Arch. Math. Logic (2016) 55:925–937
DOI 10.1007/s00153-016-0503-x

Mathematical Logic

Subgroups of $SF(\omega)$ and the relation of almost containedness

Barbara Majcher-Iwanow[1]

Received: 2 October 2015 / Accepted: 22 August 2016 / Published online: 6 September 2016

Abstract The relations of almost containedness and orthogonality in the lattice of groups of finitary permutations are studied in the paper. We define six cardinal numbers naturally corresponding to these relations by the standard scheme of $P(\omega)$. We obtain some consistency results concerning these numbers and some versions of the Ramsey theorem.

Keywords Subgroups of finitary permutations · van Douwen diagram

Mathematics Subject Classification 03E35 · 03E02

1 Preliminaries

1.1 Introduction

The paper is motivated by investigations of various versions of van Douwen's diagram, i.e. the set of relations between six cardinals referring to simple properties of almost disjointness and almost containedness, for example see [1–3,5,6,14,17]. The following theorem proved by Matet in [13] became one of the motivating results in this direction:

> Let $(\omega)^\omega$ be the set of all partitions of ω having infinitely many classes. Let $\leq$ be the order on $(\omega)^\omega$ defined by: $E_1 \leq E_2$ if E_2 is finer than E_1. Then assuming the continuum hypothesis there is a filter $F \subset (\omega)^\omega$ such that for every $(\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1)$-

✉ Barbara Majcher-Iwanow
bmajcher@math.uni.opole.pl

1 Institute of Mathematics and Informatics, University of Opole, ul.Oleska 48, 45-052 Opole, Poland

coloring $\delta : (\omega)^{\omega} \to 2$ there is a partition $E \in F$ such that δ is constant on the set of all infinite partitions coarser than E.

The statement is a variant (and a consequence) of the dualized version of Ramsey's theorem proved by T. Carlson and S. Simpson in [4]. The argument of Matet uses the observation that the tower cardinal (we denote it by $\mathfrak{t}_d$) for the ordering of infinite partitions is uncountable. Here $\mathfrak{t}_d$ is defined by the same scheme as $\mathfrak{t}$ for the lattice $P(\omega)$ of all subsets of ω (see [6]). Moreover it is proved in [13] that the tower cardinal for partitions is ω_1 in **ZFC** and it is proved in [5] that the size of a maximal almost orthogonal family of partitions must be 2^{ω}.

The lattice of partitions under the order reversing $\leq$ was studied in [2,11]. It is shown there that the corresponding cardinal invariants look differently: each of them, except $\mathfrak{a}$, is equal to the analogous cardinal in the lattice $P(\omega)$.

Note that this lattice can be also defined to be the lattice of 1-closed subgroups of $Sym(\omega)$ (i.e. the automorphism groups of structures with unary predicates only). Indeed, the corresponding isomorphism of these lattices maps a partition E to the group G_E of all permutations preserving the E-classes. On the other hand, for any E the group G_E is uniquely determined by the subgroup $G_E \cap SF(\omega)$ of the group $SF(\omega)$ of all finitary permutations of ω. The obtained embedding $E \to G_E \cap SF(\omega)$ maps almost trivial equivalence relations into the ideal IF of all finite subgroups of $SF(\omega)$.

This motivates further questions. For example it is interesting to find a variant of the result of Matet in the lattice of all subgroups of the group $SF(\omega)$. Since the corresponding tower cardinal is involved in this question, the general problem of description of the corresponding van Douwen's diagram for this lattice seems relevant. The paper is devoted to these questions.

1.2 Almost containedness

Van Douwen's diagrams are due to [6,17], where the case of $P(\omega)$ was considered. The term was used in [5] where the case of partitions was studied. The general idea can be described as follows.

Let L be a lattice with 0 and 1, and let I be an ideal of L. We say that $a, b \in L \backslash I$ are *orthogonal* if $a \wedge b \in I$. The element a is *almost contained* in b (we denote it by $a \leq_a b$) if $a \leq b \vee c$ for some $c \in I$. We write $a =_a b$ if $a \leq_a b$ and $b \leq_a a$. For any $a \in L$ we put $a_I = \{b : b =_a a\}$. It is clear that the relation $\leq_a$ becomes the usual almost containedness if we consider the lattice $(P(\omega), \subseteq)$ with respect to the ideal of finite subsets of ω.

In general, to characterize a lattice L under these relations we need some further notions. We say that *a splits b* if there are $c, d \leq b$ not in I such that $c \leq a$ and d, a are orthogonal. A family $\Gamma \subset L \backslash I$ is a *splitting family* if for every $b \in L \backslash I$ there exists $a \in \Gamma$ that splits b. We say that Γ is a *reaping family* if for each $a \in L \backslash I$ there is some $b \in \Gamma$ such that $b \leq_a a$ or a, b are orthogonal. We also define a family $\Gamma \subset L \backslash I$ to be *$\leq$-centered* if any finite intersection of its elements is not in I.

We can now associate to L the following cardinals. Define $\mathfrak{a}_I$ to be the least cardinality of an infinite maximal family of pairwise orthogonal elements from $L \backslash 1_I$. Let

$\mathfrak{p}_I$ be the least cardinality of a $\leq$-centered family Γ such that there is no $b \in L\setminus I$ such that b is a lower bound of Γ under $\leq_a$ and the family $\Gamma \cup \{b\}$ is still $\leq$-centered. Similarly, define $\mathfrak{t}_I$ (the tower cardinal) as the least cardinality of a $\leq_a$-decreasing $\leq$-centered chain without lower $\leq_a$-bound consistent (in the sense of $\leq$-centeredness) with the family. The cardinals $\mathfrak{s}_I$, $\mathfrak{r}_I$ are the corresponding (least) cardinals for splitting families and reaping families respectively. It is worth noting that $\mathfrak{p}_I$ and $\mathfrak{t}_I$ can be undefined (for example if for any $a \in L$ the set $\{b : b \leq a\}$ is finite). Also, (L, I) does not necessarily have a splitting family (for example if L is an atomic boolean algebra and I is trivial). So $\mathfrak{s}_I$ can be undefined too. On the other hand, it is clear that $\mathfrak{p}_I \leq \mathfrak{t}_I$ if they are defined.

