Indeed \mathcal{M}^* can be arranged to be of any cardinality $\kappa \geq \max\{|M|, |\mathcal{L}(\mathcal{M})|\}$.*
- (B) *There is a family of proper elementary extensions $\{\mathcal{M}_i : i \in \omega\}$ of \mathcal{M} whose intersection is M , such that for all i , \mathcal{M}_i is a proper elementary extension of \mathcal{M}_{i+1} .*
- (C) *Let I be any set. Then there is a family $\{\mathcal{M}_S : S \subseteq I\}$ of proper elementary extensions of \mathcal{M} such that $\mathcal{M}_{S_0} \prec \mathcal{M}_{S_1}$ whenever $S_0 \subseteq S_1 \subseteq I$ with the additional property that $M_{S_0} \cap M_{S_1} = M$ for all disjoint S_0, S_1 .³*

Proof Observe that by taking reducts back to $\mathcal{L}(\mathcal{M})$ it suffices to demonstrate Theorem 2.3 for some expansion of \mathcal{M} . To this end, we may assume without loss of generality that \mathcal{M} is replaced by the expansion \mathcal{M}_1 of Theorem 1.4. We begin each proof by applying Theorem 1.4 to obtain an elementary extension \mathcal{M}^* of \mathcal{M} with a set of tight indiscernibles generating \mathcal{M}^* over \mathcal{M} . We now apply the Tightness Lemma to obtain each result in turn, as follows.

Proof of (A) We first prove the theorem when $\kappa = \max\{|M|, |\mathcal{L}(\mathcal{M})|\}$ and then we will explain how to handle larger values of κ by a minor variation. Let $(I, <)$ be the ordered set \mathbb{Z} of integers. Without loss of generality assume that \mathcal{M} and I are disjoint (else proceed below using an isomorphic copy of I). We define an automorphism α of \mathcal{M}^* as follows: given $x \in M^*$, by property (2) of tight indiscernibles we may write $x = f(i_1, \dots, i_n)$ for some $i_1 < \dots < i_n$ from I and some parametrically \mathcal{M} -definable function f . Then we define $\alpha(x) = f(i_1 + 1, \dots, i_n + 1)$. Then α is well-defined and is an automorphism of \mathcal{M}^* , by properties (1) and (2) of tight indiscernibles. Moreover, α fixes each $m \in M$ since $m = f_m(i_1)$, where $i_1 \in I$ and f_m is the constant function with range $\{m\}$ and therefore $\alpha(m) = f_m(i_1 + 1) = m$. Finally suppose $x = \alpha(x)$ for x as above; we show $x \in M$. Let α^k be the k -fold composition of α . A trivial induction shows that for all positive integers k , α^k is an automorphism of \mathcal{M}^* and $\alpha^k(x) = f(i_1 + k, \dots, i_n + k)$. Taking $k = i_n - i_1 + 1$, we have $f(i_1, \dots, i_n) = f(i_1 + k, \dots, i_n + k)$, where $i_n < i_1 + k$; so by property (3) of tight indiscernibles, $x \in M$. It should be clear that every finite iterate of α

³ The proof of (C) shows that we may conclude a bit more: for each $S \subseteq I$, S forms a set of tight indiscernibles in \mathcal{M}_S . However, we find the theorem to be of interest even without that embellishment.

pointwise fixes M and moves every element of $M^* \setminus M$. Finally, to handle the case when $\kappa > \{|M|, |\mathcal{L}(\mathcal{M})|\}$, let $(I, <)$ be the lexicographically ordered set $\kappa \times \mathbb{Z}$, where κ carries its natural order, and take advantage of the automorphism $(\gamma, i) \mapsto (\gamma, i + 1)$ of $\kappa \times \mathbb{Z}$. \square

Proof of (B) Let $(I, <)$ be the ordered set of non-negative integers, and for each i let \mathcal{M}_i be the submodel of \mathcal{M}^* generated by $M \cup \{n \in I : n \geq i\}$. (Thus, $\mathcal{M}_0 = \mathcal{M}^*$.) Property (3) clearly implies that $i \notin M_{i+1}$; hence \mathcal{M}_{i+1} is a proper subset of \mathcal{M}_i . It remains to show that every element of the intersection of the \mathcal{M}_i is an element of M . So consider an arbitrary element $m = f(i_1, \dots, i_k)$ of that intersection, where $i_1 < \dots < i_k$ from I and f is an \mathcal{M} -definable function. Let $p = i_k + 1$. Since $m \in M_p$, we may choose $j_0 < \dots < j_n$ with $p \leq j_0$ such that $m = g(j_0, \dots, j_n)$, for some \mathcal{M} -definable function g . Since $i_k < j_0$, we have $m \in M$ by the Tightness Lemma. \square

Proof of (C) Without loss of generality, assume that I is ordered without endpoints, as we can always restrict to the model generated by the original set I (which can thus even be supplied with a given order). For each $S \subseteq I$, let \mathcal{M}_S be the submodel of \mathcal{M}^* generated by S . So $\mathcal{M}_{S_0} < \mathcal{M}_{S_1}$ whenever $S_0 \subseteq S_1 \subseteq I$. Now suppose that S_0 and S_1 are disjoint subsets of I , and suppose $m \in M_{S_0} \cap M_{S_1}$; it remains only to prove that $m \in M$. But since S_0 and S_1 are disjoint and generate \mathcal{M}_{S_0} and \mathcal{M}_{S_1} respectively, this is immediate by the Tightness Lemma. \square

Remark 2.4 As pointed out to us by Jim Schmerl, Theorem 2.3(A) can be derived from a powerful result due to Duby [5] who proved that every structure \mathcal{M} has an elementary extension that carries an automorphism α such that α and all its finite iterates α^n are ‘maximal automorphisms’, i.e., they move every nonalgebraic element of \mathcal{M} (an element m_0 of structure \mathcal{M} is algebraic in \mathcal{M} iff there is some unary $\mathcal{L}(\mathcal{M})$ -formula $\varphi(x)$ such that $\varphi^{\mathcal{M}}$ is finite and contains m_0).⁴ Also, Theorem 2.3(B) appears as Exercise 3.3.9 of Chang and Keisler’s textbook [4], albeit a rather challenging one.⁵

Corollary 2.5 *If T is a theory that has an infinite algebraic model (i.e., an infinite model in which every element is algebraic), then T has a nonalgebraic model of every cardinality $\kappa \geq \max\{\aleph_0, |\mathcal{L}(\mathcal{M})|\}$ that carries an automorphism α such that α and all its finite iterates α^n are maximal automorphisms.*

Proof Let \mathcal{M} be an infinite algebraic model of T , and then use Theorem 2.3(A). \square

⁴ Duby’s result is a generalization of a key result of Körner [24], who proved that if both $\mathcal{L}(\mathcal{M})$ and M are countable, then \mathcal{M} has a countable elementary extension that carries a maximal automorphism. Körner’s result, in turn, was inspired by and generalizes a theorem due to the joint work of Kaye, Kossak, and Kotlarski [21] that states that every countable model of PA has a countable elementary extension that carries a maximal automorphism (indeed it is shown in [21] that countable recursively saturated models of PA with maximal automorphisms are precisely the countable arithmetically saturated ones).

⁵ There are two reasons we included Theorem 2.3(B): (1) We know of at least two competent logicians (a set theorist and a model theorist) who were stumped by this exercise after assigning it to their students in a graduate model theory course since they assumed when assigning the exercise that the result follows easily from the usual formulation of the Ehrenfeucht–Mostowski theorem on the existence of order indiscernibles; (2) John Baldwin has informed us that a special case of Theorem 2.3(B) is contained in a result of Shelah that appears as [1, Lemma 7.5].

Remark 2.6 Every consistent theory T with definable Skolem functions has an algebraic model since if \mathcal{M}_1 is a model of T with definable Skolem functions, then by Tarski’s test for elementarity the submodel \mathcal{M} of \mathcal{M}_1 whose universe consists of the pointwise definable elements is an elementary submodel of \mathcal{M}_1 , and thus is an algebraic model of T . Corollary 2.5 can also be derived from Duby’s theorem mentioned in Remark 2.4.

Let $\text{Aut}(\mathcal{S})$ be the automorphism group of the structure \mathcal{S} . The usual proof of the Ehrenfeucht–Mostowski theorem on the existence of order indiscernibles makes it clear that given an infinite structure \mathcal{M} and a linear order $(I, <)$ there is an elementary extension \mathcal{M}^* of \mathcal{M} and a group embedding $\alpha \mapsto \widehat{\alpha}$ from $\text{Aut}(I, <)$ into $\text{Aut}(\mathcal{M}^*)$ such that $\widehat{\alpha}$ fixes every element of M . The following theorem refines this embedding result.

Theorem 2.7 *Suppose \mathcal{M} is an infinite structure. Given any linear order $(I, <)$ there is a proper elementary extension \mathcal{M}^* of \mathcal{M} and a group embedding $\alpha \mapsto \widehat{\alpha}$ from $\text{Aut}(I, <)$ into $\text{Aut}(\mathcal{M}^*)$ such that $\widehat{\alpha}$ moves every element of $M^* \setminus M$ and fixes every element of M whenever α is fixed point free.*

Proof The proof is an elaboration of the proof of Theorem 2.3(A). Use Theorem 1.4 to obtain $\mathcal{M}^* \succ \mathcal{M}$ such that $(I, <)$ is a set of tight indiscernibles generating \mathcal{M}^* over \mathcal{M} . Given $\alpha \in \text{Aut}(I, <)$ let $\widehat{\alpha} \in \text{Aut}(\mathcal{M}^*)$ be given by:

$$\widehat{\alpha}(f(i_1, \dots, i_k)) = f(\alpha(i_1), \dots, \alpha(i_k)).$$

The key observation is that for any fixed point free automorphism α of $(I, <)$ and any finite $i_1, \dots, i_k \in I$, there is some $m \in \omega$ for which α^m has the property:

$$\{\alpha^m(i_1), \dots, \alpha^m(i_k)\} \cap \{i_1, \dots, i_k\} = \emptyset.$$

To see this, it suffices to note that if α is fixed point free, then given i and j in I , if $\alpha^{m_0}(i) = j$ for some m_0 , then $\alpha^m(i) \neq j$ for all $m > m_0$ (because for any $i \in I$ either $\alpha^m(i) < \alpha^n(i)$ whenever $m < n \in \omega$; or $\alpha^m(i) > \alpha^n(i)$ whenever $m < n \in \omega$).

If $\widehat{\alpha}(f(i_1, \dots, i_k)) = f(i_1, \dots, i_k)$ for some f and some i_1, \dots, i_n , then by indiscernibility $f(\alpha^n(i_1), \dots, \alpha^n(i_k)) = f(i_1, \dots, i_k)$ for all $n \in \omega$. The above observation now allows us to apply the Tightness Lemma to conclude $f(i_1, \dots, i_k) \in M$.⁶ □

3 Dimensional Skolem ultrapowers

In this section we construct *dimensional (Skolem) ultrapowers*, typically called *iterated ultrapowers* in the literature, and show how they give rise to tight indiscernibles.

⁶ The key observation used in the proof is a special case of P. M. Neumann’s Separation Lemma [22], which states that if G is a group of permutations of an infinite set A with the property that for each $a \in A$ the G -orbit $\{\alpha(a) : \alpha \in G\}$ of each $a \in A$ is infinite, then for all finite subsets X and Y of A , there is some $\alpha \in G$ such that $\alpha(X) \cap Y = \emptyset$. Note that if α is a fixed point free automorphism of a linear order I , and $G = \{\alpha^m : m \in \mathbb{Z}\}$, then the G -orbit of each $i \in I$ is infinite.