The last cardinal $\mathfrak{h}_I$ is defined as follows. A family Σ of maximal families of pairwise orthogonal elements in $L\setminus 1_I$ is *shattering* if for every $a \in L\setminus I$ there are $\Gamma \in \Sigma$ and distinct $b, c \in \Gamma$ which are not orthogonal to a. Let $\mathfrak{h}_I$ be the least cardinality of a shattering family in L.

The following lemma seems to be folklore.

Lemma 1.1 *If $\mathfrak{s}_I$ is defined, then $\mathfrak{h}_I \leq \mathfrak{s}_I$.*

Proof Take a splitting family $\Gamma = \{c_\nu : \nu < \mathfrak{s}\}$. For each $\nu < \mathfrak{s}$ choose Ψ_ν to be a maximal family of pairwise orthogonal elements such that $c_\nu \in \Psi_\nu$. Let us check that the set of these families is shattering. Let $c \in L\setminus I$. Since Γ is a splitting family there is ν and $a, b \leq c$ such that $a \leq c_\nu$ and b is orthogonal to c_ν. By our construction there is $d \in \Psi_\nu$ not orthogonal to b. So Ψ_ν shatters c by c_ν and d. □

Remark In the case of the lattice $(P(\omega), \cup, \cap)$ and the ideal $[\omega]^{<\omega}$ of all finite subsets of ω the introduced numbers are exactly the classical cardinals $\mathfrak{a}, \mathfrak{h}, \mathfrak{p}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}$ (all of them occur in [17]). Indeed, our definitions of $\mathfrak{a}_I, \mathfrak{r}_I, \mathfrak{s}_I, \mathfrak{h}_I$ are formulated as the corresponding classical ones in [6] and in [17]. The classical $\mathfrak{t}$ is the least cardinality of a $\leq_a$-decreasing chain in $P(\omega)$ without $\leq_a$-bound. The classical $\mathfrak{p}$ is defined as follows. We say that a family $\Gamma \subseteq [\omega]^\omega$ is $\leq_a$-centered if every finite $\Gamma' \subseteq \Gamma$ has an infinite pseudointersection: a set $X \in [\omega]^\omega$ almost contained in each element of Γ'. Then the classical $\mathfrak{p}$ is the least cardinality of a $\leq_a$-centered family from $P(\omega)$ without lower $\leq_a$-bound. So, there is no assumption on $\subseteq$-centeredness as in the definitions of $\mathfrak{p}_I$ and $\mathfrak{t}_I$. On the other hand we do not need such assumptions because any $\leq_a$-centered family from $P(\omega)$ is centered. So $\mathfrak{p} = \mathfrak{p}_{[\omega]^{<\omega}}$ and $\mathfrak{t} = \mathfrak{t}_{[\omega]^{<\omega}}$. □

Note that the definitions of the above cardinals make sense if we consider $L/=_a$ under the reverse order $\geq_a$ replacing the ideal I by 1_I. In the case of $P(\omega)$ the converse cardinals are equal to the corresponding cardinals for $\subset$ because $P(\omega)$ is a Boolean algebra. The fact that this is not true in general is quite important for the lattice of subgroups of $SF(\omega)$ and for partitions.

In the latter case we can consider partitions as subsets of ω^2 under the inclusion (denoted by $\subset_{pairs}$). The lattice that we get (with operations $\vee_{pairs}$ and $\wedge_{pairs}$) is converse to the lattice $((\omega), \leq)$. Let IF be the ideal of partitions (in $((\omega), <_{pairs})$) obtained from id_ω by adding a finite set of pairs. Then the class 1_{IF} is exactly $(\omega)^{<\omega}$. Note that cardinal invariants of this lattice are studied in papers [2,11]. On the other hand, the relation of almost containedness of partitions analyzed in [5,13] can be defined as follows:

$$Y \leq^* X \leftrightarrow (\exists Z \in IF)(X \subset_{pairs} Y \vee_{pairs} Z).$$

As a result we see that the cardinals $\mathfrak{a}_d, \mathfrak{p}_d, \mathfrak{t}_d, \mathfrak{h}_d, \mathfrak{s}_d, \mathfrak{r}_d$ examined in [5,13] are the converse cardinals for the pair $(((\omega), <_{pairs}), IF)$.

1.3 The lattice of subgroups of $SF(\omega)$

Let $SF(\omega)$ be the group of all finitary permutations of ω. This means that the elements of $SF(\omega)$ are exactly the permutations g with finite support, where $supp(g) = \{x : g(x) \neq x\}$. The algebraic structure of subgroups of $SF(\omega)$ is described in [15,16]. The aim of our paper is to study the van Douwen's invariants of the lattice of subgroups of $SF(\omega)$.

Throughout the paper LF is the lattice of all subgroups of $SF(\omega)$ and IF is the ideal of all finite subgroups. We say that G_1 and G_2 from $LF \backslash IF$ are *orthogonal* if their intersection is in IF. The group G_1 is *almost contained* in G_2 ($G_1 \leq_a G_2$) if G_1 is a subgroup of a group finitely generated over G_2 by elements of $SF(\omega)$. Let $SF(\omega)_{IF} = \{G \leq SF(\omega) : SF(\omega)$ is finitely generated over $G\}$. As in Sect. 1.2 we define the cardinal numbers $\mathfrak{a}_{SF}, \mathfrak{p}_{SF}, \mathfrak{t}_{SF}, \mathfrak{r}_{SF}, \mathfrak{h}_{SF}$ and $\mathfrak{s}_{SF}$. For example, $\mathfrak{a}_{SF}$ is the least cardinality of a maximal family of pairwise orthogonal elements from $LF \backslash SF(\omega)_{IF}$ and $\mathfrak{p}_{SF}$ is the least cardinality of a $\leq$-centered family of elements in $LF \backslash IF$ with no lower $\leq_a$-bound $\leq$-consistent (in the sense of $\leq$-centeredness) with the family.