Iterated ultrapowers are typically built by means of a direct limit construction; however, we take a different route here that is more in tune with the construction of ordinary ultrapowers. A high-level description of dimensional ultrapowers will be given in Sect. 3.1 to give a context for the nitty-gritty details of the remaining sections. Finite dimensional ultrafilters are treated in Sect. 3.2. Then in Sect. 3.3 we focus on two-dimensional ultrapowers; and in Sect. 3.4 we build I -dimensional ultrapowers for any linear order I and give a proof of Theorem 1.4.

3.1 Motivation and overview

Fix a structure \mathcal{M} with definable Skolem functions; we will consider non-principal ultrafilters \mathcal{U} on the Boolean algebra of parametrically \mathcal{M} -definable subsets of M . The *Skolem ultrapower* of \mathcal{M} modulo \mathcal{U} , here denoted $\text{Ult}(\mathcal{M}, \mathcal{U})$, will be familiar to those who work with models of Peano arithmetic: it is the restriction of the usual ultrapower to equivalence classes of parametrically \mathcal{M} -definable functions from M to M . As for ordinary ultrapowers, it is easy to prove a Łoś theorem for Skolem ultrapowers: for a function f from M to M , a first-order property holds of the equivalence class $[f]$ of a parametrically \mathcal{M} -definable function in the Skolem ultrapower $\text{Ult}(\mathcal{M}, \mathcal{U})$ if and only if it holds of $f(i)$ in \mathcal{M} for almost all i —that is, it holds on a set in the ultrafilter (the definability of Skolem functions is invoked in the existential case of the inductive proof of the Łoś theorem to ensure that the Skolem ultrapower includes the equivalence class of the appropriate witnessing function). Clearly $\text{Ult}(\mathcal{M}, \mathcal{U})$ is generated over M by the equivalence class of the identity function, $[id]$, since for every definable unary function f , we have $\text{Ult}(\mathcal{M}, \mathcal{U}) \models [f] = f([id])$ (for example, by applying the above Łoś theorem for Skolem ultrapowers). Note that if \mathcal{U} is an ultrafilter over $\mathcal{P}(M)$, then the ordinary (full) ultrapower of \mathcal{M} modulo \mathcal{U} coincides with the $\mathcal{L}(\mathcal{M})$ -reduct of the Skolem ultrapower of $\mathcal{M}^\#$ (the full expansion of \mathcal{M}) modulo \mathcal{U} .

We will extend the Skolem ultrapower construction so that instead of a single generator, we obtain an elementary extension \mathcal{M}^* of \mathcal{M} generated by a set of order indiscernibles $(I, <)$, where I is disjoint from M . In the degenerate case, where I has a single element i , this *dimensional Skolem ultrapower* is isomorphic to the usual Skolem ultrapower, where $[i]$ corresponds to $[id]$. Moreover, just as $[id]$ forms a single generator of the Skolem ultrapower over M , the set of all $[i]$ (for $i \in I$) forms a set of generators for the dimensional Skolem ultrapower over M . We can identify $[i]$ with i (really, just renaming). Then we will show that I is an ordered set of tight indiscernibles. Thus, we will define a notion of *almost all* for finite sequences from M , where the single generator, $[id]$, is replaced by the generators $[i]$ for i in I , so that the following Łoś theorem holds: for $i_1 < \dots < i_k$ from I , a first-order property holds of $\langle [i_1], \dots, [i_k] \rangle$ in \mathcal{M}^* if and only if for almost all sequences $m_1 < \dots < m_k$ from M^k , the property holds in \mathcal{M} . It follows immediately (again, identifying $[i]$ with i , for each $i \in I$) that I is a set of order indiscernibles in the dimensional Skolem ultrapower.

3.2 Finite dimensional ultrafilters

Suppose \mathcal{U} provides a notion of ‘almost all’ as an ultrafilter on the parametrically \mathcal{M} -definable subsets of M , and n is a positive integer. We wish to introduce a notion of ‘almost all’, \mathcal{U}^n , as an ultrafilter on the parametrically \mathcal{M} -definable subsets of M^n . Before treating the general case, we warm-up with the important case of $n = 2$. Thus, let I be the two-element order $\{0, 1\}$ with $0 < 1$. Then a parametrically \mathcal{M} -definable set $X \subseteq M^2$ belongs to \mathcal{U}^2 precisely when for almost all $m_1 \in M$, it is the case that for almost all $m_2 \in M$ the pair $\langle m_1, m_2 \rangle$ is in X .⁷ To make that precise, it is convenient to introduce notation for a *section* of X , so let:

$$X|m_1 = \{m_2 \in M : \langle m_1, m_2 \rangle \in X\}.$$

Then we define \mathcal{U}^2 as consisting of parametrically \mathcal{M} -definable subsets of M^2 such that:

$$\{m_1 \in M : X|m_1 \in \mathcal{U}\} \in \mathcal{U}.$$

Of course, $X|m_1$ is a reasonable candidate for membership in \mathcal{U} since $X|m_1$ is parametrically \mathcal{M} -definable, by parametric \mathcal{M} -definability of X . But for the definition above to yield an ultrafilter, we also need $\{m_1 \in M : X|m_1 \in \mathcal{U}\}$ to be parametrically \mathcal{M} -definable; else neither that set nor its complement would be in \mathcal{U} . We will need that definition to be suitably uniform in m_1 in our formation of $\mathcal{U}^3, \mathcal{U}^4$, and so on. We capture this requirement in the following definition.

Definition 3.1 Let \mathcal{M} be a structure and \mathcal{U} be a non-principal ultrafilter over the parametrically \mathcal{M} -definable subsets of M . \mathcal{U} is *\mathcal{M} -amenable* if for every first-order $\mathcal{L}(\mathcal{M})$ -formula $\varphi(x, y_1, \dots, y_k)$, there is a corresponding $\mathcal{L}(\mathcal{M})$ -formula $U_\varphi(y_1, \dots, y_k)$ such that for all $m_1, \dots, m_k \in M$:

$$\varphi_1^{\mathcal{M}} \in \mathcal{U} \text{ iff } \mathcal{M} \models U_\varphi(m_1, \dots, m_k),$$

where $\varphi_1(x)$ denotes $\varphi(x, m_1, \dots, m_k)$. \mathcal{M} is *amenable* if \mathcal{M} has definable Skolem functions and there is an \mathcal{M} -amenable ultrafilter.

Remark 3.2 Note that if \mathcal{U} is a nonprincipal ultrafilter over $\mathcal{P}(M)$, and $\mathcal{M}^\#$ is a full expansion of \mathcal{M} , then \mathcal{U} is $\mathcal{M}^\#$ -amenable.

Theorem 3.3 (Amenable expansions) *Every infinite structure \mathcal{M} has an amenable expansion \mathcal{M}' with $|\mathcal{L}(\mathcal{M}')| = \max\{|\mathcal{L}(\mathcal{M})|, \aleph_0\}$. Indeed given any nonprincipal ultrafilter $\mathcal{U} \subseteq \mathcal{P}(M)$ there is an expansion \mathcal{M}' of \mathcal{M} with definable Skolem functions such that $|\mathcal{L}(\mathcal{M}')| = \max\{|\mathcal{L}(\mathcal{M})|, \aleph_0\}$, and some $\mathcal{U}' \subseteq \mathcal{U}$ such that \mathcal{U}' is an \mathcal{M}' -amenable ultrafilter.*

⁷ Even though a definable pairing function is available in the archetypical case of models of arithmetic, our development here does not require that \mathcal{M} has a definable pairing function.

Proof Consider the two-sorted structure $\overline{\mathcal{M}} = (\mathcal{M}, \mathcal{P}(M), \pi, \in, \mathcal{U})$, where π is a pairing function on M , i.e., a bijection between M and M^2 ; \in is the membership relation between elements of M and elements of $\mathcal{P}(M)$, and \mathcal{U} is construed as a unary predicate on $\mathcal{P}(M)$. Note that the presence of π assures us that for each positive $k \in \omega$ the structure $\overline{\mathcal{M}}$ carries a definable bijection between M and M^k . Since we are assuming that ZFC holds in our metatheory, $\overline{\mathcal{M}}$ satisfies the schemes $\Sigma = \{\sigma_k : k \in \omega\}$ and $A = \{\alpha_k : k \in \omega\}$, where σ_k is the $\mathcal{L}(\overline{\mathcal{M}})$ -sentence expressing:

$$\forall R \subseteq M^{k+1} \exists f : M^k \rightarrow M \\ \forall y_1 \in M \dots \forall y_k \in M (\exists x R(x, y_1, \dots, y_k) \rightarrow R(f(y_1, \dots, y_k), y_1, \dots, y_k)),$$

and α_k is the $\mathcal{L}(\overline{\mathcal{M}})$ -sentence expressing:

$$\forall R \subseteq M^{k+1} \exists U \subseteq M^k \forall y_1 \in M \dots \forall y_k \in M \\ \mathcal{U}(\{x : R(x, y_1, \dots, y_k)\}) \leftrightarrow \langle y_1, \dots, y_k \rangle \in U.$$

By the Löwenheim–Skolem theorem there is some $\mathcal{P}_0(M) \subseteq \mathcal{P}(M)$ of cardinality $\max\{|\mathcal{L}(\overline{\mathcal{M}})|, \aleph_0\}$ such that:

$$\overbrace{\left(\mathcal{M}, \mathcal{P}_0(M), \pi, \in, \underbrace{\mathcal{U} \cap \mathcal{P}_0(M)}_{\mathcal{U}_0} \right)}^{\overline{\mathcal{M}}_0} \prec \overline{\mathcal{M}}.$$

In particular $\overline{\mathcal{M}}_0$ satisfies both schemes Σ and A , and therefore the \mathcal{M} -expansion $\mathcal{M}' = (\mathcal{M}, \pi, X)_{X \in \mathcal{P}_0(M)}$ has definable Skolem functions, and \mathcal{U}_0 is \mathcal{M}' -amenable. □

Exercise 3.4 If \mathcal{U} is an \mathcal{M} -amenable ultrafilter, then \mathcal{U}^2 is an ultrafilter on the parametrically \mathcal{M} -definable subsets of M^2 .

Remark 3.5 Every model of Peano arithmetic is amenable; this fact is implicit in the usual proof of the MacDowell–Specker theorem [23, Theorem 2.2.8]. Another theory with this property is $ZF + V = OD$, i.e., Zermelo–Fraenkel set theory plus the axiom expressing that every set is definable from some ordinal. This follows from three well-known facts: (1) models of ZF with parametrically definable Skolem functions are precisely those in which $ZF + V = OD$ holds, (2) the (global) axiom of choice is provable in $ZF + V = OD$, and (3) within ZFC there is a nonprincipal ultrafilter on any prescribed infinite set. We elaborate (3): Let $\mathcal{M} = (M, E) \models ZFC$, where $E = \in^{\mathcal{M}}$. Fix an infinite $s \in M$. Then there is some $u \in M$ such that $\mathcal{M} \models$ “ u is a nonprincipal ultrafilter on the power set of s ”. The desired \mathcal{M} -amenable ultrafilter \mathcal{U} is the collection of parametrically \mathcal{M} -definable subsets X of M such that $s_E \cap X = a_E$ for some $a \in u_E$, where $x_E := \{y \in M : yEx\}$.

Our next task is to define \mathcal{U}^n for arbitrary positive integers n . The lemma following this definition shows that amenability extends appropriately to powers $n > 2$.