We put a topology on LF in the following way. Let $H \leq SF(\omega)$ be a finite group and $A \subset \omega$ be a finite set containing the union of the supports of the elements of H. Let $[H, A]$ be the set of all subgroups of $SF(\omega)$ such that the groups they induce on A are equal to H (we think of H as a permutation group on A). The topology that we consider is defined by the base consisting of all sets $[H, A]$. This topology is metrizable: fix an enumeration $A_0, A_1, \ldots$ of all finite subsets of ω and define

$d(G_1, G_2) = \sum\{2^{-n}$: the groups induced by G_1 and G_2 on A_n are not the same$\}$.

Note that the space LF is complete. A function $\delta : LF \to n, n \in \omega$, is then called a Borel (respectively $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$) coloring if $\delta^{-1}(i)$ is Borel (respectively analytic or coanalytic) for every $i < n$ (where $n \in \omega$ is viewed as $\{0, \ldots, n-1\}$).

Consider the set LF_1 of all groups of the form $SF(\omega) \cap G$ where G is 1-closed. We identify elements of LF_1 with elements of $2^{\omega \times \omega}$ (the corresponding partitions). Then it is easily seen that the topology on LF_1 induced by the topology above becomes the restriction of the product topology on $2^{\omega \times \omega}$ where 2 is considered discrete. A theorem of T. Carlson and S. Simpson from [4] can be restated as follows: for every $(\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1)$-coloring $\delta : LF_1 \to 2$ there exists $G \in LF_1 \backslash SF(\omega)_{IF}$ such that δ is constant on the elements of $LF_1 \backslash SF(\omega)_{IF}$ containing G. The corresponding theorem for $P(\omega)$ proved by Galvin and Prikry in [7] is stated as follows: for every $(\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1)$-coloring (originally: Borel coloring; see Remark 2.6 in [4]) $\delta : P(\omega) \to 2$ there exists an infinite $A \in P(\omega)$ such that δ is constant on the set of all infinite subsets of A. It is shown in [4] that this theorem is a consequence of the Carlson–Simpson theorem.

2 The diagram in (LF, IF)

In this section we describe the diagram for the lattice (LF, IF). First we compare the coefficients of this diagram with their classical analogues.

2.1 Comparing (LF, IF) with $(\mathcal{P}(\omega), Fin)$

The following result specifies basic relations between the coefficients for (LF, IF) and the coefficients for $(P(\omega)/fin, \subseteq_*)$.

Proposition 2.1 *The following inequalities are true in* **ZFC**.

$$\mathfrak{t}_{SF} \leq \mathfrak{t},\ \mathfrak{p}_{SF} \leq \mathfrak{p},\ \mathfrak{r}_{SF} \leq \mathfrak{r},\ \mathfrak{s} \leq \mathfrak{s}_{SF}\ \textit{and}\ \mathfrak{h} \leq \mathfrak{h}_{SF}.$$

Proof Fix an arbitrary group $G^* \subseteq SF(\omega)$ generated by the set $\{\sigma_i : i \in \omega\}$ of pairwise disjoint cycles with consequtive prime orders, i.e. $|supp(\sigma_i)| = p_i$, for $i \in \omega$. Such a G^* is isomorphic to the direct sum of all $\mathbb{Z}/p_i\mathbb{Z}$, $i \in \omega$. Denote by $LF_{G^*} \leq LF$ the sublattice of all elements below G^*. To each $A \subset \omega$ we associate the subgroup $G_A < G^*$ generated by all σ_i with $i \in A$. It is clear that the map $A \to G_A$ induces an isomorphism from $(\mathcal{P}(\omega)/Fin, \subseteq_a)$ to $(LF_{G^*}/=_a, \leq_a)$.

Now the first two inequalities are obvious.

To see that $\mathfrak{r}_{SF} \leq \mathfrak{r}$, observe that for any family $\mathcal{R} \subseteq LF_{G^*}$ reaping for LF_{G^*}, the family $\mathcal{R} \cup \{G^*\}$ is reaping for LF.

To prove $\mathfrak{s} \leq \mathfrak{s}_{SF}$, note that for any family $\mathcal{S} \subseteq LF$ splitting for LF, the family $\{K \cap G^* : K \in \mathcal{S}\} \setminus IF$ is splitting for LF_{G^*}.

For $\mathfrak{h} \leq \mathfrak{h}_{SF}$ we argue as follows. If $\mathcal{A}$ is a maximal family of pairwise orthogonal elements from $(LF \setminus SF(\omega)_{IF})$, then the family $\mathcal{A}_{G^*} = \{K \cap G^* : K \in \mathcal{A}\} \setminus IF$ is maximal for LF_{G^*}, although it is not necessarily infinite. Nevertheless, for any family $\mathcal{H}$ shattering for LF, the family $\mathcal{H}_{G^*} = \{\mathcal{A}_{G^*} : \mathcal{A} \in \mathcal{H}, |\mathcal{A}_{G^*}| > 1\}$ is nonempty.

For every finite $\mathcal{B} \in \mathcal{H}_{G^*}$, where $\mathcal{B} = \{H_0, H_1, \ldots, H_{n-1}, H_n\}$, choose an arbitrary infinite maximal pairwise orthogonal family $\mathcal{B}' \subseteq LF_{G^*}$ such that $\{H_0, H_1, \ldots, H_{n-1}\} \subseteq \mathcal{B}'$. It is easy to see that the family

$$\{\mathcal{B} \in \mathcal{H}_{G^*} : \mathcal{B} \text{ is infinite}\} \cup \{\mathcal{B}' : \mathcal{B} \in \mathcal{H}_{G^*} \text{ and } \mathcal{B} \text{ is finite}\}$$

is shattering for LF_{G^*}. □

The next result reduces the first two inequalities to the following equality.