Definition 3.6 For an \mathcal{M} -amenable ultrafilter \mathcal{U} , we recursively define \mathcal{U}^n for positive integers n : \mathcal{U}^1 is \mathcal{U} , and

$$\mathcal{U}^{n+1} = \left\{ X \subseteq M^{n+1} : X \text{ is parametrically } \mathcal{M}\text{-definable and} \right. \\ \left. \{m \in M : X|m \in \mathcal{U}^n\} \in \mathcal{U} \right\}.$$

Lemma 3.7 (Extended amenability) *Let \mathcal{U} be an \mathcal{M} -amenable ultrafilter. Then for every first-order $\mathcal{L}(\mathcal{M})$ -formula $\varphi(x_1, \dots, x_n, y_1, \dots, y_k)$, there is a corresponding $\mathcal{L}(\mathcal{M})$ -formula $U_\varphi(y_1, \dots, y_k)$ such that for all $m_1, \dots, m_k \in M$:*

$$\varphi_n^{\mathcal{M}} \in \mathcal{U}^n \text{ iff } \mathcal{M} \models U_\varphi(m_1, \dots, m_k),$$

where $\varphi_n(x_1, \dots, x_n)$ denotes $\varphi(x_1, \dots, x_n, m_1, \dots, m_k)$.

Proof By induction on n . The case $n = 1$ is just the definition of \mathcal{M} -amenable. Suppose the lemma holds for n , and consider the formula $\varphi(x_0, x_1, \dots, x_n, y_1, \dots, y_k)$. By the inductive hypothesis, there is a formula $U_\varphi(x_0, y_1, \dots, y_k)$ such that for all $m, m_1, \dots, m_k \in M$, we have:

$$\varphi_m^{\mathcal{M}} \in \mathcal{U}^n \text{ iff } \mathcal{M} \models U_\varphi(m, m_1, \dots, m_k),$$

where $\varphi_m = \varphi(m, x_1, \dots, x_n, m_1, \dots, m_k)$. By amenability, there is a formula $V_\varphi(y_1, \dots, y_k)$ such that for all $m_1, \dots, m_k \in M$ and for $\varphi_0(x_0) = U_\varphi(x_0, m_1, \dots, m_k)$:

$$\varphi_0^{\mathcal{M}} \in \mathcal{U} \text{ iff } \mathcal{M} \models V_\varphi(m_1, \dots, m_k).$$

Then setting $\varphi^+(x_0, x_1, \dots, x_n) = \varphi(x_0, x_1, \dots, x_n, m_1, \dots, m_k)$, we have the following equivalences:

$$\begin{aligned} &(\varphi^+)^{\mathcal{M}} \in \mathcal{U}^{n+1} \\ &\quad \Updownarrow \text{ (by the definition of } \mathcal{U}^{n+1}, \varphi^+, \text{ and } \varphi_m) \\ &\quad \left\{ m \in M : \varphi_m^{\mathcal{M}} \in \mathcal{U}^n \right\} \in \mathcal{U} \\ &\quad \Updownarrow \text{ (by the above choice of } U_\varphi) \\ &\quad \left\{ m \in M : \mathcal{M} \models U_\varphi(m, m_1, \dots, m_k) \right\} \in \mathcal{U} \\ &\quad \Updownarrow \text{ (by the above choice of } V_\varphi) \\ &\quad \mathcal{M} \models V_\varphi(m_1, \dots, m_k). \end{aligned}$$

□

Exercise 3.8 (Finite dimensions provide ultrafilters) Let \mathcal{U} be an \mathcal{M} -amenable ultrafilter on the parametrically \mathcal{M} -definable subsets of M . Then for all positive integers n , \mathcal{U}^n is an ultrafilter on the parametrically \mathcal{M} -definable subsets of M^n .

3.3 Warm-up: two-dimensional Skolem ultrapowers

- We assume throughout the rest of this section that \mathcal{M} has definable Skolem functions, and \mathcal{U} is an \mathcal{M} -amenable ultrafilter.

Now that we have defined \mathcal{U}^n for \mathcal{M} -amenable ultrafilters \mathcal{U} , we want to use them to construct the desired extension of \mathcal{M} by tight indiscernibles $(I, <)$. Let us begin by taking a close look at the simple but important case of $n = 2$. Consider the ordered set $I = 2 = \{0, 1\}$, where $0 < 1$. To form the 2-dimensional Skolem ultrapower $\mathcal{M}^* = \text{Ult}(\mathcal{M}, \mathcal{U}, 2)$, instead of taking equivalence classes of *unary* functions as we would when constructing an ordinary Skolem ultrapower, we take equivalence classes of parametrically \mathcal{M} -definable *binary* functions, where the equivalence relation at work is:

$$f(x, y) \sim g(x, y) \text{ iff } \{\langle x, y \rangle : f(x, y) = g(x, y)\} \in \mathcal{U}^2.$$

We write $[f(0, 1)]$ to denote the equivalence class of f , where here 0 and 1 refer to elements of the order, $\{0, 1\}$. The universe of the desired dimensional ultrapower \mathcal{M}^* is the set of all such $[f(0, 1)]$. Two distinguished examples of such objects are the case of $f = id_0$ and $f = id_1$ where $id_0(x, y) = x$ and $id_1(x, y) = y$. For convenience, we may write $[0]$ and $[1]$ for these respective equivalence classes. One can check that $[f(0, 1)] = f^{\mathcal{M}^*}([0], [1])$; hence, the set $\{[0], [1]\}$ is a set of generators for \mathcal{M}^* . But before we can do that, we need to define functions and relations on \mathcal{M}^* , i.e., on this set of equivalence classes. For each function symbol g and arguments $[f_i(0, 1)]$ of \mathcal{M}^* whose length is the arity of g , interpret g in \mathcal{M}^* on these arguments by applying g pointwise in \mathcal{M} :

$$g([f_0(0, 1)], \dots, [f_{k-1}(0, 1)]) = [h(0, 1)],$$

where $h(x, y) = g(f_0(x, y), \dots, f_{k-1}(x, y))$. This is well-defined: h is parametrically \mathcal{M} -definable, and if $[f_i(0, 1)] = [f'_i(0, 1)]$ for $i < k$, then $f_i(x, y) = f'_i(x, y)$ on a set in the ultrafilter \mathcal{U}^2 for each such i , hence for all i in some $X \in \mathcal{U}^2$; and on this set X of pairs, $h(x, y)$ is unchanged if we replace each f_i by f'_i in the definition of h .