Proposition 2.2

$$\mathfrak{p}_{SF} = \mathfrak{p} = \mathfrak{t}_{SF}.$$

Proof Fix an arbitrary one-to-one enumeration $SF(\omega) = \{\rho_i : i \in \omega\}$.

We have to prove $\mathfrak{p} \leq \mathfrak{p}_{SF}$. Take a $\leq$-centered family $\Gamma \subset LF \setminus IF$ of cardinality $\mathfrak{p}_{SF}$ without a $\leq_a$- bound $H \in LF \setminus IF$ such that $\Gamma \cup H$ is still $\leq$-centered. Then to every $G \in \Gamma$ assign the set $A_G = \{i : \rho_i \in G\}$. It is obvious that $\{A_G : G \in \Gamma\}$ is a $\subseteq$–centered family of subsets. We claim that it does not have an infinite $\subseteq^*$-bound A such, that the family $\{A_G : G \in \Gamma\} \cup \{A\}$ is $\subseteq$-centered. Suppose the contrary and let A be such a bound. Then the group G_A generated by the set $\{\rho_i : i \in A\}$ is almost contained in every $G \in \Gamma$ and the family $\Gamma \cup \{G_A\}$ is $\leq$-centered. This contradicts the assumption. Therefore, $\mathfrak{p} \leq \mathfrak{p}_{SF}$.

Now it suffices to use the recent theorem of Melliaris and Shelah that $\mathfrak{p} = \mathfrak{t}$ (see [12]) and Proposition 2.1 to complete the proof. □

2.2 The diagram

We need a couple of lemmas.

Lemma 2.3 *Let $G \in LF\backslash IF$ and $m \in \omega$. Then:*

(i) there exists a non-trivial $\rho \in G$ such that $supp(\rho) \cap m = \emptyset$,
(ii) moreover, for any $H \subset Sym(m)$ and any sequence $G_0, G_1, \ldots, G_n \in LF\backslash IF$ of groups orthogonal to G the above ρ can be chosen so that additionally $\langle H, \rho\rangle \cap G_i = \langle H\rangle \cap G_i$, for $i \leq n$.

Proof (i). Suppose that the lemma is not true. Choose a minimal $A = \{a_0, \ldots, a_k\} \subset m$ such that there are infinitely many $g \in G$ satisfying $m \cap supp(g) = A$. Then $A \neq \emptyset$. We fix some non-trivial g_0 with that property and consider all tuples $g(\bar{a}) = (g(a_0), \ldots, g(a_k))$ for the above g's. If each of these tuples has non-empty intersection with $supp(g_0)$, then there is $i \leq k$ such that $g(a_i)$ is the same for infinitely many g's. Clearly, for such g and g' the set $m \cap supp(g^{-1} \cdot g')$ is a subset of $A\backslash\{a_i\}$. This contradicts the minimality of A. When A is a singleton, the latter set is empty and statement (i) is satisfied.

Thus we can choose g as above with $g(\bar{a}) \cap supp(g_0) = \emptyset$ additionally. It is easily seen that $g^{-1} \cdot g_0 \cdot g$ fixes m pointwise. This contradicts our assumption.

(ii). Suppose the contrary. By (i) we can find $i \leq n$ such that for infinitely many $\rho \in G$ with $supp(\rho) \cap m = \emptyset$ there is $g \in \langle H\rangle$ satisfying $g \cdot \rho \in G_i$. Since $\langle H\rangle$ is finite, there is $g_0 \in \langle H\rangle$ such that $g_0 \cdot \rho \in G_i$ for infinitely many $\rho \in G$. Hence, for infinitely many $\rho, \rho' \in G$, $\rho^{-1} \cdot \rho' \in G_i$, which contradicts orthogonality. □

As a consequence of the above lemma we get the following easy statement which is also a corollary of Proposition 2.2.

Lemma 2.4 *For any countable sequence $G_0 > G_1 > \ldots$ of elements of $LF\backslash IF$ there is a group $G \in LF\backslash IF$ such that $G \leq_a G_i$, for every $i \in \omega$, and the family $\{G_i : i \in \omega\} \cup \{G\}$ is $\leq$-centered.*

Proof Assume we have a decreasing sequence $G_0 > G_1 > \ldots$ in $LF\backslash IF$. For every $i \in \omega$ choose non-trivial $g_i \in G_i$ such that $supp(g_i)$ is disjoint from the supports of the previous elements. We can do this by Lemma 2.3(i). Let G be the group generated by all these g_i. Then $G \in LF\backslash IF$, the family $\{G\} \cup \{G_i : i \in \omega\}$ is centered and $G \leq_a G_i$, for every $i \in \omega$. □

Lemma 2.5 *Let $G_0, \dots, G_{n-1}$ be a sequence of infinite groups from LF not a-equivalent to $SF(\omega)$. Then for any $k, m \in \omega$, $k > 0$, and $H \subset Sym(m)$ there is a non-trivial finitary permutation ρ consisting of $(k+1)$-cycles such that $supp(\rho) \subset \omega \backslash m$ and for every $i < n$,*

$$\langle H, \rho \rangle \cap G_i = \langle H \rangle \cap G_i .$$

Proof For each $i < n$ set

$$S_i = \{g \in G_i : \exists g_0, g_1 (g_0 \in \langle H \rangle \wedge (m \cap supp(g_1) = \emptyset) \wedge (g = g_0 \cdot g_1))\}$$

It is easily seen that each S_i is a group. Choose a family $\{D_{j0} : 0 \leq j < n\}$ of pairwise disjoint finite sets such that for every j, $D_{j0} \subset \omega \backslash m$ and S_j does not induce $Sym(D_{j0})$. Let $D_{j1}, \dots, D_{jk}$ be sets from $\omega \backslash m$ of the same size as D_{j0}. We may assume that every pair from $\{D_{ji} : 0 \leq i \leq k; j < n\}$ has empty intersection. For every $0 < i \leq k$ and $j < n$ we choose a bijection f_{ji} from D_{j0} onto D_{ji} such that it is not induced by any element of S_j. The existence of such f_{ji} is a consequence of the fact that for any bijections $f, g : D_{j0} \to D_{ji}$ induced by S_j, the bijection $g^{-1} \cdot f$ defines a permutation on D_{j0} induced by S_j.

We now define a permutation ρ with the support $\bigcup \{D_{ji} : 0 \leq i \leq k, 0 \leq j < n\}$ as follows. If $x \in D_{ji}, 0 < i < k$, then $\rho(x) = f_{j(i+1)}(f_{ji}^{-1}(x))$. If $x \in D_{j0}$, then $\rho(x) = f_{j1}(x)$. For $x \in D_{jk}$ we put $\rho(x) = f_{jk}^{-1}(x)$. Let us check that ρ satisfies the conclusion of the lemma. It is clear that ρ consists of cycles of length $k+1$. Suppose, that for some $g_0 \in \langle H \rangle$ the element $g = g_0 \cdot \rho^l, 0 < l \leq k$, is contained in some G_j. Thus $g \in S_j$ and, by our construction, g maps D_{j0} onto D_{jl} by f_{jl}. Since S_j does not induce f_{jl}, we have a contradiction. □

The van Douwen cardinals for (LF, IF) are described in the following theorem.

Theorem 2.6 *(i) The following inequalities are true in (LF, IF):*

$$\omega_1 \leq \mathfrak{p}_{SF} = \mathfrak{t}_{SF} \leq \mathfrak{h}_{SF} \leq \mathfrak{s}_{SF} \leq 2^{\omega},$$
$$\omega_1 \leq \mathfrak{a}_{SF}, \mathfrak{r}_{SF} \leq 2^{\omega};$$

(ii) All the coefficients are equal to continuum under Martin's Axiom;
*(iii) Each of the following equalities is consistent with $\{$**ZFC** $+ \omega_1 < 2^{\omega}\}$:*

$$\mathfrak{a}_{SF} = \omega_1, \mathfrak{s}_{SF} = \omega_1, \mathfrak{p}_{SF} = \mathfrak{t}_{SF} = \mathfrak{r}_{SF} = \omega_1.$$

Proof (i). The inequality $\mathfrak{h}_{SF} \leq \mathfrak{s}_{SF}$ is shown in Lemma 1.1. The inequality $\omega_1 \leq \mathfrak{p}_{SF}$ follows from Proposition 2.2 and $\mathfrak{t}_{SF} \leq \mathfrak{h}_{SF}$ is a consequence of Propositions 2.1 and 2.2 together with the classical inequality $\mathfrak{t} \leq \mathfrak{h}$.

To prove $\omega_1 \leq \mathfrak{r}_{SF}$, it suffices to show that if a family $\Psi \subseteq LF \backslash IF$ is countable then there exists $G \in LF \backslash IF$ such that for every $G' \in \Psi$ the groups G, G' are not orthogonal and $G' \not\leq_a G$. Let $\{G_0, G_1, ...\}$ be an enumeration of Ψ. Assume that each member of Ψ occurs infinitely often. We construct two sequences $g_0, g_1, \dots$ and

$h_0, h_1, \ldots$ of finitary permutations with pairwise disjoint supports such that for all $i, j \in \omega$ we have $supp(g_i) \cap supp(h_j) = \emptyset$ and $g_i, h_i \in G_i$. It is easily seen that Lemma 2.3(i) implies the existence of such sequences. Let $\hat{G}_1 = \langle \{g_i : i \in \omega\} \rangle$ and $\hat{G}_2 = \langle \{h_i : i \in \omega\} \rangle$. Clearly, $\hat{G}_1$ and $\hat{G}_2$ are orthogonal but they are not orthogonal to any G_i (since each member of Ψ is enumerated infinitely often). Now it is easy to see that $G = \hat{G}_1$ satisfies the conditions that we need.

To prove the inequality $\omega_1 \leq \mathfrak{a}_{SF}$ take a countable $\Psi \subset LF \backslash SF(\omega)_{IF}$. We construct a group G by induction. Fix an enumeration of Ψ: $G_0, G_1, \ldots$. Let H be the set of the elements which have been constructed at the first $n-1$ steps. At the n-th step we choose a permutation ρ as in Lemma 2.5 with respect to $G_0, \ldots, G_n$ and m large enough. It is easily seen that that the group generated by this sequence is orthogonal to any group from Ψ.

(ii). Assume **MA**. By Propositions 2.1, 2.2 and the classical result **MA** $\models \mathfrak{p} = 2^\omega$ we have

$$\mathfrak{p}_{SF} = \mathfrak{t}_{SF} = \mathfrak{s}_{SF} = \mathfrak{h}_{SF} = 2^\omega.$$

To prove $\mathfrak{r}_{SF} = 2^\omega$ we introduce a *ccc* forcing notion $\mathbf{P}_r$ as follows. Consider the family of all pairs (H, H') where $H, H' \subset SF(\omega)$ are finite and the supports of any two elements of $H \cup H'$ have empty intersection. The order is defined as follows $(H, H') \leq (F, F')$ iff $F \subseteq H$ and $F' \subseteq H'$. Let $\Psi \subset LF \backslash IF$ have cardinality $< 2^\omega$. For any $k \in \omega$ and $G \in \Psi$ the family

$$\{(H, H') \in \mathbf{P}_r : k < |H' \cap G|, k < |H \cap G|\}$$

is dense in $\mathbf{P}_r$ by Lemma 2.3(i) (see also the previous part of the proof). For a generic Φ define $G_0 = \langle \bigcup \{H : (H, H') \in \Phi\} \rangle$. It is easy to see that for any $G \in \Psi$, the groups G and G_0 are not orthogonal and G is not contained in G_0 under $\leq_a$. Thus Ψ is not reaping.