Exercise 3.9 (a) For the case of \mathcal{U}^2 above, complete the definition of \mathcal{M}^* (by interpreting relation symbols) and then prove the Łoś theorem. (b) Use part (a) to show that $\{[0], [1]\}$ forms a set of order indiscernibles in \mathcal{M}^* over \mathcal{M} .

3.4 Dimensional Skolem ultrapowers

Assume $(I, <)$ is an ordered set disjoint from M , and \mathcal{U} is an \mathcal{M} -amenable ultrafilter. In order to define $\mathcal{M}^* = \text{Ult}(\mathcal{M}, \mathcal{U}, I)$ (the I -dimensional Skolem ultrapower of \mathcal{M} with respect to \mathcal{U}) we extend the notion of dimensional ultrapower from $I = \{0, 1\}$ to arbitrary ordered sets I . For all non-empty finite subsets I_0 of I , we call the function space $I_0 \rightarrow M$, also denoted M^{I_0} , the *set of I_0 -sequences (from M)*. Given a finite

subset I_0 of I , where $\langle i_0, \dots, i_k \rangle$ enumerates I_0 in increasing order, we can take advantage of the order isomorphism j from $\{0, \dots, k\}$ to I_0 to form a ‘copy’ \mathcal{U}^{I_0} of $\mathcal{U}^{|I_0|}$ on the set of I_0 -sequences, where as usual $||$ denotes cardinality. More specifically, for $X \subseteq M^{I_0}$, let $X[j] \subseteq M^{k+1}$ be $\{s \circ j : s \in X\}$, and then define:

$$\mathcal{U}^{I_0} = \left\{ X \subseteq M^{I_0} : X[j] \in \mathcal{U}^{k+1} \right\}.$$

For finite $I_0 \subseteq I$ and function f with arity $|I_0|$, we write $f(I_0)$ as an abbreviation for the *syntactic entity*

$$f(i_0, \dots, i_k),$$

where f is a $k + 1$ -ary parametrically \mathcal{M} -definable function and $\langle i_0, \dots, i_k \rangle$ enumerates I_0 in increasing order; we call this a *generalized term* (over I_0). Moreover, if u is an I_1 -sequence for some I_1 containing I_0 , we write $f(i_0, \dots, i_k)[u]$ to denote the value $f^{\mathcal{M}}(u(i_0), \dots, u(i_k))$, where $f^{\mathcal{M}}$ is the interpretation of f in \mathcal{M} . Given finite $I_0, I_1, I_2 \subseteq I$, with $I_2 = I_0 \cup I_1$, and generalized terms $f(I_0)$ and $g(I_1)$, where f and g are allowed to have different arities, we say that $f(I_0)$ and $g(I_1)$ are \mathcal{U}^* -equivalent if

$$\left\{ u \in M^{I_2} : f(I_0)[u] = g(I_1)[u] \right\} \in \mathcal{U}^{I_2}.$$

The universe M^* of the I -dimensional ultrapower \mathcal{M}^* with respect to \mathcal{U} is the set of all \mathcal{U}^* -equivalence classes $[f(I_0)]$, where $I_0 \subseteq I, |I_0| = n \in \omega$, and f is an n -ary parametrically \mathcal{M} -definable function. Note that M naturally can be injected into M^* by:

$$\varepsilon : M \rightarrow M^*,$$

where $\varepsilon(m)$ is defined as $[f_m(I_0)]$, f_m is the constant function whose range is $\{m\}$, and I_0 is any finite subset of I . Functions and relations interpreted in \mathcal{M} extend naturally to \mathcal{M}^* : for each $n + 1$ -ary function symbol g of $\mathcal{L}(\mathcal{M})$, by defining

$$g^{\mathcal{M}^*}([f_0(I_0)], [f_1(I_1)], \dots, [f_n(I_n)]) = [h(I_{n+1})],$$

where I_0, \dots, I_n are allowed to overlap, $I_{n+1} = \bigcup_{0 \leq j \leq n} I_j$, and h is a parametrically \mathcal{M} -definable function such that for all $u \in M^{I_{n+1}}$,

$$h(I_{n+1})[u] = g^{\mathcal{M}}(f_0(I_0)[u], \dots, f_n(I_n)[u]).$$

Exercise 3.10 Define the interpretation of a relation symbol in \mathcal{M}^* in analogy to how function symbols are interpreted (as above). Then verify that ε is an isomorphism between \mathcal{M} and $\varepsilon(\mathcal{M})$, where $\varepsilon(\mathcal{M})$ is the submodel of \mathcal{M}^* whose universe is $\varepsilon(M)$.

The isomorphism of \mathcal{M} with $\varepsilon(\mathcal{M})$ allows us—as is commonly done in model theory—to replace \mathcal{M}^* with an isomorphic copy so as to arrange \mathcal{M} to be *submodel*

of \mathcal{M}^* . Therefore we may identify any element of \mathcal{M}^* of the form $[f_m(I_0)]$ with m itself. With \mathcal{M}^* defined, many of its properties now fall naturally into place, as explained below.

It is useful to extend the notation $X|x$ to $X|s$ for an ordered sequence s . Assume that $I_0, I_1 \subseteq I$ such that $\max(I_0) < \min(I_1)$; then let $I_2 = I_0 \cup I_1$, and assume that $X \subseteq M^{I_2}$ and $s \in M^{I_0}$. We want to define a subset $X|s$ of M^{I_1} to be the result of collecting, for each sequence in X that starts with s , its restriction to I_1 . The recursive definition of $X|s$ is as follows, where $\langle \rangle$ is the empty sequence. $X|\langle \rangle$ is X . Now suppose I_0 is ordered as $i_0 < \dots < i_k$ and let $I'_0 = \{i_1, \dots, i_k\}$. Then $X|I_0$ is $(X|i_0)|I'_0$. The following exercise generalizes the recursive definition of \mathcal{U}^n by stripping off initial subsequences rather than merely single initial elements.