To show $\mathfrak{a}_{SF} = 2^\omega$, given an infinite family $\Gamma \subset LF$ of infinite groups define a forcing notion $\mathbf{P}_a$ as follows. Let $\mathbf{P}_a$ be the set of all pairs (H, F) where F is a finite subset of Γ and H is a finite set of permutations such that their supports are pairwise disjoint. We define $(H, F) \leq (H', F')$ iff $H' \subset H$, $F' \subset F$ and each $h \in \langle H \rangle \backslash \langle H' \rangle$ is not contained in any $G \in F'$. It is easily verified that $\mathbf{P}_a$ is a *ccc* forcing notion.

Consider $\mathbf{P}_a$ with respect to $\Psi \subset LF \backslash SF(\omega)_{IF}$ of cardinality $< 2^\omega$. Clearly, the following sets are dense in $\mathbf{P}_a$ (apply Lemma 2.5 in the second case):

$$\Sigma_G = \{(H, F) : G \in F\}, G \in \Psi, \text{ and}$$
$$\Sigma_l = \{(H, F) : \text{ the number of the elements of } H \text{ is greater than } l\}, l \in \omega.$$

By **MA** we have a filter $\Phi \subset \mathbf{P}_a$ meeting all these Σ's. It is easy to see that the group $G_0 = \langle \bigcup \{H : (H, F) \in \Phi\} \rangle$ is orthogonal to any group from Ψ.

(iii). Using *(i)*, Proposition 2.1 and the classical result that

$$Con\left(\mathbf{ZFC} + \omega_1 < 2^\omega + \mathfrak{t} = \mathfrak{r} = \omega_1\right)$$

we have

$$Con\left(\mathbf{ZFC} + (\mathfrak{p}_{SF} = \mathfrak{t}_{SF} = \mathfrak{r}_{SF} = \omega_1 < 2^{\omega})\right).$$

To prove $Con(\mathbf{ZFC} + \mathfrak{a}_{SF} = \omega_1 < 2^{\omega})$, we start with an arbitrary countable family $\Psi_0 \subset LF \backslash SF(\omega)_{IF}$ of parwise orthogonal groups. Take a sequence

$$\Psi_0 \subset \Psi_1 \subset \cdots \subset \Psi_\gamma \subset \ldots, \gamma < \omega_1,$$

by a finite support iteration

$$(\mathbf{P}_\gamma, Q_\gamma : \gamma < \omega_1)$$

of the forcing $\mathbf{P}_a$ (from the previous part of the proof) applied to potential Ψ_γ's. The canonical name for $\mathbf{P}_\gamma$ of $\Psi_{\gamma+1}$ is obtained from the canonical name of Ψ_γ by adding the canonical name of the group G_γ defined by Q_γ as G_0 by $\mathbf{P}_a$ above. Let Φ be generic for $\mathbf{P}_{\omega_1}$ and Φ_γ be the corresponding restriction to $\mathbf{P}_\gamma$. It is easily seen that $\mathbf{P}_{\omega_1}$ fulfils the *ccc*. Since ω_1 is regular and each group G in $LF[\Phi]$ is defined by a countable set of finitary permutations, it is contained in some $LF[\Phi_\gamma]$. Suppose that some G is orthogonal to each group from Ψ_γ. Thus by Lemma 2.3(ii) every set

$$D_n = \{(H, F) \in Q_\gamma[\Phi_\gamma] : n < |H \cap G|\}$$

is dense in $Q_\gamma[\Phi_\gamma]$. So the group

$$G_\gamma = \left\langle \bigcup \{H : (H, F) \in \Phi_{\gamma+1}/\Phi_\gamma\}\right\rangle$$

is not orthogonal to G. This shows that the set $\bigcup\{\Psi_\gamma : \gamma < \omega_1\}$ is a maximal family of pairwise orthogonal groups in $LF[\Phi]$.

The case $Con(\mathbf{ZFC} + \mathfrak{s}_{SF} = \omega_1 < 2^{\omega})$ can be handled in a similar way - constructing G_γ we apply the forcing $\mathbf{P}_r$. □

We conjecture that $\mathfrak{h}_{SF} = \mathfrak{h}$, $\mathfrak{s}_{SF} = \mathfrak{s}$ and $\mathfrak{r}_{SF} = \mathfrak{r}$. Note that the corresponding equalities hold for the lattice of partitions under $\leq_{pairs}$ [2]. At the moment we cannot adapt the arguments of [2] to our case. The case of $\mathfrak{a}$ is also open.

3 Two variants of Matet's theorem

3.1 The first version of Matet's theorem

The following theorem is formulated for the context described in the previuos section.

The proof of the theorem of Matet stated in Section 1.1 (this is Proposition 8.1 from [13]) uses the Carlson-Simpson's theorem and Proposition 4.2 from [13] asserting that $\mathfrak{t}_d$ is uncountable. We will use the same strategy.

Theorem 3.1 *Assuming* **MA** *there is a filter $F \subset LF \setminus IF$ such that for every $(\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1)$-coloring $\delta : LF \to 2$ there is $G \in F$ such that δ is constant on the set of all infinite subgroups of G.*

Proof Let G^* be a group generated by an infinite family $\{\sigma_i : i \in \omega\}$ of finite permutations with pairwise disjoint supports and distinct prime orders. For example we can take the group described in the proof of Proposition 2.1. Then every $G \leq G^*$ is generated by a subset of the set $\{\sigma_i : i \in \omega\}$ and the lattice of all subgroups of G^* is isomorphic to $(P(\omega), \subseteq)$. We identify $G \leq G^*$ with the corresponding subset of ω. Notice that then the topology defined in Sect. 1.3, on $\{G : G \leq G^*\}$ becomes the product topology on 2^ω. Also, $\{G : G \leq G^*\}$ is a closed subset of LF.

We now use the strategy of Proposition 8.1 from [13]. Let $\langle \delta_\alpha : \alpha < 2^\omega \rangle$ be an enumeration of all $(\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1)$-colorings $\delta : LF \to 2$. We construct a descending tower of subgroups of G^*. Supposing that G_β, $\beta < \alpha$, have already been selected, use Theorem 2.6 (ii) to find $G_\alpha \leq G^*$ such that the family $\{G_\gamma : \gamma \leq \alpha\}$ is $\leq$-centered and $G_\alpha \leq_a G_\gamma$ for all $\gamma < \alpha$. By the Galvin–Prikry theorem ([7]) there is an infinite subset of the set of generators of G_α such that all its infinite subsets have the same color with respect to the coloring induced by δ_α. This shows that G_α can be chosen such that all its infinite subgroups have the same color with respect to δ_α.