Exercise 3.11 Let I_0 be a finite subset of I , and suppose $I_0 = I_1 \cup I_2$, where $\max(I_1) < \min(I_2)$. Then:

$$X \in \mathcal{U}^{I_0} \text{ iff } \left\{ s \in M^{I_1} : X|s \in \mathcal{U}^{I_2} \right\} \in \mathcal{U}^{I_1}.$$

We next establish a useful lemma.

Lemma 3.12 Suppose I_0 is a finite subset of I , and J is a finite subset of I with $I_0 \subseteq J$. If $X \subseteq M^{I_0}$ and

$$X' = \left\{ s \cup t \in M^J : s \in X \wedge t \in M^{J \setminus I_0} \right\},$$

then

$$X \in \mathcal{U}^{I_0} \text{ iff } X' \in \mathcal{U}^J.$$

Proof We prove the Lemma when $|J \setminus I_0| = 1$. The Lemma in full generality then follows by induction on $|J \setminus I_0|$. So, suppose $J \setminus I_0 = \{i\}$. Let $X \subseteq M^{I_0}$, and let

$$X' = \{s \cup t \in M^J : s \in X \wedge t \in M^{J \setminus I_0}\}.$$

There are three cases to consider:

Case $i < \min I_0$: By Exercise 3.11,

$$X' \in \mathcal{U}^J \text{ if and only if } \{s \in M^{\{i\}} : X'|s \in \mathcal{U}^{I_0}\} \in \mathcal{U}^{\{i\}}.$$

Now, for all $s \in M^{\{i\}}$, $X'|s = X$, so

$$\{s \in M^{\{i\}} : X'|s \in \mathcal{U}^{I_0}\} = \begin{cases} M^{\{i\}} & \text{if } X \in \mathcal{U}^{I_0} \\ \emptyset & \text{otherwise} \end{cases}$$

Therefore, $X' \in \mathcal{U}^J$ if and only if $X \in \mathcal{U}^{I_0}$.

Case $\max I_0 < i$: By Exercise 3.11,

$$X' \in \mathcal{U}^J \text{ if and only if } \{s \in M^{I_0} : X'|_s \in \mathcal{U}^{(i)}\} \in \mathcal{U}^{I_0}.$$

Now, for all $s \in M^{I_0}$,

$$X'|_s = M^{(i)} \text{ if and only if } s \in X.$$

Therefore, $X' \in \mathcal{U}^J$ if and only if $X \in \mathcal{U}^{I_0}$.

Case $(\exists i', i'' \in I_0)(i' < i < i'')$: Let $I_0 = I_1 \cup I_2$ with $\max I_1 < i < \min I_2$. By Exercise 3.11,

$$X' \in \mathcal{U}^J \text{ if and only if } \{s \in M^{I_1} : X'|_s \in \mathcal{U}^{(i) \cup I_2}\} \in \mathcal{U}^{I_1}.$$

By the first case that we considered, for all $s \in M^{I_1}$,

$$X'|_s \in \mathcal{U}^{(i) \cup I_2} \text{ if and only if } X|_s \in \mathcal{U}^{I_2}.$$

Therefore, using Exercise 3.11,

$$\begin{aligned} X' \in \mathcal{U}^J \text{ if and only if } \{s \in M^{I_1} : X|_s \in \mathcal{U}^{I_2}\} \in \mathcal{U}^{I_1} \\ \text{if and only if } X \in \mathcal{U}^{I_0}. \end{aligned}$$

□

Exercise 3.13 Use Lemma 3.12 to show that the definition of \mathcal{U}^* -equivalence above is unchanged if I_2 is replaced by any finite superset of I_2 (i.e., of $I_0 \cup I_1$) contained in I .

Exercise 3.14 Formulate and prove the Łoś theorem for \mathcal{M}^* , and use it to verify that ε is an elementary embedding.

Exercise 3.15 Prove property (1) of tight indiscernibles for \mathcal{M}^* and $(I, <)$.

It remains to prove properties (2) and (3) of tight indiscernibles for \mathcal{M}^* and $(I, <)$. Property (2) is easy to see since if $[f(i_0, \dots, i_k)] \in M^*$, then by the very manner in which functions symbols of $\mathcal{L}(\mathcal{M})$ are interpreted in \mathcal{M}^* , we have:

$$[f(i_0, \dots, i_k)] = f^{\mathcal{M}^*}(i_0, \dots, i_k).$$

Finally, to verify property (3) of tight indiscernibles for \mathcal{M}^* and $(I, <)$ suppose I_0 and I_1 are distinct finite subsets of I where $\max(I_0) < \min(I_1)$ of I_1 , and suppose that $f(I_0)$ and $f(I_1)$ are generalized terms such that $[f(I_0)] = [f(I_1)]$; we show that this common value is in M . Let us first consider the simple case that $I_0 = \{i_0\}$ and $I_1 = \{i_1\}$, with $i_0 < i_1$. Let

$$X = \left\{ \langle x, y \rangle \in M^2 : f^{\mathcal{M}}(x) = f^{\mathcal{M}}(y) \right\};$$

then from $[f(i_0)] = [f(i_1)]$ and the definition of \mathcal{U}^* -equivalence, $X \in \mathcal{U}^{(i_0, i_1)}$; thus,

$$\{x_0 \in M : X|x_{x_0} \in \mathcal{U}\} \in \mathcal{U}.$$

In particular, we may choose $x_0 \in M$ such that $X|x_{x_0} \in \mathcal{U}$. Let $r = f^{\mathcal{M}}(x_0)$; then $r \in M$ and for all $y \in X|x_{x_0}$, $f(y) = r$. Therefore by definition, $[f(I_1)] \in M$, so $m \in M$.

Enumerate I_0 as $i_1 < \dots < i_k$ and enumerate I_1 as $j_1 < \dots < j_k$, where $I_0, I_1 \subseteq I$ and $i_k < j_1$, and assume that f is a parametrically \mathcal{M} -definable function such that $f(i_1, \dots, i_k) = f(j_1, \dots, j_k)$. Let $I_2 = I_0 \cup I_1$ and let

$$X = \left\{ \langle x_1, \dots, x_k, y_1, \dots, y_k \rangle \in M^{I_2} : f(x_1, \dots, x_k) = f(y_1, \dots, y_k) \right\};$$

then by definition of \mathcal{U}^* -equivalence and the equality of $f(i_1, \dots, i_k)$ and $f(j_1, \dots, j_k)$, $X \in \mathcal{U}^{I_2}$. By Exercise 3.11 and the fact that each element of \mathcal{U}^{I_0} is non-empty, we may choose $s \in M^{I_0}$ such that $X|_s \in \mathcal{U}^{I_1}$. Let $r \in M$ such that $r = f(s)$. Thus

$$\left\{ \langle y_1, \dots, y_k \rangle \in M^{I_1} : f(y_1, \dots, y_k) = r \right\} \in \mathcal{U}^{I_1},$$

hence $[f(j_1, \dots, j_k)] = r \in M$. □

4 History and other applications

Skolem [30] introduced the definable ultrapower construction to exhibit a nonstandard model of arithmetic. Full ultraproducts/powers were invented by Łoś in his seminal paper [26]. The history of iterated ultrapowers is concisely summarized by Chang and Keisler in their canonical textbook [4, Historical Notes for 6.5] as follows:

Finite iterations of ultrapowers were developed by Frayne, Morel, and Scott [13]. The infinite iterations were introduced by Gaifman [14]. Our presentation is a simplification of Gaifman’s work. Gaifman used a category theoretic approach instead of a function which lives on a finite set. Independently, Kunen [25] developed iterated ultrapowers in essentially the same way as in this section, and generalized the construction even further to study models of set theory and measurable cardinals.

A thorough treatment of (full) iterated ultrapowers, including the many important results that are relegated to the exercises can be found in [4, Sec. 6.5]; e.g., [4, Exercise 6.5.27] asks the reader to verify that indiscernible sequences arise from the iterated ultrapower construction. In the general treatment of the subject, one is allowed to use a different ultrafilter (for the formation of the ultrapower) at different stages of the iteration process, but we decided to focus our attention in this paper on the conceptually simpler case, where the same ultrafilter is used at all stages of the iteration process. It should be noted, however, that the framework developed in this paper naturally lends

itself to a wider framework to handle iterations using different ultrafilters at each stage of the iteration.

Our treatment of iterated (Skolem) ultrapowers over amenable structures has two sources of inspiration: Kunen's aforementioned adaptation of iterated ultrapowers for models of set theory (where the ultrafilters at work are referred to as \mathcal{M} -ultrafilters⁸), and Gaifman's adaptation [15] of iterated ultrapowers for models of arithmetic. Gaifman's adaptation is recognized as a major tool in the model theory of Peano arithmetic; see [8, 9, 23] and [10] for applications and generalizations in this direction. The referee has also kindly reminded us of the fact that Gaifman's work has intimate links—at the conceptual, as well as the historical level—with the grand subdiscipline of model theory known as stability theory. For example, Definition 3.1 can be recast in the jargon of stability theory to assert that an ultrafilter \mathcal{U} on the parametrically \mathcal{M} -definable sets is amenable precisely when \mathcal{U} is a 'definable type' when viewed as a 1-type.⁹ According to Poizat [28, p. 237]:

Some trustworthy witnesses assert that the notion of definable types was not introduced in 1968 by Shelah, but by Haim Gaifman, in order to construct end extensions of models of arithmetic, see [15].

Furthermore, in his review of Gaifman's paper [15], Ressayre [29] writes:

There is a paradoxical link [between Gaifman's paper] with stable first-order theories: although the notion of definable type was introduced by Gaifman in the study of PA, which is the most unstable theory, this notion turned out to be a fundamental one for stable theories (...) I expect (i) that it will not be possible to "explain" this similarity by a (reasonable) common mathematical theory; and (ii) that this similarity is not superficial. Although they cannot be captured mathematically, such similarities do occur repeatedly and not by chance in the development of two opposite parts of logic, namely model theory in the algebraic style on one hand, and the theory (model, proof, recursion, and set theory) of the basic universes (e.g. arithmetic, analysis, \mathbb{V} , etc.) and their axiomatic systems on the other.

See also [15, Remark 0.1], describing the relationship between Gaifman's notion of 'end extension type' with the Harnik–Ressayre notion of 'minimal type'.

Iterated ultrapowers have proved indispensable in the study of inner models of large cardinals ever since Kunen's work [25]; see Jech's monograph [18, Chapter 19] for the basic applications, and Steel's lecture notes [31] for the state of the art. The applications of iterated ultrapowers in set theory are focused on well-founded models of set theory, however iterated ultrapowers are also an effective tool in the study of automorphisms of models of set theory (where well-founded models are of little interest since no

⁸ This terminology is not standard: Kunen's \mathcal{M} -ultrafilters are also referred to as *iterable* ultrafilters or as (*weakly*) *amenable* ultrafilters by other authors.

⁹ The referee has also commented that Shelah honed in on the property that every type is definable, and showed that a complete theory T is stable iff given any $\mathcal{M} \models T$, every ultrafilter over the parametrically \mathcal{M} -definable sets is amenable when viewed as a 1-type. However, it does not follow that every stable structure is amenable, as adding Skolem functions can destroy stability.

well-founded model of the axiom of extensionality has a nontrivial automorphism); see, e.g., [7, 11]. Iterated ultrapowers have also found many applications elsewhere, e.g., in the work of Hrbáček [17], and Kanovei and Shelah [19] on the foundations of nonstandard analysis. More recently, iterated ultrapowers have found new applications to the model theory of infinitary logic by Baldwin and Larson [2], and by Boney [3].

Acknowledgements We are grateful to Jim Schmerl, John Baldwin, and an anonymous referee for valuable feedback on earlier drafts of this paper. Matt Kaufmann thanks the Department of Philosophy, Linguistics and Theory of Science at the University of Gothenburg for its hospitality in Summer 2015.

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