Let F be the filter generated by the above tower. It follows from the construction that F satisfies the conditions of the theorem. □

3.2 Another version of Matet's theorem

As we noted in the introduction the lattice of partitions under the reverse order is a sublattice of LF. This suggests that in the lattice LF the most natural variant of the theorem of Matet cited there (Proposition 8.1 from [13]) is the folowing one.

Theorem 3.2 *Assuming the continnuum hypothesis there is an ideal $I \subset LF \setminus SF(\omega)_{IF}$ such that for every $(\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1)$-coloring $\delta : LF \to 2$ there is $G \in I$ such that δ is constant on the set of all supergroups of G which do not belong to $SF(\omega)_{IF}$.*

In the proof of the statement we shall apply the result below.

Lemma 3.3 *Let $P_1, \ldots, P_i, \ldots$ be a sequence of pairwise disjoint infinite subsets of ω defining a partition E_0 of ω. Let $G_0 = Aut(\omega, P_1, \ldots, P_i, \ldots) \cap SF(\omega)$ be the subgroup of $SF(\omega)$ corresponding to a 1-closed subgroup of $Sym(\omega)$ defined by P_i, $i \in \omega \setminus \{0\}$. Then any proper supergroup of G_0 has this form for a partition coarser than E_0.*

Proof Let g be a finitary permutation such that $g(a) = b \in P_j$, for $a \in P_i, i \neq j$. Let $a' \in P_i \setminus supp(g)$ and $b' \in P_j \setminus supp(g)$. Below we denote the transposition of x and y by (x, y). It is clear that the element $(a, a') \cdot g^{-1} \cdot (b, b') \cdot g \cdot (a, a')$ (which belongs to $\langle G_0, g \rangle$) is the transposition (a', b'). This yields that the group inducing $SF(P_i \cup P_j)$ and acting trivially on $\omega \setminus (P_i \cup P_j)$, is a subgroup of $\langle G_0, g \rangle$. The rest is clear. □

Proof of the theorem Let $P_1, \ldots, P_i, \ldots$ be a sequence of pairwise disjoint infinite subsets of ω defining a partition E_0 of ω. Then $G_0 = Aut(\omega, P_1, \ldots, P_i, \ldots) \cap SF(\omega)$ is the subgroup of $SF(\omega)$ obtained from the corresponding 1-closed subgroup of $Sym(\omega)$. By Lemma 3.3 any proper supergroup of G_0 has this form for a partition coarser than E_0.

We may now consider the set $L_0 = \{G : G_0 \leq G \leq SF(\omega)\}$ as a sublattice of partitions coarser than E_0. Notice that then the topology defined in Sect. 1.3, on L_0 becomes the product topology on $2^{\omega\times\omega}$. This follows from the fact that any finite permutation group (on a finite subset of ω) induced by a group G from L_0 is a finite 1-closed permutation group and can be identified with a partition induced by the partition corresponding to G. Moreover, it is easy to see that L_0 is closed in LF.

We now use the Matet's theorem. Take an ideal I_0 of L_0 provided by this theorem. Then I_0 generates an ideal of LF. This ideal works as I in the statement. □

4 Remarks

4.1 The dual diagram for LF

Theorem 3.2 suggests investigation of the reverse ordering of LF. Using Lemma 3.3 we get a result analogous to Proposition 2.1.

Proposition 4.1 *Let $\mathfrak{h}_d$, $\mathfrak{p}_d$, $\mathfrak{r}_d$, $\mathfrak{s}_d$, $\mathfrak{t}_d$ be dual cardinal invariants of the lattice of partitions defined by the scheme of Sect. 1.2 (defined as in [5]). Let us consider $LF/=_a$ with respect to the converse ordering $\geq_a$ and let $\mathfrak{h}^d_{SF}$, $\mathfrak{p}^d_{SF}$, $\mathfrak{r}^d_{SF}$, $\mathfrak{s}^d_{SF}$, $\mathfrak{t}^d_{SF}$ be the corresponding sequence of cardinal invariants defined with respect to $SF(\omega)_{IF}$ as an ideal of this converse lattice.*

Then $\mathfrak{h}_d \leq \mathfrak{h}^d_{SF}$, $\mathfrak{r}^d_{SF} \leq \mathfrak{r}_d$, $\mathfrak{s}_d \leq \mathfrak{s}^d_{SF}$ and $\mathfrak{t}^d_{SF} = \mathfrak{p}^d_{SF} = \omega_1$.

Proof We take any one-closed group defined by a partition into infinitely many classes and use Lemma 3.3 to argue as in the proof of Proposition 2.1 to obtain

$$\mathfrak{h}_d \leq \mathfrak{h}^d_{SF}, \mathfrak{r}^d_{SF} \leq \mathfrak{r}_d, \mathfrak{s}_d \leq \mathfrak{s}^d_{SF}, \mathfrak{t}^d_{SF} \leq \mathfrak{t}_d, \mathfrak{p}^d_{SF} \leq \mathfrak{p}_d.$$

Since $\mathfrak{p}_d = \mathfrak{t}^d_{SF} = \omega_1$ (see [13]), we have the statement of the lemma. □

Using this proposition and the material of papers [1,5,13,14] we obtain the following relations:

$$\mathfrak{r}^d_{SF} \leq \mathfrak{r}_d \leq min(\mathfrak{r}, \mathfrak{d}, non(\mathcal{M}), non(\mathcal{N}))$$
$$\text{and } max(cov(\mathcal{N}), cov(\mathcal{M}), \mathfrak{s}, \mathfrak{b}) \leq \mathfrak{s}_d \leq \mathfrak{s}^d_{SF}.$$

Moreover the following relations are consistent with **ZFC**:

$$\mathfrak{r}_d \leq add(\mathcal{M}), \mathfrak{r}_d > \mathfrak{b}, \mathfrak{s}_d > cof(\mathcal{M}), \mathfrak{s}_d \leq \mathfrak{r}, \mathfrak{s}_d < \mathfrak{d}.$$

We mention the following questions:

1. Is $\mathfrak{a}^d_{SF} = 2^\omega$?
2. What is relation between $\mathfrak{h}^d_{SF}$ and $\mathfrak{h}$?
3. Are the following relations consistent with **ZFC**:

$$\mathfrak{r}^d_{SF} > \mathfrak{b}, \mathfrak{s}^d_{SF} \leq \mathfrak{r}, \mathfrak{s}^d_{SF} < \mathfrak{d}?$$

4.2 Other remarks

Our results suggest the investigation of van Douwen's cardinals for the lattice of all closed subgroups of $Sym(\omega)$. The definition of the a-order in this case must be as follows: $G \leq_a G'$ iff there exists a finite set X of finitary permutations such that G is a subgroup of the closed group generated by G' and X. It is worth noting that some results of [9] can be interpreted in this vein for some converse coefficients (for example, see Observation 3.3 in [9]).

However, one can notice that the lattice of all closed subgroups admits several constructions which make some of the van Douwen's cardinals trivial. For example, the group $\mathbb{Z}$ with the natural action on itself can be considered as a closed subgroup of $Sym(\omega)$. It is clear that for every $n \in \omega$ no closed subgroups splits $n\mathbb{Z}$. So, $\mathfrak{s}_I$ is undefined. On the other hand it is worth noting that for every $n \in \omega$, $n\mathbb{Z} =_a Sym(\omega)$. Indeed, fixing some representatives a_i of all the orbits, add the transpositions of the pairs $a_i, a_i + n$. This induces all permutations on every orbit. Adding transpositions of some pairs from distinct orbits we get $Sym(\omega)$.

Another easy observation is that $\mathfrak{r}_I = 1$ in this case. Indeed, the Prüfer group $\mathbb{Z}(p^\infty)$ with the natural action on itself forms a reaping family.

It is interesting to compare the lattices that we consider here with the lattice $P(\omega)$ of all subsets of ω and the ideal of finite subsets. Since $=_a$ is a congruence of $P(\omega)$, the orthogonality of infinite a and b means the absence of c such that $c \leq_a a$ and $c \leq_a b$. So the van Douwen's cardinals can be defined only in terms of $\leq_a$ (and originally it was so). On the other hand, this does not hold in lattices of subgroups of $Sym(\omega)$. Indeed, let σ be a transposition of some pair in ω. Then $\mathbb{Z}^\sigma$ induces a closed subgroup of $Sym(\omega)$ which is a-equivalent to $\mathbb{Z}$ with the above action. Clearly, the intersection of these groups is trivial.

In the case of (LF, IF) the corresponding example is as follows. Let infinite $A, B, C \subset \omega$ define a partition of ω and R be a bijection between A and B. Let $E_0 = A^2 \cup B^2 \cup id_{C\times C}$ and $E_1 = R \cup id_{C\times C}$. It is easily seen that E_0 and E_1 are orthogonal equivalence relations, but $E_1 \leq_a E_0$. The groups $G_{E_0} \cap SF(\omega)$ and $G_{E_1} \cap SF(\omega)$ have the same properties.

Acknowledgments The author is grateful to the referee for helpful remarks.

References

1. Brendle, J.: Martin's axiom and the dual distributivity number. Math. Logic Q. **46**, 241–248 (2000)
2. Brendle, J., Zhang, Sh.: Converse dual cardinals. J. Symb. Logic **71**, 22–34 (2006)
3. Brendle, J.: Van Douwen's diagram for dense sets of rationals. Ann. Pure Appl. Logic **143**, 54–69 (2006)
4. Carlson, T.J., Simpson, S.G.: A dual form of Ramsey's theorem. Adv. Math. **53**, 265–290 (1984)
5. Cichoń, J., Krawczyk, A., Majcher-Iwanow, B., Węglorz, B.: Dualization of the van Douwen diagram. J. Symb. Logic **65**, 959–968 (2000)
6. van Dowen, E.K.: The integers and topology. In: Kunen, K., Vaughan, J. (eds.) Handbook of Set-Theoretic Topology, pp. 111–167. North-Holland, Amsterdam (1984)
7. Galvin, F., Prikry, K.: Borel sets and Ramsey's theorem. J. Symb. Logic **38**, 193–198 (1973)
8. Kunen, K.: An Introduction to Independence Proofs. North-Holland, Amsterdam (1980)
9. Macpherson, H.D., Neumann, P.M.: Subgroups of infinite symmertic groups. J. London Math. Soc. (2) **42**, 64–84 (1990)
10. Majcher, B.: Orthogonal partitions. Acta Univ. Carolinae. Math. et Phis. **31**, 59–63 (1990)
11. Majcher-Iwanow, B.: Cardinal invariants of the lattice of partitions. Comment. Mathem. Univ. Carol. **41**, 543–558 (2000)
12. Malliaris, M., Shelah, S.: Cofinality spectrum theorems in model theory, set theory and general topology. Journ. Amer. Math. Soc. **29**, 237–297 (2016)
13. Matet, P.: Partitions and filters. J. Symb. Logic **51**, 12–21 (1986)
14. Minami, H.: Around splitting and reaping for partitions of ω. Arch. Math. Logic **49**, 501–518 (2010)
15. Neumann, P.M.: The lawlessness of groups of finitary permutations. Arch. Math. **26**, 561–566 (1975)
16. Neumann, P.M.: The structure of finitary permutation groups. Arch. Math. **27**, 3–17 (1976)
17. Vaughan, J.: Small uncountable cardinals and topolgy. In: van Mill, J., Reed, G.M. (eds.) Open Problems in Topology, pp. 195–218. North-Holland, Amsterdam (1990)